POROSITIES AND DIMENSIONS OF MEASURES SATISFYING THE DOUBLING CONDITION

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This paper explains the implications of a mathematical theory (by the same authors) of holes in fractals and their relation to dimension for measurements. The novelty of our approach is to consider the fractal measure on a set rather than just the support of that measure. This should take into account in a more precise way the distribution of data points in measured sets, such as the distribution of galaxies.

1 Introduction

The aim of this talk is to describe a mathematics paper [2] written by the same authors to a community of scientists not necessarily specialized in the theory of fractals. We hope that the concepts developed here can help to understand some issues which have occurred in connection with the debate about the dimension of the galaxy distribution.

We want to explain the intuitive motivations of our definitions and results in the context of actual physical problems. These motivations will be underlined by simple but relevant examples.

The observation and study of fractal sets, *i.e.*, sets with possibly non-integer dimension, is of course widespread. The present work deals with two new aspects of fractal sets:

- Can the presence of voids be used to say something about the dimension of a set?
- How do voids which are almost empty account for the dimension of a set?

We will give some answers in the case of sets which are not too wild, namely satisfying a certain doubling condition.

Our interest in the question of voids in fractals (porosity) was raised by questions about the fractal dimension of galaxy distributions. In view of the heated debate in this subject, see *e.g.* [1,10], it seems adequate to provide as many analytical tools as possible with which the fractal dimension can be estimated. This paper provides a new such tool, namely the *porosity of the measure on the set.* The general idea is that sets with large voids must have small dimension. These voids, *e.g.* in galaxy distributions, are taken as indicators of small dimension in a sense we make precise below, and our theory allows a systematic way to disregard occasional points (galaxies) inside a (large) void. We present here a formalism which is tailored for this situation, by presenting algorithms for measures rather than for their supports.

2 Measures and sets

The main idea [6,11] describing the relation between the porosity and the dimension goes about as follows, and we present some intuitive examples which can guide the reader unfamiliar with this problem. Take the well-known middle third Cantor set C. Clearly, if we consider the voids in this set, every point in C is close to a relatively large void (namely to the middle third which has been taken out). We will give a more precise definition below. Another way to view porosity, which is closer to the actual definition is as follows: Suppose we have found a void of diameter ρ inside some minimal ball of radius r' around a given point of the fractal. Then the question is: How big a radius we have to take in order to see for the first time a bigger void than the one we have already seen? See also Fig. 3 below.

It is clear that if we take out more of the middle, we make the dimension of the set smaller, since the dimension depends in a well-known fashion on the length ratio of voids to non-voids, namely, if we remove an interval of length $\frac{k-2}{k}$ from the middle, *i.e.*, leave two intervals of length $\frac{1}{k}$, then the dimension is (in $\mathbf{R}$) $\log 2/\log k$. For example for the middle third Cantor set the dimension is $\log 2/\log 3$. Thus, *large voids imply small dimension.* However, the contrary is *not* true, as we shall explain in Example 3 below. One can construct a sequence of regular fractals, all of the same dimension, but with porosity decreasing to 0. Thus, *a set can have small dimension without any porosity.* It is this aspect which is connected with the controversy about the dimension of the galaxy distribution.

The requirement of obtaining information about experimentally measurable objects leads us to consider measures, or mass distributions, rather than sets. This issue was addressed earlier [3] in the context of dimension measurements. For example, one can compute the dimension of the *support* of a set, *i.e.*, of the complement of the open sets of zero measure. But, as explained in [3], the so-called correlation dimension which is based on the *mass* in balls seems to be the natural quantity for questions of experimental nature and is commonly used in the Grassberger-Procaccia method [5]. Therefore, we shall study here the porosity of a *measure*, and not only the porosity of the support of the measure [3]. We then show that large porosity implies a non-trivial upper bound on the dimension (in fact on all multi-fractal dimensions D_q, $q > 1$). Finally, we explain how porosity is estimated for a given set of experimental points.

