Simultaneous quantization of edge and bulk Hall conductivity

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Abstract

The edge Hall conductivity is shown to be an integer multiple of e^2/h which is almost surely independent of the choice of the disordered configuration. Its equality to the bulk Hall conductivity given by the Kubo-Chern formula follows from K-theoretic arguments. This leads to quantization of the Hall conductance for any redistribution of the current in the sample. It is argued that in experiments at most a few percent of the total current can be carried by edge states.

Soon after the discovery of the integer quantum Hall effect (QHE) [19], several geometric interpretations of the observed quantization of the Hall conductance of a two dimensional electron gas were put forward in the framework of non-relativistic quantum mechanics. Laughlin proposed an adiabatic Gedankenexperiment in order to calculate the Hall conductance [15], Halperin and later on Büttiker studied the conduction by edge channels [13, 10], while Thouless, Kohmoto, Nightingale and den Nijs investigated the Hall conductivity as given by the Kubo formula [20]. Laughlin's argument was rigorously analyzed by Avron, Seiler and Simon even for multiparticle Hamiltonians and in presence of a disordered potential [3, 4, 5]. Bellissard, recently joint by van Elst and Schulz-Baldes, generalized the TKN$_2$-work in order to show quantization of the Hall conductivity also in presence of a disordered potential as long as the Fermi level lies in a region of dynamically localized states [6, 7], a result that was also obtained by Aizenman and Graf [1]. All these beautiful mathematical approaches exhibit the Hall conductance and conductivity respectively to have a deep geometrical meaning and allow to calculate them as an index of a certain Fredholm operator. In neither of them the edges of the sample play a particular rôle.

Recently there has been a revived interest in edge states of magnetic Schrödinger operators. Hatsugai linked an edge state winding number to the Chern numbers for Harper's equation with rational flux [14]. Akkermans, Avron, Narevich and Seiler introduced spectral boundary conditions giving rise to a linear dispersion relation for edge states and a natural setting for the Laughlin wave function as a many body bulk state [2]. The stability of the absolutely continuous spectrum associated to edge states under the perturbation with a random potential was studied by several authors with Mourre's positive commutator estimates [16, 8, 12].

Our first main result is a rigorous proof of the edge current quantization in the sense of Halperin for a discrete magnetic half-plane operator containing a disordered potential, notably we show quantization of what we call the edge Hall conductivity. Our second mathematical result is its equality to the bulk Hall conductivity as calculated by the Kubo-Chern formula [20, 6, 7]. The proof of this equality unveals a deep connection between the plane and edge geometry as it is based on Bott periodicity, the heart of K-theory [9]. Using these results, we reproduce Halperin's argument explaining why the Hall conductance of a Hall bar is quantized no matter what proportion of the current is actually carried by the edge or the bulk states respectively.

Finally we present a simple theoretical reasoning showing that in a typical experimental situation at most 10% of the current flows by edge states.

For the definition of the edge Hall conductivity, we consider a gas of charged independent particles in the (discrete) upper half plane $\Gamma = \{(x, y) \in \mathbf{Z}^2 | y \geq 0\}$ submitted to a perpendicular magnetic field B. Let $\hat{H}$ denote the one-particle Hamiltonian acting on $\ell^2(\Gamma)$. All operators on the half-plane space carry a hat from now on. Typically $\hat{H}$ is the projection onto $\ell^2(\Gamma)$ of an operator $H = H_H + V$ acting on $\ell^2(\mathbf{Z}^2)$ where H_H is Harper's magnetic Hamiltonian and V is the sum of a periodic and a disordered potential. As the edge of the plane intercepts the cyclotron orbits, there may be a net electric current along the edge. In order to calculate it, let J be a spectral interval lying in a gap of the plane Hamiltonian H. Such an interval typically contains extended edge states of $\hat{H}$ [14], even in presence of a weak disordered potential [13, 16, 8, 12]. If $\hat{P}_J$ is the spectral projection of $\hat{H}$ on J, then the electric edge current in x-direction carried by the eigenstates in J is equal to $q\hat{\mathcal{T}}(\hat{P}_J \nabla_x(\hat{H}))$. Here q is the charge of the particles, $\nabla_x(\hat{H}) = \imath[X, \hat{H}]$ is the current operator given by the commutator of the Hamiltonian and the X-position operator and finally the trace $\hat{\mathcal{T}} = \mathcal{T}_x \mathrm{Tr}_y$ is the trace per unit volume [6, 7] in the x-direction and the usual trace in y-direction. Now given an energy E in a gap of extended states of H, we define

