POROSITIES AND DIMENSIONS OF MEASURES

JEAN-PIERRE ECKMANN[1], ESA JÄRVENPÄÄ[2], AND MAARIT JÄRVENPÄÄ[2]

University of Geneva, Departments of Physics and Mathematics,
1211 Geneva 4, Switzerland[1]
University of Jyväskylä, Department of Mathematics,
P.O. Box 35, FIN-40351 Jyväskylä, Finland[2]
Jean-Pierre.Eckmann@physics.unige.ch, esaj@math.jyu.fi, and amj@math.jyu.fi

ABSTRACT. We introduce a concept of porosity for measures and study relations between dimensions and porosities for two classes of measures: measures on $\mathbb{R}^n$ which satisfy the doubling condition and strongly porous measures on $\mathbb{R}$.

1. INTRODUCTION

The aim of this paper is to relate porosity, as it can be measured, to dimension. The requirement of obtaining information about experimentally measurable objects leads us to consider measures, or mass distributions, rather than sets. For sets a relation between porosity and dimension has been established by Mattila [M1] and Salli [S] using the following definition of porosity:

Definition 1.1. *The porosity of a set $A \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is defined by*

$$\operatorname{por}(A,x) = \liminf_{r \downarrow 0} \operatorname{por}(A,x,r) ,$$

where

$$\operatorname{por}(A,x,r) = \sup\{p \geq 0 : \text{ there is } z \in \mathbb{R}^n \text{ such that } B(z,pr) \subset B(x,r) \setminus A\} .$$

Here $B(x,r)$ is the closed ball with radius r and with centre at x. The porosity of $A \subset \mathbb{R}^n$ is

$$\operatorname{por}(A) = \inf\{\operatorname{por}(A,x) : x \in A\} .$$

Clearly $0 \leq \operatorname{por}(A,x,r) \leq \frac{1}{2}$ for $x \in A$, and so $0 \leq \operatorname{por}(A) \leq \frac{1}{2}$. The quantity $\operatorname{por}(A,x,r)$ gives the relative radius of the largest disk which fits into $B(x,r)$ and which does not intersect A. In this sense it gives the size of the biggest hole in A.

For Hausdorff dimension, $\dim_{\mathrm{H}}$, it is not difficult to see that there exists a function $d : (0,\frac{1}{2}] \to (0,1]$ such that $\dim_{\mathrm{H}}(A) \leq n - d(\operatorname{por}(A))$ for all $A \subset \mathbb{R}^n$ (see [S, Introduction]). However, this bound obtained using simple methods is very crude when the porosity is close to $\frac{1}{2}$. The following theorem by Salli [S] gives a better connection between dimensions and porosities for sets. For the definition of packing dimension, $\dim_{\mathrm{p}}$, see [M2, Chapter 5] or [Fa, Chapter 2].

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Theorem 1.2. *There is a non-decreasing function* $\Delta_n : [0, \frac{1}{2}] \to [0, 1]$ *satisfying*

$$\lim_{p \uparrow \frac{1}{2}} \Delta_n(p) \;=\; 1$$

such that

$$\dim_{\mathrm{p}}(A) \;\leq\; n - \Delta_n(\mathrm{por}(A)) \tag{1.1}$$

for all $A \subset \mathbb{R}^n$.

According to Theorem 1.2 the packing dimension of any set in $\mathbb{R}^n$ with porosity close to $\frac{1}{2}$ can be only a little bit bigger than $n - 1$. There is an explicit expression for the function Δ_n in [S]:

$$\Delta_n(p) \;=\; \max\{1 - \frac{c_n}{\log(1/(1-2p))}\,, 0\}$$

where c_n is a constant depending only on n. Salli also proved that this function gives the optimal convergence rate by constructing for all $\frac{1}{4} < p < \frac{1}{2}$ sets A_p with $\mathrm{por}(A_p) \geq p$ and $\dim_{\mathrm{p}}(A_p) \geq n - 1 + \frac{b_n}{\log(1/(1-2p))}$ for some constant $b_n < c_n$. Salli's proof works for box-counting dimension as well (for the definition see [M2, Chapter 5] or [Fa, Chapter 2]), but then one has to assume that $A \subset \mathbb{R}^n$ is uniformly porous in the following sense: there is $R > 0$ such that

$$\mathrm{por}(A, x, r) \geq p \text{ for all } x \in A \text{ and for all } 0 < r \leq R\,.$$

In an earlier work by Mattila [M1] the analogue of Theorem 1.2 was proved for Hausdorff dimension using different methods than those of Salli's.

In this paper we address the problem of studying analogues of Theorem 1.2 for measures. After introducing porosities of measures (see Definition 2.2) we prove that in $\mathbb{R}^n$ an analogue to Theorem 1.2 holds for measures which satisfy the doubling condition (see Definition 2.3). We also consider the class of strongly porous measures (see Proposition 5.2) in $\mathbb{R}$. This article is organized as follows. In addition to the necessary notation and definitions we discuss some basic properties of porosities and state our main theorem in Section 2. The next section is dedicated to the proof of the main results. In Section 4 we consider the role of the doubling condition and in the last section we study the situation in the real line.

2. NOTATION AND MAIN RESULTS

We define the quantities we are working with. We begin with the definitions of Hausdorff and packing dimensions for measures in terms of local dimensions:

Definition 2.1. *Let* μ *be a finite Borel measure on* $\mathbb{R}^n$. *The lower and upper local dimensions of* μ *at a point* $x \in \mathbb{R}^n$ *are*

$$\underline{d}(\mu, x) \;=\; \liminf_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r}$$

and

$$\overline{d}(\mu,x) = \limsup_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r} .$$

If $\underline{d}(\mu,x) = \overline{d}(\mu,x)$*, the common value is called the local dimension of* μ *at* x *and is denoted by* $d(\mu,x)$*. The Hausdorff and packing dimensions of* μ *are defined by*

$$\dim_{\mathrm{H}}(\mu) = \sup\{s \geq 0 \ : \ \underline{d}(\mu,x) \geq s \textit{ for } \mu\textit{-almost all } x \in \mathbb{R}^n\} \tag{2.1}$$

and

$$\dim_{\mathrm{p}}(\mu) = \sup\{s \geq 0 \ : \ \overline{d}(\mu,x) \geq s \textit{ for } \mu\textit{-almost all } x \in \mathbb{R}^n\} . \tag{2.2}$$

The local dimensions describe the power law behaviour of μ-measure of balls with small radius. For μ-almost all points the lower local dimension is at least $\dim_{\mathrm{H}}(\mu)$ and the upper one is at least $\dim_{\mathrm{p}}(\mu)$. Clearly $\dim_{\mathrm{H}}(\mu) \leq \dim_{\mathrm{p}}(\mu)$.

Remark. We will need the following equivalent definitions of Hausdorff and packing dimensions of measures in terms of dimensions of sets (see [Fa, Proposition 10.2]). In fact,

$$\dim_{\mathrm{H}}(\mu) = \inf\{\dim_{\mathrm{H}}(A) \ : \ A \text{ is a Borel set with } \mu(A) > 0\} \tag{2.3}$$

and

$$\dim_{\mathrm{p}}(\mu) = \inf\{\dim_{\mathrm{p}}(A) \ : \ A \text{ is a Borel set with } \mu(A) > 0\} . \tag{2.4}$$

The porosity of a finite Borel measure μ on $\mathbb{R}^n$ is defined using the following quantities: for $x \in \mathbb{R}^n$ and $r, \varepsilon > 0$ set

$$\begin{aligned} \operatorname{por}(\mu,x,r,\varepsilon) = \sup\{p \geq 0 \ : \ & \text{there is } z \in \mathbb{R}^n \text{ such that } B(z,pr) \subset B(x,r) \\ & \text{and } \mu(B(z,pr)) \leq \varepsilon\mu(B(x,r))\} . \end{aligned}$$

Definition 2.2. *Let* μ *be a finite Borel measure on* $\mathbb{R}^n$*. The porosity of* μ *at a point* $x \in \mathbb{R}^n$ *is defined by*

$$\operatorname{por}(\mu,x) = \lim_{\varepsilon\downarrow 0}\liminf_{r\downarrow 0} \operatorname{por}(\mu,x,r,\varepsilon) . \tag{2.5}$$

The porosity of μ *is*

$$\operatorname{por}(\mu) = \inf\{s \geq 0 : \operatorname{por}(\mu,x) \leq s \textit{ for } \mu\textit{-almost all } x \in \mathbb{R}^n\} . \tag{2.6}$$

In (2.5) the limit as $\varepsilon \downarrow 0$ exists since $\liminf_{r\downarrow 0} \operatorname{por}(\mu,x,r,\varepsilon)$ is non-decreasing and bounded.

