Discontinuity of the Magnetization in Diluted $O(n)$–Models *

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Abstract

We study the annealed site–diluted versions of the classical $O(n)$ Heisenberg ferromagnets. It is shown that if the temperature is low enough, then at some value of the chemical potential there is phase coexistence between a magnetized, high–density state (liquid-crystal state) and a low density state (gaseous state) with no magnetic order.

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1. Introduction

A subject of continuing interest is the behavior of (annealed) dilute spin–systems with continuous symmetry. The prototypical model is the diluted XY-model in d-dimensions that is described by the formal Hamiltonian

$$\mathcal{H} = -J \sum_{\langle \underline{r}, \underline{r}' \rangle} n_{\underline{r}} n_{\underline{r}'} \cos(\theta_{\underline{r}} - \theta_{\underline{r}'}) - \mu \sum_{\underline{r}} n_{\underline{r}} - \lambda \sum_{\langle \underline{r}, \underline{r}' \rangle} n_{\underline{r}} n_{\underline{r}'}, \tag{1.1}$$

where the sum $\sum_{\langle \underline{r}, \underline{r}' \rangle}$ runs over all n.n. bonds, $\sum_{\underline{r}}$ is the sum over all sites, $n_{\underline{r}} = 0$ or 1 denotes the presence or absence of a particle at the site $\underline{r} \in \mathbb{Z}^d$ and θ is the usual angular variable: $0 \leq \theta < 2\pi$. Such models are of physical relevance e. g. to nematic liquid crystals, cf. the brief discussion in [Z], [1] a standard reference is [dGP]. Here, let us discuss the case $J > 0$, $\lambda \geq 0$, $\beta \gg 1$ while μ varies between minus and plus infinity. For μ large and negative it is clear that the interaction is completely analytic. Hence one has a dilute paramagnetic phase. In [AZ], it was proved that in $d \geq 3$, for μ sufficiently large and positive there is non–vanishing spontaneous magnetization. The results of [CKS 2] are of a different nature. There it was shown that there is a discontinuous transition in the particle density at some μ_t $(= \mu_t(\beta, \lambda))$. Such results are not peculiar to the XY–model; both sets of results hold for models of the above type with $O(n)$–symmetry. The principal purpose of this paper is to combine these results. That is to say if $d \geq 3$, λ is nonnegative and the temperature is low, the two transitions are one and the same. In other words, there is just one transition, which occurs between the low density non-magnetized phase and high density magnetized one.

We caution the reader that these results are by no means "generic". For negative values of λ, there is a crystalline nonmagnetized phase (featuring preferential occupation of the even or odd sublattice) which may separate the region of complete analyticity from the magnetized phase as μ is varied [CKS 2]. Further, if $\lambda \gg J \sim 1$, it is not hard to show (as we do here) that at fairly high values of β there is a phase transition in the particle density but no possibility of magnetization. In these cases there are two transitions, along certain curves in the (β, λ) plane.

The rest of this paper is organized along the following lines: In Section II we discuss our principal tools, namely various standard estimates and results using

[1] For such systems, due to the symmetry of the molecules that the spins represent, the angular interactions are usually given by $\cos 2(\theta_{\vec{r}} - \theta_{\vec{r}'})$ rather than what appears in Equ. (1). However, by change of variables, it is easy to see that the "$\cos\theta$" and "$\cos 2\theta$" models are equivalent. For $O(n)$ models with $n > 2$ one presumes that the differences are more substantial.

reflection positivity. Most of the material in this Section II appears elsewhere; this section is included for completeness and is intended for readers already familiar with reflection positivity arguments. Those who are unfamiliar with these methods are encouraged to consult the standard references e.g. [FILS], [FL], [FSS], [S]. On the basis of these tools, we can immediately establish that (under the conditions described at the end of the first paragraph) there is a μ_t above which the magnetization – and hence the particle density – is close to one, and below which the particle density – and hence the magnetization – is close to zero. This result is not completely satisfactory since, on physical grounds, one expects that for $\mu < \mu_t$, the magnetization should actually vanish. The vanishing of the magnetization for $\mu < \mu_t$ is established in Section III. Finally, in Section IV, we discuss the cases where λ/J is large.

