Decay of correlations for piecewise invertible maps in higher dimensions

Jérôme Buzzi[1]
Véronique Maume-Deschamps[2]

[1] C.N.R.S. / Institut de Mathématiques de Luminy
[2] Université de Bourgogne / Laboratoire de Topologie

November 24, 1999

Abstract

We study the mixing properties of equilibrium states μ of non-Markov piecewise invertible maps $T : X \to X$, especially in the multidimensional case. Assuming mainly Hölder continuity and that the topological pressure of the boundary is smaller than the total topological pressure, we establish exponential decay of correlations, i.e.:

$$\left| \int_X \varphi \cdot \psi \circ T^n \, d\mu - \int_X \varphi \, d\mu \cdot \int_X \psi \, d\mu \right| \le C \cdot e^{-\alpha n}$$

for all Hölder functions $\varphi, \psi : X \to \mathbb{R}$, all $n \ge 0$ and some $C < \infty$, $\alpha > 0$.

If the smoothness assumption is further weakened, we get subexponential speeds.

Résumé

Nous étudions les propriétés de mélange des états d'équilibre μ d'applications $T : X \to X$ inversibles par morceaux et non-markoviennes, plus particulièrement dans le cas multi-dimensionnel. En supposant (essentiellement) la continuité Hölder et que la pression topologique du bord est strictement majorée par la pression totale, nous montrons la décroissance exponentielle des corrélations:

$$\left| \int_X \varphi \cdot \psi \circ T^n \, d\mu - \int_X \varphi \, d\mu \cdot \int_X \psi \, d\mu \right| \le C \cdot e^{-\alpha n}$$

pour toutes les fonctions Hölder $\varphi, \psi : X \to \mathbb{R}$, tout $n \ge 0$ et des constantes $C < \infty$, $\alpha > 0$.

Une diminution de la régularité fait apparaître des vitesses sous-exponentielles.

AMS classifiction 1991: Primary 58F11
key words: absolutely continuous invariant measures; equilibrium states; decay of correlations; transfer operator; multi-dimensional dynamics; tower extension.

INTRODUCTION

Let $(X, \mathcal{Z}, T, g)$ be a *weigthed d-dimensional piecewise invertible map*, i.e.:

- $X = \overline{\bigcup_{Z \in \mathcal{Z}} Z} \subset \mathbb{R}^d$.
- $\mathcal{Z}$ is a finite collection of pairwise disjoint, bounded and open subsets of $\mathbb{R}^d$.
- each restriction $T|Z$, $Z \in \mathcal{Z}$, is the restriction of a homeomorphism $T_Z : U \to V$ with U, V open sets in $\mathbb{R}^d$ such that $U \supset \bar{Z}$, $V \supset \overline{T(Z)}$.
- $g : X \to \mathbb{R}$.

Such systems have usually plenty of invariant probability measures, some of them rather irrelevant as the ones living on any periodic orbit. A classical way to select "interesting" measures is through the following variational principle. One considers *equilibrium states*, i.e., invariant probability measures which maximize the *measure-theoretic pressure*:

$$h(\mu, T) - \int_Y \log g \, d\mu$$

($h(\mu, T)$ is the entropy of μ — see [DKS]).

In this *piecewise, multidimensional* setting, existence and uniqueness of these measures have been studied in [Bu2] and [BPS]. Furthermore in [Bu2] such measures have been given the alternative and more "geometric" characterization of being exactly the measures absolutely continuous w.r.t. the conformal measure of the system (see below).

The crucial condition in these papers is the following inequality on the topological pressure of the boundary of the pieces of the map (see below for the definitions):

$$P(\partial\mathcal{Z}, T) < P(X, T).$$

In this paper, we show in a similar setting that these equilibrium states have exponential decay of correlations in the case of Hölder smoothness of g and uniform expansion. For less-than-Hölder smoothness we get subexponential speeds.

Acknowledgements: The authors are grateful to *program ESF/PRODYN* which has partially supported the *International Conference on Dynamical Systems, Abbey of "La Bussière"* where most of this work was carried through.

Statement of Results

We use indifferently $\mathcal{Z}^n$ to denote the set of words $A_0 \ldots A_{n-1}$ of length n on $\mathcal{Z}$ and the set of n-cylinders:

$$[A_0 \ldots A_{n-1}] \overset{\text{def}}{=} A_0 \cap \ldots \cap T^{-n+1} A_{n-1}.$$

The *topological pressure* of a subset S of X is:

$$P(S,T)=\limsup_{n\to\infty}\frac{1}{n}\log\sum_{\substack{A\in\mathcal{Z}^n\\ \bar{A}\cap S\neq\varnothing}} g^{(n)}(A)$$

where $g^{(n)}(A)=\sup_{x\in A} g(x)g(Tx)\dots g(T^{n-1}x)$.

As in papers [Bu2], [BPS] our main condition is:

$$P(\partial\mathcal{Z},T)<P(X,T) \tag{$*$}$$

where $\partial\mathcal{Z}=\bigcup_{Z\in\mathcal{Z}}\partial Z$.

The modulus of continuity of g will be controlled by the following, dynamics-dependent sequence:

$$\omega_n(g)=\sup_{Z\in\mathcal{Z}^n}\sup_{x,y\in Z}\log\frac{g(x)}{g(y)}.$$

We shall also assume "the discontinuities to be in general position", i.e., for all $d+1$-uples of integers $0\leq k_1<k_2<\ldots<k_{d+1}$, for all $Z\in\mathcal{Z}^{k_{d+1}+1}$:

$$\bigcap_{i=1}^{d+1} T_Z^{-k_i}\partial Z_{k_i}=\varnothing \tag{$\dagger$}$$

where $T_Z^k=T_{Z_{k-1}}\circ T_{Z_{k-2}}\circ\ldots\circ T_{Z_0}$. Remark that this is basically a transversality assumption. Thus, one expects this condition to be almost always satisfied (see [Bu1] and [Co]).

Finally, we shall need a *conformal measure* ν, i.e., a not necessarily invariant, probability measure on X, such that the Jacobian of T w.r.t. this measure is equal to $e^{-P(X,T)g}$. By [Bu2], the ν-absolutely continuous invariant probability measures coincide with the equilibrium states.

Replacing g by $g-P(X,T)$, we may, and shall, assume that $P(X,T)=0$.

Main Theorem. *Let $(X,\mathcal{Z},T,g)$ be a weighted piecewise invertible dynamical system. Assume:*

H1. T is non-contracting: $d(Tx,Ty)\geq d(x,y)$ for all x,y in the same element of $\mathcal{Z}$; T $\mathcal{Z}$-separates: $\lim_{n\to\infty}$ diam$(\mathcal{Z}^n)\to 0$ and T satisfies the transversality condition ($\dagger$).

H2. g is such that the pressure condition ($*$) holds and it is smooth enough w.r.t. the expansion of T in the sense that:

$$\sum_{n\geq 1}\omega_n(g)<\infty. \tag{$\ddagger$}$$

H3. there is a conformal measure ν such that $T^nX\subset\operatorname{supp}(\nu)\pmod 0$ for large enough n.

Then, T admits an invariant measure, absolutely continuous with respect to ν (ν-a.c.i.m. for short). For any ergodic ν-a.c.i.m., μ, up to replacing T by some iterate, μ is $(u_n)_{n\in\mathbb{N}}$-mixing, i.e., for all $n \geq 1$,

$$\left| \int_X \varphi \circ T^n \cdot \psi \, d\mu - \int_X \varphi \, d\mu \int_X \psi \, d\mu \right| \leq C \cdot (\sup |\varphi| + K(\varphi)) \cdot \|\psi\|_{L^1(\mu)} \, u_n.$$

with $C < \infty$ and for any mesurable functions $\varphi, \psi : X \to \mathbb{R}$ such that ψ is bounded and:

$$\sup_{Z\in\mathcal{Z}^p} \sup_{x,y\in Z} |\varphi(x) - \varphi(y)| \leq K(\varphi) \sum_{q>p} \omega_q(g) \qquad \forall p \geq 1.$$

The sequence u tends to zero with a definite speed:

- *if $\omega_n(g) = O(\rho^n)$ for some $0 < \rho < 1$, then $u_n = \kappa^n$ for some $0 < \kappa < 1$,*
- *if $\omega_n(g) = O(n^{-\alpha})$ for some $\alpha > 1$, then $u_n = O(n^{-\alpha+1})$.*

Remarks

- *The observable φ above is only required to be as regular as (in fact a little less regular than) the logarithm of the weight.*
- *[BPS] proved the existence of a conformal measure ν with full support (implying H3 above) under the following additional assumptions: each cylinder has only finitely many connected components and T is covering, i.e., for all non-empty open sets $V \subset X$, $\bigcup_{k=0}^n T^k(V) \supset Y$ for n large enough.*
- *If T is uniformly expanding, i.e., if $d(Tx,Ty) \geq \lambda d(x,y)$ for some $\lambda > 1$ and all x, y in the same element of $\mathcal{Z}$, then one can replace condition H1 above by the well-known "summable modulus of continuity" condition (see [Sc], [Go]):*

$$\int_O^1 \frac{1}{t} \sup \left\{ \log \frac{g(x)}{g(y)} : d(x,y) \leq t, \ x,y \in Z, \ Z \in \mathcal{Z} \right\} dt < \infty.$$

- *If $g = |\det T'|^{-1}$, then condition $(*)$ is always fulfilled under the transversality condition $(\dagger)$ according to [Bu2, Prop. 5.2] and therefore it can be removed from the Theorem.*

Outline of the paper

The rest of the paper is devoted to the proof of the above Theorem.