Remark. 1. We show now that the porosity of a measure has the same upper bound than that of a set, that is, $\operatorname{por}(\mu) \leq \frac{1}{2}$ for all finite Borel measures μ on $\mathbb{R}^n$. By [C, (1.10)] for μ-almost all $x \in \mathbb{R}^n$ we have $\overline{d}(\mu,x) \leq n$ giving

$$\mu(B(x,r)) \geq r^{2n} \tag{2.7}$$

for all sufficiently small $r > 0$. Assume that there is such a point x with $\mathrm{por}(\mu, x) > \frac{1}{2}(1+\delta) > \frac{1}{2}$ for some $0 < \delta < 1$. Let $\varepsilon \leq \delta^{3n}$ be sufficiently small. Then for all sufficiently small $r > 0$ there is $z \in \mathbb{R}^n$ such that $B(z, \frac{r}{2}(1+\delta)) \subset B(x, r)$ and $\mu(B(z, \frac{r}{2}(1+\delta))) \leq \varepsilon\mu(B(x, r))$. Hence for all such r we have $\mu(B(x, \delta r)) \leq \varepsilon\mu(B(x, r))$. Iterating this k times we obtain for all positive integers k

$$\mu(B(x, \delta^k r)) \leq \varepsilon^k \mu(B(x, r)) .$$

From (2.7) we obtain

$$\delta^{k2n} r^{2n} \leq \varepsilon^k \mu(B(x, r)) ,$$

implying the contradiction

$$2n \geq \frac{k \log \varepsilon + \log \mu(B(x, r))}{k \log \delta + \log r} \xrightarrow[k \to \infty]{} \frac{\log \varepsilon}{\log \delta} \geq 3n .$$

Hence $\mathrm{por}(\mu, x) \leq \frac{1}{2}$ for μ-almost all $x \in \mathbb{R}^n$ giving the claim.

2. For sets it is obvious that if $\mathrm{por}(A) \geq p$ and $B \subset A$, then $\mathrm{por}(B) \geq p$. The corresponding property holds for finite Radon measures: if B is a Borel set with $\mu(B) > 0$, then $\mathrm{por}(\mu|_B, x) \geq \mathrm{por}(\mu, x)$ for μ-almost all $x \in B$. Indeed, according to the density point theorem [M2, Corollary 2.14] we have for μ-almost all $x \in B$ that $\mu(B(x, r) \cap B) \geq \frac{1}{2}\mu(B(x, r))$ for all sufficiently small $r > 0$. For all such x and r we have for all $\varepsilon, p > 0$ and $z \in \mathbb{R}^n$ with $B(z, pr) \subset B(x, r)$ and $\mu(B(z, pr)) \leq \varepsilon\mu(B(x, r))$ that

$$\mu|_B(B(z, pr)) \leq \mu(B(z, pr)) \leq \varepsilon\mu(B(x, r)) \leq 2\varepsilon\mu|_B(B(x, r)) .$$

This implies the claim.

We denote by $\mathrm{spt}(\mu)$ the support of μ which is the smallest closed set such that the complement of it has μ-measure zero. Clearly

$$\mathrm{por}(\mathrm{spt}(\mu)) \leq \mathrm{por}(\mu) .$$

As illustrated by the following examples this inequality can be strict. In fact, it is precisely this difference which makes the definition of porosity important for physical measurements because it allows to neglect systematically dust which is visible in porosities of sets but not in those of measures.

Example 1. Let δ_0 be the Dirac measure at the origin, that is, $\delta_0(A) = 1$ if $0 \in A$ and $\delta_0(A) = 0$ if $0 \notin A$. Let μ be the sum of δ_0 and the Lebesgue measure $\mathcal{L}^n$ restricted to $B(0, 1)$, that is, $\mu = \delta_0 + \mathcal{L}^n|_{B(0,1)}$. Clearly $\mathrm{por}(\mu, 0) = \frac{1}{2}$ and $\mathrm{por}(\mu, x) = 0$ for all $x \neq 0$ with $|x| < 1$. Thus $\mathrm{por}(\mu) = \frac{1}{2}$. However, $\mathrm{por}(\mathrm{spt}(\mu)) = \mathrm{por}(B(0, 1)) = 0$.

Example 2. Enumerate the rational numbers in the closed unit interval $[0, 1]$. Let δ_i be the Dirac measure on the i^{th} rational point x_i. Define $\mu = \sum_{i=1}^{\infty} 2^{-i}\delta_i$. Then $\mathrm{por}(\mu, x_i) = \frac{1}{2}$ for all i since for all ε there exists $r > 0$ such that all the rationals in the r-neighbourhood of x_i have bigger index than $k+i$ for a fixed positive integer k with $2^{-k} < \varepsilon$. Hence $\mathrm{por}(\mu) = \frac{1}{2}$. Clearly $\mathrm{por}(\mathrm{spt}(\mu)) = \mathrm{por}([0,1]) = 0$.

For all $x \in \mathbb{R}^n$ and $r > 0$ we have $\lim_{\varepsilon \downarrow 0} \mathrm{por}(\mu, x, r, \varepsilon) = \mathrm{por}(\mathrm{spt}(\mu), x, r)$. In particular,

$$\mathrm{por}(\mathrm{spt}(\mu), x) \;=\; \liminf_{r \downarrow 0} \lim_{\varepsilon \downarrow 0} \mathrm{por}(\mu, x, r, \varepsilon) \;.$$

Thus, changing the order of taking the limits in (2.5) gives the porosity of the support of the measure.

We will need later the following measurability property:

Remark. We will prove that for all $r > 0$ and $\varepsilon > 0$ the function $x \mapsto \mathrm{por}(\mu, x, r, \varepsilon)$ is upper semi-continuous, that is,

$$\mathrm{por}(\mu, x, r, \varepsilon) \geq \limsup_{i \to \infty} \mathrm{por}(\mu, x_i, r, \varepsilon) \tag{2.8}$$

whenever $x_i \in \mathbb{R}^n$ are such that $\lim_{i \to \infty} x_i = x$. We use the notations $p_i = \mathrm{por}(\mu, x_i, r, \varepsilon)$ and $p = \limsup_{i \to \infty} p_i$. Let $\delta > 0$. For all i there exists $z_i \in \mathbb{R}^n$ such that $B(z_i, (p_i - \frac{\delta}{2})r) \subset B(x_i, r)$ and

$$\mu(B(z_i, (p_i - \frac{\delta}{2})r)) \leq \varepsilon \mu(B(x_i, r)) \;. \tag{2.9}$$

By choosing i so large that $|x - x_i| \leq \frac{\delta r}{2}$ we have $B(z_i, (p_i - \delta)r) \subset B(x, r)$. Further, taking suitable subsequences we may assume that the sequence $B(z_i, (p_i - \delta)r)$ converges with respect to the Hausdorff metric in the space of compact subsets of $\mathbb{R}^n$ (see [R, Chapter 2.6]) and $p = \lim_{i \to \infty} p_i$. Then there is $z \in \mathbb{R}^n$ such that $B(z, (p - 2\delta)r) \subset \cap_i B(z_i, (p_i - \delta)r) \subset B(x, r)$. Since the function $x \mapsto \mu(B(x, r))$ is upper semi-continuous [M2, Remark 2.10], we obtain from (2.9)

$$\varepsilon\mu(B(x,r)) \geq \limsup_{i \to \infty} \varepsilon\mu(B(x_i, r)) \geq \limsup_{i \to \infty} \mu(B(z_i, (p_i - \delta)r)) \geq \mu(B(z, (p - 2\delta)r)) \;.$$

Thus $\mathrm{por}(\mu, x, r, \varepsilon) \geq p - 2\delta$. Since $\delta > 0$ is arbitrary, this implies (2.8).

We will consider the class of measures which satisfy the doubling condition:

Definition 2.3. *A finite Borel measure μ on $\mathbb{R}^n$ satisfies the doubling condition at a point $x \in \mathbb{R}^n$ if*

$$\limsup_{r \downarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} \;<\; \infty \;. \tag{2.10}$$

We say that μ satisfies the doubling condition if (2.10) holds for μ-almost all $x \in \mathbb{R}^n$.

