Power Law Subordinacy and Singular Spectra.
II. Line Operators.

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Abstract

We study Hausdorff-dimensional spectral properties of certain "whole-line" quasiperiodic discrete Schrödinger operators by using the extension of the Gilbert-Pearson subordinacy theory that we previously developed in [17].

1 Introduction

In this paper we apply the power-law extension of the Gilbert-Pearson theory of subordinacy [10, 9], developed in [17], to certain popular models of one-dimensional quasiperiodic Schrödinger operators. Namely, we are interested in Hausdorff-dimensional properties of spectral measures of those operators. While we also formulate and prove here some statements for operators defined on the "half-line," and the "half-line" proofs are simpler and somewhat more transparent, our main interest here is in the "whole-line" case. Therefore, we treat here some technical issues of adapting theorems and methods of [17], best suited for the half-line, to the whole-line case. In this framework, we prove some statements that are quite general and that can be applied to examples other than those that we consider. However, we do not really attempt to develop a general theory for applying subordinacy methods to whole-line problems, but rather focus on handling some specific examples. Our main object are discrete operators on $\ell^2(\mathbb{Z})$, defined by

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n). \tag{1.1}$$

The potential $V = \{V(n)\}_{n=-\infty}^{\infty}$ is a sequence of real numbers. As in [17], some of our technical statements are also valid for continuous analogs of those operators, of the form $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R})$, as long as the potential $V(x)$ is such that we are in the limit point case (so the operator is essentially self-adjoint). The proofs for the discrete and continuous cases are essentially the same. Same is true for more general tridiagonal discrete operators (see the discussion in [17]). However, our actual discussion here will only involve operators of the form given by (1.1).

One family of models that we consider here is that of quasiperiodic potentials of the form

$$V(n) = f(\alpha n + \theta), \tag{1.2}$$

where f is a trigonometric polynomial of arbitrary degree and α is an irrational. Before we formulate our main result, we'll need to recall the notion of the Lyapunov exponent. We denote the n-step transfer-matrix of

$$Hu = Eu \tag{1.3}$$

by $\Phi_n(E)$,

$$\Phi_n(E) \equiv M_n(E)M_{n-1}(E)\ \cdots\ M_1(E),$$

where

$$M_n(E) \equiv \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.4}$$

In the case of an ergodic potential $V(n) = f(T^n\theta)$, where T is an ergodic transformation of a phase space (X, μ), $\theta \in X$, $f : X \to \mathbb{R}$, the corresponding operator H, and therefore the transfer-matrices Φ_n and M_n, will depend on θ, and we have that $M_n(E, \theta) = M(E, T^n\theta)$, where

$$M(E, \theta) \equiv \begin{pmatrix} E - f(\theta) & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.5}$$

In the ergodic case, the Lyapunov exponent, $\gamma(E)$, is defined as

$$\gamma(E) \equiv \lim_{k\to\infty} \frac{\int_X \ln \|\Phi_k(E,\theta)\| d\mu(\theta)}{k} = \inf_k \frac{\int_X \ln \|\Phi_k(E,\theta)\| d\mu(\theta)}{k}. \tag{1.6}$$

The celebrated Pastur-Ishii theorem (see, e.g., [7]) states that positive Lyapunov exponents imply a.e. absence of absolutely continuous spectrum for ergodic operators (everywhere absence for quasiperiodic operators, by a recent result of Last-Simon [23]). It turns out that, at least in some cases, one can say significantly more.

Let $T\theta = T_\alpha\theta = \theta + \alpha \pmod 1$. For any irrational α, T_α is an ergodic transformation of $[0,1)$. We will sometimes drop the α-dependence from the notations if there is no ambiguity. Let $V_\theta(n) = f(T_\alpha^n\theta)$, where α is any irrational, and $f(\theta) = \sum_{j=0}^p a_j \cos^j 2\pi\theta + b_j \sin^j 2\pi\theta$ is a trigonometric polynomial.

THEOREM 1 *Let H_θ be an operator of the form (1.1) with a quasiperiodic potential, $V = V_\theta$, as above. Suppose that $\gamma(E) > 0$ for every E in some Borel set $A \subset \mathbb{R}$. Then for any $\theta \in [0,1)$, the restriction $\mu(A \cap \cdot\,)$, of the spectral measure of H_θ, is zero-dimensional, namely, it is supported on a set of zero Hausdorff dimension.*

Recently, Bourgain and Goldstein [4] have proven that under exactly the same assumptions, for any fixed θ, the spectrum is pure point for a.e. α. Both Theorem 1 and the result of [4] are probably true for the case of a real analytic function f as well, however, some further arguments are needed for the proofs. Thus, for quasiperiodic operators of this type, positive Lyapunov exponents lead to pure point spectrum for a.e. (but not all! [1, 18]) α, θ, and to zero-dimensional spectrum for all α, θ. It had been proven by Herman [12], with a result later generalized and significantly improved by Sorets-Spencer [24] (also see [4] for a new proof), that if $f(\theta)$ is a trigonometric polynomial, α irrational, and $V(n) = \lambda f(\theta + \alpha n)$, then for λ sufficiently large, the Lyapunov exponents $\gamma(E)$ are positive for any $E \in \mathbb{R}$. Therefore, our theorem implies the zero-dimensionality of the spectral measures in all those cases.

In particular, Theorem 1 applies to the extensively-studied case of the Almost Mathieu operator, $H_{\alpha,\lambda,\theta}$, which is the operator of the form (1.1) with $V(n) = \lambda\cos(2\pi\alpha n + \theta)$, where $\lambda \in \mathbb{R}$, $\theta \in [0, 2\pi]$, and α is any irrational. In the regime $|\lambda| > 2$, this operator has spectral properties known to depend very delicately on the arithmetics of both α and θ (see [14] for a review). While pure point spectrum for a.e. α, θ had been recently proven [15], the singular continuous spectrum is known to occur for certain α's and/or θ's, and the full picture is rather complicated and is far from being established. It turns out, however, that if we concentrate on Hausdorff-dimensional properties of the spectral measures, rather than on distinguishing between pure-point and singular-continuous spectra, the situation becomes simpler. As an immediate corollary of Theorem 1 and Herman's Theorem [12], we obtain:

COROLLARY 2 *For $|\lambda| > 2$, every irrational α, and every θ, the spectral measure of the Almost Mathieu operator $H_{\alpha,\lambda,\theta}$ is zero-dimensional.*

It should be noted that for other values of the parameters the dimensional characteristics of spectral measures are mostly known. For rational α's, the spectrum is absolutely continuous by periodicity; also, for $|\lambda| < 2$ and any α, θ, a large absolutely continuous component exists [20, 23] (the spectrum is actually now known to be purely absolutely continuous for a.e. α [15], but this result has not yet been established for all α), so in all these cases the spectral measures are of dimension 1. Thus, besides the critical value $|\lambda| = 2$, where the situation is expected to be more delicate (see [11, 21]), this theorem provides a full description of the Hausdorff-dimensional characteristics of the spectral measures of $H_{\alpha,\lambda,\theta}$.

