On the Stochastic Kuramoto-Sivashinsky Equation

Jinqiao Duan* and Vincent J. Ervin
Department of Mathematical Sciences
Clemson University
Clemson, S.C. 29634-1907
U.S.A.

Abstract

In this article we study the solution of the Kuramoto–Sivashinsky equation on a bounded interval subject to a random forcing term. We show that a unique solution to the equation exists for all time and depends continuously on the initial data.

Keywords. Random forcing, Kuramoto–Sivashinsky

1 Introduction

In this paper we investigate the existence and uniqueness of the solution to the Kuramoto-Sivashinsky (K–S) equation subject to a random forcing term. Specifically, the solution of

$$du + (u_{xxxx} + u_{xx} + uu_x)dt - dw = 0, \tag{1.1}$$

where w is a Q–Wiener process in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Wiener process w takes value in a Hilbert space to be specified later. The distributional derivative of $w(t)$ represents an external random force.

The usual K–S equation ((1.1) without the dw term) has been studied as a prototypical example for an infinite dimensional dynamical system. It possesses a finite dimensional maximal attractor ([15, 3, 12, 11]) and inertial manifold ([10, 13, 19, 17]).

Equation (1.1) arises in the modeling of surface erosion via ion sputtering in amorphous materials [4]. The random forcing term in the model accounts for the fluctuations in the flux of the bombarding particles.

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Herein we confine our attention to the case of u restricted to the interval $I := (-l, l)$, subject to the given initial condition u_0, and homogeneous Dirichlet boundary conditions, i.e.

$$u(0,x) \;=\; u_0(x)\,, -l < x < l\,, \text{ and } u(t,-l) \;=\; u(t,l) \;= 0 \text{ for } t > 0. \tag{1.2}$$

We show that for any $T > 0$ there exists a unique solution to (1.1),(1.2) for $0 < t < T$, and establish a priori estimates for the solution. The approach we follow is similar to that for establishing existence and uniqueness for parabolic differential equations. Firstly we establish local existence (with respect to time) and then show that the solution remains bounded for any $T > 0$. For (1.1) these steps are preceeded by the introduction of a change of variable which enables us to consider, instead of the stochastic differential equation, a related deterministic equation. Local existence and uniqueness is then established via an application of a fixed point argument over a suitably defined space.

The application of the fixed point theorem necessitaties expressing the nonlinear solution operator of the derived deterministic equation as a mapping from a space $\mathcal{E}$ into itself. To achieve this we must show that the extensions of two operators, which arise in the analysis, are well defined. This effort is the major part of section 3.

In section 4 we show that the local solution remains bounded for any $T > 0$ which implies global existence of the solution.

We begin our discussion by presenting in the next section several definitions and basic results which we use later in our analysis.

2 Preliminaries

As usual, we denote by $L^p(I)$, $p = 1, 2, \ldots$ the closure of $C^\infty(I)$ (the space of infinitely differentiable functions on I) with respect to the $L^p(I)$ norm:

$$\|f\|_{L^p(I)} \;=\; \left(\int_I |f|^p \, dx\right)^{1/p} \;.$$

Also, $H^k_0(I)$, $k = 1, 2, \ldots$, denotes the closure of $C^\infty_0(I)$ (the space of infinitely differentiable functions on I which vanish at the the endponts) with respect to the $H^k_0(I)$ norm:

$$\|f\|_{H^k_0(I)} \;=\; \left(\int_I |f|^2 \, dx \;+\; \int_I |f'|^2 \, dx \;+\; \ldots \;+\; \int_I |f^{(k)}|^2 \, dx\right)^{1/2} \;.$$

For convenience we use

$$H \;:=\; L^2(I) \;\text{ and }\; V \;:=\; H^1_0(I)\,.$$

To account for the temperal dependence we use the Banach spaces $L^p(0,T;L^q(I))$, with the associated norm:

$$\|f\|_{L^p(0,T;L^q(I))} \ := \ \left(\int_0^T \left(\int_I |f|^q \, dx\right)^{p/q} dt\right)^{1/p} \ .$$

Note: The spaces $L^p(0,T;H_0^k(I))$ are defined analogously.