3 Porosities of measures

Let μ be a probability measure on $\mathbf{R}^n$. We define for $x \in \mathbf{R}^n$ and $r, \varepsilon > 0$:

$$\begin{aligned} \operatorname{por}\ (\mu, x, r, \varepsilon) \ = \ \sup\{p \geq 0 \ : \text{there is } z \in \mathbf{R}^n \text{ such that} \\ B(z, pr) \subset B(x, r) \text{ and } \mu(B(z, pr)) \leq \varepsilon\mu(B(x, r))\} \ . \end{aligned}$$

In other words, we consider the ball $B(x, r)$ of radius r centered at $x \in \mathbf{R}^n$ and observe the mass $\mu(B(x, r))$ contained in it. We now look for the largest ball of radius $p \cdot r$ (fully contained in $B(x, r)$) around a point z such that the mass of that ball does not exceed ε times the mass $\mu(B(x, r))$ of $B(x, r)$.[a] See Fig. 1. One then defines

$$\operatorname{por}(\mu, x) \ = \ \lim_{\varepsilon \downarrow 0} \ \liminf_{r \downarrow 0} \operatorname{por}(\mu, x, r, \varepsilon) \ , \tag{1}$$

and finally

$$\operatorname{por}(\mu) \ = \ \inf\{s \ : \ \operatorname{por}(\mu, x) \leq s \text{ for } \mu\text{-almost all } x \in \mathbf{R}^n\} \ . \tag{2}$$

Note that when x is not in the support of the measure, the definition is not very interesting since for small enough r the measure of $\mu(B(x, r))$ is zero and then any ball in it is also empty.

Definition 1 *The quantity* $\operatorname{por}(\mu)$ *is called the porosity of the measure* μ.

It is not difficult to see that the definition of the *porosity of a set* (see Definition 5 below) amounts to using Eq.(1) with the limits taken in the opposite order, that is,

$$\operatorname{por}(\operatorname{spt}(\mu), x) \ = \ \liminf_{r \downarrow 0} \ \lim_{\varepsilon \downarrow 0} \operatorname{por}(\mu, x, r, \varepsilon) \ . \tag{3}$$

Because the central point in $B(x, r)$ is in the support $\operatorname{spt}(\mu)$ and hence occupied when we compute $\operatorname{por}(\operatorname{spt}(\mu), x)$, one finds that $\operatorname{por}(\operatorname{spt}(\mu)) \leq \frac{1}{2}$. One can also show, using density arguments, that $\operatorname{por}(\mu) \leq \frac{1}{2}$. We also note that por is determined *first* with "dust" of relative weight ε and only then ε is taken to 0.

The two porosities we consider satisfy clearly $\operatorname{por}(\operatorname{spt}(\mu)) \leq \operatorname{por}(\mu)$. In other words the porosity of a measure is larger than that of the support of the measure, precisely because the former neglects occasional dust.

Example 1 Let δ_0 be the Dirac measure at the origin, that is, $\delta_0(A) = 1$ if $0 \in A$ and $\delta_0(A) = 0$ if $0 \notin A$. Let μ be the sum of δ_0 and the Lebesgue measure $\mathcal{L}^n$ restricted to $B(0, 1)$, that is, $\mu = C(\delta_0 + \mathcal{L}^n|_{B(0,1)})$ where C is the normalization constant. Clearly $\operatorname{por}(\mu, 0) = \frac{1}{2}$ and $\operatorname{por}(\mu, x) = 0$ for all $x \neq 0$ with $|x| < 1$. Thus $\operatorname{por}(\mu) = \frac{1}{2}$. However, $\operatorname{por}(\operatorname{spt}(\mu)) = \operatorname{por}(B(0, 1)) = 0$.

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[a] The conventional porosity asks for the largest ball in $B(x, r)$ which does not contain *any* point of the set in question; see also below.