Another "whole-line" example that we study here is the Fibonacci Hamiltonian H_λ, which is the operator of the form (1.1) on $\ell^2(\mathbb{Z})$ with potential $V(n) = \lambda([(n+1)\omega] - [n\omega])$, where $\omega = (\sqrt{5}-1)/2$ is the golden mean, and $[x] \equiv \max\{m \in \mathbb{Z} \,|\, m \leq x\}$. H_λ is the most studied of all one-dimensional quasicrystal models. It is known [2, 27, 28] that for every $\lambda \neq 0$, it has purely singular-continuous spectrum, and moreover, its spectrum (as a set) is a Cantor set of zero Lebesgue measure. We will show

THEOREM 3 *For every λ there exists an $\alpha > 0$ such that the Fibonacci Hamiltonian H_λ has purely α-continuous spectrum, namely, its spectral measures do not give weight to sets of Hausdorff dimension less than α.*

Remark. Raymond [25] have recently shown that for large λ, the spectrum of H_λ (as a set) has Hausdorff dimension strictly less than 1. This implies that for such λ's, the spectrum must also be β-singular (see below) for some $\beta < 1$.

A version of the main results of this paper had been previously announced in [16], (Theorems 3 and 4 of [16]). However, Theorem 1 generalizes the result announced in [16], in that it only requires positivity of the Lyapunov exponents. While being a second part, this paper can be read independently of [17], as we reintroduce the notations, and precisely quote the necessary facts from [17] (inequality (2.13).) However, we refer the reader to [17] for certain general discussions.

The rest of this paper is organized as follows. In Section 2, we review some facts concerning Hausdorff measures and dimensional spectral properties, and introduce some notations. In Section 3, we prove some technical statements needed for the treatment of the whole-line case. In section 4, we prove Theorem 1. We also formulate and prove a corresponding half-line version. Finally, in Section 5, we prove Theorem 3.

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2 Preliminaries.

Here we discuss briefly some definitions and preliminary results, particularly those related to dimensional properties of spectral measures. A somewhat more detailed account is given in [17], and for more information see [22]. Recall that for any subset S of $\mathbb{R}$ and $\alpha \in [0,1]$, the α-dimensional Hausdorff measure, h^α, is given by

$$h^\alpha(S) \equiv \lim_{\delta\to 0} \inf_{\delta\text{-covers}} \sum_{\nu=1}^{\infty} |b_\nu|^\alpha \ , \tag{2.1}$$

where a δ-cover is a cover of S by a countable collection of intervals, $S \subset \bigcup_{\nu=1}^{\infty} b_\nu$, such that for each ν the length of b_ν is at most δ. Given any $\emptyset \neq S \subseteq \mathbb{R}$, there exists a unique $\alpha(S) \in [0,\ 1]$ such that $h^\alpha(S) = 0$ for any $\alpha > \alpha(S)$, and $h^\alpha(S) = \infty$ for any $\alpha < \alpha(S)$. This unique $\alpha(S)$ is called the Hausdorff dimension of S.

Given α, a measure μ is called α-continuous if $\mu(S) = 0$ for every set S with $h^\alpha(S) = 0$. It is called α-singular if it is supported on some set S with $h^\alpha(S) = 0$. We say that μ is zero-dimensional if it is α-singular for every $\alpha > 0$.

Given a (positive, finite) measure μ and $\alpha \in [0,1]$, we define

$$D_\mu^\alpha(x) \equiv \limsup_{\epsilon\to 0} \frac{\mu((x-\epsilon,\ x+\epsilon))}{(2\epsilon)^\alpha} \tag{2.2}$$

and $T_\infty \equiv \{x \mid D_\mu^\alpha(x) = \infty\}$. The restriction $\mu(T_\infty \cap \cdot\,) \equiv \mu_{\alpha s}$ is α-singular, and $\mu((\mathbb{R}\setminus T_\infty)\cap\,\cdot\,) \equiv \mu_{\alpha c}$ is α-continuous. Thus, each measure decomposes uniquely into an α-continuous part and an α-singular part: $\mu = \mu_{\alpha c} + \mu_{\alpha s}$. Moreover, an α-singular measure must have $D_\mu^\alpha(x) = \infty$ a.e. (with respect to it) and an α-continuous measure must have $D_\mu^\alpha(x) < \infty$ a.e.. It is important to note that $D_\mu^\alpha(x)$ is defined with a lim sup. The corresponding limit need not exist.

The study of the spectral measure μ of an operator of the form (1.1) is related to the study of the Weyl-Titchmarsh m-function. We will briefly list the necessary related facts. More details can be found, for example, in [5].

Let $\mathbb{Z}^+ = \{1,2,3,\ldots\}$, and $\mathbb{Z}^- = \{\ldots,-2,-1,0\}$. Consider the family of phase boundary conditions:

$$\psi(0)\cos\beta + \psi(1)\sin\beta = 0, \tag{2.3}$$

where $-\pi/2 < \beta < \pi/2$. Let $z = E + i\epsilon$, and consider equation

$$Hu = zu \tag{2.4}$$

It is known that for $\epsilon > 0$, (2.4) has unique (up to normalization) solutions $\hat{u}_z^\pm$ which are ℓ^2 at, correspondingly, $\pm\infty$. Let $\hat{u}_{\beta,z}^\pm$ be $\hat{u}_z^\pm$ normalized by $\hat{u}_{\beta,z}^\pm(0)\cos\beta + \hat{u}_{\beta,z}^\pm(1)\sin\beta = 1$. We denote by $u_{1,\beta,z}^\pm$ the solutions of (2.4) on $\mathbb{Z}^\pm$ satisfying the boundary conditions (2.3) and by $u_{2,\beta,z}^\pm$ the solutions of (2.4) on $\mathbb{Z}^\pm$ satisfying the orthogonal boundary conditions, and normalized by $|u^{\pm}_{i,\beta,z}(0)|^2 + |u^{\pm}_{i,\beta,z}(1)|^2 = 1,\ i = 1, 2$; that is, $u^{\pm}_{2,\beta,z} = u^{\pm}_{1,\pi/2+\beta,z}$. For z in the upper half-plane, the right and left Weyl-Titchmarsh m-functions are uniquely defined by $\hat{u}^{\pm}_{\beta,z} = u^{\pm}_{2,\beta,z} \mp m^{\pm}_{\beta}(z) u^{\pm}_{1,\beta,z}$. We denote by m^+ the m-function m^+_0, corresponding to the Dirichlet boundary conditions, and by m^- the m-function m^-_0. The functions m^+ are related to m^+_β by a simple change of variables formula:

$$m^+ = \frac{m^+_\beta \cos\beta - \sin\beta}{m^+_\beta \sin\beta + \cos\beta}. \tag{2.5}$$