Although physical motivations for various details of the interactions may be of importance in certain contexts, it seems to the authors that the results presented here hold in a variety of systems. The important features are a ferromagnetic interaction *and* a mechanism for proving a ferromagnetic transition. We will forego great generality and treat the simplest continuous cases, namely the $O(n)$ versions of the Hamiltonian in (1.1) with $n \geq 2$.

2. Basic tools and basic results

Here we recapitulate the results in [CKS 1], [CKS 2], [AZ] and [Z].

Consider the $O(n)$ version of the Hamiltonian in (1.1) on the d–dimensional torus Λ_L of side L written in the form

$$H_L = -\sum_{\underline{r},\underline{r}'} M_{\underline{r},\underline{r}'}(n_{\underline{r}}\vec{s}_{\underline{r}} \cdot n_{\underline{r}'}\vec{s}_{\underline{r}'}) - \mu \sum_{\underline{r}} n_{\underline{r}} - \lambda \sum_{\langle \underline{r},\underline{r}'\rangle} n_{\underline{r}} n_{\underline{r}'}. \tag{2.1}$$

In the above, $M_{\underline{r},\underline{r}'} = -2d$ if $\underline{r} = \underline{r}'$, $M_{\underline{r},\underline{r}'} = 1$ if $|\underline{r} - \underline{r}'| = 1$ and vanishes otherwise, the spins $\vec{s}_{\underline{r}}$ are unit n–dimensional vectors and we assume that $\lambda \geq 0$. Since it is not strictly necessary, we have omitted J as an independent parameter.

Assuming further that L is even, let $\mathbb{P}$ denote any "plane between sites" (i.e. $\{\underline{r} \mid x_i = k + 1/2\} \cup \{\underline{r} \mid x_i = k + 1/2 + L/2\}$) and $\vartheta_{\mathbb{P}}$ the reflection in $\mathbb{P}$. It is easy to see that $-H$ is of the form $\sum_k A_k \vartheta_{\mathbb{P}} A_k + \sum_\ell [B_\ell + \vartheta_{\mathbb{P}} B_\ell]$ and hence is reflection positive with respect to $\vartheta_{\mathbb{P}}$.

The well known argument for the infrared bounds goes as follows: Let $\mathbf{h} =$

$(\vec{h}_{\underline{r}} \mid \underline{r} \in \Lambda_L)$ denote a spatially varying magnetic field and let

$$H_L(\mathbf{h}) = -\sum_{\underline{r},\underline{r}'} M_{\underline{r},\underline{r}'}(n_{\underline{r}}\vec{s}_{\underline{r}} - \vec{h}_{\underline{r}}) \cdot (n_{\underline{r}'}\vec{s}_{\underline{r}'} - \vec{h}_{\underline{r}'}) - \mu \sum_{\underline{r}} n_{\underline{r}} - \lambda \sum_{\langle \underline{r},\underline{r}'\rangle} n_{\underline{r}} n_{\underline{r}'}. \quad (2.2)$$

(Notice, that due to our definition of $M_{\underline{r},\underline{r}'}$, a constant $\mathbf{h}$ is equivalent to $\mathbf{h} = 0$.) Finally let

$$Z_L(\mathbf{h}) = \operatorname{Tr} e^{-\beta H_L(\mathbf{h})}. \quad (2.3)$$

Using the reflection positivity, one can show that $Z_L(\mathbf{h})$ is maximized by a constant field, see [FILS], Theorem 4.7, which is equivalent to the statement that $Z_L(\mathbf{h}) \leq Z_L(\mathbf{0})$.