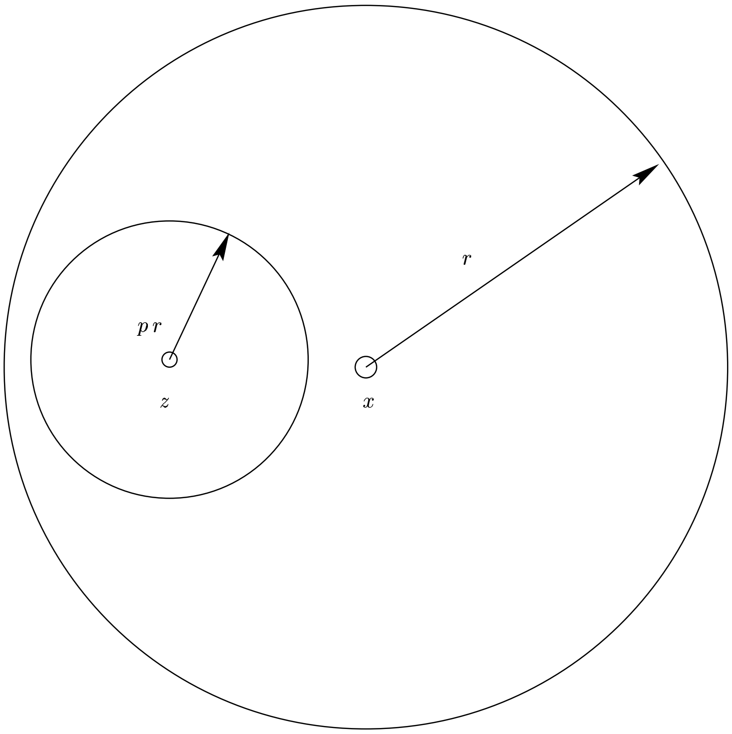

Figure 1. The ball $B(z, pr)$ inside $B(x, r)$.

Example 2 We next want to argue that voids are accounted for in a more reasonable way in the measure theoretic definition of porosity. To illustrate this with a concrete example, consider the celebrated middle third Cantor set which is obtained by starting with the interval $[0,1]$ and taking out the open interval $(\frac{1}{3}, \frac{2}{3})$. Then each of the remaining two intervals is divided into three pieces and the middle one (of length $\frac{1}{9}$) is discarded. Going on recursively (and indefinitely) in this fashion, we get the middle third Cantor set. Its dimension is $d \equiv \frac{\log 2}{\log 3}$. Clearly, this set C has voids and its porosity equals in fact $\frac{1}{4}$. We next set $C_x = \{y \ : \ y = x + z, z \in C\}$, in other words, C_x is the translate of C by x. Clearly each C_x has again dimension d. From the general theory of fractals [4,7] we get that any countable union of such sets has still dimension d. In particular, we can enumerate the rationals in $[0,1]$, for example calling them x_j, $j = 1, 2, \ldots$ and construct then the set

$$D = \bigcup_{j=1}^{\infty} C_{x_i} \ .$$

From what we said before, and from the way we constructed it, the set D has dimension $d < 1$ and *no voids*. Thus, we see that a *set* can have small dimension and no voids. The example we have just given does *not work* in the case of porosity of measures and this is one of the reasons why the porosity of measures is a more useful concept than that of sets. However, we will construct regular fractals in Example 3 where the porosity of the measure is arbitrarily small but the dimension is always $\frac{1}{2}$.

In fact, one shows easily, see [2], that the following measure has porosity $\frac{1}{2}$. We construct a measure on the set D by giving successively lower weight to the translates of C. Let μ be the usual measure associated with the Cantor set C, *i.e.*, the measure which gives equal weight to all the pieces in the recursive construction. Then we define

$$\nu = \sum_{i=1}^{\infty} 2^{-i} \mu_{x_i} ,$$

where μ_x is the translate of the measure μ by x. Clearly, the support of this measure is at least all of the interval $[0, 1]$ (with some overhangs from the translation) and has therefore no voids. But the porosity $\mathrm{por}(\nu)$, as defined in Eq.(2), is strictly positive. The reason for this is that if we consider a void of the original set C and look for a point in one of the C_{x_i} very close to the boundary of this void, then we must take in general a high index i to find such a point. But then the associated weight is smaller than 2^{-i}, and if i is large enough, then this is smaller than any ε which was given in the definition of Eq.(1). Therefore, the set C_{x_i} is not counted in this consideration, and the measure will have the same porosity as C itself.