It is also immediate from the definition that the following relation is true:

$$m^{\pm}(z) = \mp \frac{\hat{u}^{\pm}_z(1)}{\hat{u}^{\pm}_z(0)}. \tag{2.6}$$

The whole-line m-function is then defined for $\Im z > 0$ by (see, e.g., [5])

$$M(z) \equiv \frac{m^+(z) m^-(z) - 1}{m^+(z) + m^-(z).} \tag{2.7}$$

$M(z)$ coincides with the Borel transform of the spectral measure μ of the operator (1.1):

$$M(z) = \int \frac{d\mu(x)}{x - z}. \tag{2.8}$$

It is shown in [8] that

$$\limsup_{\epsilon \to 0} \epsilon^{1-\alpha} |M(E + i\epsilon)| = \infty \quad \Leftrightarrow \quad D^{\alpha}_{\mu}(E) = \infty \tag{2.9}$$

Therefore, the study of dimensional spectral properties can be reduced to the study of the behavior of $M(E+i\epsilon)$ as $\epsilon \to 0$, which in turn reduces to the study of the $\epsilon \to 0$ behavior of $m^{\pm}(E + i\epsilon)$. The main technical achievement of the Gilbert-Pearson theory of subordinacy [9, 10] (and the power-law extension of it [17]) is in relating such behavior to properties of solutions of equation (1.3). We now recall the main construction of [17]. Given an operator H of the form defined by (1.1), we fix $E \in \mathbb{R}$, and let $u^{\pm}_{1,\beta}$ be $u^{\pm}_{1,\beta,E}$, the normalized solution of (1.3), defined on, correspondingly, $\mathbb{Z}^+$ or $\mathbb{Z}^-$, and obeying the boundary conditions (2.3). Let $u^{\pm}_{2,\beta}$ be $u^{\pm}_{2,\beta,E}$, the solution obeying the orthogonal boundary conditions.

For any function $u : \mathbb{Z}^+ \to \mathbb{R}$, we denote by $\|u\|_L$ the norm of u over a lattice interval of length L, that is

$$\|u\|_L \equiv \left[\sum_{n=1}^{[L]} |u(n)|^2 + (L - [L]) |u([L] + 1)|^2 \right]^{1/2}, \tag{2.10}$$

where $[L]$ denotes the integer part of L. Similarly, for functions $u : \mathbb{Z}^- \to \mathbb{R}$, we define

$$\|u\|_L \equiv \left[\sum_{n=0}^{[L]-1} |u(-n)|^2 + (L - [L]) |u(-[L])|^2 \right]^{1/2}. \tag{2.11}$$

Now given any $\epsilon > 0$, we define lengths $L_\beta^\pm(\epsilon) \in (0, \infty)$, by requiring the equality

$$\|u_{1,\beta}^\pm\|_{L_\beta^\pm(\epsilon)} \|u_{2,\beta}^\pm\|_{L_\beta^\pm(\epsilon)} = \frac{1}{2\epsilon}. \tag{2.12}$$

The functions $L_\beta^\pm(\epsilon)$ are well defined (by (2.12)), and are monotonely decreasing continuous functions which go to infinity as ϵ goes to 0. Our main technical tool will be the following inequality which relates the Weyl-Titchmarsh m-function (for z in the upper half plane) to the solutions u_1 and u_2 (Theorem 1.1 of [17]):

$$\frac{5-\sqrt{24}}{|m_\beta^+(E+i\epsilon)|} < \frac{\|u_{1,\beta}^+\|_{L_\beta^+(\epsilon)}}{\|u_{2,\beta}^+\|_{L_\beta^+(\epsilon)}} < \frac{5+\sqrt{24}}{|m_\beta^+(E+i\epsilon)|}. \tag{2.13}$$

Finally, we will need one more general statement, that is the existence of generalized eigenfunctions [3, 23, 26], from which it is known that for a.e. E with respect to the spectral measure μ, there should exist $\beta \in (-\pi/2, \pi/2]$ such that the solutions $u_{1,\beta}^\pm$ both obey

$$\limsup_{L\to\infty} \frac{\|u\|_L}{L^{1/2}\ln L} < \infty \tag{2.14}$$

3 Whole-line operators.

In this section we discuss some technical issues that we need for dimensional analysis of the whole-line case. In particular, we will study the "right" and "left" transfer-matrices on similar scales. Given an operator of the form (1.1), define $\Phi_n^+(E) \equiv \Phi_n(E)$, and $\Phi_n^-(E) \equiv M_{-n+1}(E)M_{-n+2}(E)\cdots M_{-1}(E)M_0(E)$. Note that the $\Phi_n^\pm(E)$'s are related to solutions of equation (1.3) by

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \Phi_n^+(E) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \tag{3.1}$$

and

$$\begin{pmatrix} u(-(n+1)) \\ u(-n) \end{pmatrix} = \Phi_{n+1}^-(E) \begin{pmatrix} u(0) \\ u(1) \end{pmatrix},\ n \geq 0, \tag{3.2}$$

and so, for any β,

$$\Phi_n^+(E) = \begin{pmatrix} u_{1,\beta}^+(n+1) & u_{2,\beta}^+(n+1) \\ u_{1,\beta}^+(n) & u_{2,\beta}^+(n) \end{pmatrix} \begin{pmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{pmatrix} \tag{3.3}$$

and

$$\Phi_{n+1}^-(E) = \begin{pmatrix} u_{1,\beta}^-(-(n+1)) & u_{2,\beta}^-(-(n+1)) \\ u_{1,\beta}^-(-n) & u_{2,\beta}^-(-n) \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ -\sin\beta & -\cos\beta \end{pmatrix}. \tag{3.4}$$

In order to relate the spectral properties of a whole-line problem with the left and right transfer-matrix behaviors, we will use the following general

LEMMA 4 *Let H be an operator of the form (1.1). Fix $\delta > 1$ and let $\alpha = \frac{2}{1+\delta}$. Suppose that for every E in some Borel set A, there exist $a, c > 0$ and sequences $k_n \to \infty$, $j_n^\pm \leq ck_n$, such that*

$$\sum_{i=1}^{j_n^\pm} \|\Phi_i^\pm\|^2 > ak_n^\delta.$$

Then, for every $\epsilon > 0$, the restriction $\mu(A \cap \cdot\,)$ is $(\alpha+\epsilon)$-singular.