Expressing Definition 2.2 in terms of porosities of sets, we will prove an analogue to Theorem 1.2 for measures that satisfy the doubling condition. We will also show that the doubling condition is necessary for the validity of the relationship between porosities of measures and sets. Using Theorem 1.2 we then obtain:

Theorem 2.4. *There is a non-decreasing function* $\Delta_n : [0, 1/2] \to [0, 1]$ *satisfying*

$$\lim_{p \uparrow \frac{1}{2}} \Delta_n(p) \;=\; 1$$

such that

$$\dim_{\mathrm{p}}(\mu) \;\leq\; n - \Delta_n(\mathrm{por}(\mu)) \tag{2.11}$$

for all finite Borel measures μ *on* $\mathbb{R}^n$ *that satisfy the doubling condition.*

Remark. 1. In Theorem 2.4 one can take the same function Δ_n as in Theorem 1.2.

2. From a practical point of view, the doubling condition is satisfied for recursively constructed physical measures. For example, in many physical applications there exist $a, b, s > 0$ such that

$$ar^s \leq \mu(B(x, r)) \leq br^s$$

for all $r > 0$ and $x \in \mathrm{spt}(\mu)$ which clearly implies the validity of the doubling condition.

If the porosity of a measure μ which satisfies the doubling condition is close to $\frac{1}{2}$, then according to Theorem 2.4 the packing dimension of μ is not much bigger than $n - 1$. One cannot expect that small porosity implies big dimension. This is illustrated by the following example.

Example 3. For all positive integers k and m there is a Borel probability measure μ on $\mathbb{R}$ such that $\dim_{\mathrm{p}}(\mu) = \frac{1}{k}$ and $\mathrm{por}(\mu) \leq \frac{1}{m}$.

Construction. Divide the closed unit interval [0, 1] into m^k subintervals of length m^{-k} and select m of them by taking every $(m^{k-1})^{\mathrm{th}}$ one. Define a Borel probability measure μ_1 by giving the same weight $\frac{1}{m}$ to each of these intervals. Continue by dividing the selected intervals into m^k subintervals of length m^{-2k} and choosing every $(m^{k-1})^{\mathrm{th}}$ of them. Define a Borel probability measure μ_2 by attaching the weight $\frac{1}{m^2}$ to each of these intervals and proceed in the same way. Then (μ_i) converges weakly to a Borel probability measure μ. Clearly $\dim_{\mathrm{p}}(\mu) = \frac{1}{k}$. It is not difficult to see that $\mathrm{por}(\mu) \leq \frac{1}{m}$. In fact, this construction is a simplified version of Example 4, and therefore we give no details here.

3. THE PROOF OF THEOREM 2.4

Let μ be a finite Borel measure on $\mathbb{R}^n$. In order to prove Theorem 2.4 we first prove that if μ satisfies the doubling condition then

$$\beta(\mu) \;\geq\; \mathrm{por}(\mu)\;, \tag{3.1}$$

where

$$\beta(\mu) \;=\; \sup\{\mathrm{por}(A) : A \text{ is a Borel set with } \mu(A) > 0\}$$

(see [MM]). We will obtain Theorem 2.4 as a consequence of (3.1) and Theorem 1.2. In Example 4 we will show that (3.1) does not necessarily hold if the doubling condition is violated. That construction also indicates that the existence of the local dimension does not guarantee that (3.1) holds.

Note that the inequality

$$\beta(\mu) \le \operatorname{por}(\mu) \tag{3.2}$$

holds for any finite Radon measure μ on $\mathbb{R}^n$. In fact, if this is not the case, there exists s such that $\operatorname{por}(\mu) < s < \beta(\mu)$. Using the density point theorem [M2, Corollary 2.14], we find a Borel set A with $\mu(A) > 0$ and $\operatorname{por}(A) > s$ such that

$$\lim_{r \downarrow 0} \frac{\mu(A \cap B(x,r))}{\mu(B(x,r))} = 1$$

for all $x \in A$. This means that for all $x \in A$ and $\varepsilon > 0$ we have $\operatorname{por}(A,x,r) > s$ and

$$\mu(A \cap B(x,r)) \ge (1-\varepsilon)\mu(B(x,r)) \tag{3.3}$$

for all sufficiently small $r > 0$. Hence for all such r there exists $z \in \mathbb{R}^n$ with $B(z,sr) \subset B(x,r) \setminus A$. By (3.3) this implies

$$\mu(B(z,sr)) \le \mu(B(x,r)) - \mu(B(x,r) \cap A) \le \varepsilon\mu(B(x,r))$$

giving $\operatorname{por}(\mu) \ge s$. Thus (3.2) holds.

While (3.2) is valid without assuming the doubling condition, it is needed for the opposite inequality:

Proposition 3.1. *Let μ be a finite Borel measure on $\mathbb{R}^n$. If μ satisfies the doubling condition, then*

$$\beta(\mu) \ge \operatorname{por}(\mu) .$$

In particular, $\beta(\mu) = \operatorname{por}(\mu)$ for all finite Radon measures μ on $\mathbb{R}^n$ satisfying the doubling condition.

Proof. Assume that $\beta(\mu) < \operatorname{por}(\mu)$. Let $s > 0$ and $\delta > 0$ be such that $\beta(\mu) < s - \delta < s < \operatorname{por}(\mu)$. Setting

$$A = \{x \in \operatorname{spt}(\mu) : \operatorname{por}(\mu,x) > s\} ,$$

we have $\mu(A) > 0$. Since $r \mapsto r \operatorname{por}(\mu,x,r,\varepsilon)$ is non-decreasing and $r \mapsto \frac{1}{r}$ is continuous, the lower limit in (2.5) does not change if r is restricted to positive rationals. Also the limit as ε goes to zero can be taken over rationals since $\liminf_{r\downarrow 0} \operatorname{por}(\mu,x,r,\varepsilon)$ is non-decreasing as a function of ε. Thus by (2.8) the function $x \mapsto \operatorname{por}(\mu,x)$ is Borel measurable, and so A is a Borel set.

For all positive and finite numbers C define

$$E_C = \{x \in \operatorname{spt}(\mu) : \mu(B(x,2r)) > C\mu(B(x,r)) \text{ for some } r > 0\} .$$

Using the monotonicity of the mapping $r \mapsto \mu(B(x,r))$ it is easy to see that the definition of E_C is not altered if r is restricted to positive rationals. Therefore the Borel measurability of the mapping $x \mapsto \mu(B(x,r))$ [M2, Remark 2.10] implies that E_C is a Borel set for all C. Since μ satisfies the doubling condition, there is a positive and finite number C such that $\mu(E_C) < \frac{\mu(A)}{2}$. Hence $\mu((\mathbb{R}^n \setminus E_C) \cap A) > \frac{\mu(A)}{2} > 0$.

Consider $x \in A$. For all sufficiently small $\varepsilon > 0$ and $r > 0$ we have $\text{por}(\mu, x, r, \varepsilon) > s$. Hence for all such r and ε, there is $z \in \mathbb{R}^n$ such that $B(z, sr) \subset B(x,r)$ and $\mu(B(z,sr)) \leq \varepsilon\mu(B(x,r))$. We will prove that

$$B(z,(s-\delta)r) \cap (\mathbb{R}^n \setminus E_C) \cap \text{spt}(\mu) \;=\; \emptyset\,. \tag{3.4}$$

This gives the claim, since the fact that

$$B(z,(s-\delta)r) \subset B(x,r) \setminus ((\mathbb{R}^n \setminus E_C) \cap A \cap \text{spt}(\mu))$$

implies

$$\text{por}((\mathbb{R}^n \setminus E_C) \cap A \cap \text{spt}(\mu), x) \;\geq\; s - \delta$$

giving $\beta(\mu) \geq s - \delta$ which is a contradiction.