Having described the definitions, we can now ask more precisely our question about the relation between the porosity of the measure μ and the packing dimension of the same measure. Our general aim is to show the following

Conjecture 2 *If the porosity of μ is large, then the packing dimension of μ is smaller than the dimension n of the ambient space.*

Our results will fall somewhat short of this conjecture. In order to be able to formulate a positive result, we need the following concept:

Definition 3 *The probability measure μ on $\mathbf{R}^n$ satisfies the local doubling condition at x if*

$$\limsup_{r \downarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty . \tag{4}$$

It satisfies the (global) doubling condition if Eq.(4) holds for μ-almost every x.

The bound need not be uniform in those x. Note that the doubling condition is also implied by the stronger condition

$$0 < ar^s \leq \mu(B(x, r)) \leq br^s < \infty . \tag{5}$$

In physics, it is generally assumed that the stronger condition (5) holds. See below for the relevance of these conditions in Nature. Our main result is the following

Theorem 4 *There is a function* Δ_n *defined for* $p \in (0, 1/2)$ *with values in* $[0, 1]$ *and satisfying*

$$\lim_{p \to \frac{1}{2}} \Delta_n(p) \;=\; 1 \;,$$

such that if a Borel probability measure μ *on* $\mathbf{R}^n$ *satisfies the global doubling condition then*

$$\dim_{\mathrm{H}}(\mu) \;\leq\; \dim_{\mathrm{p}}(\mu) \;\leq\; n - \Delta_n(\mathrm{por}(\mu)) \;. \tag{6}$$

Here, $\dim_{\mathrm{H}}(\mu)$ is the Hausdorff dimension of the measure and $\dim_{\mathrm{p}}(\mu)$ is its packing dimension. The inequality among those two is obvious from their definition:

$$\begin{aligned}
\underline{d}(\mu, x) &= \liminf_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} \;, \\
\overline{d}(\mu, x) &= \limsup_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} \;, \\
\dim_{\mathrm{H}}(\mu) &= \sup\{s \geq 0 \;:\; \underline{d}(\mu, x) \geq s \text{ for } \mu\text{-almost all } x \in \mathbf{R}^n\} \;, \\
\dim_{\mathrm{p}}(\mu) &= \sup\{s \geq 0 \;:\; \overline{d}(\mu, x) \geq s \text{ for } \mu\text{-almost all } x \in \mathbf{R}^n\} \;.
\end{aligned}$$

There is an explicit lower bound for the function Δ_n in [11]:

$$\Delta_n(p) \;\geq\; \max\{1 - \frac{c_n}{\log(\frac{1}{1-2p})}, 0\} \;,$$

where $c_n > 0$ is a constant depending only on n. According to Theorem 4 if the porosity of a measure μ which satisfies the doubling condition is close to $\frac{1}{2}$, then the packing dimension of μ is not much bigger than $n - 1$.

Remark For sets, a relation between porosity and dimension has been established by Mattila [6] and Salli [11] using the following definition of porosity:

Definition 5 *The porosity of a set* $A \subset \mathbf{R}^n$ *at a point* $x \in \mathbf{R}^n$ *is defined by*

$$\mathrm{por}(A, x) \;=\; \liminf_{r \downarrow 0} \mathrm{por}(A, x, r) \;,$$

where

$$\mathrm{por}(A, x, r) = \sup\{p \geq 0 : \; \textit{there is } z \in \mathbf{R}^n \textit{ such that } B(z, pr) \subset B(x, r) \setminus A\}.$$

Here, $B(z, \alpha)$ *is the closed ball with radius* α *and with center at* z*. The porosity of* $A \subset \mathbf{R}^n$ *is*

$$\mathrm{por}(A) = \inf\{\mathrm{por}(A, x) : x \in A\} \;.$$

We do not know any example where

- The porosity of the measure is $\frac{1}{2}$,
- and the dimension of the measure is n (in the ambient space $\mathbf{R}^n$).