Remark. Note that the assumptions of Lemma 4 imply $(\alpha+\epsilon)$-singularity of both the right and the left half-line problems for any boundary condition. They require more than just that, however, as they also require some correlation between the two sides. This type of requirement is truely necessary, since, in general, the whole-line spectral problem can have much greater continuity than the corresponding half-line problems. Indeed, it is possible to construct potentials for which each of the half-line problems would have zero-dimensional spectrum for any boundary phase β, and yet the whole-line problem would have purely one-dimensional spectrum. Potentials of this type can be constructed as sparse barrier potentials of the type which we studied in [17]. By making the barriers high enough, one can insure that each of the half-line problems has zero-dimensional spectrum, but then by making them sparse enough and correlating the two sides appropriately (one needs the scales where barriers occur on the right to be very different from the scales where barriers occur on the left), one can cause the whole-line problem to have one-dimensional spectrum.

It follows from the discussion in Section 2, particularly (2.9), that Lemma 4 will be proven if we establish that $\limsup_{\epsilon\to 0} \epsilon^{1-\alpha}|M(E+i\epsilon)| = \infty$, where $M(z)$ is defined by (2.7). Since our main technical tool that allows us to study the $\epsilon \to 0$ behavior of m through the behavior of solutions (and therefore, transfer-matrices), inequality (2.13), is formulated in [17] and proven only for m^+ of operators acting on $\mathbb{Z}^+$, it will be convenient for us to find an expression for $M(z)$ that uses only m^+.

We observe that the space $\ell^2(\mathbb{Z}^-)$ is naturally identified with $\ell^2(\mathbb{Z}^+)$ by a unitary operator $U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, defined by $(U\psi)(n) = \psi(-n+1)$, $n \in \mathbb{Z}$. Clearly, U^2 is the identity operator. For any operator H on $\ell^2(\mathbb{Z})$, we define an operator $\tilde{H}$ on $\ell^2(\mathbb{Z})$ by $\tilde{H} = UHU^{-1}$. For operators of the form (1.1), $\tilde{H}$ will be H with a potential reflected with respect to the point 1/2. Let $\tilde{m}^+$, $\tilde{m}_\beta^+$, $\tilde{u}_\beta^\pm$, $\tilde{L}_\beta^\pm$, denote, correspondingly, m^+, m_β^+, $u_\beta^\pm$ and $L_\beta^\pm$ of the operator $\tilde{H}$. Then it is immediate from (2.6) that

$$m^- = -\frac{1}{\tilde{m}^+}. \tag{3.5}$$

Using (2.5),(2.7), and (3.5), we obtain by an elementary computation that for any $\beta \in (-\pi/2, \pi/2]$,

$$\frac{m_\beta^+(z)\tilde{m}_{\pi/2-\beta}^+(z) - 1}{m_\beta^+(z) + \tilde{m}_{\pi/2-\beta}^+(z)} = M(z). \tag{3.6}$$

Also, we will use the fact that the operator U, applied formally, sends sequences $u : \mathbb{Z}^{\pm} \to \mathbb{R}$ into sequences $(Uu) : \mathbb{Z}^{\mp} \to \mathbb{R}$. It follows immediately from the definition that

$$Uu^{-}_{i,\beta} = \tilde{u}^{+}_{i,\pi/2-\beta},\ i = 1, 2. \tag{3.7}$$

It is also clear from (2.10),(2.11),(2.12) that

$$\tilde{L}^{+}_{\pi/2-\beta} = L^{-}_{\beta} \tag{3.8}$$

and

$$\|u\|_L = \|Uu\|_L. \tag{3.9}$$

LEMMA 5 *Suppose that there exists a* $\beta \in (-\pi/2, \pi/2]$ *and a sequence* $\epsilon_j \to 0$, *such that for every* E *in some Borel set* A, *we have that* $\epsilon_j^{1-\alpha} m^{+}_{\beta}(E+i\epsilon_j) \to \infty$ *and* $\epsilon_j^{1-\alpha} \tilde{m}^{+}_{\pi/2-\beta}(E+i\epsilon_j) \to \infty$ *as* $j \to \infty$. *Then the restriction* $\mu(A \cap \cdot\,)$ *is* α*-singular.*

Proof. From (3.6) we immediately obtain that under the conditions of the Lemma we have $\epsilon_j^{1-\alpha} M(E + i\epsilon_j) \to \infty$ as $j \to \infty$. Thus, (2.9) yields the result. ∎

The rest of the proof uses key ingredients from the proofs of Corollaries 4.1 and 4.2 of [17]. From (3.3),(3.4) we see that for any β, we have

$$\|\Phi^{\pm}_n(E)\|^2 \leq |u^{\pm}_{1,\beta}(\pm(n+1))|^2 + |u^{\pm}_{1,\beta}(\pm(n))|^2 + |u^{\pm}_{2,\beta}(\pm(n+1))|^2 + |u^{\pm}_{2,\beta}(\pm(n))|^2,$$

and therefore we have $\|u^{\pm}_{1,\beta}\|^2_L + \|u^{\pm}_{2,\beta}\|^2_L \geq \frac{1}{2}\sum_{n=1}^{L} \|\Phi^{\pm}_n(E)\|^2$. Thus , under the conditions of Lemma 4, for any $L \geq j^{\pm}_n$ and any $\beta \in (-\pi/2, \pi/2]$, we obtain

$$\|u^{\pm}_{1,\beta}\|^2_L + \|u^{\pm}_{2,\beta}\|^2_L \geq \frac{a}{2} k_n^{\delta}. \tag{3.10}$$