To prove (3.4), we assume that there exists $y \in B(z,(s-\delta)r) \cap (\mathbb{R}^n \setminus E_C) \cap \text{spt}(\mu)$. Let n be a positive integer such that $2^{-n+1} \leq \delta$. Then

$$\begin{aligned}\mu(B(y,\delta r)) \;&\leq\; \mu(B(z,sr)) \;\leq\; \varepsilon\mu(B(x,r)) \;\leq\; \varepsilon\mu(B(y,2r))\\ &\leq\; \varepsilon C^n \mu(B(y,2^{-n+1}r)) \;\leq\; \varepsilon C^n \mu(B(y,\delta r))\,.\end{aligned}$$

This gives a contradiction because we can choose ε as small as we wish. □

Using (2.4) and Proposition 3.1 we can estimate both the packing dimensions and porosities of measures satisfying the doubling condition in terms of corresponding quantities of sets. This gives an easy way to prove Theorem 2.4 using Theorem 1.2:

Proof of Theorem 2.4. Let $\Delta_n : [0,\frac{1}{2}] \to [0,1]$ be as in Theorem 1.2. Consider a finite Borel measure μ on $\mathbb{R}^n$ which satisfies the doubling condition. Since $\beta(\mu) \geq \text{por}(\mu)$ by Proposition 3.1, we find for all $\delta > 0$ a Borel set $A \subset \mathbb{R}^n$ with $\mu(A) > 0$ such that $\text{por}(A) \geq \text{por}(\mu) - \delta$. Now (2.4) and Theorem 1.2 give

$$\dim_{\text{p}}(\mu) \leq \dim_{\text{p}}(A) \leq n - \Delta_n(\text{por}(A)) \leq n - \Delta_n(\text{por}(\mu) - \delta).$$

The claim follows using the continuity of the function Δ_n. □

4. THE ROLE OF THE DOUBLING CONDITION

In this section we show that Proposition 3.1 is not generally valid unless the measure μ satisfies the doubling condition.

Example 4. There exists a Borel probability measure μ on $\mathbb{R}$ with the following properties:

$$\beta(\mu) \;=\; 0\,, \tag{4.1}$$

$$\text{por}(\mu) \;=\; \frac{1}{3}\,, \tag{4.2}$$

$$\mu\Big(\Big\{x : \limsup_{r\downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty\Big\}\Big) \;=\; 0, \text{ and} \tag{4.3}$$

$$\dim_{\text{p}}(\mu) \;=\; 0\,. \tag{4.4}$$

Construction. For all $i = 1, 2, \dots$ we first define a Borel probability measure μ_i such that its restriction to any closed dyadic subinterval of the closed unit interval of length 2^{-i} is a constant multiple of Lebesgue measure. For $i = 1, 2, \dots$ let J_i be the set of all i-term sequences of integers 0 and 1 and let J_∞ be the corresponding set of infinite sequences, that is,

$$J_i = \{(j_1, j_2, \dots, j_i) : j_m \in \{0, 1\} \text{ for all } m = 1, \dots, i\}$$

and

$$J_\infty = \{(j_1, j_2, \dots) : j_m \in \{0, 1\} \text{ for all } m = 1, 2, \dots\}.$$

We denote by $I_{j_1 \dots j_i}$ the closed dyadic interval of length 2^{-i} whose left endpoint in binary representation is $0, j_1 j_2 \cdots j_i$. Let (p_i), $0 < p_i < 1$, be a decreasing sequence of real numbers tending to zero. The measure μ_i is defined by requiring that

$$\mu_i(I_{j_1 \dots j_i}) \; = \; \prod_{k=1}^{i} (1 - p_k)^{j_k} p_k^{1-j_k}$$

for all $(j_1, \dots, j_i) \in J_i$. It is easy to see that (μ_i) converges weakly to a Borel probability measure μ such that $\text{spt}(\mu) = [0, 1]$.

Equivalently one can think of the measure μ as the projection of a natural product measure on the code space. In fact, defining $\nu_k(\{0\}) = p_k$ and $\nu_k(\{1\}) = 1 - p_k$ for all $k = 1, 2, \dots$, the product measure $\prod_{k=1}^{\infty} \nu_k$ is a Borel probability measure on the code space J_∞ (equipped with the product topology) and the measure μ is its image under the projection $\pi : J_\infty \to [0, 1]$, where $\pi((j_1, j_2, \dots)) = \sum_{m=1}^{\infty} j_m 2^{-m}$, that is, the binary representation of a point in [0, 1].

The measure μ has the following property:

Lemma 4.1. *Let* $0 < \delta < 1$. *Given any* $\varepsilon > 0$ *the following property holds for all sufficiently large positive integers* k*: for all closed dyadic subintervals* $[x_1, x_2]$ *of the unit interval* [0, 1] *of length* 2^{-k} *we have*

$$\mu([x_1, x_2 - 2^{-k}\delta]) \; \le \; \varepsilon \mu([x_1, x_2]) \, . \tag{4.5}$$

Proof of Lemma 4.1. Consider the positive integer ℓ such that $2^{-\ell} \le \delta < 2^{-\ell+1}$. Since the interval $[x_1, x_2 - 2^{-k}\delta]$ is contained in the union of $2^\ell - 1$ closed dyadic subintervals of $[x_1, x_2]$ of length $2^{-k-\ell}$ and of measure at most $p_{k+1}\mu([x_1, x_2])$ (we take all subintervals of $[x_1, x_2]$ of length $2^{-k-\ell}$ except the right most one as covering sets) we have

$$\mu([x_1, x_2 - 2^{-k}\delta]) \le (2^\ell - 1) p_{k+1} \mu([x_1, x_2]).$$

Choosing k so large that $(2^\ell - 1) p_{k+1} \le \varepsilon$ gives the claim. □

Lemma 4.1 is essential when proving properties (4.1) – (4.4):

Proof of properties (4.1) – (4.4). For (4.1) we assume that $\beta(\mu) > 0$. Then there exist a positive integer k, a real number R with $0 < R < 1$, and $E \subset [0, 1]$ with $\mu(E) > 0$ such that for all $x \in E$ we have

$$\text{por}(E, x, r) \ge 2^{-k} \tag{4.6}$$

for all $0 < r \leq R$. Set $N = 2^{k+4}$. Let i_0 be a positive integer with $2^{-i_0} \leq 2^{-k-2}R$. We will first show that if i is a positive integer with $i \geq i_0$, then, given any family $\{D_1, \ldots, D_N\}$ of successive closed dyadic subintervals of [0, 1] of length 2^{-i}, there is $1 \leq j \leq N$ such that

$$E \cap D_j = \emptyset. \tag{4.7}$$

If this were not the case, then $D_j \cap E \neq \emptyset$ for all $j = 1, \ldots, N$. Let $M = N/2$. Consider $x \in E \cap D_M$ and set $r_i = 2^{2+k-i}$. Denote by d_1 the left-hand end-point of D_1 and by d_N the right-hand end-point of D_N. Now $|x - d_1| \geq (M-1)2^{-i}$, $|x - d_N| \geq (M-1)2^{-i}$, and $(M-1)2^{-i} \geq N2^{-i-2} = r_i$, and therefore we obtain

$$B(x, r_i) \subset \bigcup_{j=1}^{N} D_j.$$

Further, since $2 \cdot 2^{-i} < r_i \leq R$ and all dyadic intervals D_j meet E, we have

$$\text{por}(E, x, r_i) \leq \frac{2^{-i}}{r_i} = 2^{-k-2}$$

which contradicts (4.6). Thus (4.7) holds.

We complete the proof of (4.1) by showing that the property (4.7) implies that $\mu(E) = 0$. Set $\ell = k + 4$. We may assume that $i_0 = m\ell$ for some $m \in \mathbb{N}$. Denote by F the set of numbers in [0, 1] whose base two expansion does not contain the sequence $j_{n\ell} = 0, j_{n\ell+1} = 0, \ldots, j_{(n+1)\ell-1} = 0$ for any integer $n \geq m$. Let $i \geq i_0$. An N-block at stage i is a family of N successive closed dyadic subintervals of [0, 1] of length 2^{-i} which belong to the same dyadic interval of length $2^{-i+\ell}$ at stage $i - \ell$. By (4.7) in each of these N-blocks there is at least one interval which does not intersect E. Since the left-most interval of each N-block has the smallest measure, we have $\mu(E) \leq \mu(F)$. Further, choosing for all i

$$p_i = \frac{1}{\log(i+2)}$$

we have

$$\begin{aligned}
\mu(F) &\leq \prod_{j=0}^{\infty} \Big(1 - p_{j\ell+i_0} \cdot \ldots \cdot p_{(j+1)\ell+i_0-1}\Big) \\
&= \exp\Big(\sum_{j=0}^{\infty} \log(1 - p_{j\ell+i_0} \cdot \ldots \cdot p_{(j+1)\ell+i_0-1})\Big) \\
&\leq \exp\Big(-\frac{1}{2}\sum_{j=0}^{\infty} p_{j\ell+i_0} \cdot \ldots \cdot p_{(j+1)\ell+i_0-1}\Big) \\
&\leq \exp\Big(-\frac{1}{2}\sum_{j=0}^{\infty} \frac{1}{(\log((j+1)\ell+i_0+1))^{\ell}}\Big) = 0.
\end{aligned} \tag{4.8}$$

Hence (4.1) holds.