A central role in the analysis below is played by the space E defined by

$$E \ := \ L^4(0,T;L^4(I)) \ .$$

We remark that this choice for E arises from the proof of lemma 3.1 and is dictated by the nonlinear term uu_x.

We begin by establishing the following embedding result which we combine with lemma 2.3 to establish the setting for the application of the Banach contraction mapping theorem (lemma 2.2).

Lemma 2.1 *For any $T > 0$ we have*

$$L^\infty(0,T;H) \cap L^2(0,T;V) \subset E \ , \tag{2.1}$$

and there exists a constant K, independent of $T > 0$, such that

$$\|u\|_E \leq K \left(\|u\|_{L^\infty(0,T;H)} \ + \ \|u\|_{L^2(0,T;V)}\right) \ , \ u \in E \ . \tag{2.2}$$

Proof: We have by the Sobolev embedding theorem, (see [1], pg. 217), that

$$H^{1/2}(D) \subset H^{1/4}(D) \subset L^4(D)$$

and there exists a constant $C_1 > 0$ such that

$$\|v\|_{L^4(D)} \ \leq \ C_1 \|v\|_{H^{1/2}(D)} \ . \tag{2.3}$$

Using the interpolation inequality for $H^{1/2}(D)$ in terms of $L^2(D)$ and $H^1(D)$ we have for some constant $C_2 > 0$ and all $t \in [0,T]$

$$\|u(t)\|_{H^{1/2}(D)} \leq C_2 \, \|u(t)\|_{L^2(D)}^{1/2} \ \|u(t)\|_{H^1(D)}^{1/2} \ . \tag{2.4}$$

Raising both sides of (2.4) to the fourth power and integrating (2.4) over the interval $[0,T]$, equation (2.2) follows using standard inequalitites and the definitions of the norms, with $K \ = \ C_1 C_2/2$.

■

Essential to establishing the local existence is the following contraction mapping theorem.

Lemma 2.2 *([6], Pg. 290) Let* $\mathcal{F}$ *denote a transformation from a Banach space* $\mathcal{E}$ *into* $\mathcal{E}$ *,* $\tilde{a}$ *an element of* $\mathcal{E}$ *and* $\alpha > 0$ *a positive number. If* $\mathcal{F}\,(0) = 0$, $\|\tilde{a}\| \leq \frac{1}{2}\alpha$ *and*

$$\|\mathcal{F}\,(z_1) - \mathcal{F}\,(z_2)\|_{\mathcal{E}} \leq \frac{1}{2}\|z_1 - z_2\|_{\mathcal{E}} \text{ for } \|z_1\|_{\mathcal{E}} \leq \alpha,\ \|z_2\|_{\mathcal{E}} \leq \alpha, \tag{2.5}$$

then the equation

$$z \;=\; \tilde{a} \;+\; \mathcal{F}\,(z), \quad z \in \mathcal{E}\ , \tag{2.6}$$

has a unique solution $z \in \mathcal{E}$ *satisfying* $\|z\|_{\mathcal{E}} \leq \alpha$.

■

Below we use the following lemma and corollary, which describe the regularity of the solution to a negative self–adjoint operator.

Lemma 2.3 *([20] pg. 424) Assume that* A *is a negative self–adjoint operator on* H *and*

$$V \;=\; D((-A)^{1/4}) \subset H \subset V'\ .$$

Then A *and* $S(t) \;=\; e^{tA}$ *has a continuous extension from* V *to* V'. *If*

$$y(t) \;=\; y(t;g) \;=\; S(t)y_0 \;+\; \int_0^t e^{(t-s)A}\, g(s)\, ds\ , t \in [0,T],$$

for $y_0 \in H$, *and* $g \in L^2(0,T;V')$, *then*

$$y \in L^\infty(0,T;H) \cap L^2(0,T;V),$$

and for some constant $L > 0$, *independent of* $T > 0$,

$$\|y\|_{L^\infty(0,T;H)} \;+\; \|y\|_{L^2(0,T;V)} \;\leq\; L\left(\|y_0\|_H \;+\; \|g\|_{L^2(0,T;V')}\right)\ . \tag{2.7}$$

