INTERVAL TRANSLATION MAPPINGS

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Abstract. Interval translation mappings are not necessarily invertible local isometries of the interval which where recently introduced by Boshernitzan and Kornfeld [1]. Our main result is a sharp upper bound on the number of minimal sets such a mapping can have.

1. Introduction

An important field in the study of dynamical systems are the investigations on volume–preserving mappings. These mappings have strong relations to Hamiltonian systems. A major feature of them is that provided these maps are invertible they can not exhibit proper attractors or repellers in the usual sense. This situation may change if the maps are no longer required to be invertible. Moreover, for invertible volume–preserving maps near integrability it is not known whether they have "chaotic" behavior on a set of positive volume. We are interested in the change of the situation for locally volume–preserving maps not necessarily invertible. More specially we want to investigate maps of the interval with singularities which locally preserve length. These maps were introduced in [1]. Clearly, they must have zero entropy. On the other hand [see [1]] there are examples of those kind of maps having a Cantor attractor. Therefore the attraction rate must be subexponential. This leaves the possibility open to have Cantor attractors of positive Lebesgue measure and hence "chaotic" behavior.

Let $I = [0, 1)$. An interval translation mapping $T : I \to I$ is defined as follows. Fix $\beta_0 = 0 < \beta_1 < \cdots < \beta_r = 1$. Let $\Delta_i = [\beta_{i-1}, \beta_i)$. Then define $Tx = x + \gamma_i$ for $x \in \Delta_i$. The only restriction on the γ_i is that $T\Delta_i \subset I$. Let $\Omega(T) = \cap_{n=0}^{\infty} T^n I$.

Similar to the way interval exchange transformations (IETs) arise in polygonal billiards ITM's arise in polygonal billiards with semipermeable walls (spy mirrors). These billiards are not invertible locally measure preserving and hence globally contract area. The main goal of studying such models is to investigate the interplay between local area

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preservation and global loss of area. This mechanism provides a lot of interesting properties.

ITM's where recently introduced by Boshernitzan and Kornfeld [1]. They defined two classes of ITM's: those of finite type and those of infinite type. They showed that is T is of finite type then $\Omega(T)$ consists of a finite union of intervals and $T|_{\Omega(T)}$ is an IET. On the other hand they constructed examples of ITM's of infinite type for which $\Omega(T)$ is a Cantor set of fractional dimension.

Our main result, theorem 2.4 is an upper bound on the number of minimal sets a ITM can have. In theorem 2.10 we show this upper bound is sharp. We also start the study of typicality of finite versus infinite type (theorems 2.5-2.8).

2. Notation and statement of results

We make the convention that the word interval will always refer to an interval which is closed on the left, open on the right.

Let $\Omega_0 = I$ and $\Omega_n = T^n\Omega_0$. Each Ω_n is a finite union of disjoint intervals Ω_n^i and $\Omega_n \supset \Omega_{n+1}$ for all n. Let $\Omega = \cap_{n\geq 0}\Omega_n$ and $X = \overline{\Omega}$. T is called an ITM of finite type if $\Omega_n = \Omega_{n+1}$ for some n. In this case $\Omega = \Omega_n$. If T is not of finite type we call it of infinite type.

If T is of finite type, then X consists of a finite union of intervals and the restriction of T to X is an interval exchange transformation[1]. This was already observed in [1]. Our first theorem states that the converse is also true.

Theorem 2.1. *If X consists of a finite union of intervals then T is an ITM of finite type.*

The rank of an ITM is the dimension of the linear space over the rationals spanned by $(\beta_1, \cdots, \beta_{r-1}, \gamma_1, \cdots, \gamma_r)$. We endow the set of all ITM's with exactly r branches with the subset topology of $\mathbb{R}^{2r-1}$ restricted to the set of all points $(\beta_1, \cdots, \beta_{r-1}, \gamma_1, \cdots, \gamma_r)$ which define an ITM. In [1] it is proved that any ITM of rank 2 is of finite type. We have the following statement.

Corollary 2.2. *Any ITM with two branches only is of finite type.*

Remark: The maximal rank which an ITM's with two branches can have is 4. Thus the previous corollary is not contained in the theorem

[1]We broaden the definition of ITMs and IETs here, we call T an ITM if T is not defined on some of the intervals Δ_i but once the image of x is defined then the forward orbit of x is defined.

of Boshernitzan and Kornfeld [1]. Boshernitzan and Kornfeld have constructed specific examples of ITM's of infinite type. In their examples X is a Cantor set. We show this is true in general.