In order to prove (4.2) let $x \in [0,1]$ and $r > 0$. Consider a positive integer i such that $2^{-i} \le r < 2^{-i+1}$. Let D_i be a closed dyadic subinterval of $[0,1]$ of length 2^{-i+1} which contains x. We denote by D_i^{L} and D_i^{R} the neighbouring closed dyadic intervals of D_i of length 2^{-i+1} situated on left and right, respectively. The interval D_i is the union of four closed dyadic intervals of length 2^{-i-1}. Let a_i, b_i, c_i, d_i, and e_i be the end-points of these four intervals from left to right (see Figure 1). Then

$$\begin{aligned}
\mu([a_i,b_i]) &= p_i p_{i+1}\, \mu(D_i)\,, \\
\mu([b_i,c_i]) &= p_i(1-p_{i+1})\, \mu(D_i)\,, \\
\mu([c_i,d_i]) &= (1-p_i)p_{i+1}\, \mu(D_i)\,, \\
\mu([d_i,e_i]) &= (1-p_i)(1-p_{i+1})\, \mu(D_i)\,,
\end{aligned} \tag{4.9}$$

giving

$$\begin{aligned}
\frac{p_i^2}{2}\mu(D_i) \;&\le\; \mu([a_i,b_i]) \;\le\; p_i^2\mu(D_i)\,, \\
\frac{p_i}{2}\mu(D_i) \;&\le\; \mu([b_i,c_i]) \;\le\; p_i\mu(D_i)\,, \\
\frac{p_i}{4}\mu(D_i) \;&\le\; \mu([c_i,d_i]) \;\le\; p_i\mu(D_i)\,, \\
\frac{1}{4}\mu(D_i) \;&\le\; \mu([d_i,e_i]) \;\le\; \mu(D_i)\,.
\end{aligned} \tag{4.10}$$

(We can assume that i is big enough such that $p_i < \frac{1}{2}$.) The following concept of scaling is used to describe the behaviour of measures of dyadic intervals. If $D \subset [0,1]$ is a dyadic interval of length 2^{-i-1} we say that $\mu(D)$ scales like p_i^k for some integer k if there is a constant c independent of i such that

$$\frac{1}{c}p_i^k\mu(D_i) \le \mu(D) \le c p_i^k \mu(D_i).$$

In particular, (4.10) implies that $\mu([a_i,b_i])$ scales like p_i^2, $\mu([b_i,c_i])$ scales like p_i, $\mu([c_i,d_i])$ scales like p_i, and $\mu([d_i,e_i])$ scales like 1 (see Figure 1).

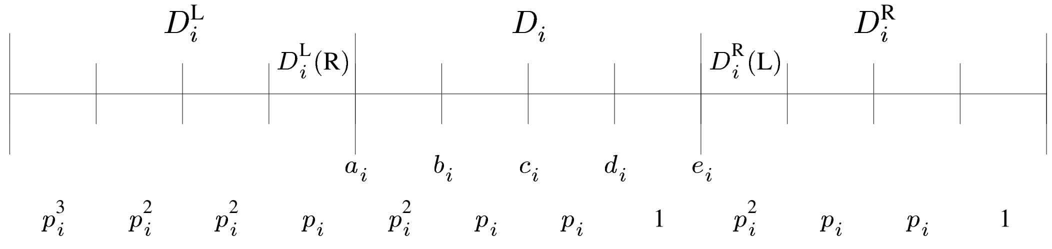

Figure 1: The scaling properties of the intervals.

We denote by $D_i^{\mathrm{R}}(\mathrm{L})$ the left-most closed subinterval of D_i^{R} of length 2^{-i-1}, and by $D_i^{\mathrm{L}}(\mathrm{R})$ the right-most closed subinterval of D_i^{L} of length 2^{-i-1} (see Figure 1). The length of the shortest possible dyadic interval containing either both $[a_i, b_i]$ and $D_i^{\mathrm{L}}(\mathrm{R})$ or both $[d_i, e_i]$ and $D_i^{\mathrm{R}}(\mathrm{L})$ is at least 2^{-i+2}. Let D be the shortest dyadic interval containing $[d_i, e_i]$ and $D_i^{\mathrm{R}}(\mathrm{L})$ and let 2^{-m}, $m \leq i-2$, be its length. Then $[d_i, e_i]$ is reached from D after stepping left at stage $m+1$ and then always right, and $D_i^{\mathrm{R}}(\mathrm{L})$ is reached after stepping first right at stage $m+1$ and after that always left, and so

$$\frac{\mu([d_i, e_i])}{\mu(D_i^{\mathrm{R}}(\mathrm{L}))} = \frac{p_{m+1}(1-p_{m+2}) \cdot \ldots \cdot (1-p_{i+1})}{(1-p_{m+1})p_{m+2} \cdot \ldots \cdot p_{i+1}}. \tag{4.11}$$

Similarly

$$\frac{\mu(D_i^{\mathrm{L}}(\mathrm{R}))}{\mu([a_i, b_i])} = \frac{p_{n+1}(1-p_{n+2}) \cdot \ldots \cdot (1-p_{i+1})}{(1-p_{n+1})p_{n+2} \cdot \ldots \cdot p_{i+1}}, \tag{4.12}$$

where $n \leq i-2$ is the biggest possible stage where $D_i^{\mathrm{L}}(\mathrm{R})$ and $[a_i, b_i]$ belong to the same dyadic interval.

We consider the case where we have the minimum relative weight for $D_i^{\mathrm{L}}(\mathrm{R})$. Other cases can be treated similarly. By (4.12) the relative weight of $D_i^{\mathrm{L}}(\mathrm{R})$ obtains the minimum for $n = i-2$. This means that $D_i^{\mathrm{L}}(\mathrm{R})$ and $[a_i, b_i]$ belong to the same dyadic interval of length 2^{-i+2}. Then (4.12) implies

$$\mu(D_i^{\mathrm{L}}(\mathrm{R})) = \frac{p_{i-1}(1-p_i)(1-p_{i+1})}{(1-p_{i-1})p_i p_{i+1}} \mu([a_i, b_i])$$

giving

$$\frac{1}{4} p_i \mu(D_i) \leq \mu(D_i^{\mathrm{L}}(\mathrm{R})) \leq 4 p_i \mu(D_i).$$

Hence $\mu(D_i^{\mathrm{L}}(\mathrm{R}))$ scales like p_i. As before we see that the other three closed dyadic subintervals of D_i^{L} of length 2^{-i-1} scale like p_i^3, p_i^2, and p_i^2 from left to right (see Figure 1).

Since $D_i^{\mathrm{L}}(\mathrm{R})$ and $[a_i, b_i]$ belong to the same dyadic interval at stage $i-2$, the stage m where $[d_i, e_i]$ and $D_i^{\mathrm{R}}(\mathrm{L})$ belong to the same dyadic interval cannot be bigger than $i-3$. Here we consider the case $m = i-3$. Again other cases are similar to this one. Using (4.11) for $m = i-3$ we obtain the maximum value for the relative weight of $D_i^{\mathrm{R}}(\mathrm{L})$

$$\mu(D_i^{\mathrm{R}}(\mathrm{L})) = \frac{(1-p_{i-2})p_{i-1}p_i p_{i+1}}{p_{i-2}(1-p_{i-1})(1-p_i)(1-p_{i+1})} \mu([d_i, e_i]).$$

This implies that the closed dyadic subintervals of D_i^{R} of length 2^{-i-1} scale like p_i^2, p_i, p_i, and 1 (see Figure 1).