■

Corollary 2.1 *For* A, $S(t)$, *and* y_0 *as described in lemma 2.3, we have that*

$$\|S(t)y_0\|_E \leq 8KT^{1/4}\left(\|S(t)y_0\|^4_{L^\infty(0,T;H)} \;+\; \|S(t)y_0\|^4_{L^\infty(0,T;V)}\right)^{1/4}\ . \tag{2.8}$$

Proof: Using (2.3), (2.4), (2.7) and, for notation convenience, $u \;:=\; S(t)y_0$, we have that

$$\begin{aligned}\int_0^T \|u\|_{L^4}^4 \, dt \;&\le\; (C_1\, C_2)^4 \int_0^T \|u\|_H^2 \, \|u\|_V^2 \, dt \\ &\le\; (C_1\, C_2)^4/2 \left(\int_0^T \|u\|_H^4 \, dt \;+\; \int_0^T \|u\|_V^4 \, dt \right) \\ &\le\; (C_1\, C_2)^4 T/2 \left(\, \|u\|^4_{L^\infty(0,T;H)} \;+\; \|u\|^4_{L^\infty(0,T;V)} \right) \; .\end{aligned}$$

Taking the fourth root of both sides yields (2.8) for $K \;=\; C_1\, C_2/2$.

■

Note: From lemma 2.3 we have that $S(t)\, y_0 \in L^\infty(0,T;H) \cap L^2(0,T;V)$ which guarantees that the right hand side of (2.8) is finite.

3 Local Existence and Uniqueness

Our first step in establishing local existence and uniqueness of the stochastic differential equation is to introduce a change of variable to reduce (1.1) to a deterministic equation.

Denote by A the self-adjoint operator

$$Au \;:=\; -u_{xxxx} \;-u_{xx} \;-c\, u \; . \tag{3.1}$$

We assume that c is chosen sufficiently large such that A is a strictly negative operator on the space $H_0^4(I)$.

Observe that as A is a strictly negative, self–adjoint, operator we can define $(-A)^\alpha$ via Fourier analysis, with domain $D((-A)^\alpha) \;=\; H_0^{4\alpha}(I)$. (See [18] pg.55 for details.)

In view of (3.1) note that (1.1) can be rewritten in the form

$$du \;=\; (Au \;-uu_x \;+cu)\; dt \;+\; dw \;, \tag{3.2}$$

where the Wiener process w takes value in the separable Hilbert space $H = L^2(I)$ and it has the covariance operator Q. With $S(t) \;:=\; e^{tA}$, $t \ge 0$, we define $w_A(t)$ via the stochastic integral

$$w_A(t) \;:=\; \int_0^t S(t-s)\, dw(s) \; . \tag{3.3}$$

Using the substitution

$$y(t,x) \;:=\; u(t,x) \;-\; w_A(t,x) \;,\; t \in [0,T] \; \mathbb{P}\text{–}a.s. \;, \tag{3.4}$$

(1.1) reduces to the deterministic problem

$$y_t \;=\; Ay - (y \;+ w_A)(y \;+ w_A)_x \;+ c(y \;+ w_A)\;, \tag{3.5}$$

subject to

$$y(0,x) \;=\; u_0(x) \;\text{ and } y(t,-l) \;=\; y(t,l) \;=\; 0\;. \tag{3.6}$$

Note: The assumption that dw in (1.1) denotes a Q–Wiener process, together with the fact that A is a strictly negative self–adjoint operator, ensures that $w_A(t)$ given by (3.3) has a version which is Hölder continuous with values in $D((-A)^\alpha)$ for $0 \le \alpha < 1/4$, with Hölder exponent less than $(1/4-\alpha)$, (see [6], pg. 60). Thus, with $\alpha \;=\; 1/8$, in view of (2.3), we conclude that $w_A(t)$ has a continuous version in $L^4(I)$. Below we take $w_A(t)$ to denote this continuous version.