$$\sigma^e_\perp(E) \;=\; \frac{q^2}{\hbar} \lim_{J \to \{E\}} \frac{1}{|J|} \hat{\mathcal{T}}(\hat{P}_J \nabla_x(\hat{H})) \;. \tag{1}$$

Because an infinite half-plane is a typical model for a mesoscopic volume with a boundary, we call $\sigma^e_\perp(E)$ the edge Hall conductivity rather than the edge Hall conductance just as the bulk Hall conductivity is calculated with an infinite planar model for a mesoscopic volume, while the conductance is always associated to a finite macroscopic sample. Both the edge and the bulk Hall conductivity are idealized quantities for which clear mathematical results can be obtained. Further we note that one could define the edge Hall conductivity for a strip geometry, but this would not lead to quantization because of backscattering, that is tunneling from upper to lower edge states [10].

Before starting the more mathematical analysis, let us consider the Harper Hamiltonian H_H on $\ell^2(\mathbf{Z}^2)$ in order to familiarize ourselves with the notion of edge Hall conductivity. It is defined by the finite difference equation $(H_H\psi)_{n,m} = \psi_{n+1,m} + \psi_{n-1,m} + e^{2\pi\imath\varphi}\psi_{n,m+1} + e^{-2\pi\imath\varphi}\psi_{n,m-1}$ and we suppose here that the magnetic flux per unit cell is rational $\varphi = p/q$. Then the spectrum of H_H is known to be a band spectrum. To analyse the half plane operator $\hat{H}_H$ on $\ell^2(\Gamma)$, we use the translation invariance in the x-direction to make a Bloch decomposition $\hat{H}_H = \int^{\oplus}_{[-\pi,\pi)} \frac{dk_x}{2\pi}\, \hat{H}_H(k_x)$ where $\hat{H}_H(k_x)$ is a Jacobi matrix on $\ell^2(\mathbf{N})$. The spectrum of each $\hat{H}_H(k_x)$ contains the bands of the corresponding periodic operator $H_H(k_x)$ on $\ell^2(\mathbf{Z})$, but there may now also be a Dirichlet eigenvalue $\hat{E}_l(k_x)$ in each gap of $H_H(k_x)$. Upon varying k_x, the eigenvalues form a finite number of continuous curves the end-points of which touch the adjacent Bloch bands of H_H (see Fig. 1). To each of these so-called *edge channels* we associate a weight $+1$ (respectively -1) if the Dirichlet eigenvalues of the channel vary from the upper towards the lower (respectively lower to upper) adjacent Bloch band as k_x increases. Let s_n be the sum of all these weights in the nth gap G_n of H_H. Then the edge current carried by the edge states in an interval J contained in G_n is equal to $s_n|J|q^2/h$ because

$$\hat{\mathcal{T}}(\hat{P}_J \nabla_x(\hat{H}_L)) \;=\; \sum_l \int_{-\pi}^{\pi} dk_x\, \chi_J(\hat{E}_l(k_x)) \frac{d\hat{E}_l(k_x)}{dk_x} \;.$$

Here χ_J denotes the indicator function on J. This implies that $\sigma^e_\perp(E) = s_n q^2/h$ for all $E \in G_n$. Hatsugai, in a beautiful paper [14], has shown that s_n is equal to the sum of the Chern numbers of the n bands below G_n. This sum multiplied by q^2/h is the bulk Hall conductivity $\sigma^b_\perp(E)$ [20].