By judicious choices of $\mathbf{h}$, we obtain the so–called infrared bounds: Let $\vec{p}$ denote a reciprocal lattice vector, $\vec{p} = (p_1, \ldots, p_d) = \frac{\pi}{L}(k_1, \ldots, k_d)$ with integer k_i, let $\vec{v}_{\underline{r}} \equiv n_{\underline{r}}\vec{s}_{\underline{r}}$ and $\hat{v}_{\vec{p}}$ the Fourier transform of $\vec{v}$,

$$\hat{v}_{\vec{p}} = \frac{1}{|\Lambda_L|^{\frac{1}{2}}} \sum_{\underline{r} \in \Lambda} \vec{v}_{\underline{r}} e^{i\vec{p}\cdot\underline{r}}.$$

Let $\langle - \rangle_{L;\beta,\mu,\lambda}$ denote expectation with respect to the Gibbs measure on Λ_L induced by the Hamiltonian in (2.1). Then for $\vec{p}$ different from zero we have

$$\langle |\hat{v}_{\vec{p}}|^2 \rangle_{L;\beta,\mu,\lambda} \leq \frac{n}{2\beta}\frac{1}{\hat{M}_{\vec{p}}} \equiv \frac{n}{2\beta}\frac{1}{\sum_{j=1}^{d}(1-\cos p_j)}.$$

Since for the sum over all $\vec{p}$-s we have $|\Lambda_L|^{-1}\sum_{\vec{p}}\langle |\hat{v}_{\vec{p}}|^2 \rangle_{L;\beta,\mu,\lambda} = \langle n_0 \rangle_{L;\beta,\mu,\lambda}$ (i.e. the particle density) we conclude

Theorem 2.1. *Let $\rho = \rho(\beta, \mu, \lambda) = \limsup_{L\to\infty}\langle n_0 \rangle_{L;\beta,\mu,\lambda}$ and suppose ρ satisfies*

$$\rho > \frac{n}{2\beta}\int \frac{d^d p}{\hat{M}_{\vec{p}}}$$

(which necessitates $d \geq 3$). Then there is positive spontaneous magnetization.

Proof. See [AZ], [Z]. ■

We now turn to the discontinuity. Let $\mathbf{P}$ denote a plane containing sites (i.e. $\{\underline{r} \mid x_i = k\} \cup \{\underline{r} \mid x_i = k + L/2\}$) and $\vartheta_{\mathbf{P}}$ be the reflection in $\mathbf{P}$; it leaves $\mathbf{P}$ invariant. Since the interaction is nearest neighbor, H_L is reflection positive with respect to the reflections $\vartheta_{\mathbf{P}}$ as well.

To establish a discontinuity in the particle density along the lines of the argument in [KS], it is sufficient to establish that the following items hold for some infinite subsequence of L: (i) For $\pm\mu = \mathcal{M}$ (where $\mathcal{M}$ may be taken as large as needed) the density $\langle n_0\rangle_{L;\beta,\mu,\lambda}$ is within some small ϵ of $\frac{1}{2} \pm \frac{1}{2}$. (ii) For any $\underline{r}$, the quantity $\langle n_0(1-n_{\underline{r}})\rangle_{L;\beta,\mu,\lambda}$ must be "small" for all $\mu \in [-\mathcal{M}, +\mathcal{M}]$.

The former item is essentially trivial. As for the latter, we may identify *contour elements* as neighboring pairs of sites with one occupied and the other vacant; indeed whenever $n_0(1-n_{\underline{r}}) = 1$, either the origin or the site at $\underline{r}$ must be "surrounded" by a contour composed of such pairs. Once we can prove that the contour elements have small weights, the chessboard estimate allows us to run a Peierls argument and establish that $\langle n_0(1-n_{\underline{r}})\rangle_{L;\beta,\mu,\lambda}$ is itself small. The estimate for these contour elements is not difficult and will be needed for future purposes:

Proposition 2.2. *Let Λ_L denote the d-dimensional torus of side L where, for convenience, we take $L = 2^\ell$ for integer ℓ. Let H_L be as defined in (2.1). Then, for β large, the quantity $\langle n_0(1-n_{\hat{r}})\rangle_{L;\beta,\mu,\lambda}$, where $\hat{r}$ is a unit vector, is uniformly small: for all $\varepsilon > 0$*

$$\langle n_0(1-n_{\hat{r}})\rangle_{L;\beta,\mu,\lambda} \leq \frac{1}{2}\mathcal{N}_\varepsilon^{-1/2} e^{-\beta(1+\lambda/2-d\varepsilon)},$$

with proper $\mathcal{N}_\varepsilon$.