Theorem 2.3. *Suppose T is an ITM of infinite type and $T|_X$ is topologically transitive. Then X is a Cantor set.*

The mapping T is not continuous. Let $A := \cup_{i=1}^{r} \cup_{n=-\infty}^{+\infty} T^n \beta_i \cup \{1\}$ and $B := I \cup A$ (disjoint union). We introduce an order on B by taking the natural linear orders on both A and I, then putting every point $x \in A$ directly to the left of its counterpart in I and putting the point $1 \in A$ at the far right. It is easy to check that the corresponding order topology makes B into a compact, metrizable space, and that T has a unique continuous extension to B [2],[1].

We say T is aperiodic if $O^+(x)$ is infinite for all $x \in I$. We say that Y is a minimal set if Y is a closed subset (of B),[2] $TY \subset Y$, and $O^+(x)$ is dense in Y for all $x \in Y$. Our main result is an upper bound on the number of minimal sets an ITM can have.

Theorem 2.4. *Suppose T is aperiodic. The number of minimal sets is less than or equal to $[r/2]$.*

The next statements are concerning the typical behavior of ITM's.

Theorem 2.5. *The set of ITM's of infinite type is a G_δ.*

This theorem has two corollaries:

Corollary 2.6. *The set of ITMs for which X has zero Lebesgue measure is a G_δ.*

Corollary 2.7. *If ITM's of infinite type are somewhere dense then they form a set of second Baire category.*

In contrast we have the following results.

Theorem 2.8. *For any $r > 1$ there is an open set of ITM's of finite type.*

Theorem 2.9. *If T is a maximal rank ITM of finite type then all ITM's in a neighborhood of T are of finite type.*

Keane has shown that a generic IET is minimal [2]. The situation changes drastically in the case of ITM's.

Theorem 2.10. *For each $r > 1$ there is an open set of ITM's having at least $[r/2]$ minimal components.*

[2]Alternatively, if $Y \subset I$ then Y should be closed from the left, and open from the right.

3. Eventual IET's

Proof of theorem 2.1: For x preperiodic define its preperiod to be $k+l$ where $T^i x$ is not periodic for $0 \le 1 < k$, while $T^k x$ is periodic with period l. Suppose X consist of N disjoint intervals. We set

$$\epsilon := \frac{1}{2} \min \left\{ \sum_{k=0}^{r} n_k \gamma_k : \quad \|n_k\| < 2N; \sum_{k=0}^{r} n_k \gamma_k \neq 0 \right\}.$$

Since $T^j x$ has the form $\sum_{k=0}^r n_k \gamma_k + x$ with $\|n_k\| \le j$ we have for x aperiodic, periodic with period greater than $2N$, or preperiodic with preperiod greater than $2N$ that $d(T^i x, T^j x) > 2\epsilon$ for all $0 \le i < j \le 2N$.

We consider an ϵ–neighborhood U of X consisting of X and a union $\cup_{k=1}^{2N} I_k$ where I_k are intervals of length at most ϵ. Take M sufficiently large that $\Omega_M \in U$.

If x is periodic then there is an interval J about x such that $J \subset X$. If x is preperiodic with period less than or equal to $2N$ then there is an interval J about x such that $T^{2N} J \subset X$ and thus $T^{2N+M} J \subset X$ as well.

Finally suppose x is aperiodic, or has (pre)period greater than $2N$. Then $T^M x \in U$. If $T^M x \in X$ we are done. So we assume $T^M x \in U \setminus X$. We claim that $T^{M+2N} x \in X$. For otherwise, $T^{M+k} x \in U \setminus X$ for $0 \le k \le 2N$ and $d(T^{M+j} x, T^{M+i} x) > 2\epsilon$ for all $0 \le i < j \le 2N$, a contradiction.

We have shown $\Omega_{M+2N} \subset X$ and thus T is of finite type.**q.e.d.**

Proof of corollary 2.2: Since T has only two branches and $TI \subset I$ $\Omega_1 = TI$ is a single interval. By induction Ω_n is a single interval for all n. Hence, X is either an interval or a point. The latter case contradicts aperiodicity. In the first case T is of finite type by theorem 2.1. **q.e.d.**

4. ITM's of infinite type

Proof of theorem 2.3: First of all X can not be a finite set. If it is finite then there is an $n \ge 1$ so that T^n fixes a point. Then it is easy to see that T^n fixes an interval of points. This contradicts the fact that X is finite.