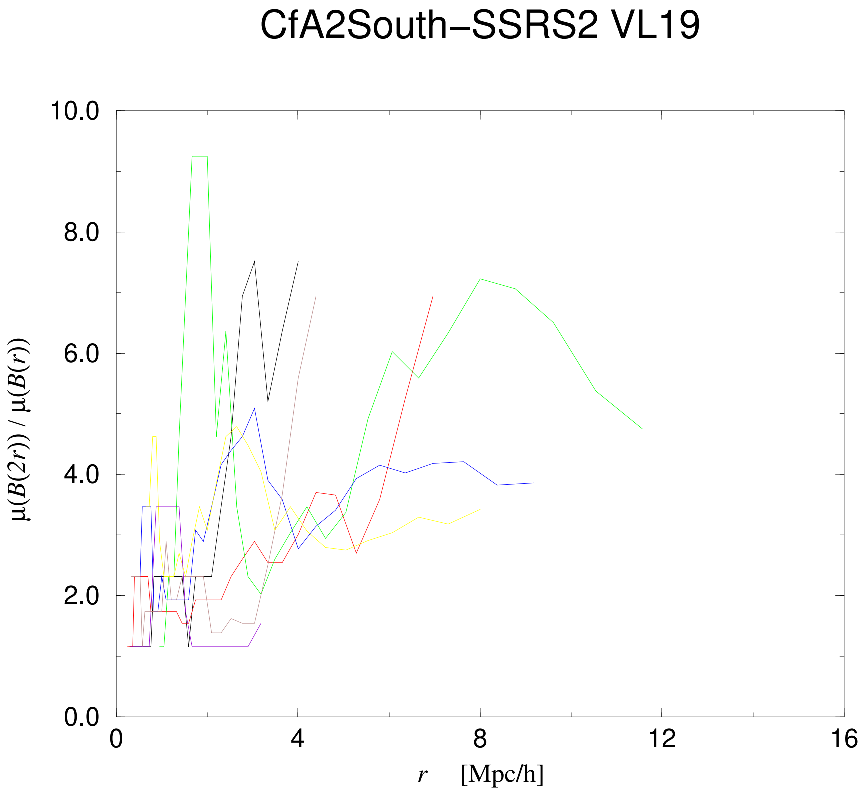

Figure 2. Verification of the doubling condition for several central points of two galaxy catalogs. We thank M. Montuori and F. Sylos-Labini for these calculations. The fluctuations seem quite bounded, except at short distances, where there are problems because one runs out of data.

Of course, this would have to be a measure which violates the doubling condition.

The doubling condition is a bound on the amplitudes of the fluctuations of the integrated density of the measure. The role of this condition in experiments in galaxy distributions is somewhat obscure. But, for example, in [9] the authors measured 2 periods of density fluctuations (of about the same amplitude) and so in a very weak sense, the doubling condition seems experimentally satisfied. Another example is given in Fig. 2 where the results of measuring the doubling condition for certain catalogs of galaxies are shown.

For regular recursively constructed fractals the doubling condition is always satisfied. Finally, we believe that fractals formed in Nature by a physical law have not only the same dimension everywhere, but also satisfy the doubling condition. The reason for this is that for example an attractor looks everywhere similar because it is created by a (smooth) physical law, which transports the structure of the fractal around in space (making at most smooth coordinate changes locally). This point of view has been advocated in [3], and has been rigorously verified for a few non-trivial examples of dynamical systems.

Another class of measurements takes the density fluctuations of the mass itself as an indicator of the dimension of the measure. This seems a mathematically inaccessible (and probably wrong) criterion for dimension measurements. It might be that this idea is a consequence of the regular oscillations one gets for regular Cantor sets. The only case where a rigorous result is known is that of integer dimension:

Theorem 6 *(Marstrand) Let s be a positive integer. Suppose that there exists a Radon measure μ on $\mathbf{R}^n$ such that the density*

$$\lim_{r\downarrow 0} \frac{\mu(B(x,r))}{r^s}$$

exists and is positive and finite in a set of positive μ-measure. Then s is an integer.

(For the proof see [7] Theorem 14.10.)

It is well-known that one cannot expect an inequality in the sense opposite to the one stated in the theorem. That is, big voids imply small dimension, but small voids do not imply big dimension. This is illustrated by the following example in $\mathbf{R}$, *i.e.*, in one dimension.