Proof. To estimate $\langle n_0(1-n_{\hat{r}})\rangle_{L;\beta,\mu,\lambda}$ we use the Schwartz inequalities that are enabled by the various reflections: For the coordinate plane orthogonal to $\hat{r}$, we use repeated reflections in its shifts passing through sites, until an alternating loop of ...–occupied–vacant–occupied–vacant–... sites circles the torus. For each coordinate plane parallel to $\hat{r}$, we use repeated reflections in its shifts passing halfway between sites. The result is alternating "stripes" of $d-1$-dimensional configurations, empty and full, of unit thickness. Thus

$$\langle n_0(1-n_{\hat{r}})\rangle_{L;\beta,\mu,\lambda} \leq \left[\frac{[Z_L^{\emptyset;d-1} Z_L^{f;d-1}]^{\frac{L}{2}}}{Z_L^d}\right]^{\frac{1}{L^d}} \leq \frac{[Z_L^{\emptyset;d-1} Z_L^{f;d-1}]^{\frac{1}{2L^{d-1}}}}{[Z_L^{\emptyset;d} + Z_L^{f;d}]^{\frac{1}{L^d}}}, \tag{2.4}$$

where Z_L^d denotes the total partition function, while $Z_L^{\emptyset;q}$ and $Z_L^{f;q}$ are the partition functions corresponding to the restriction of the Hamiltonian (2.1) to the subsets of configurations on a q-dimensional subtorus Λ_L^q of the d-dimensional torus Λ_L. Namely, the partition function $Z_L^{\emptyset;q}$ is calculated over the subset of configurations with no particles (i.e. over a subset with just one configuration), while the partition function $Z_L^{f;q}$ is obtained by summing over fully occupied configurations. Note that due to our definition (2.1), the partition functions $Z_L^{f;q}$ are still d-dependent. Clearly, $Z_L^{\emptyset;q} = 1$ and $Z_L^{f;q} = e^{\beta\mu L^q} e^{q\beta\lambda L^q} Z_L^{u;q}$ where $Z_L^{u;q}$ is the partition function of the uniform $O(n)$ spin system. Thus we must estimate $Z_L^{u;q}$ from above and below.

For the upper bounds, we take evident estimates

$$Z_L^{u;d-1} \leq e^{-2\beta L^{d-1}} \text{ and } Z_L^{u;d} \leq 1. \tag{2.5}$$

For the lower bounds, we restrict the summation to spin configurations where each spin is "close" to a given direction $\vec{e}$. Namely, we choose an ε and we consider the configurations $\vec{s}_{\underline{r}}$ where for each $\underline{r}$ we have $\vec{s}_{\underline{r}} \cdot \vec{e} > 1 - \varepsilon/4$. For such configurations we have $\vec{s}_{\underline{r}} \cdot \vec{s}_{\underline{r}'} > 1 - \varepsilon$. Introduce also $\mathcal{N}_\varepsilon = \int_{\vec{s}_{\underline{r}} \cdot \vec{e} > 1-\varepsilon/4} d\vec{s}_{\underline{r}}$. Then we have

$$Z_L^{u;d-1} \geq \left(\mathcal{N}_\varepsilon e^{-\beta(2+\varepsilon(d-1))}\right)^{L^{d-1}} \text{ and } Z_L^{u;d} \geq \left(\mathcal{N}_\varepsilon e^{-2\varepsilon d\beta}\right)^{L^d}. \tag{2.6}$$

Combining these estimates we find

$$\begin{aligned} \langle n_0(1-n_{\hat{r}})\rangle_{L;\beta,\mu,\lambda} &\leq \frac{\left(e^{\beta\mu} e^{(d-1)\beta\lambda} e^{-2\beta}\right)^{1/2}}{\left[1 + \left(\mathcal{N}_\varepsilon e^{\beta\mu} e^{d\beta\lambda} e^{-2d\beta\varepsilon}\right)^{L^d}\right]^{\frac{1}{L^d}}} \\ &= \frac{\mathcal{N}_\varepsilon^{-1/2} e^{-\beta(1+\lambda/2-d\varepsilon)}}{\left[\left(\mathcal{N}_\varepsilon e^{\beta\mu} e^{\beta(d\lambda-2d\varepsilon)}\right)^{-L^d/2} + \left(\mathcal{N}_\varepsilon e^{\beta\mu} e^{\beta(d\lambda-2d\varepsilon)}\right)^{L^d/2}\right]^{\frac{1}{L^d}}}. \end{aligned} \tag{2.7}$$