The solution y satisfying (3.5) may be expressed in integral form as

$$\begin{aligned} y(t) \;&=\; S(t)u_0 \;+ \\ &\int_0^t S(t-s)\left[-(y(s) \;+\; w_A(s))(y(s) \;+\; w_A(s))_x \;+\; c(y(s) \;+\; w_A(s))\right]\,ds \qquad (3.7)\\ &=\; S(t)u_0 \;+\; F(y \;+\; w_A)(t)\;,\;\; t \in [0,T]\;. \qquad (3.8)\end{aligned}$$

In the following we show the existence and uniqueness of the solution y to this integral equation (3.8). This gives a so-called (mild) solution u for the stochastic Kuramoto-Sivashinsky equation (1.1). This is the definition of 'solution' used in this paper.

In (3.8) $F : E \to E$ is a continuous extension of the operator

$$F_0 : C^1([0,T];V) \to E$$

defined by

$$(F_0u)(t) \;=\; \int_0^t S(t-s)(G_0u)(s)\;ds\;, t \in [0,T], \tag{3.9}$$

where

$$G_0 : C^1([0,T];V) \to E$$

is given by

$$(G_0u)(t) \;=\; -uu_x(t) \;+\; cu(t)\;, t \in [0,T]. \tag{3.10}$$

In view of (3.8), and assuming that F is well defined — a non–trivial point whose discussion occupies the later part of this section — , on applying lemma 2.2 we have local existence of the solution to (1.1),(1.2).

Note: The value of τ for which we establish local existence and uniqueness of the solution on the interval $[0,\tau]$, depends upon the particular realization.

Theorem 3.1 *For u_0 in H there exists a random variable τ taking values $\mathbb{P}$–a.s. in $(0,T]$ such that equations (1.1)(1.2) have a unique solution u on the interval $[0,\tau]$.*

Note that by a general result in [5], page 72, the solution u has a measurable modification. In the following the solution u refers to this measurable version.

Proof: Observe that with $z(t) = y(t) + w_A(t) - S(t)\,u_0$, equation (3.8) may be rewritten as

$$z = \tilde{a} + \mathcal{F}\,(z) \tag{3.11}$$

for $\tilde{a} = w_A(t)$, and $\mathcal{F}\,(z) = F(z + S(t)\,u_0)$. Thus the existence and uniqueness of the solution to (3.8) is equivalent to that for (3.11).

Let $\alpha = 1/6M$, and τ_1 be given by

$$\tau_1 = \left(6M\,[c\,(2l)^{1/4} + 16K\,(\|S(t)u_0\|^4_{L^\infty(0,T;H)} + \|S(t)u_0\|^4_{L^\infty(0,T;V)})^{1/4}\,]\right)^{-4}, \tag{3.12}$$

for K defined in lemma 2.1, and M defined in lemma 3.2.

As $w_A(t)$ is continuous with $w_A(0) = 0$, there exists τ_2 such that

$$\int_0^t \|w_A(s)\|^4_{L^4(I)}\,ds \leq \alpha/2\,, \quad \text{for } 0 \leq t \leq \tau_2\ .$$

Let $\tau := \min\{\tau_1\,,\,\tau_2\}$ and analogous to the definition for E introduce $\mathcal{E}$ as

$$\mathcal{E} := L^4(0,\tau;L^4(I))\ .$$

With z_1 and z_2 satisfying $\|z_i\|_{\mathcal{E}} \leq \alpha$ ($= 1\,/\,6$ M) for $i = 1,2$, we have using lemma 3.2, (2.8), and the definition of τ

$$\begin{aligned}
\|\mathcal{F}\,(z_1) - \mathcal{F}\,(z_2)\|_{\mathcal{E}} &\leq M\left(\|z_1 + S(t)u_0\|_{\mathcal{E}} + \|z_2 + S(t)u_0\|_{\mathcal{E}} + c(2l\tau)^{1/4}\right)\|z_1 - z_2\|_{\mathcal{E}} \\
&\leq M\left(\|z_1\|_{\mathcal{E}} + \|z_2\|_{\mathcal{E}} + 2\|S(t)u_0\|_{\mathcal{E}} + c(2l\tau)^{1/4}\right)\|z_1 - z_2\|_{\mathcal{E}} \\
&\leq \frac{1}{2}\|z_1 - z_2\|_{\mathcal{E}}\ .
\end{aligned}$$

Finally, applying lemma 2.2 we establish the existence and uniqueness of $z(t)$, and consequently $y(t)$, on the interval $[0,\tau]$.