To begin with, we define a Markov extension $(\hat{X}, \hat{T})$ *à la* Hofbauer-Keller. We check that the return time $\hat{R}$ with respect to a slightly enlarged basis $\hat{X}_*$ of this tower satisfies an exponential estimate with rate $P(\partial\mathcal{Z}, T)$.

We then state a version of our Main Theorem for abstract towers and check its hypothesis under the assumptions of the Main Theorem.

We finally recall standard facts about Birkhoff metrics and put them to use for the proof of the abstract Theorem.

Comments

- Our Main Theorem generalizes well-known results from the *globally expanding* case (see, e.g., [Bo]) to our *piecewise* setting.
- Even in the one-dimensional case, our assumptions are essentially standard (see [LSV] for recent references). The exception is the transversality assumption (†) (which in the one-dimensional case is the exclusion of periodic critical points). We think that it is technical but we could not remove it.
- We believe our result to be the first considering less-than-Hölder smoothness *outside of the Markov setting* (see [KMS] where non-Hölder potentials are considered in the Markov setting). Also we provide an answer to a question of L.S. Young [Y2] about *controlling the upper floors* of the tower: it is done by a thermodynamical estimate (Proposition 1.1) in the spirit of [Bu2].
- As we go to a Markov extension, we avoid having to deal with spaces of discontinuous functions, i.e., spaces of functions with bounded *multi-dimensional variation.* This is a significant technical gain.
- Our abstract theorem is much in the spirit of L.-S. Young (see [Y1]). But our proof is quite different, based on Birkhoff cone techniques and previous works [KMS], [M].

1 CONSTRUCTION OF THE TOWER

We define a Markov tower extension $(\hat{X},\hat{T})$ in the vein of F. Hofbauer [Ho] and G. Keller [Ke] and then prove that the pressure of the boundary provides an exponential estimate on the tail statistics of the return times to the basis.

Let $(X,\mathcal{Z},T,g,\nu)$ be as in the Main Theorem. Let $\hat{X}_0=\{(x,Z):x\in Z$ and $Z\in\mathcal{Z}\}$. For $x\in Y\cap T^{-1}Y$ and $A\subset X$, set:

$$\hat{T}(x,A)=(Tx,T(A)\cap\mathcal{Z}[Tx])$$

where $\mathcal{Z}[x]$ is that element of $\mathcal{Z}$ which contains x.
Set $\hat{X}=\bigcup_{n\geq 0}\hat{T}^n(\hat{X}_0)$. Remark that

$$\hat{X}=\bigcup_{D\in\mathcal{D}} D\times\{D\} \text{ with } \mathcal{D}=\{T^nZ:n\geq 0 \text{ and } Z\in\mathcal{Z}^{n+1}\}.$$

Thus, $\hat{X}$ is a countable, disjoint union of (sets naturally isomorphic to) subsets of X.

Proposition 1.1 *For each $\delta > 0$, if $N_* < \infty$ is large enough, then for*

$$\hat{X}_* = \hat{T}^{N_*}\hat{X}_0,$$

the return time to $\hat{X}_*$, $\hat{R}(\hat{x}) = \inf\{n \geq 1 : \hat{T}^n(\hat{x}) \in \hat{X}_*\}$, satisfies, for each $D \in \mathcal{D}$,

$$\nu\left(\{x \in D : \hat{R}(x, D) > n\}\right) \leq const \cdot \exp[-n(-P(\partial\mathcal{Z}, T) - \delta)]$$

for some positive number const $< \infty$.

Proof:

Let $\delta > 0$. Observe that D being of the form $T^k(A_1 \cap \ldots \cap T^{-k}A_k)$, $P(\partial D, T) \leq P(\partial\mathcal{Z}, T)$. In particular,

$$\sum_{\substack{Z\in\mathcal{Z}^n\\ \bar{Z}\cap\partial D\neq\varnothing}} g^{(n)}(Z) \leq \text{const} \cdot e^{(P(\partial\mathcal{Z},T)+\delta/10)n} \qquad (\forall n \geq 0)$$

where const depends only on D.

Take an integer N_0 so large that the following holds:

$$\sum_{\substack{A\in\mathcal{Z}^n\\ \bar{A}\cap\partial\mathcal{Z}\neq\varnothing}} g^{(n)}(A) \leq e^{(\delta/10d)N_0} e^{(P(\partial\mathcal{Z},T)+\delta/10)n} \qquad (\forall n \geq 0)$$

$$d\left(\frac{n}{N_0} + 2\right)\binom{n}{d(n/N_0 + 1)} \leq e^{\delta n/10}. \qquad (\forall n \geq N_0)$$

Observe that the "general position" assumption implies that there exists $\varepsilon_* > 0$ such that there is no "ε_*-near, $d+1$-order intersection up to time N_0" in the following sense: for all $x \in \bigcap_{k=0}^{N_0-2} T^{-k}(X \setminus \partial\mathcal{Z})$,

$$\#\{0 \leq k < N_0 : d(T^k x, \partial\mathcal{Z}) < \varepsilon_*\} \leq d. \tag{1.1}$$

Fix $N_* < \infty$ so large that the diameter of any N_*-cylinder is smaller than ε_*. For $n \geq 1$, set:

$$\rho_n(x) = \sup\{\rho \in [0, \varepsilon_*] : B(T^n x, \rho) \subset T^n(D \cap \mathcal{Z}^n(x))\}.$$

Recall that a Bowen ball is:

$$B_n(x, \varepsilon_*) := \{y \in X : \forall k = 0, \ldots, n-1 \ d(T^k x, T^k y) < \varepsilon_*\}.$$

Remark that $\rho_n(x)$ is also the interior diameter at $T^n x$ of $T^n(D \cap \mathcal{Z}^n(x) \cap B_{n+1}(x, \varepsilon_*))$ as T is non-contracting. Also,

$$\hat{R}(x, D) \leq R(x) + N_*$$

with $R(x) \overset{\text{def}}{=} \min\{n \geq 1 : \rho_n(x) = \varepsilon_*\}$. Indeed,

$$\begin{aligned}
\hat{T}^{R(x)+N_*}(x,D) &= \big(T^{R(x)+N_*}x, T^{N_*}(\mathcal{Z}^{N_*}[T^{R(x)}x] \cap T^{R(x)}(D \cap \mathcal{Z}^{R(x)}[x]))\big) \\
&= (T^{R(x)+N_*}x, T^{N_*}\mathcal{Z}^{N_*}[T^{R(x)}x]) \\
&= \hat{T}^{N_*}(T^{R(x)}x, \mathcal{Z}[T^{R(x)}x]) \in T^{N_*}\hat{X}_0 = \hat{X}_*.
\end{aligned}$$

Thus it is enough to prove the proposition for $R(x) > n$ instead of $\hat{R}(x,D) > n$.

The *n-skeleton* of a point $x \in D$ with $R(x) > n$ will be the finite decreasing integer sequence $(s_i)_{i=0,\dots,r}$, $r \in \mathbb{N}$, defined as follows. Set $s_0 = n$ and define inductively s_{i+1} to be the smallest integer $0 \leq s < s_i$ (if it exists) such that:

$$\overline{[A_s \dots A_{s_i-1}]} \cap \partial\mathcal{Z} \cap \overline{B(T^s x, \varepsilon_*)} \neq \varnothing$$

where $A_i = \mathcal{Z}(T^i x)$. Finally set r to be maximum such that $s_0, \dots, s_r$ exist.

We claim that:

$$\overline{[A_0 \dots A_{s_r-1}]} \cap \partial D \neq \varnothing.$$

If $s_r = 0$, the claim reduces to $\partial D \neq \varnothing$: it is clear. So, assume that $s_r > 0$ and that the above intersection is empty. Hence points in the boundary of $T^{s_r}(D \cap [A_0 \dots A_{s_r-1}] \cap B_{s_r+1}(x, \varepsilon_*))$ do not come from ∂D. They do not come either from $\partial[A_0 \dots A_{s_r-1}]$ because of the defintion of r. Hence this boundary is a subset of $\partial T^{s_r} B_{s_r+1}(x, \varepsilon_*)$.

T being non-contracting, $T^{s_r} B_{s_r+1}(x, \varepsilon_*) = B(T^{s_r}x, \varepsilon_*)$. Hence, $\rho_{s_r}(x) = \varepsilon_*$. Thus, $R(x) \leq s_{r+1} \leq n$, a contradiction.

