A Gibbsian description of $P(\phi)_1$-processes

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Abstract

We consider Brownian motions perturbed by suitable potentials and ask the question whether their stationary path measures can be described as Gibbs measures. Our main result is that for a rich set of potentials and a large subset of boundary conditions Gibbs measures do exist and are moreover unique. The set of allowed boundary path configurations is identified explicitely. We conclude by illustrating these results on some specific examples.

1 Introduction

$P(\phi)_1$-stochastic processes appear within the context of Euclidean quantum field theory and they can be viewed as a 0 space-dimensional case of it. In this case there are no pair or many-body interactions present, but even the self-potential model turns out to be of interest. This class has been studied mathematically in the book [12] and in papers such as [11, 7, 10]. By the known relationships between Euclidean field theory and statistical mechanics one would expect that $P(\phi)_1$-processes allow a description in terms of Gibbs measures, however this question seems not to have been addressed in sufficient generality. In [12] there is a wealth of material on various aspects of these processes constructed, however, slightly differently than we will do it below. Also, the existence of $P(\phi)_1$-processes is tackled only in terms of what will be called below constant stochastic boundary conditions. In [10] the main goal was studying the existence and uniqueness of Gibbs measures for diffusions related to pair potentials, and therefore results on $P(\phi)_1$-processes are only marginal and less than optimal. Closer to the context of infinite dimensional stochastic differential equations the problem has been considered in [7, 8]. In particular, [7] used the class of potentials divergent at infinity and showed the existence of Gibbs measures concentrated on a set of subexponentially growing boundary configurations.

Our principal aim here is to show that the stationary measures of $P(\phi)_1$-processes can in fact be constructed as Gibbs measures for a large class of interesting potentials and for a suitable subset of path configurations. This goal is framed in our broad interest in giving Gibbsian descriptions to more general diffusions than only those associated with self-potentials, but here we only concentrate on $P(\phi)_1$-processes.

The first natural problem to settle is the existence of such Gibbs measures. Our first finding is that for Kato-class potentials Gibbs measures do exist and can be constructed by using constant boundary conditions. This class includes potentials growing at infinity and also those that decay to zero, possibly having also some singularities. Next we show that, for the same class of potentials, by taking more general boundary conditions this limiting procedure still goes through provided the boundary paths do not grow too fast at infinite time. We actually obtain an explicit condition on the asymptotics of allowed boundary path configurations and identify them as a full-measure subset of all possible configurations. Moreover, we conclude that for this choice of boundary conditions the same Gibbs measure appears in the limit as the one obtained for constant boundary paths, hence we show actually uniqueness of the $P(\phi)_1$-measure supported by this set. Understandably, not every Brownian path can be a "good" configuration but only those getting readily enough "in tune" with the stationarity of the $P(\phi)_1$-process. The situation of working with a restricted set of configurations is reminiscent of similar constructions for lattice spin systems with noncompact state spaces; below it will be seen that our results are, however, of apparently different flavour than those involving superstability and temperedness arguments.

The paper is organized as follows. In the next section we fix the notations, define the basic ingredients and point out some basic facts. In Section 3 we state and prove our main results, and in Section 4 we conclude by looking at some examples.

2 Preliminaries

We start with some basic definitions. Consider $\Omega = C(\mathbb{R}, \mathbb{R}^d)$, the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, endowed with the σ-field $\mathcal{F} = \sigma(p_t : t \in \mathbb{R})$ generated by the point evaluations

$$p_t : \Omega \to \mathbb{R}^d, \qquad \omega \mapsto p_t(\omega) = \omega(t) \tag{2.1}$$

$(\Omega, \mathcal{F})$ is the measurable space we will use in the following. For a subset $I \subset \mathbb{R}$ we define the σ-algebra $\mathcal{F}_I \subset \mathcal{F}$ by $\mathcal{F}_I = \sigma(p_t : t \in I)$. We use the notation $I^c = \mathbb{R} \setminus I$ for any $I \subset \mathbb{R}$.

Several different measures will be considered on $(\Omega, \mathcal{F})$. The first is Wiener measure which will be denoted by $\mathcal{W}$. Although this is a standard notion, we give an explicit definition here suitable for our context. Consider the transition density for Brownian motion

$$s_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \qquad (x, y) \mapsto \frac{1}{(2\pi t)^{d/2}}\, e^{-\frac{(x-y)^2}{2t}} \tag{2.2}$$

This gives rise to the operator

$$S_t : L^2(d\lambda^d) \to L^2(d\lambda^d), \quad f \mapsto \int s_t(\cdot, y) f(y)\, d\lambda^d(y) \tag{2.3}$$

where λ^d is the d-dimensional Lebesgue measure. $\mathcal{W}$ is then the unique measure on $(\Omega, \mathcal{F})$ such that for all $n \in \mathbb{N}$, $t_1 < t_2 < \ldots < t_n \in \mathbb{R}$, $A_1, \ldots, A_n \in \mathbb{R}^d$ with $\lambda^d(A_i) < \infty$ $(i = 1, \ldots, n)$ and with $A = \cap_{i=1}^n p_{t_i}^{-1}(A_i)$ the following holds:

$$\mathcal{W}(A) = \langle 1_{A_1}, S_{t_2-t_1} 1_{A_1} S_{t_3-t_2} 1_{A_2} \ldots S_{t_n-t_{n-1}} 1_{A_n} \rangle_{L^2(d\lambda^d)} \tag{2.4}$$

Here 1_{A_i} is the indicator function for A_i, and the scalar product is taken in the Hilbert space $L^2(d\lambda^d)$. Also, here and henceforth we use the same symbol for a function and the multiplication operator induced by it. Recall that $\mathcal{W}$ is an infinite measure. Since (2.4) is not the standard way of introducing Wiener measure, we point out that the existence of $\mathcal{W}$ is not a mere consequence of Kolmogorov's Existence Theorem, but follows by using the existence of Brownian motion (for details see [12]). Uniqueness, on the other hand, can be shown by standard uniqueness results for measures, see for instance [2]. Furthermore, $\mathcal{W}$ is a priori only a measure on $(\mathbb{R}^d)^{\mathbb{R}}$, however, by the continuity of Brownian paths we can define it as a measure on Ω.