■

What remains is to establish the regularity result used for F in the proof of theorem 3.1. However we must first show that F is well defined by showing G_0 and F_0 defined by (3.10) and (3.9) have appropriate extensions.

Lemma 3.1 *The operator G_0 defined by (3.10) can be continuously extended to*

$$G: \ E \ \rightarrow \ L^2(0,T;V') \ ,$$

satisfying

$$\|G(u) \ - \ G(v)\|_{L^2(0,T;V')} \leq 27^{1/4}(\|u\|_E \ + \|v\|_E \ + c(2lT)^{1/4})\|u-v\|_E \ \ u,v \in E \ .$$

Proof: Let $u, \ v, \ \psi \in L^2(0,T;V)$. Denoting the duality mapping between $L^2(0,T;V)$ and $L^2(0,T;V')$ by $\langle \cdot, \cdot \rangle$, we have

$$\begin{aligned} \langle G_0(u) - G_0(v), \psi \rangle &= \int_0^T \int_I (-uu_x \ + vv_x)\,\psi \ + \ c(u-v)\psi \ dx \ dt \\ &= \int_0^T \int_I \frac{1}{2}(u+v)(u-v)\psi_x \ + \ c(u-v)\psi \ dx \ dt \ , \end{aligned}$$

i.e.

$$\begin{aligned} |\langle G_0(u) - G_0(v), \psi \rangle| &\leq \left(\int_0^T \int_I (|u| \ + \ |v| + \ c)^2 (u \ - \ v)^2 \ dx \ dt \right)^{1/2} \cdot \\ &\qquad \left(\int_0^T \int_I (\frac{1}{2}|\psi_x| \ + \ |\psi|)^2 \ dx \ dt \right)^{1/2} \\ &\leq \left(\int_0^T \int_I (|u| \ + \ |v| + \ |c|)^4 \ dx \ dt \right)^{1/4} \cdot \\ &\qquad \left(\int_0^T \int_I (u \ - \ v)^4 \ dx \ dt \right)^{1/4} \ \|\psi\|_{L^2(0,T;V)} \\ &\leq \left(27(\|u\|_E^4 \ + \ \|v\|_E^4 \ + \ \|c\|_E^4) \right)^{1/4} \cdot \\ &\qquad \|(u \ - \ v)\|_E \ \|\psi\|_{L^2(0,T;V)} \\ &\leq 27^{1/4}(\|u\|_E \ + \ \|v\|_E \ + \ c(2lT)^{1/4}) \cdot \\ &\qquad \|(u \ - \ v)\|_E \ \|\psi\|_{L^2(0,T;V)} \ , \end{aligned}$$

from which the result follows. ■

For the extension of F_0 and the regularity of F we have:

Lemma 3.2 *The transformation F_0 described by (3.9) can be continuously extended to $F: \ E \rightarrow E$. Moreover there exists a constant $M > 0$, independent of $T > 0$, such that*

$$\|F(u) \ - \ F(v)\|_E \leq M(\|u\|_E \ + \ \|v\|_E \ + \ c(2lT)^{1/4})\|(u \ - \ v)\|_E \ , \ u, \, v \in E \ . \tag{3.13}$$