Let $\delta, \varepsilon > 0$. We may assume that i is so large, that is, $[a_i, e_i]$ is so short, that Lemma 4.1 holds. Assume first that $a_i \in B(x, r)$. Since $r \geq 2^{-i}$ and $x \in [a_i, e_i]$, we obtain

$[a_i, c_i] \subset B(x, r)$. By Lemma 4.1 we have $\mu([a_i, c_i - 2^{-i}\delta]) \leq \varepsilon\mu([a_i, c_i]) \leq \varepsilon\mu(B(x, r))$. If $B(x, r) \subset (a_i - 2^{-i}, e_i)$, then $r \leq 3 \cdot 2^{-i-1}$, and so

$$\text{por}(\mu, x, r, \varepsilon) \geq \frac{\frac{1}{2}(c_i - a_i - 2^{-i}\delta)}{3 \cdot 2^{-i-1}} = \frac{1}{3}(1 - \delta). \tag{4.13}$$

If $B(x, r)$ is not contained in $(a_i - 2^{-i}, e_i)$, then either $e_i \in B(x, r)$ or $a_i - 2^{-i} \in B(x, r)$. Consider first the case where $e_i \in B(x, r)$. Then $[a_i, e_i] \subset B(x, r)$. Thus Lemma 4.1 implies that $\mu([a_i, e_i - 2^{-i+1}\delta]) \leq \varepsilon\mu(B(x, r))$ giving

$$\text{por}(\mu, x, r, \varepsilon) \geq \frac{\frac{1}{2}(e_i - a_i - 2^{-i+1}\delta)}{2^{-i+1}} = \frac{1}{2}(1 - \delta). \tag{4.14}$$

In the case where $a_i - 2^{-i} \in B(x, r)$ and $e_i \notin B(x, r)$ we have

$$\frac{x - a_i}{r} \leq \min\left\{1 - \frac{2^{-i}}{r}, \frac{2^{-i+1}}{r} - 1\right\} \leq \frac{1}{3}. \tag{4.15}$$

Using Lemma 4.1 and the previously mentioned scaling properties of μ we find a constant C independent of i and ε such that $\mu([x - r, a_i - 2^{-i+1}\delta]) \leq C\varepsilon\mu(B(x, r))$. From (4.15) it follows that

$$\text{por}(\mu, x, r, C\varepsilon) \geq \frac{\frac{1}{2}(a_i - 2^{-i+1}\delta - x + r)}{r} \geq \frac{1}{2} - \frac{x - a_i}{2r} - \frac{2^{-i}\delta}{r} \geq \frac{1}{3} - \delta. \tag{4.16}$$

Finally we consider the remaining case where $a_i \notin B(x, r)$. Then $e_i \in B(x, r)$, and so by Lemma 4.1 and the scaling properties of μ there is a constant C independent of i and ε such that $\mu([x - r, e_i - 2^{-i+1}\delta]) \leq C\varepsilon\mu(B(x, r))$. Since $x \leq e_i$ we obtain

$$\text{por}(\mu, x, r, C\varepsilon) \geq \frac{\frac{1}{2}(e_i - 2^{-i+1}\delta - x + r)}{r} \geq \frac{\frac{1}{2}(r - 2^{-i+1}\delta)}{r} \geq \frac{1}{2} - \delta. \tag{4.17}$$

By (4.13) – (4.17) we have $\text{por}(\mu, x) \geq \frac{1}{3}$ for all $x \in [0, 1]$, and so $\text{por}(\mu) \geq \frac{1}{3}$.

For the opposite inequality consider a sequence (δ_k) of positive real numbers tending to zero. Then for all k there is a positive integer m_k and a sequence $(1, 0, 0, 1, 1, \ldots, 1, 1) \in J_{m_k}$ such that if the base two expansion of a point $x \in D_i$ contains this sequence from the $(i-1)^{\text{th}}$ place, then D_i^{L} and D_i belong to the same dyadic subinterval of $[0, 1]$ of length 2^{-i-2} and $0 \leq b_i - x \leq 2^{-i-1}\delta_k$. For all k let A_k be the set of points $x \in [0, 1]$ whose base two expansion contains the sequence $(1, 0, 0, 1, 1, \ldots, 1, 1) \in J_{m_k}$ infinitely many times and let

$$B_k = \{x \in [0, 1] : \text{por}(\mu, x) \leq \frac{1}{3} + \delta_k\}.$$

Then $A_k \subset B_k$ for all k. In fact, as in (4.13) we see that for all $\varepsilon > 0$ and for all $x \in A_k$

$$\text{por}(\mu, x, 2^{-i-1}(3 - \delta_k), \varepsilon) \leq \frac{\frac{1}{2}(c_i - a_i)}{2^{-i-1}(3 - \delta_k)} \leq \frac{1}{3} + \delta_k$$

for all i large enough implying that $x \in B_k$. Further, as in (4.8) we obtain that $\mu(A_k) = 1$ for all k, giving $\mu(\cap_{k=1}^{\infty} B_k) = 1$. Hence (4.2) holds.

As in (4.8) it can be shown that μ-almost every point has the sequence 01 infinitely many times in its expansion. Thus μ-almost every point belongs for arbitrarily large positive integers i to the second one of the four dyadic subintervals of a dyadic interval of length 2^{-i+1}. This implies that $\mu(B(x, 2^{-i-1}))$ scales like p_i and $\mu(B(x, 3 \cdot 2^{-i-1}))$ scales like 1 giving

$$\limsup_{r \downarrow 0} \frac{\mu(B(x, 3r))}{\mu(B(x, r))} = \infty$$

for μ-almost all $x \in [0, 1]$. Thus (4.3) is proved.

It remains to show that (4.4) holds. By [C, Lemma 2.3] it is enough to prove that for μ-almost all $x \in [0, 1]$

$$\lim_{i \to \infty} \frac{1}{i} \log \mu(I_{j_1 \cdots j_i}(x)) = 0,$$

where $I_{j_1 \cdots j_i}(x)$ is the dyadic subinterval of [0, 1] of length 2^{-i} which contains x. Note that

$$\frac{1}{i} \log \mu(I_{j_1 \cdots j_i}(x)) = \frac{1}{i} \sum_{m=1}^{i} \big(\delta_{j_m,0} \log p_m + \delta_{j_m,1} \log(1 - p_m)\big) =: A_i,$$

where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$. Let Y_i be a random variable such that $Y_i = \log p_i$ with probability p_i and $Y_i = \log(1 - p_i)$ with probability $1 - p_i$. Then the expectation of Y_i is

$$E_i = p_i \log p_i + (1 - p_i) \log(1 - p_i).$$

Clearly the variance

$$V_i = p_i(\log p_i)^2 + (1 - p_i)(\log(1 - p_i))^2 - \big(p_i \log p_i + (1 - p_i) \log(1 - p_i)\big)^2$$

goes to zero as i tends to infinity, and so there exists a constant C such that $|V_i| \leq C$ for all i. According to Kolmogorov's Criterion [Fe, (X.7.2)] the strong law of large numbers is valid [Fe, (X.7.1)], that is, for μ-almost all $x \in [0, 1]$

$$\lim_{i \to \infty} A_i = \lim_{i \to \infty} \frac{1}{i} \sum_{m=1}^{i} E_m.$$

Since $|(1 - p_m) \log(1 - p_m)| \leq p_m$ and the sums $\frac{1}{i} \sum_{m=1}^{i} p_m$ and $\frac{1}{i} \sum_{m=1}^{i} p_m \log p_m$ go to zero as i goes to infinity, we obtain the claim. □

Remark. For all $A \subset \mathbb{R}^n$ define

$$\overline{\mathrm{por}}(A) = \inf\{\overline{\mathrm{por}}(A, x) : x \in A\}$$

where

$$\overline{\mathrm{por}}(A, x) = \limsup_{r \downarrow 0} \mathrm{por}(A, x, r).$$

For all finite Borel measures μ on $\mathbb{R}^n$ set

$$\overline{\beta}(\mu) = \sup\{\overline{\mathrm{por}}(A) : A \text{ is a Borel set with } \mu(A) > 0\}.$$

According to [MM, Theorem 1.1] the measure μ satisfies the doubling condition if $\overline{\beta}(\mu) < \frac{1}{2}$. Example 4 shows that the assumption $\underline{\beta}(\mu) < \frac{1}{2}$ does not necessarily guarantee this.

5. ONE DIMENSIONAL CASE

In this section we study the situation in $\mathbb{R}$. By considering the class of strongly porous measures we prove that the doubling condition (Definition 2.3) is not necessary for the validity of Theorem 2.4 although without it Proposition 3.1 is not true.

Definition 5.1. *Let μ be a finite Borel measure on $\mathbb{R}^n$. We say that μ is uniformly p-porous if for all $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for μ-almost all $x \in \operatorname{spt}(\mu)$*

$$\operatorname{por}(\mu, x, r, \varepsilon) \geq p \tag{5.1}$$

for all $0 < r \leq R_\varepsilon$. Further, μ is called strongly p-porous if $\operatorname{por}(\mu) \geq p$ and if the following property holds for all $q < p$: given any Borel set $A \subset \mathbb{R}^n$ with $\mu(A) > 0$ such that $\operatorname{por}(\mu, x) > q$ for all $x \in A$ there exists a Borel set $B \subset A$ with $\mu(B) > 0$ such that $\mu|_B$ is uniformly q-porous.