The second measure we consider is defined on $(C([T_1,T_2],\mathbb{R}^d),\mathcal{F}_{[T_1,T_2]})$, for $T_1 < T_2 \in \mathbb{R}$, instead of Ω. For $\xi,\eta \in \mathbb{R}^d$ let $\mathcal{W}^{\xi,\eta}_{[T_1,T_2]}$ be the unique finite measure on the above space with

$$\mathcal{W}^{\xi,\eta}_{[T_1,T_2]}(A) = (S_{t_1-T_1}1_{A_1}S_{t_2-t_1}1_{A_2}...S_{t_n-t_{n-1}}(1_{A_n}s_{T_2-t_n}(\cdot,\eta)))(\xi) \qquad (2.5)$$

where $T_1 < t_1 < ... < t_n < T_2$, and A_i, A, n are as above. $\mathcal{W}^{\xi,\eta}_{[T_1,T_2]}$ is called conditional Wiener measure or Brownian bridge on $[T_1,T_2]$ starting at ξ and ending in η. This terminology comes from the fact that $\mathcal{W}^{\xi,\eta}_{[T_1,T_2]}$ is concentrated on functions $\omega \in C([T_1,T_2],\mathbb{R}^d)$ with $\omega(T_1)=\xi$ and $\omega(T_2)=\eta$. Since S_t maps $L^2(d\lambda^d)$ into $L^\infty(d\lambda^d)$, definition (2.5) makes sense and $\mathcal{W}^{\xi,\eta}_{[T_1,T_2]}$ is a finite (though not necessarily a probability) measure. The connection between Wiener measure and Brownian bridge is given by the following useful formula. Let A be as before and write $\mathcal{W}^{\xi,\eta}_{[T_1,T_2]}$ for $\mathcal{W}^{\xi,\eta}_{[T_1,T_2]}(C([s,t]),\mathbb{R}^d)$. We then have

$$\mathcal{W}(A) = \int 1_{A_1}(\xi_1)...1_{A_n}(\xi_n)\,\mathcal{W}^{\xi_1,\xi_2}_{[t_1,t_2]}\cdots\mathcal{W}^{\xi_{n-1},\xi_n}_{[t_{n-1},t_n]}\,d\xi_1...d\xi_n \qquad (2.6)$$

The third measure we want to consider is a variant of Brownian bridge. Fix $\bar\omega \in \Omega$ and put $\mathcal{W}^{\bar\omega}_{[T_1,T_2]}$ for the measure on Ω with

$$d\mathcal{W}^{\bar\omega}_{[T_1,T_2]}(\omega) = \begin{cases} d\mathcal{W}^{\bar\omega(T_1),\bar\omega(T_2)}_{[T_1,T_2]}(\omega) & \text{if } \bar\omega=\omega \text{ on } [T_1,T_2]^c \\ 0 & \text{otherwise} \end{cases} \qquad (2.7)$$

By path continuity we have

$$\mathcal{W}^{\bar\omega}_{[T_1,T_2]} = \mathcal{W}_{[T_1,T_2]} \otimes \delta_{\bar\omega}|_{[T_1,T_2]^c} \qquad (2.8)$$

where $\mathcal{W}_{[T_1,T_2]}$ denotes Wiener measure restricted to $C([T_1,T_2],\mathbb{R}^d)$ and $\delta_{\bar\omega}|_{[T_1,T_2]^c}$ is Dirac point measure on $C([T_1,T_2]^c,\mathbb{R}^d)$. We will make use of this measure later on when looking at $P(\phi)_1$-measures as Gibbs measures. Note also that if $x \in \Omega$ is a constant function, then $\mathcal{W}^x_{[0,\infty)}$ is exactly the measure of Brownian motion started at $x \in \mathbb{R}^d$.

Finally, the fourth measure on Ω we need to introduce is the $P(\phi)_1$-measure itself. Consider the Kato-class of functions

$$K_d = \{f:\ \mathbb{R}^d \to \mathbb{R} : \lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\int\left(\int_0^t f(\omega(s))\,ds\right)d\mathcal{W}^x_{[0,\infty)}(\omega)=0\} \qquad (2.9)$$

and its "local" version

$$K_d^{\mathrm{loc}} = \{f:\ \mathbb{R}^d \to \mathbb{R} : f1_A \in K_d,\ \text{ for all compact } A \subset \mathbb{R}^d\} \qquad (2.10)$$

Take a function V, called here potential, such that $V^+ = \sup\{0, V\} \in K_d^{\mathrm{loc}}$ and $V^- = \inf\{0, -V\} \in K_d$. Such potentials will be called below of Kato-class. Pick $E_0 \in \mathbb{R}$ and define the Schrödinger operator

$$H = -\frac{1}{2}\Delta + V(x) - E_0 \tag{2.11}$$

as a sum of quadratic forms in $L^2(d\lambda^d)$. Then the following properties hold:

1. $C_0^\infty(\mathbb{R}^d)$ is a form core for H, and it becomes an operator core if in addition $V^+ \in L^2_{\mathrm{loc}}$ (see Theorems A.2.8 and A.2.9 in [13]). H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ and bounded from below. We choose E_0 such a way that $\inf \mathrm{Spec}(H) = 0$.

2. By using functional calculus we define the semigroup $\{e^{-tH} : t \geq 0\}$ of bounded operators on $L^2(d\lambda^d)$. Then for every $t > 0$ this semigroup has the smoothing property that e^{-tH} is a bounded operator from L^p to L^q, for every $1 \leq p \leq q \leq \infty$ (see Theorem B.1.1. of [13]).

3. For every $t > 0$, e^{-tH} is an integral operator with continuous, uniformly bounded kernel K_t. Moreover, the map $(t, x, y) \mapsto K_t(x, y)$ is jointly continuous on $(0, \infty] \times \mathbb{R}^d \times \mathbb{R}^d$ (see Theorem B.7.1, [13]).

4. The Feynman-Kac formula holds, i.e. for every $\xi, \eta \in \mathbb{R}^d$ and every $s < t \in \mathbb{R}$
$$K_{t-s}(\xi, \eta) = \int e^{-\int_s^t V(\omega(u))du} d\mathcal{W}_{[s,t]}^{\xi,\eta}(\omega) \tag{2.12}$$

5. Using (4) above and the assumption $V^+ \in K_d^{\mathrm{loc}}$ it is clear that e^{-tH} is positivity improving for all $t > 0$. Therefore 0, the bottom of the spectrum, is a simple eigenvalue and the corresponding eigenfunction ψ_0 is strictly positive. We take ψ_0 normed to 1 in L^2 and call it ground state of H.

6. Suppose H has compact resolvent, and $H\psi_n = E_n\psi_n$. Then (see [13], Theorem C.3.3.) for any $\alpha_n > 0$ there is a $c_n > 0$ such that
$$|\psi_n(x)| \leq c_n\, e^{-\alpha_n |x|} \tag{2.13}$$

A $P(\phi)_1$-*measure* is the unique probability measure μ on Ω such that

$$\mu(A) = \langle \psi_0 1_{A_1}, e^{-(t_2-t_1)H} 1_{A_2} e^{-(t_3-t_2)H} 1_{A_3} \ldots e^{-(t_n-t_{n-1})H} 1_{A_n} \psi_0 \rangle_{L^2(d\lambda^d)} \tag{2.14}$$

where A_i, t_i and n are as in the definition of the Wiener measure above. $\mu(\Omega) = 1$ follows from $||\psi_0||_2 = 1$ and $e^{-tH}\psi_0 = \psi_0$. $P(\phi)_1$-processes derive their name from quantum field theory which was kept by tradition, even though "Schrödinger processes" might be a more suggestive term to use.