Proof: In view of the definition of F_0 in (3.9) and lemma 2.3 we have that

$$F(u)(t) \;=\; y(t; G(u)) \in L^\infty(0,T;H) \cap L^2(0,T;V) \;.$$

Moreover from lemma 2.1 we have

$$\begin{aligned}
\|F(u) \;-\; F(v)\|_E \;&\leq\; K\left(\|y(\cdot; G(u)) \;-\; y(\cdot; G(v))\|_{L^\infty(0,T;H)} \;+\right. \\
&\qquad\qquad \left.\|y(\cdot; G(u)) \;-\; y(\cdot; G(v))\|_{L^2(0,T;V)}\right) \\
&\leq\; KL\|G(u) - G(v)\|_{L^2(0,T;V')} \;,\; \text{using lemma 2.3} \;, \\
&\leq\; M(\|u\|_E \;+\; \|v\|_E \;+\; c(2lT)^{1/4})\|(u \;-\; v)\|_E \;,\quad \text{using lemma 3.1} \;,
\end{aligned}$$

for $M = 27^{1/4}KL$.

■

4 Global Existence

We now extend the local existence of theorem 3.1 established in the previous section to global existence. Local existence establishes that the solution, u, lies in the solution space, E, for some initial time period. We establish global existence by showing that for any time T the E–norm of u is finite and hence u still lies in the space E. To do this we first establish that the solutions are continuous with respect to the initial data. This enables us to restrict our attention to showing that strong solutions remain bounded in E.

Following directly the proof of lemma 3.1 with the inner product (only) taken over the spatial domain, I, we have:

Corollary 4.1 *The operator G defined in lemma 3.1 satisfies for $u,\ v \in E$*

$$\|G(u) \;-\; G(v)\|_{V'} \leq 27^{1/4}(\|u\|_{L^4(I)} \;+\|v\|_{L^4(I)} \;+c(2l)^{1/4})\;\|u - v\|_{L^4(I)} \;. \tag{4.1}$$

■

Lemma 4.1 *The solution, $y(t)$, of (3.8) depends continuous on the initial data $u_0 \in H$, and the random forcing term $w_A(t) \in E$.*

Note that the continuous dependence on $w_A(t)$ is needed in the proof of the next lemma where we approximate $w_A(t)$ by regular processes.

Proof: Let y_0, and y_1 denote solutions of (3.8) generated by u_0, $w_A^0(t)$, and u_1, $w_A^1(t)$, respectively. Then, on the common existence interval of y_0 and y_1,

$$\begin{aligned} y_0 \; - \; y_1 \;\; &= \;\; S(t)(u_0 \; - \; u_1) \; + \\ & \int_0^t \Big[S(t-s)G(y_0 + w_A^0)(s) \; - \; S(t-s)G(y_1 + w_A^1)(s) \Big] \; ds \; . \end{aligned}$$

From lemma 2.3 we have y_0, and $y_1 \in L^2(0,T;V)$ thus $(y_0 \; - \; y_1)(t) \in V, \mu$ *a.e.*, (i.e. for almost all $t \in (0,T)$). Using the continuity of $S(t)$ and lemma 4.1 we have that there exits constants L_1 and C_1 such that

$$\begin{aligned} \|y_0 \; - \; y_1\|_V \;\; &\leq \;\; L_1 \|u_0 \; - \; u_1\|_H \; + \; \int_0^t C_1 \|G(y_0 + w_A^0)(s) \; - \; G(y_1 + w_A^1)\|_{V'} \; ds \\ &\leq \;\; L_1 \|u_0 \; - \; u_1\|_H \; + \\ C_1 \int_0^t 27^{1/4} (\|y_0 + w_A^0\|_{L^4(I)} \; &+ \|y_1 + w_A^1\|_{L^4(I)} \; + c(2l)^{1/4}) \|(y_0 + w_A^0) \; - \; (y_1 + w_A^1)\|_{L^4(I)} \; ds \; , \\ & \mu \; a.e. \end{aligned}$$

Using the Sobolev embedding theorem, the existence of y_0, and $y_1 \in E$, implies there exists constants C_2 and C_3 such that

$$\|y_0 \; - \; y_1\|_{L^4(I)} \leq C_2 \Big(\|u_0 \; - \; u_1\|_H \; + \; \|w_A^0 \; - \; w_A^1\|_E \Big) \; + \; C_3 \int_0^t \|y_0 \; - \; y_1\|_{L^4(I)} \; ds \;\; \mu \; a.e.$$

Applying Gronwall's inequality then yields

$$\|y_0 \; - \; y_1\|_{L^4(I)} \leq C_2 \Big(\|u_0 \; - \; u_1\|_H \; + \; \|w_A^0 \; - \; w_A^1\|_E \Big) \; e^{C_3 t} \quad \mu \; a.e.$$

from which the stated conclusion follows. ■

Next we establish appropriate norm estimates for the solution.