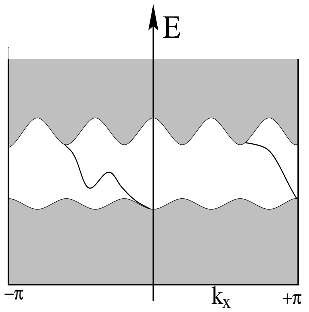

Figure 1: *Schematic representation of the spectrum of $\hat{H}_H$ in a given gap of H_H, the solid lines are the Dirichlet bands and the shaded regions are the Bloch bands (taken from* [14]*).*

Hence we obtain $\sigma^e_\perp(E) = \sigma^b_\perp(E)$ for all energies in the gaps of H_H, which is a particular case of Theorem 2 below. Note that the equivalent result for the Landau Hamiltonian simply states that there are n edge channels in the gap between the nth and $(n+1)$th Landau bands [13].

Now we would like to add a disordered potential V. First of all, if V is sufficiently small, sufficiently large gaps of H_H remain open for $H = H_H + V$. It follows further from Mourre estimates on the current operator that the spectrum remains absolutely continuous in the gaps of H for a weak potential whenever the current of the edge states of $\hat{H}_H$ has a definite sign [8]. Whereas the latter condition is always satisfied for the Landau Hamiltonian, it may not hold in the discrete case (*cf.* Fig. 1 and the numerical studies in [14] where edge channels having edge states with group velocity both to the left and to the right are exhibited). In this situation, the positive commutator methods cannot be applied. Nevertheless, we shall be able to show that the current remains constant. However, but we cannot deduce that the spectrum is still absolutely continuous once a small perturbation is added.

In order to treat the situation with broken translation invariance, we parallel Bellissard's non-commutative generalization of the TKN$_2$ work [6, 7]. No particular structure of the Hamiltonian H on $\ell^2(\mathbf{Z}^2)$ is needed except for its homogenuity in the sense of [6, 7]. The main mathematical tool in [6, 7] is the C*-algebra $\mathcal{A}$ of homogeneous observables in the plane. It has the structure of a crossed product algebra $\mathcal{A} = C(\Omega) \times \mathbf{Z}_x \times \mathbf{Z}_y$ associated to the dynamical system given by the magnetic translations $\mathbf{Z}_x$ and $\mathbf{Z}_y$ in the x and y-direction respectively acting on the compact space of disorder configurations Ω which is the hull of H. Each such configuration $\omega \in \Omega$ induces a representation π_ω of the observable algebra $\mathcal{A}$ on physical Hilbert space $\ell^2(\mathbf{Z}^2)$. There exists an $H \in \mathcal{A}$ such that $\pi_\omega(H)$ is precisely the Hamilton operator with disordered configuration $\omega \in \Omega$. Physical quantities as the Hall conductivities hence carry a supplementary index ω. We now consider the Toeplitz extension $T(\mathcal{A})$ with respect to the crossed product structure of $\mathbf{Z}_y$ [18]. Its physical representations give operators in the half plane. This naturally gives rise to an exact sequence of C^*-algebras [18]

$$0 \;\rightarrow\; \mathcal{E} \;\overset{i}{\rightarrow}\; T(\mathcal{A}) \;\overset{\pi}{\rightarrow}\; \mathcal{A} \;\rightarrow\; 0\;. \tag{2}$$

Here $\mathcal{E}$ is the C*-algebra of observables localized near the edge $y = 0$; it is isomorphic to the C*-tensor product of $C(\Omega) \times \mathbf{Z}_x$ with the compact operators K. The exact sequence (2) induces two six-term exact sequences, one for K-theory groups [18, 9] and one for the cyclic cohomology groups [17], and we shall use their duality [17] to prove the equality of bulk and edge Hall conductivities.