Before turning to the Gibbsian interpretation of $P(\phi)_1$-processes we point out another two properties of $P(\phi)_1$-measures. First, a $P(\phi)_1$-process is a stationary Markov process. To see this consider the probability measure $d\nu = \psi_0^2\, d\lambda^d$ on $\mathbb{R}^d$, and define an isometry

$$j : L^2(d\nu) \to L^2(d\lambda^d), \qquad f \mapsto f\psi_0 \tag{2.15}$$

Consider now in $L^2(d\nu)$ the operator H_ν with $\mathrm{Dom}(H_\nu) = j^{-1}(\mathrm{Dom}(H))$ and

$$H_\nu f = j^{-1}Hjf = \frac{1}{\psi_0}H(\psi_0 f) \tag{2.16}$$

for all $f \in \mathrm{Dom}(H_\nu)$. We have by functional calculus that

$$e^{-tH_\nu}f = \frac{1}{\psi_0}e^{-tH}\psi_0 f \qquad \forall f \in L^2(d\nu),\ \forall t \geq 0 \tag{2.17}$$

and therefore the $P(\phi)_1$-measure can be described in terms of H_ν as

$$\mu(A) = \langle 1_{A_1}, e^{-(t_2-t_1)H_\nu}1_{A_2}e^{-(t_3-t_2)H_\nu}1_{A_3}...e^{-(t_n-t_{n-1})H_\nu}1_{A_n}\rangle_{L^2(d\nu)} \tag{2.18}$$

with A_i, t_i and n as before. Since $\{e^{-tH_\nu} : t \geq 0\}$ is a strongly continuous Markov semigroup, that is $0 \leq e^{-tH_\nu}f \leq 1$, whenever $0 \leq f \leq 1$, and $e^{-tH_\nu}1 = 1$, by general arguments the stochastic process induced by μ is a stationary Markov process with transition probability

$$\mu(\omega(t) \in A \mid \omega(s) = \cdot) = e^{-(t-s)H_\nu}1_A \tag{2.19}$$

for every Borel set A, and with stationary distribution $\psi_0^2 d\lambda^d$.

Secondly, the process associated with μ can be identified as the solution of a particular stochastic differential equation. Computing H_ν in terms of Δ and V by (2.16), and using the fact that $(V - E_0)\psi_0 = \frac{1}{2}\Delta\psi_0$, we get

$$H_\nu f = -\frac{1}{2}\Delta f - \left\langle \frac{\nabla\psi_0}{\psi_0}, \nabla f \right\rangle_{\mathbb{R}^d} \tag{2.20}$$

for all $f \in \mathrm{Dom}(H_\nu)$. Thus we obtain that the $P(\phi)_1$-process is the solution of the stochastic differential equation

$$d\omega(t) = dB_t - b(\omega(t))dt \tag{2.21}$$

Here B_t denotes Brownian motion, the drift term is given by $b = \nabla\psi_0/\psi_0$ and the diffusion matrix is $\mathrm{id}_{\mathbb{R}^d}$. Solutions of such stochastic differential equations are also called Itô-diffusions.

We note in passing that the particular case $V(x) = kx^2$ received more attention in the literature and is relatively well understood (its lattice version corresponds to a harmonic crystal, while other choices of the potential describe anharmonic crystals). As in [12], one could look at associated processes generated by $kx^2 + \tilde{V}(x)$, with some additional potential $\tilde{V}$, and then choose the Ornstein-Uhlenbeck process for reference. Up to appropriate transformations the results obtained for the associated process go then through for the $P(\phi)_1$-processes as defined above, but we will not adopt this approach here.

We are now prepared to define Gibbs measures. For the configuration space we take the space of all continuous paths Ω. We consider the measure space $(\Omega, \mathcal{F})$ as defined before, and use the composite of Wiener measure and Dirac measure defined in (2.8) as reference measure. Note that it is not a product measure and it is also not a probability measure. For the space of observables (or test functions) we set $L^2(\Omega, d\mathcal{W})$.

A self-potential is a function $V : \mathbb{R}^d \to \mathbb{R}$. We will use throughout Kato-class potentials, i.e. $V^+ = \sup\{0, V\} \in K_d^{\mathrm{loc}}$ and $V^- = \inf\{0, -V\} \in K_d$. This space in particular includes continuous potentials bounded from below, potentials decaying to zero, and potentials having also singularities. It is useful to have an idea about how singular these potentials are allowed to be, therefore we quote a result from [1] giving equivalent representations of K_d according to the dimension d (see also useful discussions also in [5, 14]):

$$d = 1 \;\Rightarrow\; K_1 = \{V : \sup_{x\in\mathbb{R}^d} \textstyle\int_{|x-y|\le 1} |V(y)| dy < \infty\}$$

$$d = 2 \;\Rightarrow\; K_2 = \{V : \lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \textstyle\int_{|x-y|\le r} |V(y)| \log \frac{1}{|x-y|} dy = 0\}$$

$$d \ge 3 \;\Rightarrow\; K_d = \{V : \lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \textstyle\int_{|x-y|\le r} |V(y)| \frac{1}{|x-y|^{d-2}} dy = 0\}$$

Next we define finite-time Hamiltonians by

$$\mathcal{H}_T^V(\omega) = \int_{-T}^{T} V(\omega(s)) ds \tag{2.22}$$

They account for the energy associated with each path ω. For Kato-class potentials Hamiltonians are well-defined objects. Since V is a self-potential, the Hamiltonian does not depend on paths outside the fixed time interval (free-boundary condition Hamiltonian).

A Gibbs measure μ for the Hamiltonian $\mathcal{H}^V$ is a probability measure on $(\Omega, \mathcal{F})$ such that for a version of its conditional probabilities

$$\mu(A|\mathcal{F}_{[-T,T]^c})(\omega) = \frac{1}{Z_T(\omega)} \int 1_A(\xi) e^{-\mathcal{H}_T^V(\xi)} d\,\mathcal{W}^\omega_{[-T,T]}(\xi) \tag{2.23}$$

holds for almost every $\omega \in \Omega$, every $A \in \mathcal{F}$, and each $0 < T < \infty$. Here

$$Z_T(\omega) = \int e^{-\mathcal{H}_T^V(\xi)} d\,\mathcal{W}^\omega_{[-T,T]}(\xi) \tag{2.24}$$

is the normalizing constant, turning μ into a probability measure. One can check by direct inspection that the right hand side of (2.23), which we will denote by $\mu_T(A, \omega)$, gives rise to a family of conditional probability kernels (specification) indexed by the intervals $[-T, T]$ (terminology cf. [9]), having the following properties:

(i) $\mu_T(\cdot, \omega)$ is a probability measure on $(\Omega, \mathcal{F})$ for every $\omega \in \Omega$

(ii) $\mu_T(\cdot, \omega)$ is $\mathcal{F}_{[-T,T]^c}$-proper, i.e. $\mu_T(A, \omega) = 1_A(\omega)$ for every $A \in \mathcal{F}_{[-T,T]^c}$

(iii) for every $\tau > 0$, $\int \mu_{T+\tau}(d\xi, \omega)\mu_T(A, \xi) = \mu_{T+\tau}(A, \omega)$, where $\mu_{T+\tau} = \mu_{[-T-\tau, T+\tau]}$, i.e. the probability kernels are compatible

(iv) $\mu_T(A, \cdot)$ is a function measurable with respect to $\mathcal{F}_{[-T,T]^c}$

We will view $\mu_T(A, \omega)$ as the finite time-interval specification for μ under boundary condition ω.