Observe that eq. (1.1) implies

$$r \leq d\left(\frac{n}{N_0} + 1\right).$$

We compute, using the conformality of ν:

$$\begin{aligned}
\nu(\{x \in D\ : R(x) \geq n\}) \leq& \sum_{\substack{A \in \mathcal{Z}^n \\ \exists x \in A \text{ s.t. } R(x) \geq n}} e^{-nP(X,T)} g^{(n)}(A) \\
\leq&\ e^{-nP(X,T)} \sum_{r=0}^{d(n/N_0+1)} \sum_{0<s_r<\dots<s_0=n} \sum_{\substack{A_0 \dots A_{s_r-1} \\ \overline{[A_0 \dots A_{s_r-1}]} \cap \partial D \neq \varnothing}} g^{(s_r)}(A_0 \dots A_{s_r-1}) \\
&\times \prod_{i=1}^{r-1} \sum_{\substack{A_{s_{i+1}} \dots A_{s_i-1} \\ \overline{[A_{s_{i+1}} \dots A_{s_i-1}]} \cap \partial\mathcal{Z} \neq \varnothing}} g^{(s_i - s_{i+1})}(A_{s_{i+1}} \dots A_{s_i-1}) \\
\leq&\ e^{-nP(X,T)} d\left(\frac{n}{N_0} + 2\right) \binom{n}{d\left(\frac{n}{N_0}+1\right)} \text{const} \cdot e^{(P(\partial\mathcal{Z},T)+\delta/10)s_1} \\
&\prod_{i=1}^{r-1} e^{\delta N_0/10d} e^{(P(\partial\mathcal{Z},T)+\delta/10)(s_i - s_{i+1})} \\
\leq&\ \text{const} \cdot e^{-n(P(X,T)-P(\partial\mathcal{Z},T)-\delta)}.
\end{aligned}$$

The proposition is proved. ■

2 ABSTRACT TOWER

To emphasize the relevant properties of the previously defined tower, we are going to state an abstract version of our Main Theorem, somewhat in the spirit of L.-S. Young [Y1]. Note that its proof will use the technical tools of [KMS], [M]. We will work on a tower which is hybrid beetween the construction of Young [Y1] and that of Hofbauer and Keller [HK].

Let N_* be such that Proposition 1.1 is satisfied and $T^{N_*}X \subset \text{supp}\nu$. Let $\Delta_0 = \hat{X}_*$ be the enlarged basis of the Markov tower previously defined. We shall write x for a point of $\Delta_0 \subset X \times \mathcal{D}$.

Let $(\Delta_{0,j})_{j\in\mathbb{N}}$ be the partition of Δ_0 obtained by dividing it first into level sets $\{x : \hat{R}(x) = r\}$ and then according to $\mathcal{Z}^{r+1}$ on the rth level set i.e. $(\Delta_{0,j})_{j\in\mathbb{N}}$ is the coarsest partition of Δ_0 such that $\hat{R}(x)$ and $Z_{\hat{R}(x)}(x)$ are constant on atoms.

We define

$$\Delta_\ell = \{(x,\ell) \ / \ x \in \Delta_0 \text{ and } \hat{R}(x) > \ell\}.$$

The tower $\Delta \subset X \times \mathcal{D} \times \mathbb{N}$ is the disjoint union of the Δ_ℓ's. We will denote by $\Delta_{\ell,j} \subset \Delta_\ell$ the set of (x,ℓ) such that $(x,0)$ belongs to $\Delta_{0,j}$. The $\Delta_{\ell,j}$'s for $\ell \in \mathbb{N}$, $j \in \mathbb{N}$, form a partition of Δ which we will denote by $\mathcal{P}$.

Let $F : \Delta \longrightarrow \Delta$ be defined in the following way:

$$\begin{cases} F(x,\ell) &= (x,\ell+1) \text{ if } \ell+1 < \hat{R}(x) \\ &= (\hat{T}^{\hat{R}(x)}x, 0) \text{ otherwise.} \end{cases}$$

For $x, y \in \Delta$ the *separation time* $s(x,y)$ is the largest integer $n \geq 0$ such that for all $j \leq n$, $F^j(x)$ and $F^j(y)$ belong to the same atom of the partition $\mathcal{P}$. Set

$$d_0(x,y) = C \cdot \sum_{j \geq s(x,y)+1} \omega_j(g) \qquad \text{with } C = \exp \sum_{j\geq 1} \omega_j(g).$$

Note that this metric is designed so that the family of functions $\log g_n$, $n \geq 1$, are uniformly Lipschitz w.r.t. it (and the constant C ensures (A.III) below).

Remark that we do not assume that the images $F^R\Delta_{0,j}$ are the whole basis Δ_0. That is why we have to introduce the finite partition $\{B_1, \dots, B_p\}$ of Δ_0 generated by $\{F^R\Delta_{0,j} : j \in \mathbb{N}\}$. Also $B_{j,\ell} \subset \Delta_\ell$ is the subset of points (x,ℓ) with $x \in B_j$.

Finally, we lift the probability measure ν on X to a measure $\hat{\nu}$ on Δ defined by:

$$\hat{\nu}(S \times \{D\} \times \{\ell\}) = \nu(S).$$

Since Δ_0 is a finite union of $\widehat{D}$, $D \in \mathcal{D}$, $\hat{\nu}(\Delta_0) < \infty$. Also, by the choice of N_*, $\text{supp}(\hat{\nu}) = \Delta$.

Let us summarize the crucial properties of the tower.

(A.I) **Exponentially small upper floors.** By Proposition 1.1,

$$\hat{\nu}\left(\{x \in \Delta_0 \;/\; \hat{R}(x) > \ell\}\right) \leq \text{const} \cdot \gamma^\ell$$

for some $0 < \gamma < 1$. In particular, $\hat{\nu}(\Delta) = \sum_\ell \hat{\nu}(\Delta_\ell) \leq \text{const} \sum_\ell \gamma^\ell < \infty$. Moreover, the support of $\hat{\nu}$ is Δ: this follows from our construction and the fact that the support of ν contains $T^{N_*}X$. In what follows, we assume that $\hat{\nu}$ has been normalized i.e. $\hat{\nu}(\Delta) = 1$.

(A.II) **Generating Partition.** The partition $\mathcal{P}$ generates under F i.e.: the partition $\bigvee_{n=0}^\infty F^{-n}\mathcal{P}$ is the partition into points, this follows directly from H1. In particular, d_0 defines a metric on Δ.

(A.III) **Bounded distortion.** Let JF be the Jacobian of F with respect to $\hat{\nu}$. Obviously, $JF(x,\ell) = 1$ if $\ell + 1 < \hat{R}(x)$ and $JF(x, \hat{R}(x) - 1) = g_{\hat{R}(x)}(x)$. If x, y are in the same B_j, we can defined their paired pre-images x' and y' as follows. $F^n x' = x$, $F^n y' = y$ and $F^k(x')$ and $F^k(y')$ belong to the same element of $\mathcal{P}$ for $0 \leq k < n$. We have:

$$\left| \frac{JF^n(x')}{JF^n(y')} - 1 \right| \leq d_0(x,y).$$

(A.IV) **Large image and Markov properties.** *Large image:* there exists $\eta > 0$ such that

$$\inf_{j \in \mathbb{N}} \hat{\nu}(F^{\hat{R}}(\Delta_{0,j})) \geq \eta.$$

Indeed, $F^{\hat{R}}(\Delta_{0,j})$ takes only finitely many distinct values, none zero as $\operatorname{supp}(\hat{\nu}) = \Delta$.

Markov property: $F^{\hat{R}}\Delta_{0,j}$ is a union of some $\Delta_{0,p}$.

Let $\mathbf{L}(\Delta)$ be the space of bounded functions on Δ that are *locally Lipschitz* with respect to the metric d_0, i.e., for all x, y in the same $B_{j,\ell}$,

$$|\varphi(x) - \varphi(y)| \leq K(\varphi) d_0(x,y),$$

for some $K(\varphi)$ which is assumed to be the smallest positive number such that the above inequality is satisfied. Let $\|\varphi\|_{\mathbf{L}(\Delta)} = K(\varphi) + \|\varphi\|_\infty$ be the norm on $\mathbf{L}(\Delta)$.

$\mathcal{P}$ has a natural graph structure: $P \to Q$ iff $F(P) \supset Q$. Its (restricted) *spectral decomposition* is $\mathcal{P} = \mathcal{P}_t \cup \bigcup_i \bigcup_{j=0}^{p_i - 1} \mathcal{P}_j^{(i)}$, where:

- $\mathcal{P}_t$ is the set of transient elements of $\mathcal{P}$, i.e., elements P such that there exists a path from P going to some $Q \in \mathcal{P}$ and there is no path from Q to P (observe that we don't separate this part into irreducible subsets).

- for each i, $\bigcup_{j=0}^{p_i-1}\mathcal{P}_j^{(i)}$ is the set of $P\in\mathcal{P}$ such that there exist paths from P to Q and Q to P, for some fixed Q (i.e., these unions are the irreducible components of $\mathcal{P}$ which are maximal w.r.t. the precedence ordering).

- if there is an arrow from $\mathcal{P}_j^{(i)}$ to $\mathcal{P}_l^{(k)}$ then $k=i$ and $l=j+1 \bmod p_i$.