We next show that X can not contain an interval. The image of an interval is a finite union of disjoint intervals, thus if it does contain an interval then X contains a union of disjoint intervals $X \supset \cup I_i$. Since $T|_X$ is topologically transitive we have $X = \cup I_i$. By theorem 2.1 the union can not be a finite one since then T is of finite type. Thus it is a countable union. We (partially) order the intervals by length and assume the enumeration of the I_i is compatible with this order, i.e. if the length $|I_i|$ of I_i is strictly greater than the length $|I_j|$ then $i < j$. If the interval I_i does not contain any discontinuity point β_j the $f|_{I_i}$ is continuous and thus f maps I_i into one of the intervals I_j which is not shorter than I_i. Only a finite number of the intervals contain a discontinuity point. Their images can be contained in shorter intervals. Since there are only finitely many discontinuity points there is a shortest interval I_{i_0} which contains the image of a longer interval. There is also a shortest interval I_{i_1} which contains a discontinuity point. Furthermore, if I_i and I_j are shorter than $\min(|I_{i_0}|, |I_{i_1}|)$ and $TI_i \subset I_j$ then $TI_i = I_j$. Thus infinitely many of the I_i have the same length, a contradiction.

Finally we show that X can not have an isolated point. Suppose the point $x \in X$ is isolated. Then each $y \in O^+(x)$ is also isolated. If $O^+(x)$ is finite we have a contradiction as before. Thus we assume $O^+(x)$ is countable. Since $y \in O^+(x)$ is isolated we can associate to y an interval I_y as follows. Denote by Ω_n^i the intervals which compromise Ω_n. The interval I_y is the maximal interval Ω_n^i containing y such that $I_y \cap X = y$. By definition the intervals I_y are all disjoint.

If the interval I_y does not contain a discontinuity point then we claim that $TI_y = I_{Ty}$. Since $I_y = \Omega_n^i$ and TI_y is connected we have $TI_y = \Omega_{n+1}^j$ for some $j = j(i,n)$. Clearly $Ty \in \Omega_{n+1}^j$ and $\Omega_{n+1}^j \cap X = Ty$ (otherwise if $z \in \Omega_{n+1}^j \cap X$ then there is a $w \in \Omega_n^i \cap X$ such that $Tw = z$, a contradiction). Thus $I_{Ty} = \Omega_{n+1}^j$.

There are only a finite number of discontinuity points. Thus for a finite number of $y \in O(x)$ we can have a discontinuity point in I_y. In particular there is a n_0 so that if $n \geq n_0$ then $I_{T^n x}$ contains no discontinuity points. Thus all these intervals have the same length, a contradiction.**q.e.d.**

5. Minimal sets

Let $D = \{\beta_0, \ldots, \beta_{r-1}\}$. For an interval J let $\mathring{J}$ be it's interior and $D(J) = \{\beta_i \in D : T^n \beta_i \in \mathring{J} \text{ for some } n \geq 0\}$. We say that $J \subset I$ is a minimal interval if $D(J) \neq \emptyset$ and $D(J) = D(J_1)$ for every subinterval $J_1 \subset J$ with $\mathring{J} \cap X \neq \emptyset$.

Lemma 5.1. *Suppose T is aperiodic and J is a minimal interval for T. Then we construct a set F satisfying the following:*

1. *$J \subset F$*
2. *There is an N such that $TF \subset F \cup (I \setminus \Omega_N)$*
3. *F is a finite union of intervals*

4. $F \cap D = D(J)$
5. $\partial F \cap X \subset D^\infty(J) = O^+(D(J)) \cup \{1\}$ *and* $\partial F \subset D^\infty(J) \cup \bigcup_{k>0} \cup_{\beta_i \in D(J)} T^{-k}\beta_i$

such that $X_F := X_J := \cap_{n\geq 0} T^n F$ *is a minimal set. Moreover for any* $J_1 \subset J$*;* $\mathring{J}_1 \cup X \neq \emptyset$ *holds* $X_{J_1} = X_J = \cup_{\beta_i \in D(J)} \omega(\beta_i)$*.*

Remark: In fact we show a stronger fact that for every $x \in F$ the orbit $O^+(x)$ of x is dense in X_F.

Proof of lemma: The proof uses some ideas of Keane [2]. Let $J = [a,b)$. Since J is a minimal interval and $D^\infty = D^\infty(J)$ is countable we can assume that $a \in X \setminus D^\infty$ and $b \in X \setminus D^\infty$, by making J a bit smaller if necessary.