In particular, inequality (3.10) holds for any $L \geq ck_n$. Since for a.e. E w.r.t. μ, there exists a phase $\beta(E) \in (-\pi/2, \pi/2]$ such that (2.14) holds for both $u^{-}_{1,\beta(E)}$ and $u^{+}_{1,\beta(E)}$, we deduce that for such E's and large n's,

$$\|u^{\pm}_{2,\beta}\|_{ck_n} \geq a_1 k_n^{\frac{\delta}{2}}, \tag{3.11}$$

for some $a_1 > 0$. Take $\epsilon_n = k_n^{-\frac{1}{\alpha}}$. Then, by (2.12),(2.14), for a.e. E w.r.t. μ and large n, we get

$$\frac{\|u^{\pm}_{2,\beta(E)}\|_{L^{\pm}_{\beta(E)}(\epsilon_n)}}{\|u^{\pm}_{1,\beta(E)}\|_{L^{\pm}_{\beta(E)}(\epsilon_n)}} > \frac{c_1}{2\epsilon_n L^{\pm}_{\beta(E)}(\epsilon_n) \ln^2 L^{\pm}_{\beta(E)}(\epsilon_n)} \tag{3.12}$$

for some positive constant c_1. If it so happens that $L^{\pm}_{\beta(E)}(\epsilon_n) < ck_n$, for either the + or − case, then (3.12) implies that for some $c_2 > 0$,

$$\frac{\|u^{\pm}_{2,\beta(E)}\|_{L^{\pm}_{\beta(E)}(\epsilon_n)}}{\|u^{\pm}_{1,\beta(E)}\|_{L^{\pm}_{\beta(E)}(\epsilon_n)}} > c_2 \frac{k_n^{\frac{1-\alpha}{\alpha}}}{\ln^2 k_n} \tag{3.13}$$

for the corresponding case. Otherwise (namely, if $L^{\pm}_{\beta(E)}(\epsilon_n) \geq ck_n$), (2.12) and (3.11) imply that for $E \in A$,

$$\frac{\|u^{\pm}_{2,\beta(E)}\|_{L^{\pm}_{\beta(E)}(\epsilon_n)}}{\|u^{\pm}_{1,\beta(E)}\|_{L^{\pm}_{\beta(E)}(\epsilon_n)}} \geq 2a_1^2 k_n^{\delta-\frac{1}{\alpha}} = 2a_1^2 k_n^{\frac{1-\alpha}{\alpha}} \tag{3.14}$$

for this case. Thus, we see that in either case, for a set of E's of full $\mu(A \cap \cdot\,)$ measure, we have positive constants $C(E)$, such that for large n,

$$\frac{\|u^{\pm}_{2,\beta(E)}\|_{L^{\pm}_{\beta(E)}(\epsilon_n)}}{\|u^{\pm}_{1,\beta(E)}\|_{L^{\pm}_{\beta(E)}(\epsilon_n)}} > C(E)\frac{k_n^{\frac{1-\alpha}{\alpha}}}{\ln^2 k_n}. \tag{3.15}$$

We thus obtain by (2.13), for any $\epsilon > 0$,

$$\epsilon_n^{1-\alpha-\epsilon} m^{+}_{\beta}(E)(E+i\epsilon_n) \sim \epsilon_n^{1-\alpha-\epsilon} \frac{\|u^{+}_{2,\beta(E)}\|_{L^{+}_{\beta(E)}(\epsilon_n)}}{\|u^{+}_{1,\beta(E)}\|_{L^{+}_{\beta(E)}(\epsilon_n)}} \geq C(E)\frac{k_n^{\frac{\epsilon}{\alpha}}}{\ln^2 k_n} \to \infty \tag{3.16}$$

as $n \to \infty$.

Similarly, using in addition (3.7),(3.8),(3.9), we get

$$\epsilon_n^{1-\alpha-\epsilon} \tilde{m}^{+}_{\pi/2-\beta(E)}(E+i\epsilon_n) \sim \epsilon_n^{1-\alpha-\epsilon} \frac{\|\tilde{u}^{+}_{2,\pi/2-\beta(E)}\|_{\tilde{L}^{+}_{\pi/2-\beta(E)}(\epsilon_n)}}{\|\tilde{u}^{+}_{1,\pi/2-\beta(E)}\|_{\tilde{L}^{+}_{\pi/2-\beta(E)}(\epsilon_n)}} = \epsilon_n^{1-\alpha-\epsilon} \frac{\|u^{-}_{2,\beta(E)}\|_{L^{-}_{\beta(E)}(\epsilon_n)}}{\|u^{-}_{1,\beta(E)}\|_{L^{-}_{\beta(E)}(\epsilon_n)}} \to \infty \tag{3.17}$$

as $n \to \infty$. Therefore, Lemma 5 implies Lemma 4. ∎

For the zero-dimensionality our main tool will be the following

COROLLARY 6 *Let H be an operator of the form (1.1). Suppose that for every E in some Borel set A, there exist $c > 0$ and sequences $k_n \to \infty$, $j_n^{\pm} \leq ck_n$, such that $\|\Phi^{\pm}_{j_n^{\pm}}\| \geq e^{ak_n^{\gamma}}$, for some $a, \gamma > 0$. Then the restriction $\mu(A \cap \cdot\,)$ is zero-dimensional.*

Remark. This Corollary is still more general than needed for our current application (in particular, we will only use it with $\gamma = 1$). However, this more general formulation may be useful in certain other situations.

Proof. Clearly, $\|\Phi^{\pm}_{j_n^{\pm}}\| \geq e^{ak_n^{\gamma}}$ implies $\sum_{i=1}^{j_n^{\pm}} \|\Phi^{\pm}_{j}\|^2 > ak_n^{\beta}$ for any $\beta < \infty$. Therefore, by Lemma 4, $\mu(A \cap \cdot\,)$ is α-singular for any $\alpha > 0$. ∎

4 Quasiperiodic operators and zero-dimensional spectrum. Proof of Theorem 1

As a warmup, we first prove a half-line version of Theorem 1. Let us consider H^+_β, operators on $\ell^2(\mathbb{Z}^+)$, defined by (1.1) along with a phase boundary condition (2.3). In case of ergodic $V(n)$, the Lyapunov exponent $\gamma(E)$ is defined as before and is, clearly, not dependent on β, θ.