Finally, $\Delta_j^{(i)}$ is the subset of Δ defined by $\mathcal{P}_j^{(i)}$. Observe that, up to trivialities, it is enough to study the dynamics of $F^{p_i}:\Delta_0^{(i)}\to\Delta_0^{(i)}$ for each i. We call this the *spectral reduction.*

Theorem 2.1 *Let $(\Delta, F, \hat\nu)$ be a tower system satisfying (A.I - IV).*

First, there exists an invariant probability measure absolutely continuous with respect to $\hat\nu$ (a $\hat\nu$-a.c.i.m. for short).

Second, any $\hat\nu$-a.c.i.m. $\hat\mu$, up to the spectral reduction, is mixing, with the following speed estimate: for all $\varphi\in\mathbf{L}(\Delta)$ and $\psi\in L^\infty(\Delta)$,

$$\left|\int\limits_\Delta \varphi\circ F^n\cdot\psi\,d\hat\mu-\int\limits_\Delta\varphi\,d\hat\mu\int\limits_\Delta\psi\,d\hat\mu\right|\le C\cdot\|\varphi\|_{\mathbf{L}(\Delta)}\|\psi\|_{L^1(\hat\mu)}\cdot u_n\qquad\textit{for all } n\ge 0$$

for some $C<\infty$ and a sequence u converging to zero and which can be explicited:

- *if $\omega_n(g)=O(\rho^n)$ for some $0<\rho<1$, then $u_n=\kappa^n$ for some $0<\kappa<1$,*

- *if $\omega_n(g)=O(n^{-\alpha})$ for some $\alpha>1$, then $u_n=O(n^{-\alpha+1})$.*

The Main Theorem follows from Theorem 2.1 using the following observations.

First, the existence of a ν-a.c.i.m. for T follows from the existence of a $\hat\nu$-a.c.i.m. for F which can be projected in the following way. Define $\pi:\Delta\to T^{N_*}X$ by: $\pi(x,D,\ell)=T^\ell x$. π is surjective and $T\circ\pi=\pi\circ F$. If $\hat\mu$ is a $\hat\nu$-a.c.i.m., it projects by π to a ν-a.c.i.m.

Second, to study a given ν-a.c.i.m. we lift it to a $\hat\nu$-a.c.i.m. as follows. If $\mu=h_0\nu$ is a ν-a.c.i.m., let $\widehat{h}_0=\text{const}\cdot h_0\circ\pi$ with const such that $\hat\nu(\widehat{h}_0)=1$. Let $\widehat{h}$ be any limit point of $1/n\sum_{i=0}^{n-1}\mathcal{L}_0^i\widehat{h}_0$ where $\mathcal{L}_0$ is the transfer operator defined in section 4, such a limit point exists and satisfies $\hat\nu(\widehat{h})=1$ by Lemma 4.2. Let $\hat\mu=\widehat{h}\hat\nu$, $\hat\mu$ is a $\hat\nu$-a.c.i.m., it projects by π on μ.

Finally, let $\varphi,\psi:X\to\mathbb{R}$ be as in the Main Theorem. Observe that $\varphi\circ\pi\in\mathbf{L}(\Delta)$ and $\psi\circ\pi\in L^\infty$. Finally remark that the integrals over Δ involving these lifted functions are equal to the integrals in the Main Theorem.

Remark 2.2 *We could replace in the theorem above assumption (A.I) (the exponentially small upper floors) by the weaker summability assumption $\sum_{\ell\in\mathbb{N}}\hat\nu(\Delta_\ell)<\infty$.*

We would get u-mixing with u given by:

$$C\cdot\max[u_n,\ \kappa^n,\ \varepsilon_n^{-1}\hat\nu(\Delta_n)]$$

where u_n is from Theorem 2.1, $0<\kappa<1$ and ε_n is any decreasing and summable sequence. (see [M]).

3 BIRKHOFF'S CONES AND PROJECTIVE METRICS

The main tool for the proof of the Theorem 2.1 will be the theory of cones and projective metrics of Garrett Birkhoff [Bi]. P. Ferrero and B. Schmitt [FS] applied it to estimate the correlation decay for random products of matrices. Recently this strategy has been used by many authors to obtain exponential decay of correlations. In [KMS] these techniques have been used in a different way to obtain sub-exponential decay of correlations.

Let us recall definitions and properties of cones and projective metrics (see [Li] for a more complete presentation).

Let B be a vector space and $C \subset B$ a cone with the following properties.

- C is convex,
- $C \cap -C = \{0\}$,
- if α_n is a sequence of real numbers such that $\alpha_n \to \alpha$ and $x - \alpha_n y \in C$ $\forall n$ then $x - \alpha y \in C$. This property is called "integral closure".

For such a cone, the pseudo-metric δ_C on C is defined in the following way. For $x, y \in C$,

$$\mu(x,y) = \inf\{\beta > 0 \text{ such that } \beta x - y \in C\}.$$

with the convention: $\mu(x,y) = \infty$ if the corresponding set is empty. Let $\delta_C(x,y) = \log \mu(x,y)\mu(y,x)$. δ_C is not necessarily finite. Moreover, it is projective: if x and x_1 are proportional then for any $y \in C$, $\delta_C(x,y) = \delta_C(x_1,y)$.

The following two results reveal the usefulness of projective metrics.

Let C and C' be two cones, P a linear operator $P : C \to C'$. Let Γ denote the diameter of PC in C':

$$\Gamma = \sup_{f,g \in C} \delta_{C'}(Pf, Pg) \leq \infty.$$

THEOREM 3.1 [Bi] *For any f, g in C, we have:*

$$\delta_{C'}(Pf, Pg) \leq \tanh\left(\frac{\Gamma}{4}\right) \delta_C(f,g).$$

This theorem implies that $P : C \to C'$ is always a contraction (in the weak sense) for the projective metrics. If $\Gamma < \infty$ then it is a strict contraction.

The following result shows that the projective metric δ_C dominates norms on B which are well-behaved: A norm $\| \ \|$ on B is *adapted to C* if for f and g in B such that both $f+g$ and $f-g$ belong to C then $\|g\| \leq \|f\|$. $\rho : C \to \mathbb{R}_+$ is a *homogeneous form adapted to C* if, for any $\lambda > 0$ and $f \in C$, $\rho(\lambda f) = \lambda \rho(f)$ and if $f - g \in C$ implies $\rho(g) \leq \rho(f)$.

THEOREM 3.2 [Bi], [Li]. *Let C be a cone, let $\| \ \|$ and ρ be adapted to C. For any f and g in C such that $\rho(f) = \rho(g) \neq 0$ we have:*

$$\|f - g\| \leq (e^{\delta(f,g)} - 1)\min(\|f\|, \|g\|).$$

4 PROOF OF THEOREM 2.1

As usual, the *transfer operator* acting on bounded functions is defined by:

$$\mathcal{L}_0 f(x) = \sum_{Fy=x} \frac{1}{JF(y)} f(y).$$

The measure $\hat{\nu}$ is conformal for $\mathcal{L}_0$ in the following sense: for any bounded function f,

$$\int \mathcal{L}_0 f d\hat{\nu} = \int f d\hat{\nu}.$$

First, we are going to construct a $\hat{\nu}$-a.c.i.m. and discuss its mixing properties (section 4.1). Then will construct a sequence of cones C_j and a sequence of integers k_j such that $\mathcal{L}^{k_j}$ maps C_{j-1} into C_j with uniformly bounded diameter where $\mathcal{L}$ is the normalized tranfer operator (sections 4.2, 4.3).

4.1 Construction of a $\hat{\nu}$-a.c.i.m. and mixing properties

For $s \in \mathbb{N}$, the *s-cylinders* are the non empty sets of the form: $\bigcap_{i=0}^{s-1} F^{-i} A_i$ with $A_i \in \mathcal{P}$. Recall that $\mathcal{P}$ is obtained by translation from the partition of Δ_0 according to the return time $\hat{R}$ and $\mathcal{Z}^{\hat{R}}$-itinerary.

For $k \in \mathbb{N}$ and $x \in \Delta$, $C_k(x)$ denotes the k-cylinder which contains x.

The following lemmas are technical tools to study $\mathcal{L}_0^n$.