For each $y \in D(J) \cup \{a,b\}$, set

$$k(y) = \inf\{n \geq 0 : T^n y \in (a,b)\}.$$

The points $\{T^{k(y)}y : y \in D \cup \{a,b\}, k(y) < \infty)\}$ partition J into intervals $J_1, \ldots, J_l$ where $l \leq r+2$. Let N be the smallest integer which is larger than $\max(|J_i|^{-1} : 1 \leq i \leq l)$. In the next argument we will only need to consider T^{-n} for $0 \leq n \leq N$.

For $0 \leq n \leq N$ the cardinality of the set $T^{-n}x$ is bounded above by a constant (depending on N). If we consider the preimage tree of length N we can label each of the finite branches by a subscript i, i.e. $T_i^{-n}x$ is a choice of preimage branches.

Set $n(i,j)$ to be the smallest n with $0 < n < N$ such that either $T_i^{-n}\mathring{J}_j \cap J \neq \emptyset$ or $T_i^{-n}\mathring{J}_j = \emptyset$. A priori it is not clear that $n(i,j)$ is defined for all i,j. To see this note that if $m < n(i,j)$ (i.e. the orbit of J_j has not yet returned to J) then $T_i^{-m}\mathring{J}_j \cap D = \emptyset$ by definition of the partition. This implies that all of the preimages $T_i^{-m}\mathring{J}_j$ for fixed j, all i and all $m < n(i,j)$ are disjoint. We also have for all i,j and $0 < m \leq n(i,j)$ either $T_i^{-m}\mathring{J}_j = \emptyset$ or $T^m(T_i^{-m}\mathring{J}_j) = J_j$. Thus we have that either $J_j \subset \Omega_N$ or $J_j \cap \Omega_N = \emptyset$. If $J_j \subset \Omega_N$ we have that for at least one of the branches $i(j)$ for all m such that $0 < m < n(i(j),j)$ the $T_{i(j)}^{-m}J_j$ have the same length, $|J_j|$. Since they are disjoint we have $n(i,j)|J_j| < 1$ and thus $n(i,j) < N$ for all i,j.

Since a and b are points from X they have an infinite preimage path contained in X. In the above construction we will disregard all preimage intervals $T^{-k}J_j$ where J_j has a or b as an endpoint and the endpoint of $T^{-k}J_j$ corresponding to the preimage of a respectively b are not in some preimage path contained in X.

Furthermore, $T_i^{-n(i,j)} \overset{\circ}{J}_j$ can not contain a or b, for if it contains one of them, say a, then $T^n a \in \overset{\circ}{J}_j$, a contradiction. Thus, if $T^{-n(i,j)} J_j \cap J \neq \emptyset$ then $T_i^{-n(i,j)} J_j \subset J$. In the case $J_j \cap \Omega_N = \emptyset$ we can not guarantee that a single branch comes back to J_j. None the less, if some branch returns, it must return completely, and all branches which do not return die out in less than N steps.

Let

$$F = F(J) = \cup_{j=1}^{l} \cup_i \cup_{m=0}^{n(i,j)} T_i^{-m} J_j.$$

It follows that F is a finite union of intervals. Also $D(F) = D(J)$. By the preceding construction, if we redefine N to be $\max n(i,j)$ we have $F \setminus T^{-1}F \subset I \setminus \Omega_N$. Hence, $TF \subset F \cup (I \setminus \Omega_{N+1})$.

Next we need the following lemma.

Lemma 5.2. *Suppose J_1 and J_2 are minimal intervals. Then exactly one of the following two options holds*

1. $D(J_1) = D(J_2)$ *and* $X_{J_1} = X_{J_2} = \cup_{\beta_i \in D(J_1)} \omega(\beta_i)$.
2. $D(J_1) \cap D(J_2) = \emptyset$ *and* $X_{J_1} \cap X_{J_2} = \emptyset$