For all L and μ, this is seen to be less than $\frac{1}{2}\mathcal{N}_\varepsilon^{-1/2} e^{-\beta(1+\lambda/2-d\varepsilon)}$. ■

As a direct consequence of Proposition 2.2, we have

Theorem 2.3. *Consider the Hamiltonian as described in (2.1). For all β sufficiently large, there is a $\mu_t = \mu_t(\beta, \lambda)$ and an $\eta = \eta(\beta, \lambda) > \frac{1}{3}$ such that for all $\mu > \mu_t$, the density exceeds $1 - \eta$ and for all $\mu < \mu_t$, the density is less than η while at $\mu = \mu_t$ both such phases are present.*

Proof. The existence of a μ_t at which there is coexistence of the two described phases follows from an application of [KS] Theorem 4 – for which we have assembled all the necessary ingredients. Next, we note that the particle density is, formally, a thermodynamic derivative. Hence, for a.e. μ, the value of the density is independent of the particular state and, if $\mu_1 > \mu_2$ (with all other parameters equal) the value of the density in any state with $\mu = \mu_2$ cannot exceed the density in any state with $\mu = \mu_1$. This implies the stated results for $\mu \neq \mu_t$. ■

Remarks. (i) If $\beta\lambda$ is large and β is *not,* the estimate (2.7) results in the bound $\approx e^{-\frac{1}{2}\beta\lambda}$, which also implies a discontinuity. Thus, not unexpectedly, if $\lambda \gg 1$, there will be a phase transition which is due to the attraction of the particles alone. We will bolster this assertion in Section IV when we show that, under these circumstances, there is *no* spontaneous magnetization accompanying the phase transition.

(ii) A consequence of the estimate in Proposition 2.2 is that below $\mu_t(\beta)$ the density is exponentially small in β – with power law corrections; cf. the Remark following Lemma 2.4 in [CKS 1]. Thus there is – with justification – no hope for a *proof* of magnetization for $\mu < \mu_t(\beta)$ since the coefficient of the spin–wave integral in Theorem 2.1 is only $[\beta]^{-1}$. However, magnetization below μ_t cannot be ruled out by these methods; we must resort to different arguments in the next section to achieve this goal.

Let us summarize what has been proved thus far:

Theorem 2.4. *Consider the Hamiltonian in (2.1) and let $M(\beta, \mu, \lambda)$ denote the spontaneous magnetization. Then for β large enough, for all $\mu \geq \mu_t$ the magnetization exceeds $1 - k_1[\beta]^{-1}$ while for $\mu \leq \mu_t$ the density does not exceed $e^{-k_2\beta}$ where k_1 and k_2 are positive numbers of the order of unity.*

Proof. This follows immediately from the results of Theorems 2.1 and 2.3. ■

3. The Magnetization

In this section we will show that in the phase of low density the spontaneous magnetization is in fact zero. To facilitate the analysis of the spontaneous magnetization, we will study our system (2.1) in non–zero external field. Thus consider

$$-H_L^h = 2\sum_{\langle \underline{r},\underline{r}'\rangle} n_{\underline{r}} n_{\underline{r}'} \vec{s}_{\underline{r}} \cdot \vec{s}_{\underline{r}'} + (\mu - 2d)\sum_{\underline{r}} n_{\underline{r}} + \lambda \sum_{\langle \underline{r},\underline{r}'\rangle} n_{\underline{r}} n_{\underline{r}'} + h\sum_{\underline{r}} n_{\underline{r}} \vec{s}_{\underline{r}} \cdot \vec{e} \quad (3.1)$$

with $h > 0$ and $\vec{e}$ a fixed unit vector.