Lemma 4.2 *Let $u_0 \in H$, w_A be given by (3.3), and y denote the solution of*

$$y(t) \; = \; S(t)u_0 \; + \; F(y \; + \; w_A)(t) \; , \quad t \in [0,T] \; . \tag{4.2}$$

Then, y satisfies

$$\sup_{t \in [0,T]} \|y(t)\|_H^2 \leq \|u_0\|_H^2 \, e^{\int_0^T f(s)\, ds} \; + \; \int_0^T e^{\int_0^T f(s)\, ds} \; g(\tau)\, d\tau \; , \tag{4.3}$$

and

$$\int_0^T \|y(t)\|_V^2 \, dt \leq \|u_0\|_H^2 \; + \; \sup_{s \in [0,T]} \|y(s)\|_H^2 \int_0^T f(\tau)\, d\tau \; + \; \int_0^T g(\tau)\, d\tau \; , \tag{4.4}$$

where

$$\begin{aligned} f(t) &= \frac{1}{2}\left(2\,c \;+\; (2 + \frac{3}{4}C_1C_2)^2 \;+\; C_1C_2\|w_A\|_{L^4}^4\right) \;, \\ &\textit{and} \\ g(t) &= c\,\|w_A\|_H^2 \;+\; \frac{1}{4}\|w_A\|_{L^4}^4 \;. \end{aligned}$$

Proof: As $D(A)$ and $C(0,T;H_0^2(I))$ are dense in H and E, respectively, and from lemma 4.1 we have established continuous dependence of the solution, it suffices to establish (4.3),(4.4) for the (strong) solution of the differential equation

$$\begin{aligned} \frac{dy(t)}{dt} &= Ay(t) \;-\; (y(t) \;+\; w_A(t))(y(t) \;+\; w_A(t))_x \;+\; c(y(t) \;+\; w_A(t)) \;, \qquad (4.5) \\ y_0 &= u_0 \;. \end{aligned}$$

We first show that

$$\frac{d}{dt}\|y(t)\|_H^2 \;+\; \|y(t)\|_V^2 \le f(t)\,\|y\|_H^2 \;+\; c\,\|w_A\|_H^2 \;+\; \frac{1}{4}\|w_A\|_{L^4}^4 \;. \qquad (4.6)$$

Multiplying (4.5) by $y(t)$ and integrating over I we obtain

$$\frac{1}{2}\frac{d}{dt}\|y\|_H^2 \;+\; \|y_{xx}\|_H^2 \;-\; \|y_x\|_H^2 \;=\; -\int_{-l}^{l} y(y \;+\; w_A)(y \;+\; w_A)_x\,dx \;+\; c\int_{-l}^{l} yw_A dx \;. \qquad (4.7)$$

We obtain a lower bound for $\|y_{xx}\|_H$ via:

$$\begin{aligned} \|y_x\|_H^2 &= \int_{-l}^{l} y_x\,y_x\,dx \;=\; -\int_{-l}^{l} y_{xx}\,y\,dx \\ &\le \left(\int_{-l}^{l} y_{xx}^2\,dx\right)^{1/2}\left(\int_{-l}^{l} y^2\,dx\right)^{1/2} \\ &\le \frac{1}{(2 \;+\; \frac{3}{4}C_1C_2)}\,\|y_{xx}\|_H^2 \;+\; \frac{(2 \;+\; \frac{3}{4}C_1C_2)}{4}\,\|y\|_H^2 \;. \end{aligned}$$