Let us illustrate these notions for the Harper Hamiltonian with arbitrary flux φ, but without a further potential. The C*-algebra $\mathcal{A}$ is then the rotation algebra generated by the two magnetic translations U_x and U_y satisfying the commutation relation $U_x U_y = e^{2\pi\imath\varphi} U_y U_x$. Thus in this case

$C(\Omega) \cong \mathbf{C}$. The Toeplitz extension is generated by $\hat{U}_x$ and $\hat{U}_y$ satisfying the same commutation relation, but, while $\hat{U}_x$ remains unitary, $\hat{U}_y$ is now only an isometry satisfying $\hat{U}_y^* \hat{U}_y = 1 - \Pi_0$ where Π_0 is the projection on the states supported by the boundary of Γ. Finally, $\mathcal{E}$ is isomorphic to the tensor product of $\mathrm{C}^*(\hat{U}_x) \cong C(S^1)$ with K. The maps in (2) are the inclusion i and the projection π given by $\pi(\hat{U}_{x,y}) = U_{x,y}$.

The traces $\mathcal{T}_{x,y}$ of physical representations π_ω of an observable are almost surely independent of ω with respect to any given invariant and ergodic measure $\mathbf{P}$ on Ω [7]. Hence they allow to define traces on the observable algebras $\mathcal{A}$ and $\mathcal{E}$. Now the definition (1) of the edge Hall conductivity remains valid as long as the projections $\hat{P}_J$ are in the Schatten ideal of traceclass operators with respect to $\hat{\mathcal{T}}$ for J sufficiently close to $\{E\}$. This is possible even though $\hat{P}_J$ is only an element of the bicommutant $\mathcal{E}''$, the envelopping von Neumann algebra. Now the crucial observation is that the current of the edge states in an interval J lying in a gap of H can be calculated using Duhamel's formula and taking into account the discontinuity of $\hat{P}_J$:

$$\hat{\mathcal{T}}(\hat{P}_J \nabla_x(\hat{H})) \;=\; \frac{|J|}{2\pi\imath}\, \hat{\mathcal{T}}(\hat{\mathcal{U}}(J)^* \nabla_x \hat{\mathcal{U}}(J)) \;, \tag{3}$$

where

$$\hat{\mathcal{U}}(J) \;=\; \exp\left(2\pi\imath \hat{P}_J \frac{\hat{H} - E'}{|J|}\right) \;, \qquad E' = \inf(J) \;. \tag{4}$$

Although $\hat{P}_J$ contains discontinuities, the construction assures that the unitary operator $\hat{\mathcal{U}}(J)$ is an element of the unitization $\tilde{\mathcal{E}}$ of the C^*-algebra $\mathcal{E}$. Further $\nabla_x \hat{\mathcal{U}}(J)$ is $\hat{\mathcal{T}}$-traceclass if $\hat{P}_J$ is (that (3) is finite also follows from the exponential map (5) and the duality theorem below). Finally, we note that $\hat{\mathcal{U}}(J)$ remains in the same homotopy class of unitaries in $\tilde{\mathcal{E}}$ as the borders of J vary in a gap G of H. Thus (4) associates to G a class in the K-group $K_1(\mathcal{E})$.

It now follows from Connes' non-commutative geometry [11] that $\frac{1}{2\pi\imath}\hat{\mathcal{T}}(\hat{\mathcal{U}}^* \nabla_x \hat{\mathcal{U}})$ is an integer for any unitary $\hat{\mathcal{U}}$ in (a suitable subalgebra of) $\tilde{\mathcal{E}}$. Actually $\zeta_1(\hat{A}, \hat{B}) = \frac{1}{2\pi\imath}\hat{\mathcal{T}}(\hat{A}\nabla_x(\hat{B}))$ defines a 1-cocycle on $\mathcal{E}$ because $\hat{\mathcal{T}}$ is invariant under ∇_x. With some calculatory effort, this cocycle can be linked to the standard 1-cocyle of the Fredholm module $(C_1 \otimes \mathcal{E}_0, \pi_\omega \oplus \pi_\omega, \ell^2(\Gamma) \oplus \ell^2(\Gamma), \sigma_2 \otimes \imath X/|X|)$ where $\mathcal{E}_0$ is the in $\mathcal{E}$ dense $*$-algebra of operators with finite support in y-direction and C_1 a two-dimensional $\mathbf{Z}_2$ graded Clifford algebra in $\mathrm{Mat}(\mathbf{C}^2)$, $\pi_\omega \oplus \pi_\omega$ is a doubling of the physical representation on the doubled physical Hilbert space $\ell^2(\Gamma) \oplus \ell^2(\Gamma)$ and the Dirac phase is constructed from the Pauli matrix σ_2 and the position operator X. Hence the odd index theorem [11, p. 291] and a treatment of the disorder configuration along the lines of [7] imply the following result.