LEMMA 4.1 *There exists $C < \infty$ such that for any $\ell \in \mathbb{N}$ and any $x \in \Delta_\ell$ and $k \in \mathbb{N}$ with $F^k x \in \Delta_0$,*

$$C^{-1}\hat{\nu}(C_k(x)) \leq \frac{1}{JF^k(x)} \leq C\hat{\nu}(C_k(x)).$$

Proof: Let $x \in \Delta_\ell$ such that $F^k(x) \in \Delta_0$. The Markov property and the large image property (A.IV) imply that $\hat{\nu}(F^k C_k(x)) \geq \eta > 0$. The bounded distortion property (A.III) gives:

$$\begin{array}{rcl} C^{-1}\dfrac{\hat{\nu}(C_k(x))}{\hat{\nu}(F^k C_k(x))} & \leq \dfrac{1}{JF^k(x)} \leq & C\dfrac{\hat{\nu}(C_k(x))}{\hat{\nu}(F^k C_k(x))} \\ C^{-1}\dfrac{\hat{\nu}(C_k(x))}{1} & \leq \dfrac{1}{JF^k(x)} \leq & C\dfrac{\hat{\nu}(C_k(x))}{\eta} \end{array}$$

The Lemma is proved. ■

LEMMA 4.2 *There exists $K < \infty$ such that:*

- *for all $x \in \Delta$, all $n \in \mathbb{N}$, $\mathcal{L}_0^n \mathbf{1}(x) \leq K$.*
- *for all x, y in a given $B_{j,\ell}$ and all $n \in \mathbb{N}$,:*

$$|\mathcal{L}_0^n \mathbf{1}(x) - \mathcal{L}_0^n \mathbf{1}(y)| \leq K d_0(x, y). \tag{4.1}$$

Proof:

The upper bound $\mathcal{L}_0^n\mathbf{1} \le K$ follows from Lemma 4.1, writing:

$$\mathcal{L}_0^n\mathbf{1}(x) = \sum_{x'\in F^{-n}x} \frac{1}{JF^n(x')} \le C \sum_{x'\in F^{-n}x} \hat\nu(C_n(x')) \le C \tag{4.2}$$

Let x and y belong to one $B_{j,\ell}$. Their preimages by F^n are paired, i.e., if $F^n x' = x$, there is exactly one $y' \in C_n(x')$ such that $F^n y' = y$. So, using (A.IV), we get:

$$\begin{aligned} |\mathcal{L}_0^n\mathbf{1}(x) - \mathcal{L}_0^n\mathbf{1}(y)| &= \left|\sum_{F^n x'=x} JF^n(x')^{-1} - \sum_{F^n x'=x} JF^n(y')^{-1}\right| \\ &= \left|\sum_{F^n x'=x} JF^n(x')^{-1}\left(\frac{JF^n(x')}{JF^n(y')} - 1\right)\right| \\ &\le \mathcal{L}_0^n\mathbf{1}(x)(e^{d_0(x,y)} - 1) \\ &\le K d_0(x,y) \end{aligned}$$

(4.1) is proved. ■

Corollary 4.3 *F admits a $\hat\nu$ a.c.i.m.*

Proof: By Lemma 4.2, the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\mathcal{L}_0^i\mathbf{1}$ is relatively compact for the topology of uniform convergence on compact subsets (this is Arzela-Ascoli theorem on separable spaces). Each limit point h of this sequence is a non zero fixed point for $\mathcal{L}_0$ (by Lebesgue's dominated theorem, $\hat\nu(h) = 1$), so that $\hat\mu = h\hat\nu$ is a $\hat\nu$-a.c.i.m. ■

The system $(\Delta, F, \hat\nu, \mathcal{P})$ has a Markov structure in the sense that for each $P \in \mathcal{P}$, $F(P)$ is a union of atom of $\mathcal{P}$. According to [ADU], we will say that F is *aperiodic* if:

$$\forall\ P,\ P' \in \mathcal{P}\ \exists N \in \mathbb{N} \text{ such that } \hat\nu(F^{-n}P \cap P') > 0\ \forall\ n \ge N. \tag{4.3}$$

The existence of a $\hat\nu$-a.c.i.m. implies that the recurent part is non empty (it contains the support of $\hat\mu$). Up to decomposition into aperiodic components, *we may and shall assume that F is aperiodic.* Remark that aperiodicity implies that any s-cylinder has positive $\hat\nu$-measure. The following lemma implies that any s-cylinder also has $\hat\mu$ positive measure.

Lemma 4.4 *Aperiodicity of F implies that $\inf_{n\in\mathbb{N}}\inf_{x\in\Delta}\mathcal{L}_0^n\mathbf{1}(x) > 0$. So we have $0 < \inf h \le \sup h < \infty$.*

Proof: Let K be given by Lemma 4.2. Remark that for n, $k \in \mathbb{N}$, $x \in \Delta$,

$$\mathcal{L}_0^{n+k}\mathbf{1}(x) \le K\mathcal{L}_0^n\mathbf{1}(x). \tag{4.4}$$

Moreover, we have for $j = 1, \ldots, p$, any $n \in \mathbb{N}$, $x, y \in B_j$,

$$\mathcal{L}_0^n\mathbf{1}(x) = \sum_{F^n x'=x} JF(x')^{-1} \le K d_0(x,y) \sum_{F^n y'=y} JF(y')^{-1} \le K\mathcal{L}_0^n\mathbf{1}(y). \tag{4.5}$$

So, for each j, either $\sup_{B_j} \mathcal{L}_0^n \mathbf{1} \to 0$ or $\mathcal{L}_0^n \mathbf{1}(x) \geq \alpha$ for any $x \in B_j$, $n \in \mathbb{N}$, for some $\alpha > 0$. On the one hand, assume $\mathcal{L}_0^n \mathbf{1}(x) \geq \alpha$ for any $x \in B_j$, $n \in \mathbb{N}$, for some $\alpha > 0$, let us consider $\mathcal{L}_0^n \mathbf{1}(x)$ for $x \in B_q$. The aperiodicity implies that there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $C_n(z) \subset F^{-n} B_j \cap B_q$ for some $z \in B_q$ i.e. $F^n z_0 = z$ for some $z_0 \in B_j$. We have:

$$\begin{aligned} \mathcal{L}_0^{n_0+p} \mathbf{1}(z) &= \sum_{F^{n_0} z' = z} g_{n_0}(z') \mathcal{L}_0^p \mathbf{1}(z') \\ &\geq g_{n_0}(z_0) \alpha := \alpha' \end{aligned}$$

Using (4.4) and (4.5), we get $\mathcal{L}_0^n \mathbf{1}(x) \geq \alpha'/K$ for any $n \in \mathbb{N}$, $x \in B_q$. This proves that either $\sup_{B_j} \mathcal{L}_0^n \mathbf{1} \to 0$ for all $j = 1, \ldots, p$ or there exists $K' > 0$ such that $\inf_{j=1,\ldots,p} \inf_{B_j} \mathcal{L}_0^n \mathbf{1} \geq K'$ for any $n \in \mathbb{N}$.
Assume $\sup_{B_j} \mathcal{L}_0^n \mathbf{1} \to 0$ for all $j = 1, \ldots, p$, let ε be fixed, let n_0 be such that for $n \geq n_0$, $\mathcal{L}_0^n \mathbf{1}(x) \leq \varepsilon$ for all $x \in \Delta_0 = \bigcup_{j=1,\ldots,p} B_j$, then $\mathcal{L}_0^n \mathbf{1}(x) < \varepsilon$ for any $x \in \Delta_\ell$, provided $n - \ell \geq n_0$, choose $n \geq n_0$ so that $\hat{\nu}(\bigcup_{\ell > n-n_0} \Delta_\ell) < \varepsilon$, we have:

$$\begin{aligned} \int_\Delta \mathcal{L}_0^n \mathbf{1} d\hat{\nu} &= \sum_\ell \int_{\Delta_\ell} \mathcal{L}_0^n \mathbf{1} d\hat{\nu} \\ &= \sum_{\ell \leq n-n_0} \int_{\Delta_\ell} \mathcal{L}_0^n \mathbf{1} d\hat{\nu} + \sum_{\ell > n-n_0} \int_{\Delta_\ell} \mathcal{L}_0^n \mathbf{1} d\hat{\nu} \\ &\leq \varepsilon(1+K). \end{aligned}$$

But, since $\hat{\nu}$ is conformal, $\hat{\nu}(\mathcal{L}_0^n \mathbf{1}) = 1$, we are lead to a contradiction. ∎

Theorems 2.5 and 3.2 in [ADU] imply that $(F, \hat{\mu})$ is then exact hence mixing and that $\hat{\mu}$ is the only $\hat{\nu}$-a.c.i.m. The following lemma is a direct consequence of mixing.

LEMMA 4.5 *There exists positive numbers A and B such that for any f, $g \in L^2(\hat{\nu})$, with $\hat{\mu}(f) > 0$, $\hat{\mu}(g) > 0$, there exists n_0 such that for $n \geq n_0$,*

$$B \leq \frac{\hat{\mu}(f \circ F^n \cdot g)}{\hat{\mu}(f)\hat{\mu}(g)} \leq A.$$

4.2 Definition of the cones

Let $\mathcal{L}$ be the normalized transfer operator: $\mathcal{L} f = \frac{1}{h} \mathcal{L}_0(fh)$, it satisfies: $\mathcal{L}\mathbf{1} = \mathbf{1}$. The Jacobian of F with respect to $\hat{\mu}$ is:

$$\frac{JF \cdot h \circ F}{h}$$

¿From now one, JF will be the Jacobian of F with respect to the invariant measure $\hat{\mu}$. Remark that the proof of Lemma 4.1 and Lemma 4.4 gives for some $C > 0$:

$$C^{-1} \hat{\mu}(C_k(x)) \leq \frac{1}{JF^k(x)} \leq C \hat{\mu}(C_k(x)) \tag{4.6}$$

For technical reasons, we need to introduce $\gamma < \gamma' < 1$ (recall that γ is given by (A.I)). We shall denote by $\hat{\mu}_\gamma$ the (finite) measure defined by $\hat{\mu}_\gamma(A) = \sum_\ell \hat{\mu}(A \cap \Delta_\ell) \cdot \gamma'^{-\ell}$. Let s and t be two integers which will be fixed later (see the proof of Proposisition 4.9). Let $\mathcal{Q} = \{P_\infty\} \cup \mathcal{Q}_1$ be the finite partition of Δ defined by:

- $P_\infty = \{(x, \ell) \in \Delta \ / \ \hat{R}(x) \geq t\}$
- $\mathcal{Q}_1$ is the partition of $\Delta \setminus P_\infty$ into s-cylinders.