THEOREM 7 *Let V be as in Theorem 1. Suppose that $\gamma(E) > 0$ for every E in some Borel set A, then for every boundary phase β and every θ, the restriction $\mu(A \cap \cdot\,)$, of the corresponding spectral measure of the operator H^+_β, is zero-dimensional.*

Proof. Recall that the upper Lyapunov exponent $\overline{\gamma}(E)$ is defined by

$$\overline{\gamma}(E) \equiv \limsup_{n\to\infty}(1/n)\ln\|\Phi_n(E)\|.$$

Note that, unlike $\gamma(E)$, the upper Lyapunov exponent of an ergodic operator H_θ is dependent on θ. We will use the following

THEOREM 8 (Corollary 4.3 of [17]) *Suppose that $\overline{\gamma}(E) > 0$ for every E in some Borel set A, then for every boundary phase, the restriction $\mu(A \cap \cdot\,)$ is zero-dimensional.*

Therefore, we need to prove that under the conditions of Theorem 7, for every θ, we have $\overline{\gamma}(E, \theta) > 0$. Note that boundedness of $V(n)$ immediately implies the following a priori upper bound:

$$\|\Phi_n(\theta, E)\| \le e^{nC} \tag{4.1}$$

for some $C < \infty$. Put $c(E) = \frac{\gamma(E)}{2C-\gamma(E)}$. Let $A_k = \{\theta \in [0,1) : \|\Phi_k(\theta, E)\| > e^{k\gamma(E)/2}\}$. We obtain, using (1.6), that

$$k\gamma(E) \le \int_0^1 \ln\|\Phi_k(\theta, E)\|d\theta = \int_{A_k} + \int_{[0,1)\setminus A_k} \le |A_k|kC + (1-|A_k|)k\gamma(E)/2,$$

where $|\cdot|$ stands for the Lebesgue measure. We, therefore, obtain $|A_k| \ge c(E)$. Since f is a trigonometric polynomial of degree p, $\|\Phi_k\|^2$ is a trigonometric polynomial of degree $2kp$, and the set A_k consists of no more than $4kp$ intervals. Therefore, there exists a segment, $\Delta_k \subset A_k$, with $|\Delta_k| \ge \frac{c(E)}{4kp}$.

We will now show that for every θ, there exists a sequence $j_n \to \infty$ such that $\|\Phi_{j_n}\|$ is exponentially large. This sequence will be related to a continued fractions expansion of α. Let p_n/q_n be the sequence of continued fraction approximants of α. We will use the following elementary properties of continued fractions, that can be found, for example, in [19]. (We assume here $\alpha > 0$. Otherwise, the signs in (4.3) will change.)

$$|q_n\alpha - p_n| < |k\alpha - p|,\ k = 1, \dots, q_{n+1}-1,\ k \ne q_n,\ p \in \mathbb{Z}; \tag{4.2}$$

$$q_n\alpha - p_n > 0 \text{ for } n \text{ odd and } q_n\alpha - p_n < 0 \text{ for } n \text{ even;} \tag{4.3}$$

$$|q_n\alpha - p_n| < \frac{1}{q_{n+1}}. \tag{4.4}$$

Let $\Delta \subset [0,1)$ be an arbitrary segment. We prove below the following simple

LEMMA 9 *Let $n \in \mathbb{N}$ be such that $|\Delta| > \frac{1}{q_n}$. Then, for any θ, there exists a k in $\{0,1,\ldots,q_n+q_{n-1}-1\}$ such that $T_\alpha^k\theta \in \Delta$.*

We will now finish the proof of Theorem 7.

Set

$$k_n = \left[\frac{c(E)q_n}{4p}\right] + 1. \tag{4.5}$$

Then, by Lemma 9, for every θ there exists a j in $\{0,1,\ldots,q_n+q_{n-1}-1\}$ such that $T_\alpha^j\theta \in A_{k_n}$, and so $\|\Phi_{k_n}(T_\alpha^j\theta)\| > e^{k_n\gamma(E)/2}$. Since $\Phi_{j+k_n}(\theta) = \Phi_{k_n}(T_\alpha^j\theta)\Phi_j(\theta)$, and each Φ_i is unimodular, we obtain that either $\|\Phi_j(\theta)\|$ or $\|\Phi_{j+k_n}(\theta)\|$ is greater than $e^{\frac{k_n\gamma(E)}{4}}$. Let

$$j_n = \min\left\{j \in \{0,\ldots,q_n+q_{n-1}-1+k_n\} : \|\Phi_j(\theta)\| \geq e^{\frac{k_n\gamma(E)}{4}}\right\}. \tag{4.6}$$

By (4.1) we have

$$j_n \geq \frac{k_n\gamma(E)}{2C}, \tag{4.7}$$

so $j_n \to \infty$ as $n \to \infty$. To finish the proof of Theorem 7 it now remains to notice that (4.5),(4.6),(4.7) yield $\|\Phi_{j_n}(\theta)\| \geq e^{j_n\frac{c(E)\gamma(E)}{4(8p+c(E))}}$, which implies $\overline{\gamma}(E,\theta) \geq \frac{c(E)\gamma(E)}{4(8p+c(E))}$. ∎

Proof of Lemma 9. Let $k^{+(-)}$ define such $s \in \{0,\ldots,q_n+q_{n-1}-1\}$ that $T_\alpha^s\theta$ is the nearest neighbor to the right (left) of $T_\alpha^k\theta$. To prove the lemma we show that for any $k \in \{0,\ldots,q_n+q_{n-1}-1\}$, we have

$$\mathrm{dist}(T_\alpha^k\theta, T_\alpha^{k^{+(-)}}\theta) < \frac{1}{q_n} < |\Delta|. \tag{4.8}$$

To this end we consider three cases.

First, suppose $k < q_{n-1}$. Without loss of generality we assume that n is even (otherwise the $+$ and $-$ will change places). Then, by (4.2) and (4.3), $k^+ = k + q_n$ and $k^- = k + q_{n-1}$. Therefore, because of (4.4), $\mathrm{dist}(T_\alpha^k\theta, T_\alpha^{k^+}\theta) = |q_n\alpha - p_n| < \frac{1}{q_{n+1}}$ and $\mathrm{dist}(T_\alpha^k\theta, T_\alpha^{k^-}\theta) = |q_{n-1}\alpha - p_{n-1}| < \frac{1}{q_n}$, and (4.8) holds.

Second, assume $q_{n-1} \leq k < q_n$. Then, similarly, $k^+ = k - q_{n-1}$ and $k^- = k + q_{n-1}$, and (4.8) holds in the same way.