Example 3 We will construct a sequence of measures $\mu^{(n)}$, all of dimension $\dim_{\mathrm{p}}(\mu^{(n)}) = \frac{1}{2}$ in $\mathbf{R}$ with porosity $\mathrm{por}(\mu^{(n)}) \leq \frac{1}{n}$. The set $A^{(n)}$ is a Cantor set obtained recursively as follows: Divide the interval $[0,1]$ into n^2 equal subintervals and select n of these subintervals, namely the 1^{st}, $n+1^{\mathrm{st}}$, and so on. The measure at this level of the construction is obtained by giving the same weight $\frac{1}{n}$ to each subinterval of $A^{(n)}$. In other words, the unit measure is uniformly distributed on the n intervals constructed so far.

Now repeat inductively the procedure for each of the n intervals, dividing it into n^2 equal pieces and selecting each n^{th} among them. Give each of these intervals weight $\frac{1}{n^2}$.

Continuing indefinitely in this fashion, one obtains the Cantor set $A^{(n)}$ and the measure $\mu^{(n)}$ on it. The dimension of this measure is $\dim_{\mathrm{p}}(\mu^{(n)}) = \log(n)/\log(n^2) = \frac{1}{2}$, and it is not difficult to check that the porosity is less than $\frac{1}{n}$. (Since the gaps become smaller for larger n, see [2].) In this case, it is easy to compute numerically the porosity of the sets $A^{(n)}$ when n is not too large. In Fig. 3 we show the quotient $\mathrm{por}(\mu, x, r, \varepsilon = 0)$ as a function of r when x is the leftmost point of $A^{(n)}$. Similar, but more irregular pictures are obtained when one chooses another point x. One can understand the origin of the oscillations by looking at Fig. 4, where we plot the radius ρ of the largest empty interval as a function of r. We see that ρ grows linearly, until a point of the Cantor set is hit, and then it stays constant until a bigger void is found. Then Fig. 3 is obtained by dividing the values obtained in Fig. 4 by r.

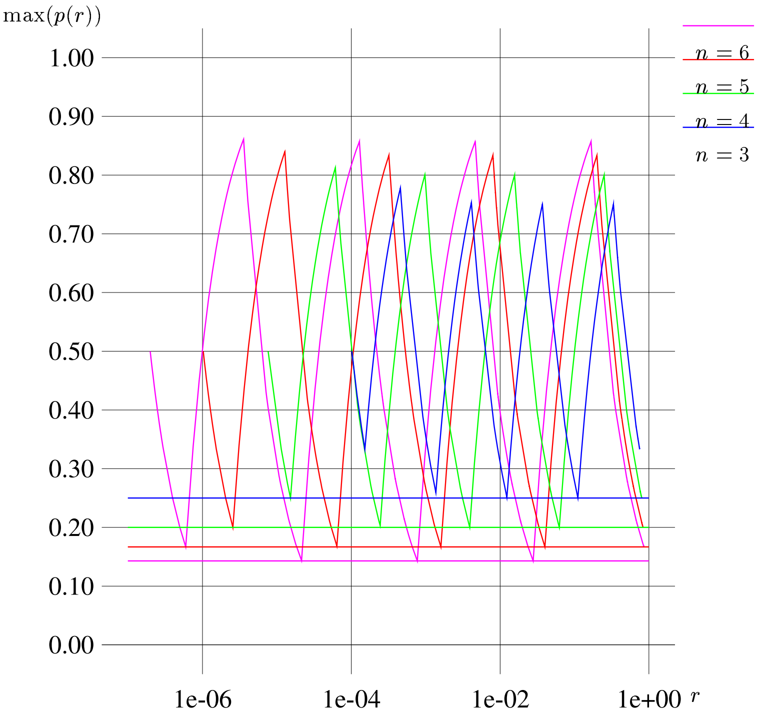

Figure 3. The oscillations of $\mathrm{por}(r)$ as a function of r (on a logarithmic scale) for the sets $A^{(n)}$, with $n = 3, \ldots, 6$.