In the next section we will discuss the existence of such Gibbs measures. Before that, we point out that the inverse uniqueness problem whether different $P(\phi)_1$-measures are obtained for different potentials is solved as follows. Consider $d\nu_1$ and $d\nu_2$, the corresponding measures for a V_1 and V_2. Then it follows by the ergodic theorem that if the difference between V_1 and V_2 is not a constant, then $d\nu_1$ and $d\nu_2$ are mutually singular (see [11, 12]). Moreover, for some polynomial potentials these measures can be distinguished by the asymptotics of paths; if the leading term of the polynomial is $a_{2m}x^{2m}$, then

$$\limsup_{t\to\infty} \frac{\omega(t)}{(\log t)^{1/(m+1)}} = a_{2m}^{-1/2m}, \quad \liminf_{t\to\infty} \frac{\omega(t)}{(\log t)^{1/(m+1)}} = -a_{2m}^{-1/2m} \tag{2.25}$$

for almost every $P(\phi)_1$ path ω.

3 Existence and uniqueness of Gibbs measures

A natural guiding principle in constructing Gibbs measures for $P(\phi)_1$ processes is to follow the signposts known from lattice spin systems (a standard reference for this is [9]). Work in this direction in our context has been reported in [10]. Now we turn to investigating the construction in terms of limits over boundary conditions, thereby also proving the existence of Gibbs measures. Even though $P(\phi)_1$-processes are generated by a self-potential, since the reference measure is not a product measure boundary conditions play a role. For more clarity we deal separately with existence and uniqueness.

Theorem 3.1 *Suppose V is a Kato-class potential. Then the $P(\phi)_1$ measure μ on $(\Omega, \mathcal{F})$ is a Gibbs measure for the Hamiltonian $\mathcal{H}^V$. Moreover, whenever $\omega(t) = \bar{\omega}$ is a constant boundary condition, we have that*

$$\lim_{T\to\infty} \mu_T(A, \bar{\omega}) = \mu(A) \tag{3.1}$$

for every $A \in \cup_{T\geq 0} \mathcal{F}_T$.

Proof: By (2.14) we have

$$\begin{aligned} \mu(p_{t_1}^{-1}(A_1) \cap ... \cap p_{t_n}^{-1}(A_n)) &= \mu(\omega(t_1) \in A_1, ..., \omega(t_n) \in A_n) \\ &= \langle \psi_0, 1_{A_1} e^{-(t_2-t_1)H} 1_{A_2} ... e^{-(t_n - t_{n-1})H} 1_{A_n} \psi_0 \rangle \end{aligned} \tag{3.2}$$

for $A_1, ..., A_n \in \mathcal{F}$ and $t_1 < t_2 < ... < t_n$. Choose $-T < t_1 < t_2 < T$ and put $A = p_{t_1}^{-1}(A_1) \cap p_{t_2}^{-1}(A_2)$; by proceeding inductively any conjunction of events appearing at the left hand side of (3.2) can be covered. We write $\bar{\omega}(-T) = \bar{\omega}_1$ and $\bar{\omega}(T) = \bar{\omega}_2$, and put $K_{t_2-t_1}(\xi, \eta)$ for the kernel of the operator $e^{-(t_2-t_1)H}(\xi, \eta)$. Thus we have

$$\begin{aligned} \mu_T(A, \bar{\omega}) &= \frac{1}{Z_T(\bar{\omega})} \int 1_{p_{t_1}^{-1}(A_1)}(\omega) 1_{p_{t_2}^{-1}(A_2)}(\omega) e^{-\mathcal{H}_T^V(\omega)} d\mathcal{W}_T^{\bar{\omega}_1, \bar{\omega}_2}(\omega) \\ &= \frac{1}{Z_T(\bar{\omega})} \int\int d\xi d\eta 1_{A_1}(\xi) 1_{A_2}(\eta) \int e^{-\int_{-T}^{t_1} V(\omega(s))ds} d\mathcal{W}_{[-T,t_1]}^{\bar{\omega}_1, \xi}(\omega) \\ &\quad \times \int e^{-\int_{t_1}^{t_2} V(\omega(s))ds} d\mathcal{W}_{[t_1,t_2]}^{\xi,\eta}(\omega) \int e^{-\int_{t_2}^{T} V(\omega(s))ds} d\mathcal{W}_{[t_2,T]}^{\eta, \bar{\omega}_2}(\omega) \\ &= \frac{1}{K_{2T}(\bar{\omega}_1, \bar{\omega}_2)} \int\int 1_{A_1}(\xi) 1_{A_2}(\eta) K_{t_1+T}(\bar{\omega}_1, \xi) \\ &\quad \times \int e^{-\int_{t_1}^{t_2} V(\omega(s))ds} d\mathcal{W}_{[t_1,t_2]}^{\xi,\eta}(\omega) K_{T-t_2}(\eta, \bar{\omega}_2) d\xi d\eta \end{aligned} \tag{3.3}$$

In the first equality we applied (2.6), the last is obtained by applying the Feynman-Kac formula (2.12). Now let $T \to \infty$. The latter expression above becomes

$$
\begin{aligned}
& \frac{1}{\psi_0(\bar\omega_1)\psi_0(\bar\omega_2)} \int\int 1_{A_1}(\xi) 1_{A_2}(\eta)\psi_0(\bar\omega_1)\psi_0(\xi) \\
& \qquad \times \int e^{-\int_{t_1}^{t_2} V(\omega(s))ds} d\mathcal{W}^{\xi,\eta}_{[t_1,t_2]}(\omega)\psi_0(\eta)\psi_0(\bar\omega_2) d\xi d\eta \\
= & \ \langle \psi_0, 1_{A_1} e^{-(t_2-t_1)H} 1_{A_2}\psi_0\rangle
\end{aligned}
$$

where again the last equality results by the Feynman-Kac formula. This proves (3.1).