The previous analysis, Theorems 2.3 and 2.4 may be performed exactly as before with the same conclusions. The only required modification is that we now have for $Z_L^{f;d} \equiv Z_L^{f;d}(\beta, \lambda, \mu, h)$ the bounds

$$\left[e^{d\beta\lambda}e^{\beta(\mu+h)}\right]^{L^d} \geq Z_L^{f;d} \geq \left[\mathcal{N}_\varepsilon e^{-2d\beta\varepsilon}e^{d\beta\lambda}e^{\beta(\mu+h)}\right]^{L^d}, \tag{3.2}$$

and analogously for $Z_L^{f;d-1}$.

For the purposes of this section, we will assume that β is large enough so that the jump in the particle density at μ_t exceeds $1/2$. This, along with standard convexity arguments provides us with a well defined phase boundary in the (μ, h) plane:

Proposition 3.1. *Consider the model defined by the Hamiltonian in (3.1) with $h < 1$. For β sufficiently large, there is a $\mu_t(h)$ such that for $\mu < \mu_t(h)$ the density is "low", i.e. less than 1/4 while for $\mu \geq \mu_t(h)$ the density is "high", i.e..greater than 3/4. Furthermore, $\mu_t(h)$ is a continuous function of h.*

Proof. The existence of $\mu_t(h)$ and the behavior of the density for $\mu \neq \mu_t(h)$ follows from identical arguments as in the previous section – the bounds on the density before and after the jump are ensured by the condition $\beta \gg 1$. Thus $\mu_t(h)$ may now be thought of as *the* point of (the big) discontinuity.

To show that $\mu_t(h)$ is continuous, consider a sequence h_j with $h_j \to h_0$ and suppose that $\mu_* = \lim_{j\to\infty} \mu_t(h_j) > \mu_t(h_0)$. Then we could construct a state at $h = h_0, \mu = \mu_*$ with a "low" density of particles. But this is impossible by the monotonicity of the particle density – for fixed $h = h_0$ – as a function of μ because a high density state already exists at $\mu = \mu_t(h_0)$. A similar argument rules out the existence of any sequence h_j tending to h_0 with $\lim_{j\to\infty} \mu_t(h_j) < \mu_t(h_0)$. ■

The principal result of this section is the following:

Proposition 3.2. *For β large enough and for any $\mu < \mu_t(0)$, the magnetization $M(h; \mu, \beta, \lambda)$ satisfies*

$$M(h; \mu, \beta, \lambda) < Dh$$

for all $h > 0$ sufficiently small (depending on μ), with D independent of h. In particular, the spontaneous magnetization vanishes at any $\mu < \mu_t(0)$.

Proof. It is sufficient to establish the stated bound in any preferred state: Indeed, once again invoking thermodynamic arguments, we note that for a.e. h, the magnetization is the same in all states and is a monotone function of h.

Let $\mu < \mu_t(0)$. Then, for h sufficiently small, $\mu < \mu_t(h)$. Our first goal will be to show that at these values of parameters, the probability of large cluster of occupied sites tends to zero rapidly with cluster size. For this we will need the fact that we are at low density and that contours are rare. Let $N = N(L)$ denote any set of increasing numbers that are small compared with L but tend to infinity with L. Let V_N denote the box of side N centered at the point $c_L = (L/2, L/2, \dots L/2)$ – the back of the torus. For future reference we denote by U_N a similar box centered at the origin. Let $\mathcal{Q}_N$ denote the event that there is a connected path of vacant sites connecting c_L and the boundary of V_N and $\mathbb{I}_{\mathcal{Q}_N}$ be the indicator of this event.

We claim that the probability $\langle \mathbb{I}_{\mathcal{Q}_N} \rangle_{L;\beta,\mu,\lambda,h}$ is uniformly bounded away from zero. Indeed, the complimentary event consists in existing an occupied contour in V_N, surrounding c_L. But the probability of the event that a given contour γ is occupied is bounded by $\exp\{-c(\beta)|\gamma|\}$ with $c(\beta)$ diverging with β. This follows from the underlying proof of Theorem 2.3 and the fact that $\mu < \mu_t(h)$. Thus, there is no occupied contour surrounding c_L with probability close to one. If, in addition this point is vacant – which also happens with probability close to one – the event is guaranteed.