Theorem 1 *Suppose that $G \subset \mathbf{R}$ is a spectral gap of the plane operator H acting on $\ell^2(\mathbf{Z}^2)$. Then, if Π denotes the projection from $\ell^2(\mathbf{Z} \otimes \mathbf{N})$ onto $\ell^2(\mathbf{N} \otimes \mathbf{N})$ and $\hat{\mathcal{U}}(G)$ is constructed by (4) from $\hat{P}_G$, the operator $\Pi\pi_\omega(\hat{\mathcal{U}}(G))\Pi$ is a Fredholm operator on $\ell^2(\mathbf{N} \otimes \mathbf{N})$ and for all $E \in G$ and every disorder configuration $\omega \in \Omega$:*

$$\sigma^e_\perp(E, \omega) \;=\; \frac{q^2}{h}\, \mathrm{Ind}\left(\Pi\, \pi_\omega(\hat{\mathcal{U}}(G))\, \Pi\right) \;.$$

Moreover, $\sigma^e_\perp(E, \omega)$ is $\mathbf{P}$-almost surely equal to a constant $\sigma^e_\perp(E)$.

We remark that the index can also be written as a relative index of a pair of projections as defined by Avron, Seiler and Simon [5], notably as the relative index of Π and $\pi_\omega(\hat{\mathcal{U}}(G))\Pi\pi_\omega(\hat{\mathcal{U}}(G))$.

Using the exact sequence (2), we now link this edge theory to the bulk theory as developed in [7]. From the above follows that $\sigma^e_\perp(E)$ actually results from a pairing between $[\hat{\mathcal{U}}(G)]_1 \in K_1(\mathcal{E})$ with the odd cyclic cohomology class defined by the 1-cocycle ζ_1 given above. Similarly, the bulk

Hall conductivity $\sigma^b_\perp(\mu)$ for a Fermi level μ in a gap of H comes from a pairing of the class of the Fermi projection $[P_\mu]_0 \in K_0(\mathcal{A})$ with the 2-cocycle ζ_2 over $\mathcal{A}$ defined by $\zeta_2(A,B,C) = 2\pi\imath\mathcal{T}_x\mathcal{T}_y(A\nabla_x B\nabla_y C - A\nabla_y B\nabla_x C)$ [7]:

$$\sigma^e_\perp(E) \;=\; \langle\zeta_1,[\hat{\mathcal{U}}(G)]_1\rangle\;, \qquad \sigma^b_\perp(\mu) \;=\; \langle\zeta_2,[P_\mu]_0\rangle\;.$$

We now first show that the image of $[P_\mu]_0$ under the exponential map in the six-term exact sequence of K-groups [9] is $[\hat{\mathcal{U}}(G)]_1$ whenever the Fermi level μ is in the gap G of H. In fact, P_μ is equal to the continuous function of the Hamiltonian $f(H) = P_{<G} - P_G(H-E')/|G|$ where $P_{<G}$ is the spectral projection of H on all energies below G and $E' = \inf(G)$. Now a self-adjoint lift of P_μ is given by $f(\hat{H})$. From $[\hat{P}_{<G},\hat{P}_G] = 0$ thus follows

$$\exp([P_\mu]_0) \;=\; [\exp(-2\pi\imath f(\hat{H}))]_1 \;=\; [\hat{\mathcal{U}}(G)]_1\;. \tag{5}$$