By compatibility we now have for every $B \in \mathcal{F}$

$$
\begin{aligned}
\int \mu_T(B,\omega) d\mu(\omega) &= \lim_{\tau\to\infty} \int \mu_T(B,\omega)\mu_{T+\tau}(d\omega,\bar\omega) \\
&= \lim_{\tau\to\infty} \mu_{T+\tau}(B,\bar\omega) \\
&= \mu(B)
\end{aligned}
$$

Hence μ is a well defined Gibbs measure for $\mathcal{H}^V$ for all constant boundary conditions. □

Our next result is that Gibbs measures can be constructed also for more general boundary conditions provided the boundary paths have a good enough asymptotics. In fact we show that for μ-almost every boundary condition a Gibbs measure exists and it is unique, in particular it agrees with the Gibbs measure obtained under constant boundary conditions.

We need to introduce here the positive number $\gamma = \inf[\mathrm{Spec}(H) \setminus \{0\}]$ called the spectral gap of operator H.

Theorem 3.2 *Suppose V is a Kato-class potential and is such that $\gamma > 0$. Consider the tail measurable subset of paths*

$$
\Omega^* = \{\omega \in \Omega : \ \exists\, 0 < \bar\gamma < \gamma, \ \lim_{T\to\infty} \frac{e^{-\bar\gamma T}}{\psi_0(\omega(\pm T))} = 0\} \tag{3.4}
$$

If $\omega \in \Omega^*$, then $\lim_{T\to\infty} \mu_T(A,\omega) = \mu(A)$, for all $A \in \cup_{T\geq 0}\mathcal{F}_T$, that is, we have existence and uniqueness of the $P(\phi)_1$-measure supported on Ω^*. Moreover, $\mu(\Omega^*) = \mu(\Omega) = 1$.

Proof: By (3.3) it suffices to show that for any $-t < \tau_1, \tau_2 < t$

$$
\frac{K_{t-\tau_1}(\omega(-t),x) K_{t-\tau_2}(y,\omega(t))}{K_{2t}(\omega(-t),\omega(t))} \longrightarrow \psi_0(x)\psi_0(y) \tag{3.5}
$$

in $L^1_{loc}(\mathbb{R}^{2d})$ as $t \to \infty$. By rearranging we write for the absolute value of the terms at the left hand side

$$|1 - \frac{\psi_0(\omega(-t))\psi_0(\omega(t))}{K_{2t}(\omega(-t),\omega(t))} \frac{K_{t-\tau_1}(\omega(-t),x)}{\psi_0(\omega(-t))\psi_0(x)} \frac{K_{t-\tau_2}(y,\omega(t))}{\psi_0(y)\psi_0(\omega(t))}|\psi_0(x)\psi_0(y) \quad (3.6)$$

and note that for (3.5) to hold it is sufficient that each of the factors in (3.6) goes to 1 locally uniformly in x and y. Pick a number $\bar{\gamma} < \gamma$ corresponding to (3.4), and define $L_t = e^{-tH} - P_{\psi_0}$, where P_{ψ_0} is the projection onto the one-dimensional subspace spanned by ψ_0. Note that

$$||L_t||_{2,2}e^{\bar{\gamma}t} \to 0 \text{ as } t \to \infty \quad (3.7)$$

The notation $||L_t||_{p,q}$ above corresponds to the norm of L_t as an operator from L^p to L^q. L_t has the kernel

$$\tilde{K}_t(x,y) = K_t(x,y) - \psi_0(x)\psi_0(y) \quad (3.8)$$

By the assumption that $\omega(\pm t) \in \Omega^*$ there is a function q with $\lim_{t\to\infty} q(t) = 0$ such that $e^{-\bar{\gamma}t}/\psi_0(\omega(\pm t)) = q(t)$, hence if we can show that the expressions at the right hand side of

$$\frac{\tilde{K}_{t-\tau}(x,y)}{\psi_0(x)\psi_0(y)} = \frac{\tilde{K}_{t-\tau}(x,y)}{\psi_0(x)}e^{\bar{\gamma}t}q(t) \quad (3.9)$$

and

$$\frac{\tilde{K}_{2t}(x,y)}{\psi_0(x)\psi_0(y)} = \tilde{K}_{2t}(x,y)e^{2\bar{\gamma}t}q(t)^2 \quad (3.10)$$

both go to zero locally uniformly as $t \to \infty$, then (3.5) holds. Now, we have that

$$\sup_{x,y\in\mathbb{R}^d} \tilde{K}_{t-\tau}(x,y) = \sup_{\substack{q\in L^1 \\ ||q||_1=1}} ||\int \tilde{K}_{t-\tau}(x,y)q(y)dy||_\infty = ||L_t||_{1,\infty} \quad (3.11)$$

The first equality above comes from the fact that L^∞ is the dual space of L^1, and for $g \in L^\infty$, $||g||_\infty = ||g||_{L^{1*}} = \sup_{||f||_1=1} \int fgdx$ holds.

In view of (3.11) we have to show that

$$\lim_{t\to\infty} ||L_{t-\tau}||_{1,\infty}e^{\bar{\gamma}t} = 0 \quad (3.12)$$

for every fixed $\tau > 0$, because then (3.9) and (3.10) follow since ψ_0 is locally strictly bounded away from zero. To obtain (3.12) it suffices to show

$\lim_{t\to\infty} ||L_t||_{1,\infty} e^{\bar{\gamma}t} = 0$, and this goes as follows. For $h \in L^1$, we have

$$\begin{aligned} e^{\bar{\gamma}t}||L_t h||_\infty &= e^{\bar{\gamma}t}||(e^{-tH} - P_{\psi_0})h||_\infty \\ &= e^{\bar{\gamma}t}||e^{-H}(e^{-(t-2)H} - P_{\psi_0})e^{-H}h||_\infty \\ &\leq e^{2\bar{\gamma}}||e^{-H}||_{2,\infty}\, ||L_{t-2}||_{2,2}\, e^{\bar{\gamma}(t-2)}\, ||e^{-H}||_{1,2}\, ||h||_1 \\ &= \text{const } e^{\bar{\gamma}(t-2)}||L_{t-2}||_{2,2}\, ||h||_1 \end{aligned}$$

In the second equality above we used the semigroup property and the fact that $e^{-\tau H}P_{\psi_0} = P_{\psi_0}e^{-\tau H} = P_{\psi_0}, \forall\tau$, and in the inequality we used Theorem B.1.1 of [13] on the existence of $||e^{-H}||_{2,\infty}$ respectively $||e^{-H}||_{1,2}$, resulting from the smoothing properties of Schrödinger semigroups with Kato-class potentials. Hence we have shown that

$$e^{\bar{\gamma}t}||L_t||_{1,\infty} \;\leq\; \text{const}||L_{t-2}||_{2,2}\, e^{\bar{\gamma}(t-2)} \to 0 \tag{3.13}$$

which follows by (3.7).