Let us consider the measures $\langle - \mid \mathcal{Q}_N \rangle_{L;\beta,\mu,\lambda,h}$. Let $\mathcal{G}_n(\underline{r})$ denote the event that the site $\underline{r}$ is occupied and belongs to an occupied (connected) cluster of size n and $\mathbb{I}_{\mathcal{G}_n(\underline{r})}$ be the indicator of this event. Our next claim is that for all L sufficiently large and n small compared to N the following holds: if $\underline{r}$ is in the vicinity of the origin (explicitly, $\underline{r} \in U_N$) we have

$$\langle \mathbb{I}_{\mathcal{G}_n(\underline{r})} \mid \mathcal{Q}_N \rangle_{L;\beta,\mu,\lambda,h} \le C_1 e^{-c_2 n^{1/d}}, \tag{3.3}$$

with the c's finite and positive. (This estimate is not optimal but sufficient for our purposes.) Indeed if $\mathcal{G}_n(\underline{r})$ and $\mathcal{Q}_N$ both occur, there must be a contour somewhere in Λ_L that separates the cluster at $\underline{r}$ from the empty path inside V_N. Summing over all such contours, and noting that the smallest contours start at size $\sim n^{1/d}$ the result follows easily.

Let $\underline{r} \in U_N$. Let us estimate the magnetization at $\underline{r}$ in the state $\langle - \mid \mathcal{Q}_N \rangle_{L;\beta,\mu,\lambda,h}$. To do this, we will employ the Wolff algorithm [W]. (See [C], [CaC] for a detailed mathematical discussion of this procedure.) The heart of the algorithm is the random map $w(\vec{s}_{\underline{r}})$ from the set of configurations into itself, which preserves the Gibbs measure. The following is the description of the probability measures $w(\vec{s}_{\underline{r}})$.

For present purposes, the magnetic field is treated as a single "ghost" site g locked into the $\vec{e}$ position. Given a spin–particle configuration $\vec{s}_{\underline{r}}$, we first define a random bond configuration $w^b(\vec{s}_{\underline{r}})$ as follows: Let $v_{\underline{r}} = n_{\underline{r}} |\vec{s}_{\underline{r}} \cdot \vec{e}|$ and let $\sigma_{\underline{r}} = \mathrm{sgn}(\vec{s}_{\underline{r}} \cdot \vec{e})$. If $\underline{r}'$ and $\underline{r}''$ are neighbors, with $\sigma_{\underline{r}'} = \sigma_{\underline{r}''}$, then we give to the bond variable $b_{\underline{r}'\underline{r}''}$ the value 1 with probability $1 - e^{\beta v_{\underline{r}'} v_{\underline{r}''}}$ and 0 with complimentary probability; if the σ'-s disagree, then the bond variable is zero. Also, if $\sigma_{\underline{r}'} = +1$, we give to the bond variable $b_{\underline{r}'g}$ the value 1 with probability $1 - e^{2\beta h v_{\underline{r}'}}$ and 0 with complimentary probability. Note that if $n_{\underline{r}'} = 0$, no bonds are attached to this site.

Next, for every site configuration $\vec{s}_{\underline{r}}$ and every bond configuration $b_{..}$ we define a random site configuration $w^s(\vec{s}_{\underline{r}}, b_{..})$ as follows: $b_{..}$-clusters of spins $\vec{s}_{\underline{r}}$, which *do not* contain the ghost site, are flipped with probability $\frac{1}{2}$. Spins of the cluster which contains the ghost site do not change.