Remark. 1. The upper semi-continuity of the function $x \mapsto \operatorname{por}(\mu, x, r, \varepsilon)$ implies that if (5.1) is true for μ-almost all $x \in \operatorname{spt}(\mu)$ then it is true for all $x \in \operatorname{spt}(\mu)$.

2. We showed in Remark 2 after Definition 2.2 that the restriction of a Radon measure to a p-porous Borel set is p-porous. However, that argument does not imply that the restriction to a uniformly p-porous Borel set would yield a uniformly p-porous measure.

Proposition 5.2. *There is a non-decreasing function $d : [0, 1/2] \to [0, 1]$ satisfying*

$$\lim_{p \uparrow \frac{1}{2}} d(p) = 1$$

such that

$$\dim_{\mathrm{H}}(\mu) \leq 1 - d(p)$$

for all finite strongly p-porous Borel measures μ on $\mathbb{R}$.

Proof. We may assume that $\dim_{\mathrm{H}}(\mu) > 0$. Since $\operatorname{por}(\mu) \geq p$, given any $q < p$, there exists a Borel set A with $\mu(A) > 0$ such that $\operatorname{por}(\mu, x) > q$ for all $x \in A$. Let $0 < s < \dim_{\mathrm{H}}(\mu)$. Then by (2.1) there are $R > 0$ and a Borel set $E \subset A$ with $\mu(E) > 0$ such that

$$\mu(B(x, r)) \leq r^s \tag{5.2}$$

for all $x \in E$ and for all $0 < r < R$. Using Definition 5.1 we find a Borel set $B \subset E$ with $\mu(B) > 0$ such that $\nu = \mu|_B$ is uniformly q-porous. In particular,

$$\nu(B(x, r)) \leq r^s \tag{5.3}$$

for all $x \in \operatorname{spt}(\nu)$ and $0 < r < R$.

Intuitively our argument below is based on the fact that if the porosity is close to $\frac{1}{2}$ then for all sufficiently small $r > 0$ there exists an interval of length close to r inside $B(x, r)$ for $x \in \operatorname{spt}(\nu)$ such that the measure of this interval is close to $\nu(B(x, r))$. Iterating this we find a ball which has a very small radius compared to r and which has measure quite close to $\nu(B(x,r))$. This forces the dimension to be small.

We may assume that ν is non-atomic since otherwise $\dim_{\mathrm{H}}(\mu) \le \dim_{\mathrm{H}}(\nu) = 0$. Assume that $\frac{63}{128} < q < p$. Let $0 < \delta < \frac{1}{64}$ with $q' = \frac{1-\delta}{2} < q < p$. Consider $0 < \varepsilon < \frac{\delta}{2}$. Let R_ε be such that

$$\operatorname{por}(\nu, x, r, \varepsilon) \ge q \tag{5.4}$$

for all $0 < r \le R_\varepsilon$ and for all $x \in \operatorname{spt}(\nu)$. Let n be the biggest integer such that $2^{-n(n+1)} > \delta$.

The following lemma is essential in our proof:

Lemma 5.3. *Let $a < b < c$ be real numbers such that $c - b \le R_\varepsilon$, $b - a \ge \frac{1-\delta}{1+\delta}(c-b)$ and $\nu([a,b]) \le \nu([b,c])$. Then one of the following properties holds:*

(1) *There is $z' \in \operatorname{spt}(\nu) \cap [b,c]$ with $\nu(B(z', 2\delta(c-b))) \ge 2^{-n}(1-5\varepsilon)\nu([b,c])$.*

(2) *There are $b \le a' < b' < c' \le c$ such that $b' - a' \ge \dfrac{1-\delta}{1+\delta}(c'-b')$,*

$$c' - b' \le \frac{1}{2}(1+\delta)(c-b), \quad \nu([a',b']) \le \nu([b',c']),$$

and $\nu([b',c']) \ge (1-2^{-n})(1-5\varepsilon)\nu([b,c])$.

Remark. Note that the choice of n guarantees that in the case (1) we gain much more in the radius than we loose in the weight.

Proof of Lemma 5.3. Since ν has no atoms there exists $y \in \operatorname{spt}(\nu) \cap (b,c)$ such that

$$2\varepsilon\nu([b,c]) < \nu([b,y]) < 3\varepsilon\nu([b,c]). \tag{5.5}$$

This gives $\nu([y,c]) \ge (1-3\varepsilon)\nu([b,c])$. Therefore the requirement $c - y \le 2\delta(c-b)$ implies (1) with $z' = y$. Thus we may now assume that $c - y > 2\delta(c-b)$. If $\nu([y + 2\delta(c-b), c]) = 0$, then (1) holds again because by (5.5) we obtain $\nu(B(y, 2\delta(c-b))) \ge \nu([y, y+2\delta(c-b)]) \ge (1-3\varepsilon)\nu([b,c])$.

It remains to consider the case $c - y > 2\delta(c-b)$ and $\nu([y+2\delta(c-b), c]) \ne 0$. Let $y' = \inf\{\operatorname{spt}(\nu) \cap [y + 2\delta(c-b), c)\}$. Suppose first that $y' \ge b + \frac{1}{2}(1-\delta)(c-b)$ (see Figure 2).

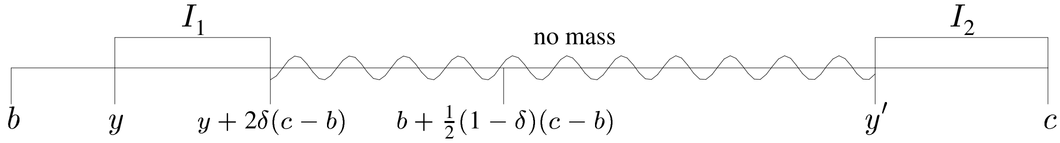

Figure 2: The case $y' \ge b + \frac{1}{2}(1-\delta)(c-b)$.

Setting $I_1 = [y, y + 2\delta(c - b)]$ and $I_2 = [y', c]$ we conclude from (5.5) that

$$\nu(I_1) + \nu(I_2) = \nu([b, c]) - \nu([b, y]) \geq (1 - 3\varepsilon)\nu([b, c]).$$

Note that (1) holds in the case when $\nu(I_1) \geq 2^{-n}(1 - 3\varepsilon)\nu([b, c])$. If the opposite inequality is valid, we have $\nu(I_2) > (1 - 2^{-n})(1 - 3\varepsilon)\nu([b, c])$ giving (2). (To check this choose $a' = b$, $b' = y'$, and $c' = c$.)

We are left with the case $y' < b + \frac{1}{2}(1 - \delta)(c - b)$. Using the fact that ν is uniformly q-porous, we find $z \in \mathbb{R}$ such that $B(z, q'(c - y')) \subset B(y', c - y')$ and $\nu(B(z, q'(c - y'))) \leq \varepsilon\nu(B(y', c - y'))$ giving

$$\nu(B(z, q'(c - y'))) \leq 2\varepsilon\nu([b, c]) \ , \tag{5.6}$$

since $B(y', c - y') \subset [a, c]$ and $\nu([a, b]) \leq \nu([b, c])$. From (5.5) and (5.6) we get $[b, y] \not\subset B(z, q'(c - y'))$ and claim that

$$B(z, q'(c - y')) \subset [b, c] \, . \tag{5.7}$$

(The possibility which is excluded here is that the whole ball is to the left of y (see Figure 3).) This being not the case gives $z - q'(c - y') < b$. Since $y - (y' - (c - y')) \leq c - y' - 2\delta(c - b) < 2q'(c - y')$ we have $B(z, q'(c - y')) \not\subset [y' - (c - y'), y]$, giving $y < z + q'(c - y')$. This implies that $[b, y] \subset B(z, q'(c - y'))$ which is a contradiction.

Now we split our study into three cases depending on the positions of $[y, y']$ and $B(z, q'(c - y'))$. First assume that $[y, y'] \subset B(z, q'(c - y'))$ as in Figure 3.

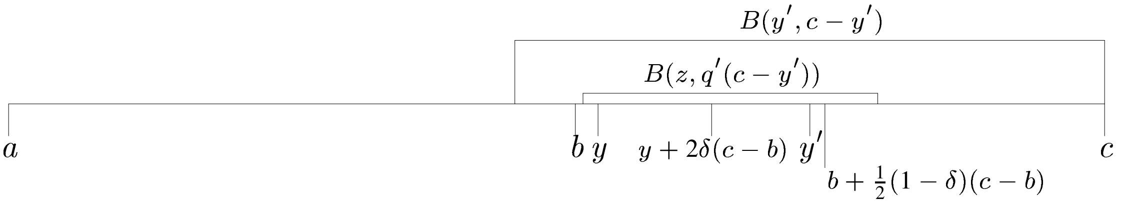

Figure 3: The case $[y, y'] \subset B(z, q'(c - y'))$.