Finally we prove that the set of allowed boundary paths carries full μ measure. It suffices to consider $t \geq 0$; by path continuity it follows that it actually suffices to take positive rational values of t. Put

$$B_{\mathbb{Q}} = \{\omega \in \Omega : \limsup_{t\to\infty,\, t\in\mathbb{Q}} \frac{e^{-\gamma t/2}}{(\psi_0 \circ \omega)(t)} < \infty\} \tag{3.14}$$

and for the sets $Q_n := \{k/2^n : k \in \mathbb{N}\}$ put

$$B_n = \{\omega \in \Omega : \limsup_{t\to\infty,\, t\in Q_n} \frac{e^{-\gamma t/2}}{(\psi_0 \circ \omega)(t)} < \infty\} \tag{3.15}$$

Clearly, $B_n \supset B_{n+1}$ and $B_n \downarrow B_{\mathbb{Q}}$; and by continuity of μ, also $\mu(B_{\mathbb{Q}}) = \lim_{n\to\infty}\mu(B_n)$. Hence we will be done once we will have shown that $\mu(B_n) = 1$ for all $n \in \mathbb{N}$. By taking complements we have

$$\begin{aligned} B_n^c &= \{\omega \in \Omega :\ \forall M \in \mathbb{N},\ \limsup_{t\to\infty,\, t\in Q_n} \frac{e^{-\gamma t/2}}{(\psi_0 \circ \omega)(t)} \geq M\} \\ &\subset \{\omega \in \Omega :\ \limsup_{k\in\mathbb{N}} \frac{e^{-\gamma k/2^{n+1}}}{(\psi_0 \circ \omega)(\frac{k}{2^n})} \geq 1\} \\ &= \limsup_{l\to\infty}\{\omega \in \Omega : \frac{e^{-\gamma l/2^{n+1}}}{(\psi_0 \circ \omega)(\frac{l}{2^n})} \geq 1\} \end{aligned}$$

By stationarity of the process it follows that

$$\begin{aligned}
\mu(\{\omega \in \Omega : \frac{e^{-\gamma l/2^{n+1}}}{(\psi_0 \circ \omega)(\frac{l}{2^n})} \geq 1\}) &= \int 1_{\{x\in\mathbb{R}^d:\, \psi_0(x)\leq e^{-\gamma l/2^{n+1}}\}} \psi_0^2(x) d\lambda^d \\
&\leq e^{-\gamma l/2^{n+1}} \int \psi_0(x) d\lambda^d \\
&= e^{-\gamma l/2^{n+1}} ||\psi_0||_1 \qquad (3.16)
\end{aligned}$$

Since thus the left hand side is summable with respect to l, it follows by the Borel-Cantelli lemma that $\mu(B_n^c) = 0$, hence $\mu(B_n) = 1$ and thus $\mu(B) = 1$. By choosing $\gamma/2 < \bar{\gamma} < \gamma$ in (3.4), it is clear that $\Omega^* \supset B$, and that completes the proof of the theorem. □

Remark 3.3 The Borel-Cantelli type of argument above can slightly be improved drawing us nearer to actually identifying the support of the $P(\phi)_1$-measure. Moreover, the argument can be extended to *any* probability measure ϱ on Ω with the property that a function $\psi_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\psi_0 > 0$ exists, such that ϱ is a stochastic process with stationary distribution $\psi_0^2 d\lambda^d$ (in particular ϱ need not be Markovian). For being specific, let us keep now to $\varrho = \mu$, i.e. the $P(\phi)_1$-measure. Indeed the argument goes through if we change the B_n above for

$$A_n = \{\omega \in \Omega : \limsup_{t\to\infty,\, t\in Q_n} \frac{1}{t^\alpha} \frac{1}{(\psi_0 \circ \omega)(t)} < \infty\} \qquad (3.17)$$

with some $\alpha > 1$. The estimate (3.16) becomes then

$$\mu(\{\omega \in \Omega : \frac{(2^n/l)^\alpha}{(\psi_0 \circ \omega)(l/2^n)} \geq 1\}) \leq \frac{2^{\alpha n}}{l^\alpha} ||\psi_0||_1 \qquad (3.18)$$

which is summable in l. Hence $\mu(\liminf_n A_n) = 1$, and by taking α arbitrarily close to 1 from above, we get the smallest set $\liminf_n A_n$ supporting μ that can be obtained by this argument. Particular examples (see more on this in Section 4 below) suggest that this set might be a near accurate approximation to the actual support. For the case of the harmonic oscillator (3.17) gives the condition $\omega(t) \lesssim \sqrt{\log t}$, which compares well with the typical large time behaviour known in this case (see Theorems 7.4 and 7.5 in [12], and the primary source by Colella and Lanford [4]). (Here and henceforth we use the symbol $f(x) \lesssim g(x)$ meaning that there is a constant c such that $\limsup_{x\to\infty} f(x)/g(x) = c$.) Also, if $V(x) = x^{2s} + V_1(x)$, where $V_1(x)/x^{2s} \to 0$ as $x \to \infty$, then (3.17) yields $\omega(t) \lesssim (\log t)^{1/(s+1)}$, which is what Rosen and Simon obtained ([11], Theorem 14 and Remark following it).

We now turn to investigating the case of stochastic boundary conditions. Consider a family $\{g_T : T \in \mathbb{R}\}$ of non-negative functions $g_T \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and define for every $T \in \mathbb{R}$ and every $A \in \mathcal{F}_T$ the probability measures

$$\mu_T^{g_T}(A) = \frac{\int g_{-T}(\xi) g_T(\eta) \mu_T(A, \xi, \eta) d\xi d\eta}{\int g_{-T}(\xi) g_T(\eta) \mu_T(\Omega, \xi, \eta) d\xi d\eta} \tag{3.19}$$

Since these measures obviously remain unchanged under multiplication by constants c_T of the g_T for arbitrary T, we can take $||g_T||_1 = 1$, for every T, and think of $g_{\pm T}$ as the density of a stochastic boundary condition. Hence $\mu_T^{g_T}(A)$ is to be interpreted as the probability of a path lying in $A \in \mathcal{F}_T$, given a distribution of boundary paths $g_{\pm T} d\lambda^d$ at the times $\pm T$. We will now prove that the $T \to \infty$ limit of measures under (3.19) exists for a large set of stochastic boundary conditions. To make notation easier, we will henceforth assume $||g_{\pm T}||_2 = 1$, for every T, instead of the similar property in L^1-norm quoted before.

Theorem 3.4 *Suppose V is a Kato-class potential and is such that $\gamma > 0$.*

1. *If for $T \to \infty$*

$$\frac{e^{-\gamma T}}{\langle \psi_0, g_{-T} \rangle} \to 0 \quad \textit{and} \quad \frac{e^{-\gamma T}}{\langle \psi_0, g_T \rangle} \to 0 \tag{3.20}$$

then

$$\lim_{T \to \infty} \mu_T^{g_T}(A) = \mu(A) \tag{3.21}$$

exists for all $A \in \cup_{T \geq 0} \mathcal{F}_T$, where μ is given by (2.14).