Rearranging yields

$$\|y_{xx}\|_H^2 \;\ge\; (2 \;+\; \frac{3}{4}C_1C_2)\,\|y_x\|_H^2 \;-\; \frac{(2 \;+\; \frac{3}{4}C_1C_2)^2}{4}\,\|y\|_H^2 \;. \qquad (4.8)$$

Next consider the first integral on the r.h.s. of (4.7) in three pieces:

$$-\int_{-l}^{l} y\, y\, y_x\, dx \;=\; -\int_{-l}^{l} \frac{1}{3}\frac{d}{dx}(y^3)\, dx \;=\; 0\,, \tag{4.9}$$

$$|-\int_{-l}^{l} y\, w_A\, w_{Ax}\, dx\,| \;=\; |\frac{1}{2}\int_{-l}^{l} y_x\, w_A^2\, dx\,| \;\le \frac{1}{2}\|y\|_V^2 \;+\; \frac{1}{8}\|w_A\|_{L^4}^4 \tag{4.10}$$

and

$$-\int_{-l}^{l} \Big(y\, y_x\, w_A \;+\; y^2\, w_{Ax}\Big)\, dx \;=\; \int_{-l}^{l} y\, y_x\, w_A\, dx\,.$$

This last term is estimated as follows.

$$\begin{aligned}
|\int_{-l}^{l} y\, y_x\, w_A\, dx| \;&\le\; \left(\int_{-l}^{l} y_x^2\, dx\right)^{1/2} \left(\int_{-l}^{l} y^2\, w_A^2\, dx\right)^{1/2}\\
&\le\; \left(\int_{-l}^{l} y_x^2\, dx\right)^{1/2} \left(\int_{-l}^{l} y^4\, dx\right)^{1/4} \left(\int_{-l}^{l} w_A^4\, dx\right)^{1/4}\\
&\le\; \|y\|_V\, \|y\|_{L^4}\, \|w_A\|_{L^4}\\
&\le\; C_1\, C_2 \|y\|_V^{3/2}\, \|y\|_H^{1/2}\, \|w_A\|_{L^4} \text{ (using (2.3),(2.4))}\\
&\le\; \frac{3\, C_1\, C_2}{4}\|y\|_V^2 \;+\; \frac{C_1\, C_2}{4}\|y\|_H^2\, \|w_A\|_{L^4}^4 \qquad (4.11)
\end{aligned}$$

(using Young's Inequality).

For the remaining term on the r.h.s. of (4.7) we have

$$\begin{aligned}
|c\int_{-l}^{l} y\, w_A\, dx| \;&\le\; c\left(\int_{-l}^{l} y^2\, dx\right)^{1/2} \left(\int_{-l}^{l} w_A^2\, dx\right)^{1/2}\\
&\le\; \frac{c}{2}\|y\|_H^2 \;+\; \frac{c}{2}\|w_A\|_H^2\,. \qquad (4.12)
\end{aligned}$$

Combining (4.7)—(4.12) yields (4.6).

The estimate for $\|y\|_H$ in (4.3) now follows from the observation that (4.6) is a first order differential inequality for $\|y\|_H$. The bound involving $\|y\|_V$ in (4.4) is established by integrating (4.6) from 0 to T.

■

We are now in a position to establish the global existence of the solution.

Theorem 4.1 *For $u_0 \in H = L^2(I)$, there exists $\mathbb{P}$ a.s. a unique solution $u(\cdot, x) \in E$ of (1.1),(1.2).*

Proof: From theorem 3.1 we have existence of the solution $u(\cdot, x) \in E$, $\mathbb{P}$ a.s., for the interval $[0, \tau]$. In view of (3.4) and lemma 4.2 we conclude that $u(t, x)$ remains bounded in E, $\mathbb{P}$ a.s. for all $t \geq 0$, which implies global existence of the solution to (1.1),(1.2). ■

Finally we remark that by following the same argument as in Brannan et al. [2], we can show that the solution is actually Hölder continuous in space with exponent less than $\frac{1}{8}$. It is also possible to consider multiplicative noise in equation (1.1). The approach in this paper should also apply to other similar parabolic type stochastic partial differential equations.

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