Since $y' < b + \frac{1}{2}(1 - \delta)(c - b)$, we have $b + 2q'(c - y') \geq b + \frac{1}{2}(1 - \delta)(c - b)$, and so we obtain from (5.7) that $[y, b + \frac{1}{2}(1 - \delta)(c - b)] \subset B(z, q'(c - y'))$. Hence by (5.5) and (5.6)

$$\nu([b + \frac{1}{2}(1 - \delta)(c - b), c]) \geq \nu([b, c]) - \nu([b, y]) - \nu(B(z, q'(c - y'))) \geq (1 - 5\varepsilon)\nu([b, c])$$

implying (2). (To verify this choose $a' = b$, $b' = b + \frac{1}{2}(1 - \delta)(c - b)$, and $c' = c$.)

Next assume that $[y, y'] \not\subset B(z, q'(c - y'))$ and $y' \in B(z, q'(c - y'))$ as in Figure 4.

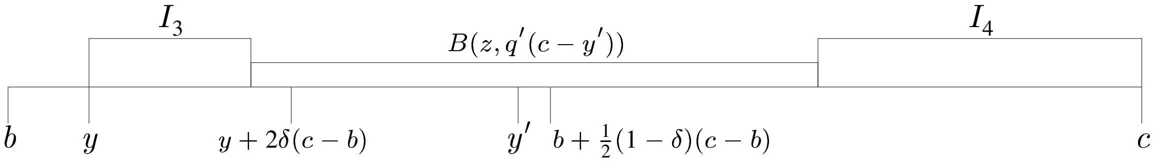

Figure 4: The case $[y,y'] \not\subset B(z,q'(c-y'))$ and $y' \in B(z,q'(c-y'))$.

Then the measure

$$\nu([b,c]) - \nu([b,y]) - \nu(B(z,q'(c-y'))) \geq (1-5\varepsilon)\nu([b,c])$$

is divided between two disjoint intervals $I_3 = [y, \min\{z - q'(c-y'), y + 2\delta(c-b)\}]$ and $I_4 = [z + q'(c-y'), c]$ contained in $[y, y+2\delta(c-b)]$ and $[b + \frac{1}{2}(1-\delta)(c-b), c]$, respectively. If $\nu(I_3) \geq 2^{-n}(1-5\varepsilon)\nu([b,c])$, then (1) holds. When $\nu(I_4) \geq (1-2^{-n})(1-5\varepsilon)\nu([b,c])$ we obtain (2). (To check this choose $a' = z - q'(c-y')$, $b' = z + q'(c-y')$, and $c' = c$.)

In the remaining case we have $[y,y'] \not\subset B(z,q'(c-y'))$ and $y' \notin B(z,q'(c-y'))$ giving $y' < z - q'(c-y')$ since $b + 2q'(c-y') \geq b + \frac{1}{2}(1-\delta)(c-b) > y'$ as in Figure 5.

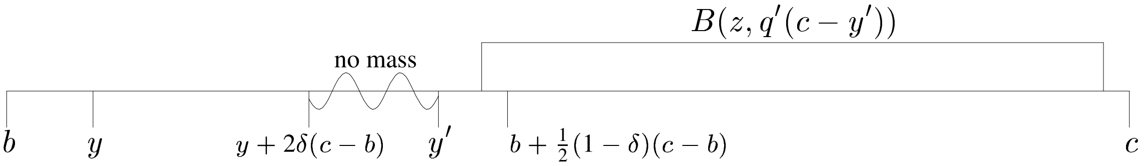

Figure 5: The case $[y,y'] \not\subset B(z,q'(c-y'))$ and $y' \notin B(z,q'(c-y'))$.

Note that $c - y' - 2q'(c-y') = \delta(c-y') \leq \delta(c-b)$ which means that the set $[y',c] \setminus B(z,q'(c-y'))$ is the union of at most two intervals of length at most $\delta(c-b)$, and so the measure

$$\nu([b,c]) - \nu([b,y]) - \nu(B(z,q'(c-y'))) \geq (1-5\varepsilon)\nu([b,c])$$

is divided between at most three intervals of length at most $2\delta(c-b)$. (Take the two above intervals and $[y, y+2\delta(c-b)]$ which has the same ν-measure as $[y,y']$.) Hence there exists an interval of length at most $2\delta(c-b)$ having ν-measure at least $\frac{1}{3}(1-5\varepsilon)\nu([b,c])$, and so (1) is satisfied. $\square$

The continuation of the proof of Proposition 5.2. Let $x \in \mathrm{spt}(\nu)$ and $0 < r < \min\{R, \frac{R_\varepsilon}{2}\}$. Since ν is uniformly q-porous, $B(x,r)$ contains an interval $[a, a+2q'r]$ such that $\nu([a,a+2q'r]) \leq \varepsilon\nu(B(x,r))$. Hence either $\nu([x-r,a]) \geq \frac{1}{2}(1-\varepsilon)\nu(B(x,r))$ or $\nu([a+2q'r, x+r]) \geq \frac{1}{2}(1-\varepsilon)\nu(B(x,r))$. Note that the length of both of these intervals is at most $r(1+\delta)$. We assume that $\nu([a+2q'r, x+r]) \geq \frac{1}{2}(1-\varepsilon)\nu(B(x,r))$. The other case can be treated similarly. Setting $b = a + 2q'r$ and $c = x + r$ Lemma 5.3 implies that either (1) or (2) holds.

Assuming the validity of (1) we find $z \in \operatorname{spt}(\nu)$ such that

$$\gamma_1\nu(B(x,r)) \equiv 2^{-n-1}(1-5\varepsilon)(1-\varepsilon)\nu(B(x,r)) \le \nu(B(z,2\delta(1+\delta)r)) \equiv \nu(B(z,\lambda_1 r)).$$

If (2) holds instead of (1) in Lemma 5.3, then the assumptions of Lemma 5.3 are again satisfied for the points given in (2). Assuming that when applying Lemma 5.3 (2) is valid n times we find $z \in \operatorname{spt}(\nu)$ such that

$$\begin{aligned}\gamma_2\nu(B(x,r)) &\equiv \frac{1}{2}(1-2^{-n})^n(1-5\varepsilon)^n(1-\varepsilon)\nu(B(x,r))\\ &\le \nu(B(z,(\frac{1+\delta}{2})^n(1+\delta)r)) \equiv \nu(B(z,\lambda_2 r)).\end{aligned}$$

In the remaining case (2) holds $0 < l < n$ times in the application of Lemma 5.3. Then there is $z \in \operatorname{spt}(\nu)$ with

$$\begin{aligned}\gamma_3\nu(B(x,r)) &\equiv (1-2^{-n})^l(1-5\varepsilon)^{l+1}2^{-n-1}(1-\varepsilon)\nu(B(x,r))\\ &\le \nu(B(z,(\frac{1+\delta}{2})^l 2\delta(1+\delta)r)) \equiv \nu(B(z,\lambda_3 r)).\end{aligned}$$

Repeating the above procedure we find $z_k \in \operatorname{spt}(\nu)$ for all $k \ge 1$ and $(\Gamma_i,\Lambda_i) \in \{(\gamma_1,\lambda_1),(\gamma_2,\lambda_2),(\gamma_3,\lambda_3)\}$ for all $1 \le i \le k$ such that

$$(\prod_{i=1}^k \Gamma_i)\nu(B(x,r)) \le \nu(B(z_k,(\prod_{i=1}^k \Lambda_i)r)) \le (\prod_{i=1}^k \Lambda_i)^s r^s$$

by (5.3). This gives for all k

$$s \le \frac{\sum_{i=1}^k \big(\log\Gamma_i + \frac{1}{k}\log\nu(B(x,r))\big)}{\sum_{i=1}^k \big(\log\Lambda_i + \frac{1}{k}\log r\big)} \equiv \frac{\sum_{i=1}^k \alpha_{i,k}}{\sum_{i=1}^k \beta_{i,k}} \le \max_{1\le i\le k}\frac{\alpha_{i,k}}{\beta_{i,k}}$$

(all terms are negative) implying

$$s \le \max_{i=1,2,3}\frac{\log\gamma_i}{\log\lambda_i}.$$

The claim follows since this upper bound goes to zero as δ tends to zero. □

6. ACKNOWLEDGEMENTS

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