2. *Pick an $\omega \in \Omega^*$ and choose a stochastic boundary condition of the form*

$$g_T(\omega(T), \varepsilon_T) = \frac{1}{2\,\varepsilon_T} 1_{[\omega(T) - \varepsilon_T, \omega(T) + \varepsilon_T]} \tag{3.22}$$

with some function $\varepsilon_T \to 0$ as $T \to \infty$, decaying sufficiently fast (and similarly for $\omega(-T)$). Then the limit

$$\lim_{T \to \infty} \mu_T^{g_T(\omega(T), \varepsilon_T)}(A) = \mu(A) \tag{3.23}$$

exists for every $\omega \in \Omega^*$ and $A \in \cup_{T \geq 0} \mathcal{F}_T$.

Before turning to the proof of the above theorem, we state and prove the following lemma.

Lemma 3.5 *Under conditions (3.20) we have*

$$\lim_{T\to\infty} \frac{e^{-(T-\tau)H} g_{\pm T}}{||e^{-(T-\tau)H} g_{\pm T}||} = \psi_0 \tag{3.24}$$

for every finite τ.

Here and henceforth we work with $L^2(d\lambda^d)$-norm without explicit labelling.

Proof: We only consider g_T; for g_{-T} the same argument applies. Denote by P_{ψ_0} the projection onto the one-dimensional subspace spanned by ψ_0, and note that $||P_{\psi_0} f|| = \langle \psi_0, f\rangle$, for every $f \in L^2(\mathbb{R}^d)$. We use the notation $L_T = e^{-TH}|_{\{\psi_0\}^\perp}$ for the component of the operator on the orthogonal complement of the ψ_0-subspace. We have then that $L_T\psi_0 = 0$, $e^{-TH} f = P_{\psi_0} f + L_T f$, and $\langle \psi_0, L_T f\rangle = 0$, for all L^2-functions f. Now let $0 < \tau < T$. Then

$$\begin{aligned} ||\psi_0 - \frac{e^{-(T-\tau)H} g_T}{||e^{-(T-\tau)H} g_T||}||^2 \quad &= \quad ||(1 - \frac{\langle \psi_0, g_T\rangle}{||e^{-(T-\tau)H} g_T||})\psi_0||^2 \\ &\quad + ||\sqrt{1 - \langle \psi_0, g_T\rangle^2} \frac{L_{T-\tau} g_T}{||e^{-(T-\tau)H} g_T||}||^2 \end{aligned} \tag{3.25}$$

On the other hand, since $||L_T|| \leq e^{-\gamma T}$ and g_T is normed for all T,

$$||e^{-(T-\tau)H} g_T||^2 = ||P_{\psi_0} g_T||^2 + ||L_{T-\tau} g_T||^2 \leq ||P_{\psi_0} g_T||^2 + e^{-2\gamma(T-\tau)} \tag{3.26}$$

holds. Hence we have the estimate

$$\frac{||e^{-(T-\tau)H} g_T||^2}{\langle \psi_0, g_T\rangle^2} \leq 1 + e^{2\tau} \left(\frac{e^{-\gamma T}}{\langle \psi_0, g_T\rangle} \right)^2 \to 1 \tag{3.27}$$

as $T \to \infty$, so the first term at the right hand side of (3.26) is vanishing in the limit. With the second term the same happens since we have then similarly

$$\frac{||e^{-(T-\tau)H} g_T||^2}{||L_{T-\tau} g_T||^2} \geq 1 + \frac{\langle \psi_0, g_T\rangle^2}{e^{-2\gamma(T-\tau)}} \to \infty \tag{3.28}$$

as $T \to \infty$. This proves the lemma. $\square$

Proof of the theorem: By the Feynman-Kac formula, for $A = p_{t_1}^{-1}(A_1) \cap \dots \cap p_{t_n}^{-1}(A_n)$, with $A_1, \dots, A_n \in \mathcal{F}$, we have

$$\mu_T^{g_T}(A) = \frac{\langle g_{-T}, e^{-(T+t_1)H} 1_{A_1} \dots 1_{A_n} e^{-(T-t_n)H} g_T\rangle}{\langle g_{-T}, e^{-2TH} g_T\rangle} \tag{3.29}$$

By simple rearrangement the right hand side can be written as

$$\frac{||e^{-(T+t_1)H}g_{-T}||}{||e^{-TH}g_{-T}||}\frac{||e^{-(T+t_1)H}g_T||}{||e^{-TH}g_T||}\frac{\langle\frac{e^{-(T+t_1)H}g_{-T}}{||e^{-(T+t_1)H}g_{-T}||},1_{A_1}...1_{A_n}\frac{e^{-(T-t_n)H}g_T}{||e^{-(T-t_n)H}g_T||}\rangle}{\langle\frac{e^{-TH}g_{-T}}{||e^{TH}g_{-T}||},\frac{e^{TH}g_T}{||e^{-TH}g_T||}\rangle} \tag{3.30}$$

From the Lemma above it follows that for any $0 \leq \tau < T$

$$\frac{||e^{-TH}g_T||}{||e^{-(T-\tau)H}g_T||} = ||e^{-\tau H}\frac{e^{-(T-\tau)H}g_T}{||e^{-(T-\tau)H}g_T||}|| \to ||e^{-\tau H}\psi_0|| = ||\psi_0|| = 1 \tag{3.31}$$

Hence, by (3.30) we get for $T \to \infty$

$$\mu_T^{g_T}(A) \to \langle\psi_0, 1_{A_1}e^{-(t_2-t_1)H}1_{A_2}...1_{A_{n-1}}e^{-(t_n-t_{n-1})H}1_{A_n}\psi_0\rangle \tag{3.32}$$

This proves the claim of the first part of the theorem.