Remark 4.6 *The $\hat{\mu}_{\gamma'}$-measure of P_∞ is as small as we want, provided t is large enough. We may choose s to get $R_0(s) := \sum_{k \geq s+1} \omega_k(g)$ as small as we want. $R_0(s)$ is the d_0-diameter of the partition $\mathcal{Q}_1$.*

The following lemma directly follows from Lemmas 4.4 and 4.5 applied to the (finitely many) functions $\mathbf{1}_P$ and $\mathbf{1}_P \cdot \psi$ with $P \in \mathcal{Q}$, $\psi(x) = \gamma'^{-\ell}$ for $x \in \Delta_\ell$.

LEMMA 4.7 *There exist $A > 0$ and $B > 0$ such that for all s and t, there exists k_0' such that for $k \geq k_0'$ and $P, P' \in \mathcal{Q}$,*

$$B \leq \frac{\hat{\mu}(F^{-k}P \cap P')}{\hat{\mu}(P)\hat{\mu}(P')} \leq A \text{ and } B \leq \frac{\hat{\mu}_{\gamma'}(F^{-k}P \cap P_\infty)}{\hat{\mu}_{\gamma'}(P_\infty)\hat{\mu}(P)} \leq A$$

Let us define inductively a sequence d_j of metrics and a sequence k_j of integers. Let $k_0 \geq k_0'$ and $D > 1$ to be chosen later (see the proof of Proposition 4.9). For $\ell \geq 1$, define:

$$k_\ell = \inf\{k \geq k_0 \ : \ R_{\ell-1}(s+k) \leq \frac{1}{D} R_0(s)\}.$$

For $x, \ y \in \Delta$, let

$$d_\ell(x,y) = D[R_{\ell-1}(k_\ell + s(x,y)) + R_0(s(x,y))].$$

Define $R_\ell(n) = \sup_{s(x,y) \geq n} d_\ell(x,y)$ for $\ell \geq 0$. The d_ℓ-diameter of $\mathcal{Q}_1$ is $R_\ell(s)$.

The following lemma follows directly from the above definitions. It proves that the d_ℓ-diameter of the partition $\mathcal{Q}_1$ is uniformly bounded (in ℓ).

LEMMA 4.8 *For all $\ell \in \mathbb{N}$, $R_\ell(s) \leq R_0(s)(D+1)$.*

We are now in position to define our cones. They will depend on three positive numbers a, b, c which will be fixed later (see the proof of Proposition 4.9).

For $j \in \mathbb{N}$, $C_j(a,b,c)$ is the cone of functions f that satisfy:

1. $\forall P \in \mathcal{Q}, \ 0 \leq \dfrac{1}{\hat{\mu}(P)} \displaystyle\int_P f d\hat{\mu} \leq a \int_\Delta f d\hat{\mu}.$
2. for all $x, \ y$ in the same $B_{q,\ell}$, $|f(x) - f(y)| \leq d_j(x,y) \cdot b \displaystyle\int_\Delta f d\hat{\mu}.$

3. $\sup_{P_\infty \cap \Delta_\ell} |f| \leq \frac{c}{\gamma'^\ell} \int_\Delta f \, d\hat{\mu}$.

Of course, $C_j(a,b,c)$ depends also on s and t (via the partition $\mathcal{Q}$).

4.3 Contraction of the cones

The crucial result on our sequence of cones is the following proposition.

Proposition 4.9 *For any a, b, c large enough, for any $D > 1$ small enough, there exist $s_0 \in \mathbb{N}$, $t_0 \in \mathbb{N}$, such that for any $j \in \mathbb{N}$, $s \geq s_0$ and $t \geq t_0$, $\mathcal{L}^{k_j}$ maps $C_{j-1}(a,b,c)$ into $C_j(D^{-1}a, D^{-1}b, D^{-1}c)$ and the projective diameter of $\mathcal{L}^{k_j} C_{j-1}(a,b,c)$ into $C_j(a,b,c)$ is bounded above by $\Gamma := 2\log\left(\frac{D+1}{D-1}\right)$.*

Before proving Proposition 4.9, let us state two properties of the cones $C_j(a,b,c)$. The first one is a bound on the supremum on Δ_ℓ of functions in the cones. The second one comes from a "Lasota-Yorke type inequality".

Lemma 4.10 *For any $j \in \mathbb{N}$, for any $f \in C_j(a,b,c)$, for any $\ell \in \mathbb{N}$, for any $P \in \mathcal{Q}_1$, we have:*

$$\inf_{x \in P} |f(x)| \geq -b\, R_j(s) \int_\Delta f \, d\hat{\mu} + \frac{1}{\hat{\mu}(P)} \int_P f \, d\hat{\mu}$$

$$\sup_{x \in P} |f(x)| \leq b\, R_j(s) \int_\Delta f \, d\hat{\mu} + \frac{1}{\hat{\mu}(P)} \int_P f \, d\hat{\mu}, \tag{4.7}$$

$$\inf_{x \in P_\infty \cap \Delta_\ell} |f(x)| \geq \frac{1}{\hat{\mu}(P_\infty)} \int_{P_\infty} f \, d\hat{\mu} - \frac{2c}{\gamma'^\ell} \int_\Delta f \, d\hat{\mu}$$

$$\sup_{x \in P_\infty \cap \Delta_\ell} |f(x)| \leq \frac{c}{\gamma'^\ell} \int_\Delta f \, d\hat{\mu}. \tag{4.8}$$

The proof of Lemma 4.10 is a direct consequence of the definition of $C_j(a,b,c)$.

Lemma 4.11 *Let f belong to $C_{j-1}(a,b,c)$, for any integers ℓ, j and x, $y \in B_{q,\ell}$,*

$$|\mathcal{L}^{k_j} f(x) - \mathcal{L}^{k_j} f(y)| \leq b \int_\Delta f \, d\hat{\mu} [R_{j-1}(k_j + s(x,y)) + \frac{cC\hat{\mu}_{\gamma'}(P_\infty) + bR_{j-1}(s) + a}{b} R_0(s(x,y))].$$

Proof: Let $f \in C_{j-1}(a,b,c)$ and x, $y \in B_{q,\ell}$. If $\ell \geq k_j$ then x and y have exactly one preimage $x' \in B_{q,\ell-k_j}$ and $y' \in B_{q,\ell-k_j}$ by F^{k_j} and

$$\begin{aligned} |\mathcal{L}^{k_j} f(x) - \mathcal{L}^{k_j} f(y)| &= |f(x') - f(y')| \\ &\leq b \int_\Delta f \, d\hat{\mu}\ R_{j-1}(s(x',y')) = b \int_\Delta f \, d\hat{\mu}\ R_{j-1}(s(x,y) + k_j). \end{aligned}$$

If $\ell < k_j$ then $\mathcal{L}^{k_j} f(x) = \mathcal{L}^{k_j-\ell} f(x')$ where $x' \in B_q$ is the preimage of x by F^ℓ. Thus, we are lead to consider $|\mathcal{L}^r f(x) - \mathcal{L}^r f(y)|$ for $r \leq k_j$, x,$y \in \Delta_0$. We have:

$$\begin{aligned}
|\mathcal{L}^r f(x) - \mathcal{L}^r f(y)| \leq & \sum_{F^r(x')=x} JF^r(x')^{-1} |f(x') - f(y')| \\
+ & \sum_{F^r(x')=x} |f(y')| JF^r(y')^{-1} \left| \frac{JF^r(y')}{JF^r(x')} - 1 \right| \\
\leq & \; bR_{j-1}(r + s(x,y)) \sup \mathcal{L}^r \mathbf{1} \int\limits_\Delta f d\hat{\mu} \\
+ & \; R_0(s(x,y)) \left[\sum_{\substack{F^r y' = y \\ y' \in P_\infty}} |f(y')| JF^r(y')^{-1} + \sum_{\substack{F^r y' = y \\ y' \notin P_\infty}} |f(y')| JF^r(y')^{-1} \right] \\
\leq & \; bR_{j-1}(r + s(x,y)) \int\limits_\Delta f d\hat{\mu} \\
+ & \; R_0(s(x,y)) \int\limits_\Delta f d\hat{\mu} [cC\hat{\mu}_{\gamma'}(P_\infty) + bR_{j-1}(s) + a],
\end{aligned}$$