Finally, for $k \geq q_n$ we have $k^+ = k - q_{n-1}$ and $k^- = k - q_n$, which also implies (4.8). ∎

Proof of Theorem 1. In the quasiperiodic case, $V(n) = f(T_\alpha^n \theta)$, we have the following relations for the left and right transfer-matrices:

$$\Phi_n^{+,\alpha}(\theta, E) = M(T_\alpha^n \theta, E) M(T_\alpha^{n-1}\theta, E) \ \cdots \ M(T_\alpha \theta, E),$$

where $M(\theta, E)$ is given by (1.4) with $f(\theta)$ as in Theorem 1, and

$$\Phi_n^{-,\alpha}(\theta, E) = \Phi_n^{+,(-\alpha)}(\theta + \alpha, E). \tag{4.9}$$

Fix $\epsilon > 0$. Note that the denominators q_n are the same for α and $-\alpha$. Then, from (4.6),(4.7), and(4.9), we obtain for n sufficiently large, that the conditions of Corollary 6 are satisfied with $A = \{E : \gamma(E) > 0\}$, $a = \frac{c(E)\gamma(E)}{4(8p+c(E))}(1-\epsilon)$, $\gamma = 1$, $c = \frac{8p}{c(E)} + 1$, the sequence k_n defined by (4.5), and the sequences $j_n^\pm$ defined by (4.6) with $\Phi_n = \Phi_n^{\pm,\alpha}$. ∎

5 The Fibonacci Hamiltonian. Proof of Theorem 3

Our proof of Theorem 3 relies on two main ingredients. First, we will show that for energies E in the spectrum of H_λ, the half-line m-functions scale in a way that corresponds to α-continuous spectrum, namely that for every β, $\limsup_{\epsilon\to 0} \epsilon^{1-\alpha} |m_\beta^+(E + i\epsilon)| < \infty$ (where α is some positive constant depending only on λ). Second, we will use the symmetry properties of the potential to conclude a corresponding result for the whole-line m-function. To prove all that, we use a number of facts that where established by Sütő in [27], as well as a result of Iochum and Testard [13]. We start with

PROPOSITION 10 *Let $\lambda \in \mathbb{R}$ and let H_λ be the corresponding Fibonacci Hamiltonian. Then there exists an $\tilde\alpha > 0$ such that for any E in the spectrum of H_λ, any solution u of $H_\lambda u = Eu$ obeys $\liminf\limits_{L\to\infty} L^{-\tilde\alpha} \|u\|_L > 0$.*

Proof. Let $\{F_n\}_{n=1}^\infty$ be the Fibonacci numbers, defined by $F_0 = 1$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$. As shown by Sütő (Proposition 1 of [27]), the Fibonacci potential obeys $V(F_n + \ell) = V(\ell)$ for $n \geq 3$ and $1 \leq \ell \leq F_n$. Thus, for n large and $1 \leq \ell \leq F_{n-2}$, we also get $V(2F_n + \ell) = V(F_{n+1} + F_{n-2} + \ell) = V(F_{n-2} + \ell) = V(\ell)$, namely, we have

$$V(\ell) = V(F_n + \ell) = V(2F_n + \ell) \tag{5.1}$$

for $1 \leq \ell \leq F_{n-2}$. By Lemma 1 of [27], we have for any 2×2 matrix B with $\det B = 1$,

$$\max\{|\operatorname{Tr} B| \cdot \|B\Psi\|, \|B^2\Psi\|\} \geq \frac{1}{2}\|\Psi\| \tag{5.2}$$

for any 2-vector Ψ. Thus, if we have a bound on $|\operatorname{Tr} B|$ of the form $|\operatorname{Tr} B| \leq C$ with $C > 1$, we get

$$\|B\Psi\|^2 + \|B^2\Psi\|^2 > \frac{1}{4C^2}\|\Psi\|^2. \tag{5.3}$$

For every $m < k$, let $\Phi_{m,k}$ denote the 2×2 transfer matrix which takes $(u(m+1), u(m))^T$ to $(u(k+1), u(k))^T$. Let $1 \leq \ell \leq F_{n-2}$. By (5.1), we have that $\Phi_{\ell,F_n+\ell} = \Phi_{F_n+\ell,2F_n+\ell}$, and moreover, we have $\operatorname{Tr}\Phi_{\ell,F_n+\ell} = \operatorname{Tr}\Phi_{0,F_n}$. As shown in the proof of Lemma 2 of [27], there exists a constant C_λ, depending only on λ, such that $|\operatorname{Tr}\Phi_{0,F_n}| < C_\lambda$ for every E in the spectrum of H_λ and every n. Thus, (5.3) implies

$$|u(F_n+\ell+1)|^2+|u(F_n+\ell)|^2+|u(2F_n+\ell+1)|^2+|u(2F_n+\ell)|^2 > \frac{1}{4C_\lambda^2}(|u(\ell+1)|^2+|u(\ell)|^2), \quad (5.4)$$

from which we easily deduce that

$$\|u\|_{F_{n+2}} > \left(1+\frac{1}{4C_\lambda^2}\right)^{1/2} \|u\|_{F_{n-2}} \quad (5.5)$$

for large n. Since the F_n's grow exponentially in n, (5.5) proves the required result. ∎

We can now get the required estimate for m_β^+.

PROPOSITION 11 *Let $\lambda \in \mathbb{R}$ and let H_λ be the corresponding Fibonacci Hamiltonian. Then there exists an $\alpha > 0$, which depends only on λ, such that for any E in the spectrum of H_λ and every β, $\limsup_{\epsilon\to 0}\epsilon^{1-\alpha}|m_\beta^+(E+i\epsilon)| < \infty$.*

Proof. By combining (2.12) and (2.13), we get

$$\epsilon^{1-\alpha}|m_\beta^+(E+i\epsilon)| \sim (\|u_{1,\beta}^+\|_{L_\beta^+(\epsilon)}\|u_{2,\beta}^+\|_{L_\beta^+(\epsilon)})^{\alpha-1}\frac{\|u_{2,\beta}^+\|_{L_\beta^+(\epsilon)}}{\|u_{1,\beta}^+\|_{L_\beta^+(\epsilon)}} = (\|u_{2,\beta}^+\|_{L_\beta^+(\epsilon)})^{\alpha}(\|u_{1,\beta}^+\|_{L_\beta^+(\epsilon)})^{\alpha-2}. \quad (5.6)$$