In the second part we want to show that by choosing a boundary condition centred and uniformly distributed around an $\omega \in \Omega^*$ the limit still exists if the stochastic boundary condition becomes asymptotically sharp. Choose now in particular g_T as in (3.22). There is an $\bar{\varepsilon}_T \in [\omega(T) - \varepsilon_T, \omega(T) + \varepsilon_T]$ such that $\langle\psi_0, g_T^{\omega(T),\varepsilon_T}\rangle = \psi_0(\omega(T) + \bar{\varepsilon}_T)$. Then by combining (3.20) and (3.22) it is enough to ensure that there is some $c > 0$ such that

$$\frac{\psi_0(\omega(T) + \bar{\varepsilon}_T)}{\psi_0(\omega(T))} \to c \tag{3.33}$$

as $T \to \infty$. But such a choice of ε_T is possible for specific ψ_0. □

4 Examples

An interesting example is the harmonic oscillator where $V(x) = x^2$ and where explicit formulae are available. The kernel of e^{-tH} is given by Mehler's formula

$$K_t(x,y) = \frac{1}{\sqrt{\pi(1-e^{-2t})}}\exp\left(\frac{4xye^{-t} - (x^2+y^2)(1+e^{-2t})}{2(1-e^{-2t})}\right) \tag{4.1}$$

and the ground state is well known to be $\psi_0(x) = \pi^{-1/4}e^{-x^2/2}$. In this case an application of Theorem (3.4) gives the information that all paths that grow in time slower than $\sqrt{t}$ are good boundary configurations, and this was seen to be a large set. However, by a direct calculation we get a stronger result showing that the allowed boundary configurations can be a much larger set than Ω^* has to offer. This, however, is not a major difference from the point of view of a Gibbsian description as Ω^* itself is a full-measure subset. Here is our result.

Proposition 4.1 *Suppose* $V(x) = x^2$ *and consider the set*

$$\Omega_{osc} = \{\omega \in \Omega : \ \exists \alpha, \ 0 < \alpha < 1 \ \text{ such that } \lim_{T \to \infty} \omega(T) e^{-\alpha T} = 0\} \tag{4.2}$$

Then we have $\lim_{T \to \infty} \mu_T(\cdot, \omega) = \mu$, *for every* $\omega \in \Omega_{osc}$.

Proof: By using (3.3) we want to find out the allowed growth in time of $\omega(-T), \omega(T) \in \Omega$ when $T \to \infty$ such that

$$\mid \frac{K_T(\omega(-T), x) K_T(y, \omega(T))}{K_{2T}(\omega(-T), \omega(T))} - \psi_0(x)\psi_0(y) \mid \to 0 \tag{4.3}$$

for every $x, y \in \mathbb{R}^d$. Then by direct computation we get (4.2), and dominated convergence applied on (3.3) shows that $\mu_T(\cdot, \omega(T)) \to \mu$, for each ω in Ω_{osc}. $\square$

Note that here the spectral gap $\gamma = 1$, which appears in the upper bound of α.

Consider now a potential having x^{2s}, $s > 1$, as the leading term. Then it is obtained [6] that in all dimensions the ground state has the asymptotics $\psi_0(x) \sim \text{const}_1 \exp(-\text{const}_2\, x^{s+1}/(s+1))$ (meaning that a lower and upper bound of this form with different constants applies). Hence we get the set of good boundary conditions defined by $\omega(t) \lesssim t^{1/(s+1)}$. This result may seem curious at first sight when comparing cases of different exponents s (since the larger s, the larger set of allowed boundary conditions should be expected). To understand it take two potentials $V_1, V_2 > 0$ with $\lim_{x \to \infty} V_1(x)/V_2(x) = 0$, and consider the corresponding Gibbs specifications $\mu_T^{(1)}$ and $\mu_T^{(2)}$, respectively the Gibbs measures μ_1 and μ_2. Since V_2 is a more confining potential than V_1, one should expect that for all $\omega \in \Omega$ for which $\mu_T^{(1)}(\cdot, \omega) \to \mu_T^{(1)}$ also $\mu_T^{(2)}(\cdot, \omega) \to \mu_T^{(2)}$. In other words, the subset of configurations for which convergence holds should be larger for V_2 than for V_1. Consider now also the corresponding ground states $\psi_0^{(1)}$ respectively $\psi_0^{(2)}$. Again, since V_2 is more confining than V_1, $\psi_0^{(2)}$ will decay faster to zero at infinity than $\psi_0^{(1)}$. So by (3.4) for the corresponding sets of good boundary conditions $\Omega_2^* \subset \Omega_1^*$ follows, just the opposite of what happens with the sets for which convergence holds. One explanation is that our results are crucially tied to the stationary density ψ_0^2 and therefore Ω^* can be viewed as the set of all paths whose limsup behaviour is not far different from the typical asymptotic behaviour.

As a concluding general comment at least for potentials of compact support we point out that "good" boundary configurations are roughly those that grow at most linearly in time with a slope dependent on the spectral gap. It is known that if $V^+(x) \leq c_1|x|^{2s}$ outside a compact set (with some positive constants $c_1 > 0, s \geq 0$), then a lower bound of the form $\psi_0(x) \geq c_2 \exp(-c_3|x|^{s+1})$ applies, see Proposition 4.1, [3] (again with suitable constants $c_2, c_3 > 0$). Thus if we restrict now to Kato-class potentials of compact support, then we can put $s = 0$, and by (3.4) it follows that $\omega(t) \lesssim t$, with a coefficient dependent on the spectral gap. By Remark 3.3 the approximation of the support is given by the condition $\omega(t) \lesssim \log t$.

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References

[1] Aizenman, M. and Simon, B.: Brownian motion and Harnack inequality for Schrödinger operators, *Commun. Pure Appl. Math.* **35**, 209-273, 1982

[2] Billingsley, P. *Probability and Measure*, Wiley, 1979

[3] Carmona, R.: Pointwise bounds for Schrödinger operators, *Commun. Math. Phys.* **62**, 97-106, 1978

[4] Colella, P. and Lanford, O.E. III: Sample field behaviour for free Markov random fields, in: Velo, G. and Wightman, A.S. : *Constructive Quantum Field Theory*, Springer, 1973, pp. 44-70

[5] Cycon, H.L., Froese, R.G., Kirsch, W. and Simon, B.: *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Springer, 1987

[6] Faris, W.G. and Minlos, R.A.: A quantum crystal with multidimensional anharmonic oscillators, *J. Stat. Phys.* **94**, 365-388, 1999

[7] Iwata, K.: Reversible measures of a $P(\phi)_1$ time evolution, in *Probabilistic Methods in Mathematical Physics, Proc. Taniguchi Symp., Katata-Kyoto, 1985*, Academic Press, 1987, pp. 195-209

[8] Iwata, K.: An infinite dimensional stochastic differential equation with state space $C(\mathbb{R})$, *Probab. Theory Rel. Fields* **74**, 141-159, 1987

[9] Georgii, H.O.: *Gibbs Measures and Phase Transitions*, Walter de Gruyter, Berlin, 1988

[10] Osada, H. and Spohn, H.: Gibbs measures relative to Brownian motion, preprint, Austin archive no. 99-57, 1999

[11] Rosen, J. and Simon, B.: Fluctuations in $P(\phi)_1$-processes, *Ann. Probab.* **4**, 155-174, 1976

[12] Simon, B.: *Functional Integration and Quantum Physics*, Academic Press, 1979

[13] Simon, B.: Schrödinger semigroups, *Bull. AMS* **7**, 447-526, 1982

[14] Sznitman, A.-S.: *Brownian Motion, Obstacles and Random Media*, Springer, 1998