Finally we define

$$w(\vec{s}_{\underline{r}}) = w^s\left(\vec{s}_{\underline{r}}, w^b(\vec{s}_{\underline{r}})\right).$$

Consider now the site $\underline{r}$ and the random variable $v_{\underline{r}}$. Under condition $\mathcal{G}_n$ the (random) bond cluster of $\underline{r}$ contains at most n "true" sites, plus, possibly, the ghost site. Given the "true" sites cluster, the probability that it is attached to the ghost site can be estimated from above by $(1 - e^{2\beta hn})$, since $v_{\underline{r}} \leq 1$. So because of invariance of the Gibbs measure under the transformation $w(\cdot)$ we conclude that the expected value of $v_{\underline{r}}$ given the event $\mathcal{G}_n$ is not greater than $(1 - e^{2\beta hn})$. Therefore we may write

$$\begin{aligned}\langle \vec{s}_{\underline{r}} \cdot \vec{e} \mid \mathcal{Q}_N \rangle_{L;\beta,\mu,\lambda,h} \leq & \sum_{n=1}^{\infty} \langle \mathcal{G}_n \mid \mathcal{Q}_N \rangle (1 - e^{2\beta hn}) \\ \leq & \sum_{n=1}^{\infty} C_1 e^{-c_2 n^{1/d}} (2\beta nh) \leq hD\end{aligned}$$

for some $D < \infty$ independent of h.

Finally, let $\langle - \rangle^*_{N;\beta,\mu,\lambda,h}$ denote the restriction of $\langle - \mid \mathcal{Q}_N \rangle_{L;\beta,\mu,\lambda,h}$ to the box U_N and

$$M^*_N(h) = \frac{1}{N^d} \sum_{\underline{r} \in U_N} \langle \vec{s}_{\underline{r}} \cdot \vec{e} \rangle^*_{N;\beta,\mu,\lambda,h} \tag{3.4}$$

the magnetization in finite volume. Then, uniformly in N we have $M^*_N(h) \leq Dh$ and the desired result is established by taking L and hence N to infinity in any chosen fashion. ■

We may now state:

Theorem 3.3. *Consider the Hamiltonian in (2.1) and let $M(\beta, \mu, \lambda)$ denote the spontaneous magnetization. Then for β large enough, for all $\mu \geq \mu_t$ the magnetization exceeds $1 - k_3[\beta]^{-1}$ while for $\mu \leq \mu_t$ the magnetization vanishes.*

Proof. Follows immediately from Theorem 2.4 and Proposition 3.2 ■

4. Cases of Strong Attraction

Let us briefly prove the assertions made in Section II:

Theorem 4.1. *Consider the models described by the Hamiltonian in (2.1) with β not large; in particular, $1-e^{-2\beta} < p_c(d)$ where $p_c(d)$ is the percolation threshold for bonds on $\mathbb{Z}^d$. Then, if $\lambda\beta$ is sufficiently large, there is a $\tilde{\mu}_t(\beta, \lambda)$ at which the particle density changes discontinuously; however, for all μ, the magnetization is zero.*

Proof. The existence of $\tilde{\mu}_t(\beta, \lambda)$ is a consequence of Theorem 2.3 and the subsequent remark. To show the absence of magnetization, we again appeal to the Wolff algorithm. Let $\Lambda \subset \mathbb{Z}^d$ denote a finite connected set with boundary $\partial\Lambda$ and consider a specification of spins and particles on $\partial\Lambda$. If $\underline{r} \in \Lambda$, it is not hard to see that the magnetization at $\underline{r}$ is bounded by the probability that $\underline{r}$ is connected to $\partial\Lambda$ by the bonds described in the proof of Proposition 3.2. However, for any bond–spin configuration, the maximum possible probability for any bond is $1 - e^{-2\beta}$. Thus, if $1 - e^{-2\beta} < p_c(d)$, the magnetization at any site, with any boundary condition goes to zero exponentially with the distance to the boundary. From this the result follows easily. ■

Remark. We thus see that in the large λ models, there are two distinct phase boundaries. Indeed, let $\lambda \gg 1$ and $\beta_1 > \beta_2$ with β_2 satisfying the conditions of Theorem 4.1 and β_1 the conditions of Theorem 3.3. Let $\overline{\mu} > \max\{\tilde{\mu}_t(\beta_2), \mu_t(\beta_1)\}$ and consider the path in the (μ, β)–plane: $(-\infty, \beta_2) \to (\overline{\mu}, \beta_2) \to (\overline{\mu}, \beta_1)$. Along the first leg of the journey, we have a transition in density alone, but by the time we get to $(\overline{\mu}, \beta_1)$ on the second leg, there is magnetization. Thus (at a minimum) two phase boundaries have been crossed.

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