The proof of our main result is based on comparing the porosity of a measure with the porosity of subsets with positive measure. For this, we use the quantity $\beta(\mu)$ introduced in [8]:

$$\beta(\mu) = \sup\{\mathrm{por}(A) : A \text{ is a Borel set with } \mu(A) > 0\} .$$

The inequality $\beta(\mu) \leq \mathrm{por}(\mu)$ holds for any Borel probability measure μ, but the converse inequality does not need to be true. We show that it holds when the measure μ satisfies the doubling condition. We have shown in [2] that the doubling condition implies $\beta(\mu) = \mathrm{por}(\mu)$. Together with the results of [11] it implies our main bound on the dimension (see Theorem 4). We also showed that there are measures violating the doubling condition for which $\beta(\mu) \neq \mathrm{por}(\mu)$. In the case of measures on the line $\mathbf{R}$, *i.e.,* in 1 dimension, we also show that a somewhat weaker condition than the doubling condition implies Theorem 4.

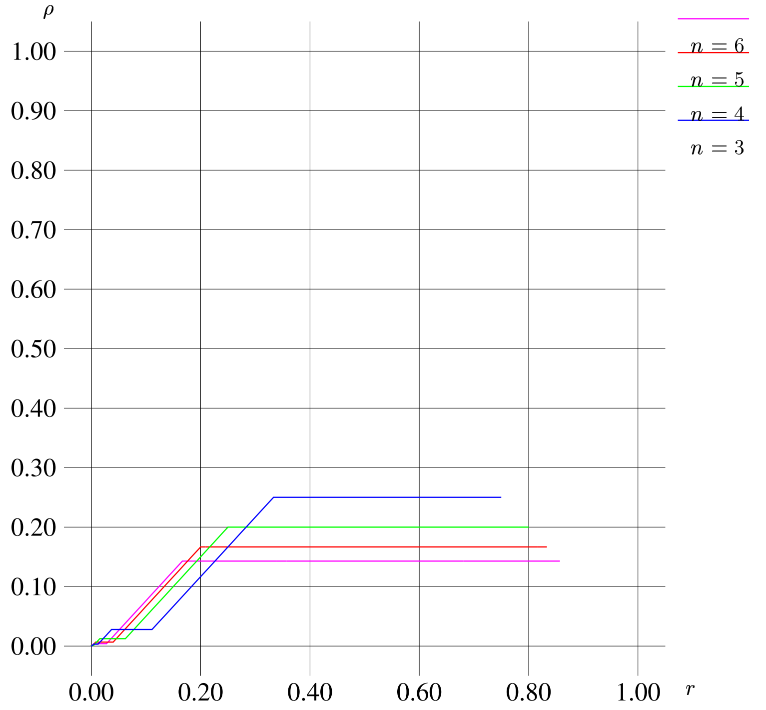

Figure 4. The largest void ρ as a function of r, measured from the left most point in $A^{(n)}$ for $n = 3, \ldots, 6$.

References

1. F. Davis, Is the universe homogeneous on large scales? In: *Critical Dialogues in Cosmology*, N. Turok, ed. (Singapore, World Scientific, 1997)
2. J.-P. Eckmann, E. Järvenpää and M. Järvenpää, Porosities and dimensions of measures. Nonlinearity, submitted.
3. J.-P. Eckmann, D. Ruelle, The ergodic theory of strange attractors Rev. Mod. Physics **57**, 617–656 (1985)
4. K. J. Falconer, *Techniques in fractal geometry* (John Wiley & Sons, 1997)
5. P. Grassberger and I. Procaccia, On the characterization of strange attractors Phys. Rev. Lett. **50**, 346 (1983)
6. P. Mattila, Distribution of sets and measures along planes. J. London Math. Soc. (2) **38**, 125–132 (1988)

7. P. Mattila, Geometry of Sets and Measures in Euclidean Spaces (Cambridge University Press, 1995)
8. M. E. Mera and M. Morán, Porosity of measures and the doubling condition (to appear).
9. G. Murante, A. Provenzale, E.A. Spiegel and R. Thieberger, Scaling regimes in the distribution of galaxies. Ann. NY Academy of Sciences **867**, 258–267 (1998)
10. L. Pietronero, M. Montuori, and F. Sylos Labini, On the fractal structure of the visible universe. In: *Critical Dialogues in Cosmology*, N. Turok, ed. (Singapore, World Scientific, 1997)
11. A. Salli, On the Minkowski dimension of strongly porous fractal sets in $\mathbf{R}^n$. Proc. London Math. Soc. (3) **62**, 353–372 (1991)