Iochum and Testard [13] proved an estimate that is analogus to our Proposition 10, but bounds the growth of solutions from above. Their result implies that for every $\lambda \in \mathbb{R}$, there exists a finite positive constant $\tilde{\alpha}_1$, such that for every E in the spectrum of H_λ, any solution u of $H_\lambda u = Eu$ obeys $\limsup_{L\to\infty} L^{-\tilde{\alpha}_1}\|u\|_L < \infty$. This means that for appropriate energies E, the solution $u_{2,\beta}^+$ of (5.6) obeys $\|u_{2,\beta}^+\|_{L_\beta^+(\epsilon)} < C_2(L_\beta^+(\epsilon))^{\tilde{\alpha}_1}$ for some constant C_2. Similarly, Proposition 10 implies that $\|u_{1,\beta}^+\|_{L_\beta^+(\epsilon)} > C_1(L_\beta^+(\epsilon))^{\tilde{\alpha}}$ for some constant C_1. Thus, for $\alpha \leq 1$, the r.h.s. of (5.6) is bounded from above by $C_2^\alpha C_1^{\alpha-2}(L_\beta^+(\epsilon))^{\alpha(\tilde{\alpha}_1+\tilde{\alpha})-2\tilde{\alpha}}$, which is bounded from above (as $\epsilon \to 0$) if we take $\alpha \leq 2\tilde{\alpha}/(\tilde{\alpha}_1+\tilde{\alpha})$. ∎

Proof of Theorem 3. By Proposition 1 of [27], the Fibonacci potential obeys $V(-n) = V(n-1)$ for any $n \geq 2$. Let $z = E + i\epsilon$, where $E \in \mathbb{R}$ and $\epsilon > 0$, then the uniqueness of the solutions $\hat{u}_z^\pm$ (introduced in Section 2) along with this symmetry, implies that

$$\frac{\hat{u}_z^+(1)}{\hat{u}_z^+(0)} = \frac{\hat{u}_z^-(-2)}{\hat{u}_z^-(-1)}. \quad (5.7)$$

For any solution u of $H_\lambda u = zu$, we clearly have for every n,

$$\frac{u(n+1)}{u(n)} = z - V(n) - \frac{u(n-1)}{u(n)}, \tag{5.8}$$

and thus

$$\frac{\hat{u}_z^-(-2)}{\hat{u}_z^-(-1)} = z - V(-1) - \left(z - V(0) - \frac{\hat{u}_z^-(1)}{\hat{u}_z^-(0)}\right)^{-1}. \tag{5.9}$$

By using (2.6) and the fact that $V(0) = 0$ and $V(-1) = \lambda$, (5.7) and (5.9) imply

$$-m^+(z) = z - \lambda - (z - m^-(z))^{-1}, \tag{5.10}$$

which can be rearanged to yield

$$m^-(z) = z - (z - \lambda - m^+(z))^{-1}. \tag{5.11}$$

Since we already know that H_λ has purely singular spectrum, its spectral measure is supported on the set of energies E for which $|M(E+i0)| = \infty$. Moreover, it had been shown by Gilbert [9] that the spectral measure is supported on the set of E's for which $H_\lambda u = Eu$ has solutions that are subordinate both to left and to the right. This implies that for such energies, $m^\pm(E+i0)$ exist and are either real or infinity (where we identify $m^\pm(E+i0) = \infty$ with $|m^\pm(E+i0)| = \infty$, namely with $\lim_{\epsilon\to 0} |m^\pm(E+i\epsilon)| = \infty$). That is, the spectral measure of H_λ is supported on a set S that is contained in its spectrum and such that for any $E \in S$, $|M(E+i0)| = \infty$ and $m^\pm(E+i0)$ both exist and are in $\mathbb{R} \cup \{\infty\}$. From (2.7) along with the fact that $m^\pm(z)$ have positive imaginary parts, we see that for each $E \in S$, we must have either $|m^+(E+i0)| = |m^-(E+i0)| = \infty$ or that $m^\pm(E+i0)$ are real and $m^+(E+i0) = -m^-(E+i0)$. But, by (5.11) we see that if $|m^+(E+i0)| = \infty$, then $m^-(E+i0)$ is finite, and so for each $E \in S$ we must have that $m^\pm(E+i0)$ are real and $m^+(E+i0) = -m^-(E+i0)$. Let $E \in S$, then by inverting (2.5) to yield m_β^+ in terms of m^+, we obtain

$$m_\beta^+ = \frac{m^+ \cot\beta + 1}{\cot\beta - m^+}, \tag{5.12}$$

from which we see that $|m_\beta^+(E+i0)| = \infty$ exactly when $\cot\beta = m^+(E+i0)$, and in that case, $|m_\beta^+(E+i\epsilon)| \sim |(\cot\beta - m^+(E+i\epsilon))^{-1}|$ as $\epsilon \to 0$. Thus, Proposition 11 implies that for small ϵ, $|\cot\beta - m^+(E+i\epsilon)|$ is much larger than ϵ. By differentiating the r.h.s of (5.11) with respect to m^+ (at the point where $z = E$ and $m^+ = \cot\beta$), we see that as $\epsilon \to 0$, $|m^+(E+i\epsilon) + m^-(E+i\epsilon)| \sim |\cot\beta - m^+(E+i\epsilon)|$ and thus, by (2.7), we also have $|M(E+i\epsilon)| \sim |m_\beta^+(E+i\epsilon)|$. By Proposition 11, this implies that

$$\limsup_{\epsilon\to 0} \epsilon^{1-\alpha} |M(E+i\epsilon)| < \infty \tag{5.13}$$

for a.e. E w.r.t. the spectral measure of H_λ, and thus, H_λ has purely α-continuous spectrum. ∎

Remark. The proof of Theorem 3 uses the symmetry of the potential to show that for an appropriate β, $|m_\beta^+(E+i\epsilon)| \sim |M(E+i\epsilon)|$ as $\epsilon \to 0$, and thus that the continuity result for the half-line problem (Proposition 11) translates to a corresponding continuity result for the whole-line problem. The crucial fact to showing this is (5.11), which has the form

$$m^-(z) = f(z, m^+(z)), \tag{5.14}$$

where f is a rational function of two variables with real coefficients. By using (5.8) and (2.6), one can show that a relation of the form (5.14) would hold whenever $V(k-n) = V(m+n)$ for some $k, m \in \mathbb{Z}$ and all $n \in \mathbb{N}$, and the implication $|m_\beta^+(E+i\epsilon)| \sim |M(E+i\epsilon)|$ can thus be extended to any such case. That is, if $V(k-n) = V(m+n)$ for some $k, m \in \mathbb{Z}$ and all $n \in \mathbb{N}$, and if there exists an $0 < \alpha \leq 1$ such that for any E in some Borel set S and every β, $\limsup_{\epsilon\to 0} \epsilon^{1-\alpha} |m_\beta^+(E+i\epsilon)| < \infty$, then the restriction of the spectral measure of the whole-line problem to S is α-continuous.

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