(we used (4.6) and (4.8) to bound $\sum_{\substack{F^r y' = y \\ y' \in P_\infty}} |f(y')| JF^r(y')^{-1}$ and then (4.7) to bound $\sum_{\substack{F^r y' = y \\ y' \notin P_\infty}} |f(y')| JF^r(y')^{-1}$.) This proves the result for x, y in Δ_0 (take $r = k_j$). For x and y in $B_{q,\ell}$ if x' and y' are the unique preimage of x and y by F^ℓ then x', $y' \in B_q$ and $R_{j-1}(k_j - \ell + s(x',y')) = R_{j-1}(k_j + s(x,y))$, $R_0(s(x',y')) \leq R_0(s(x,y))$. So,

$$\begin{aligned}
|\mathcal{L}^{k_j} & f(x) - \mathcal{L}^{k_j} f(y)| = |\mathcal{L}^{k_j-\ell} f(x') - \mathcal{L}^{k_j-\ell} f(y')| \\
& \leq b \int\limits_\Delta f d\hat{\mu} [R_{j-1}(k_j - \ell + s(x',y')) + b^{-1} R_0(s(x',y'))(cC\hat{\mu}'_\gamma(P_\infty) + bR_{j-1}(s) + a)] \\
& \leq b \int\limits_\Delta f d\hat{\mu} [R_{j-1}(k_j + s(x,y)) + \frac{cC\hat{\mu}_{\gamma'}(P_\infty) + bR_{j-1}(s) + a}{b} R_0(s(x,y))].
\end{aligned}$$

■

Proof of Proposition 4.9: Let f belong to $C_{j-1}(a,b,c)$. We are going to check that each of the three conditions involved in the definition of the cones is contracted by $\mathcal{L}^{k_j}$.
<u>Condition 1.</u> Let $P \in \mathcal{Q}$.

$$\frac{1}{\hat{\mu}(P)} \int\limits_P \mathcal{L}^{k_j} f d\hat{\mu} = \sum_{P' \in \mathcal{Q}} \frac{1}{\hat{\mu}(P)} \int\limits_{F^{-k_j} P \cap P'} f d\hat{\mu}$$

$$
\begin{aligned}
&\le \sum_{P'\in\mathcal{Q}_1} \frac{\hat\mu(P'\cap F^{-k_j}P)}{\hat\mu(P')\hat\mu(P)} \left[\int_{P'} f\,d\hat\mu \;+\; bR_{j-1}(s)\int_{\Delta} f\,d\hat\mu\;\hat\mu(P')\right] \qquad \text{using (4.7)}\\
&+\; c\int_{\Delta} f\,d\hat\mu \sum_{\substack{\ell\in\mathbb{N}\\ \Delta_\ell\cap P_\infty\neq\varnothing}} \frac{\hat\mu(P_\infty\cap F^{-k_j}P\cap\Delta_\ell)\cdot\hat\mu_{\gamma'}(P_\infty)}{\hat\mu(P)\cdot\hat\mu_{\gamma'}(P_\infty)\cdot\gamma'^{\ell}} \qquad \text{using (4.8)}\\
&\le\; B\int_{\Delta} f\,d\hat\mu\,[1+bR_{j-1}(s)+c\hat\mu_{\gamma'}(P_\infty)] \qquad \text{using Lemma 4.7.}
\end{aligned}
\tag{4.9}
$$

Similarly,

$$
\frac{1}{\hat\mu(P)}\int_{P} \mathcal{L}^{k_j} f\,d\hat\mu \ge \int_{\Delta} f\,d\hat\mu\,[A-B(bR_{j-1}(s)+2c\hat\mu_{\gamma'}(P_\infty))] \tag{4.10}
$$

Condition 2. Lemma 4.11 gives:

$$
\begin{aligned}
|\mathcal{L}^{k_j}f(x)-\mathcal{L}^{k_j}f(y)| \;&\le\; b\int_{\Delta} f\,d\hat\mu[R_{j-1}(k_j+s(x,y))\\
&+\; \frac{cC\hat\mu_{\gamma'}(P_\infty)+bR_{j-1}(s)+a}{b}R_0(s(x,y))].
\end{aligned}
\tag{4.11}
$$

Condition 3. Let x belongs to $\Delta_\ell\cap P_\infty$. If $\ell\ge k_j$ then x has only one preimage x' by F^{k_j}, the definition of P_∞ implies that x' belong also to P_∞, so

$$
\begin{aligned}
|\mathcal{L}^{k_j}f(x)| = |f(x')| \;&\le\; \frac{c}{\gamma'^{\ell-k_j}}\int_{\Delta} f\,d\hat\mu\\
&\le\; \frac{c}{\gamma'^{\ell}}\gamma'^{k_0}\int_{\Delta} f\,d\hat\mu.
\end{aligned}
$$

If $\ell<k_j$ then $|\mathcal{L}^{k_j}f(x)| = |\mathcal{L}^{k_j-\ell}f(x')|$ where $x'\in\Delta_0$ is the unique preimage of x by F^{ℓ}. So, it suffices to consider $x\in\Delta_0$ and $p\le k_j$.

$$
\begin{aligned}
|\mathcal{L}^{p}f(x)| \;&\le\; \sum_{\substack{F^p x'=x\\ x'\in P_\infty}} \frac{|f(x')|}{JF^p(x')} \;+\; \sum_{\substack{F^p x'=x\\ x'\notin P_\infty}} \frac{|f(x')|}{JF^p(x')}\\
&\le\; c\int_{\Delta} f\,d\hat\mu\cdot C\hat\mu_{\gamma'}(P_\infty) + bR_{j-1}(s)\int_{\Delta} f\,d\hat\mu + a\int_{\Delta} f\,d\hat\mu\\
&\le\; \int_{\Delta} f\,d\hat\mu\,[cC\hat\mu_{\gamma'}(P_\infty)+bR_{j-1}(s)+a]\,.
\end{aligned}
$$

So, for any $\ell<k_j$, $x\in\Delta_\ell\cap P_\infty$,

$$
|\mathcal{L}^{k_j}f(x)| \le \int_{\Delta} f\,d\hat\mu\,[cC\hat\mu_{\gamma'}(P_\infty)+bR_{j-1}(s)+a]\,. \tag{4.12}
$$

Now choose the parameters a, b, c, D, k_0, s and t as follows:

- $D > 1$ such that $\dfrac{D+1}{D-1} \geq \dfrac{4B[1+\frac{A}{2B}]}{A}$,
- a such that $a \geq D[B + \frac{A}{2}]$,
- b such that $b \geq (2+a)$,
- $\sigma' < D^{-1}$, c such that $c > \dfrac{1+a}{D^{-1} - \sigma'}$,
- s such that $R_{j-1} \leq R_0(s)(D+1) \leq \frac{1}{b}$ and $R_{j-1} \leq R_0(s)(D+1) \leq \frac{A}{4Bb}$
- t such that $\hat{\mu}_{\gamma'}(P_\infty) \leq \frac{\sigma'}{C}$, $\hat{\mu}_{\gamma'}(P_\infty) \leq \frac{1}{cC}$ and $\hat{\mu}_{\gamma'}(P_\infty) \leq \frac{A}{4cBC}$,
- $k_0 \geq k_0'$ such that $D \leq \gamma'^{-k_0}$ (recall that k_0' is given by Lemma 4.7 and that $\gamma' < 1$).

With these choices, using (4.9, 4.11, 4.12), we show that $\mathcal{L}^{k_j} f$ belongs to the cone $C_j(D^{-1}a, D^{-1}b, D^{-1}c)$. Moreover, (4.10) leads to: $\forall P \in \mathcal{Q}$,

$$\frac{1}{\hat{\mu}(P)} \int_P \mathcal{L}^{k_j} f \, d\hat{\mu} \geq \frac{1}{4} A. \tag{4.13}$$

It remains to prove that the projective diameter of $\mathcal{L}^{k_j} C_{j-1}(a,b,c)$ into $C_j(a,b,c)$ is bounded by $\Gamma := 2 \log \frac{D+1}{D-1}$. A standard computation (see [KMS], [M]) gives that the projective diameter is bounded by

$$\begin{aligned} & 2 \log \max \left[\frac{D+1}{D-1}, \frac{\sup\limits_{f \in C_{j-1}(a,b,c)} \sup\limits_{P \in \mathcal{Q}} \hat{\mu}(P)^{-1} \int_P f \, d\hat{\mu}}{\inf\limits_{f \in C_{j-1}(a,b,c)} \inf\limits_{P \in \mathcal{Q}} \hat{\mu}(P)^{-1} \int_P f \, d\hat{\mu}} \right] \\ \leq \ & 2 \log \max \left[\frac{D+1}{D-1}, \frac{4B[1+\frac{A}{2B}]}{A} \right] \qquad \text{using (4.9, 4.13).} \end{aligned}$$

The choice of D concludes the proof.

■

4.4 Conclusion

To conclude the proof of Theorem 2.1, we need to derive from the projective metric bound obtained above, bounds on the correlations. The following lemma may be proved in a standard way (see [KMS] page 687, [M] Lemmas 3.9-3.10). For $d > 0$, let $\| \ \|_d$ be the norm on bounded functions defined by:

$$\|f\|_d = \max \left[d \left| \int_\Delta f d\hat\mu \right| , \ \sup_{P\in\mathcal{Q}} \hat\mu(P)^{-1} \left| \int_P f d\hat\mu \right| , \ \|f\|_\infty \right] .$$

Lemma 4.12 *The norm $\| \ \|_d$ and the homogeneous form $f \mapsto \hat\mu(f)$ are adapted to the cones $C_j(a,b,c)$, provided $d \geq \max[c, \ R_j b]$.*
For all $p \in \mathbb{N}$, the function $\mathcal{L}^p \mathbf{1}$ belongs to all the cones $C_j(a,b,c)$.
For any $f \in \mathbf{L}(\Delta)$, there exists $R(f) > 0$ such that $f + R(f)\mathbf{1} \in C_0(a,b,c)$, moreover $R(f) \leq C\|f\|_{\mathbf{L}(\Delta)}$.