Next one can verify by an elementary calculation that the 1-cocycle ζ_1 over $\mathcal{E}$ is mapped to the 2-cocyle ζ_2 over $\mathcal{A}$ under the mapping # defined in [17, Sec. 8]. For this map, the duality theorem of the pairing holds, notably $\langle\zeta_1,\exp([P]_0)\rangle = \langle\#\zeta_1,[P]_0\rangle$ for any projection $P\in\mathcal{A}$ [17, Sec. 12]. Hence we obtain:

Theorem 2 *$\sigma^e_\perp(E) = \sigma^b_\perp(E)$ for all energies E in a spectral gap of H.*

At this point let us comment on generalization of these results. Just as one does not need the existence of a gap G in order to prove the quantization of the bulk Hall conductivity [7, 1], it is possible to prove Theorem 1 and and probably also Theorem 2 under the weaker hypothesis that the interval G only contains dynamically localized states of H in the sense of [7]. Furthermore, the whole theory should have a continuous analogon for a disordered Landau Hamiltonian. As both of these generalizations ask for more lengthy and detailed proofs, they will be subject of a forthcoming publication.

We now sketch how the above results lead to the desired explanation of a QH regime measurement in a QH bar. Following Halperin [13], we suppose that the measured Hall voltage $V_\perp$ across the bar is the sum of the the potential drop V^b due to an electrostatic field and the (relativ) chemical potential difference $\Delta\mu/q = (\mu_{\rm u}-\mu_{\rm l})/q$ between the upper and the lower edge. Furthermore, let the interval $[\mu_{\rm l},\mu_{\rm u}]$ be contained in a gap G of H (the above generalization only needs the weaker condition that G is dynamically localized). In linear response approximation, the electric field leads to a bulk current $I^b = \sigma^b_\perp V^b$. Now both the upper and the lower edge may carry a current. In absence of backscattering, we can treat them as two separate half-plane problems. But actually the lower edge can be seen as an upper edge with reversed magnetic field, which is equivalent to a time reversal. This changes the orientation of its current so that the net current carried by both edges comes from the upper edge states with energies in $[\mu_{\rm l},\mu_{\rm r}]$. From the above thus results a net edge current $I^e = \sigma^e_\perp\Delta\mu/q$. Hence the Hall conductance of the sample given by the quotient of the total current $I = I^e + I^b$ and the voltage $V_\perp$ is equal to the integer $\sigma^e_\perp = \sigma^b_\perp$ for any value of $V^b/V_\perp$.

An interesting question which has led to considerable theoretical and experimental work (see [21] and references therein) is how much current is carried by either edge or bulk states in a typical QH experiment. Let us argue that at most 10% of the current is carried by the edge states. This agrees with recent experimental studies [21]. For the edge current of $[\mu_{\rm l},\mu_{\rm u}]$ to be equal to an integer times $\Delta\mu$, the difference of chemical potentials $\Delta\mu$ clearly has to be smaller than the energetic distance $\hbar\omega_c(1-q)$ (here ω_c is the cyclotron frequency so that $\hbar\omega_c$ is the distance between two Landau levels, and q is the quotient of the energetic width of the plateaux and $\hbar\omega_c$). Hence the proportion of edge current has to be smaller than $\Delta\mu/qV_\perp$. In order to estimate this condition and the temperature corrections below, we use the experimental data from [19, Chapter 2] for the $\sigma_\perp = 4$ plateau: $B \approx 6\,T$, $V_\perp \approx 170\,mV$ and $T \approx 1.2\,K$ and $q \approx 0.6$. Using the data for the effective electron mass ($m_* \approx 0.07 m_e$) and the electron charge, we obtain $\hbar\omega_c \approx 48\,meV$ and a maximal proportion of edge currents of 10%.

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