Let $f \in C_0(a,b,c)$, by Proposition 4.9 (remark that $\mathbf{1} \in C_0(a,b,c)$), for any ℓ, $\mathcal{L}^{k_1+\cdots+k_\ell} f$ and $\mathcal{L}^{k_1+\cdots+k_\ell}\mathbf{1} = \mathbf{1}$ belong to C_ℓ. Applying $\ell - 1$ times Theorem 3.1, we get:

$$\delta_{C_\ell}(\mathcal{L}^{k_1+\cdots+k_\ell} f, \mathbf{1}) \leq [\tanh \frac{\Gamma}{4}]^{\ell-1} \cdot \delta_1(\mathcal{L}^{k_1} f, \mathbf{1}) \leq [\tanh \frac{\Gamma}{4}]^{\ell-1} \cdot \Gamma,$$

where Γ is given by Proposition 4.9. Remark that $\tanh \frac{\Gamma}{4} = D^{-1}$. Since the norm $\| \ \|_d$ is adapted to the cones $C_j(a,b,c)$ and is equivalent to the uniform norm, Theorem 3.2 gives for f in $C_0(a,b,c)$ with $\hat\mu(f) = 1$,

$$\|\mathcal{L}^{k_1+\cdots k_\ell} f - \mathbf{1}\|_\infty \leq \text{const} D^{-\ell}.$$

For $n \in \mathbb{N}$, let $\ell(n)$ be defined by:

$$n = k_1 + \cdots k_{\ell(n)} + r \text{ with } r < k_{\ell(n)+1},$$

we have

$$\|\mathcal{L}^n f - \mathbf{1}\|_\infty \leq \|\mathcal{L}^r \mathbf{1}\|_\infty \ \|\mathcal{L}^{k_1+\cdots+k_\ell} f - \mathbf{1}\|_\infty \leq \text{const} D^{-\ell(n)}.$$

For any function $f \in \mathbf{L}(\Delta)$, applying the above inequality to $\frac{f+R(f)\mathbf{1}}{\hat\mu(f)+R(f)}$ gives

$$\|\mathcal{L}^n f - \hat\mu(f)\|_\infty \leq \text{const} \cdot D^{-\ell(n)} \|f\|_{\mathbf{L}(\Delta)}.$$

The decay of correlations follows: for $f \in L$ and $g \in L^1(\hat\mu)$,

$$\begin{aligned} \left| \int_\Delta g \circ F^n \cdot f d\hat\mu - \int_\Delta f d\hat\mu \int_\Delta g d\hat\mu \right| &= \left| \int_\Delta g[\mathcal{L}^n(f) - \hat\mu(f)] d\hat\mu \right| \\ &\leq \text{const} \cdot D^{-\ell(n)} \|f\|_{\mathbf{L}(\Delta)} \|g\|_1. \end{aligned}$$

Thus we can take $u_n = D^{-\ell(n)}$.

We now prove that u_n has the announced behavior for exponential or polynomial sequences $\omega_n(g)$.

Estimates on u_n

To estimate the rate of mixing u_n, we have to analyse the asymptotic behavior of the sequence k_j. Recall that:

$$k_{\ell+1} = \inf\{k \geq k_0 : R_\ell(s+k) \leq \frac{R_0(s)}{D}\}$$

and observe that an easy induction using the definitions gives:

$$R_\ell(n) = D^\ell R_0(k_1 + \cdots + k_\ell + n) + \sum_{i=1}^{\ell} D^{\ell+1-i} R_0(k_{i+1} + \cdots + k_\ell + n).$$

where $R_0(m) = \sum_{k>m} \omega_k(g)$.
¿From these remarks we get the following results.

- If there exists $0 < \rho < 1$ such that $\omega_n = O(\rho^n)$ then $R_0(n) = O(\rho^n)$ and one can take $k_j = p = \log D / |\log \rho| + 10$. This implies $u_n = D^{-\ell(n)} = D^{-n/p}$.
- If there exists $\alpha > 1$ such that $\omega_n = O(n^{-\alpha})$ then $R_0(n) = O(n^{-\alpha+1})$ and $R_\ell(s + k_{\ell+1}) = O\left(R_0(s + k_{\ell+1}) \cdot D^\ell\right)$ so, the k_ℓ's may be taken to be $\sim \text{const} D^{\frac{\ell}{\alpha-1}}$ so $\ell(n) \sim (\alpha-1)\frac{\log n}{\log D} + \text{const}$ and thus

$$u_n = D^{-\ell(n)} = O(e^{-(\alpha-1)\log n}) = O(n^{-\alpha+1}).$$

References

[ADU] J. Aaronson, M. Denker & M. Urbanski *Ergodic theory for Markov fibered systems and parabolic rational maps.* Trans. Amer. Math. Soc. (1993), **337** (2), 495-548.

[Bi] G. Birkohff *Extensions of Jentzch's theorem.* Trans. Amer. Math. Soc. (1957), **85**, 219-227.

[Bo] R. Bowen *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Lecture Notes in Mathematics, Vol. **470**. Springer-Verlag, Berlin, 1975.

[Bu1] J. Buzzi *Absolutely continuous invariant measures for generic piecewise affine and expanding maps* Int. J. Chaos & Bif. (1999), **9** (9), 1743-1750.

[Bu2] J. Buzzi *A.c.i.m.'s as equilibrium states for piecewise invertible maps* pré-publication I.M.L. **98-23** (1998) (see `http://iml.univ-mrs.fr/~buzzi`)

[BPS] J. Buzzi, F. Paccaut, B. Schmitt *Conformal measures for multidimensional expanding maps.* Preprint Laboratoire de Topologie #196 (Dijon, 1999) (see `http://iml.univ-mrs.fr/~buzzi`)

[Co] W. Cowieson, *Piecewise smooth expanding maps in* $\mathbb{R}^d$, Ph.D. Thesis, U.C.L.A., Berkeley (see `http://math.usc.edu/~cowieson`).

[DKS] M. Denker, C. Grillenberger, K. Sygmund *Ergodic theory on compact spaces.* Lecture notes in mathematics **527**, Springer, Berlin, 1976.

[FS] P. Ferrero, B. Schmitt *Ruelle Perron Frobenius theorems and projective metrics.* Colloque Math. Soc. J. Bolyai Random Fields. Estergom (Hungary) (1979).

[Go] P. Gora *Properties of invariant measures for piecewise expanding one-dimensional transformations with summable oscillations of derivative.* Erg. Th. and Dyn. Syst. (1994), **14**, 475-492.

[Ho] F. Hofbauer *Piecewise invertible dynamical systems.* Probab. Theory Relat. Fields **72** (1986), no. 3, 359–386.

[HK] F. Hofbauer, G. Keller *Ergodic properties of invariant measures for piecewise monotonic transformations* Math. Z. **180**, (1982), 119–140.

[Ke] G. Keller *Markov extensions, zeta functions, and Fredholm theory for piecewise invertible dynamical systems.* Trans. Amer. Math. Soc. **314** (1989), no. 2, 433–497.

[KMS] A. Kondah, V. Maume & B. Schmitt *Vitesse de convergence vers l'état d'équilibre pour des dynamiques markoviennes non höldériennes.* Ann. Inst. Poincaré Sec. Prob. Stat. (1997) **33** (6) 675-695.

[Li] C. Liverani *Decay of correlations.* Ann. of Math. (1995), **142** (2), 239-301

[LSV] C. Liverani, B. Saussol, S. Vaienti *Conformal measure and decay of correlations for covering weighted systems* Ergodic th. & dynam. syst. **18**, 1998, 1399–1420.

[M] V. Maume-Deschamps *Projective metrics and mixing properties on towers.* Preprint (see `http://www.unige.ch/math/biblio/preprint/pp99.html`).

[R] M. Rychlik *Bounded variation and invariant measures* Studia Math. **76**, (1983), 69–80.

[Sc] B. Schmitt *Existence of acip for expanding maps of the interval.* Dynamical Systems and Ergodic Theory Banach Center Publications. Warszawa (1989).

[Y1] L.-S. Young *Return times and rates of mixing* Israel J. Math **110**, (1999), 153–188.

[Y2] L.-S. Young *Decay of Correlations*, talk given at the International Conference on Dynamical Systems, *La Bussière*, July 1999.

e-mail: `buzzi@iml.univ-mrs.fr`
I.M.L.
163, av. de Luminy - case 930
13288 Marseille cedex 09 / FRANCE

e-mail: `vmaume@topolog.u-bourgogne.fr`
LABORATOIRE DE TOPOLOGIE
Université de Bourgogne
9,Avenue Alain Savary - B.P. 47870
21078 Dijon Cedex FRANCE