If $J' \in T^n F$ is an interval whose interior has non–empty intersection with X then J' is minimal. Thus the lemma shows that for each interval intersecting X we have $D(J') = D(J)$ so that $X_J \subset \cup_{\beta_i \in D(J)} \omega(\beta_i)$. On the other hand, since F contains at least one orbit point of all $\beta_i \in D(J)$ and clearly $\omega(\beta_i) \subset (I \setminus \Omega_n)$ for all n we conclude

$$X_j = \cup_{\beta_i \in D(J_1)} \omega(\beta_i).$$

If x is a boundary point of F not belonging to D^∞, then for each $m \geq 0$ T^{-m} is continuous at x, provided this map is defined. This implies that T^{-1} must fix the part of the boundary of F which is in $X \setminus D^\infty$, however since this is a finite set this contradicts the fact that $O^+(x)$ is infinite for all x. Thus the boundary of F cannot contain a or b or preimages of them (Note that we considered only particular preimage intervals for a and b.). The other endpoints are all preimages of the β_i's from $D(J)$. Hence, $\partial F \subset D^\infty \cup \bigcup_{k>0} \cup_{\beta_i \in D(J)} T^{-k} \beta_j$.

If $O^+(x)$ is not dense in X_F for some $x \in F$, then there is an interval $J_1 \subset F$ which has nonempty intersection with X_F but $O^+(x) \cap J_1 = \emptyset$. Applying the above construction to the interval J_1 yields a set F_1 which is invariant $T^{-1}F_1 \subset F_1$. Since J is a minimal interval we must have $X_{F_1} = X_F$, a contradiction. Thus $O^+(x)$ must have been dense. **q.e.d.**

Proof of lemma 5.2: Suppose J_1, J_2 are minimal intervals. Let $D_i = D(J_i)$ and $F_i = F(J_i)$. Suppose that $D_1 \cap D_2 \neq \emptyset$. Suppose $\beta_i \in D_1 \cap D_2$ and $T^{n_1}\beta_i \in J_1$ and $T^{n_2}\beta_i \in J_2$. If $n_1 \geq n_2$ then $T^{n_2 - n_1} J_1 \cap J_2$ contains an interval J_3 with $T^{n_2}\beta_i \in \overset{\circ}{J}_3$. By minimality of J_2 we have $D_3 = D_2$. Also since $J_4 = T^{n_1-n_2}J_3 \subset J_1$ we have $D_4 = D_1$. Thus, we have $D_2 = D_3 \subset D_4 = D_1$. But since J_2 is a minimal interval we can find a $n_3 > n_2$ such that $T^{n_3}\beta_i \in \overset{\circ}{J}_2$. Thus arguing as above with the J_i reversing roles shows that $D_1 \subset D_2$ and thus $X_{J_1} = X_{J_2}$.

If $D(J_1) \cap D(J_2) = \emptyset$ then $X_{J_1} \cap X_{J_2} = \emptyset$. Thus $X_{J_1} \subset \cup_{\beta_i \in D(J_1)} \omega(\beta_i)$. On the other hand, since F contains at least one orbit point of all $\beta_i \in D(J)$ and clearly $\omega(\beta_i) \subset (I \setminus \Omega_n)$ for all n we conclude

$$X(J_1) = \cup_{\beta_i \in D(J_1)} \omega(\beta_i).$$

q.e.d

Proof of theorem 2.4: Lemma 5.2 shows that minimal intervals induced a partition of D. The number of minimal sets is simply the cardinality of this partition. Since D is finite there can only be a finite number of minimal intervals, and thus only a finite number of minimal sets.

We claim that there can be no minimal interval J for which $D(J)$ consists of a single $\beta := \beta_i$. If this would happen we would have

$$d(\omega(\beta), \omega(D \setminus \beta)) = \delta > 0.$$

Hence there is an N such that for all $n > N$

$$d(T^n\beta, \overline{O^+(D \setminus \beta)}) > \frac{\delta}{2}.$$

Let n, m be natural numbers larger than N such that the interval $J_1 = (T^n\beta, T^m\beta)$ has length less than $\frac{\delta}{2}$. Since $|T^l J_1| \leq |J_1|$ no image $T^l J_1$ can contain a point from $\overline{O^+(D \setminus \beta)}$, in particular no other singularity but β is in the orbit of J_1. Therefore the first return map of J_1 is completely determined by β and thus is an ITM with two branches. By theorem 2.2 this reduces to an IET with two branches. Obviously, the endpoints of the IET are images of the endpoints of J_1 and hence contained in $O^+(\beta)$. The same holds for the singularity β' of the first return map. This means that for this IET (with two branches only) both endpoints and the singularity lie on one orbit. This is only possible if it is periodic. But then T itself has to have an periodic orbit what contradicts our assumptions.

Therefore any partition of D induced by minimal intervals has at most $[r/2]$ atoms. **q.e.d.**

6. Open subsets of ITM's

In this section we fix the number of branches r.

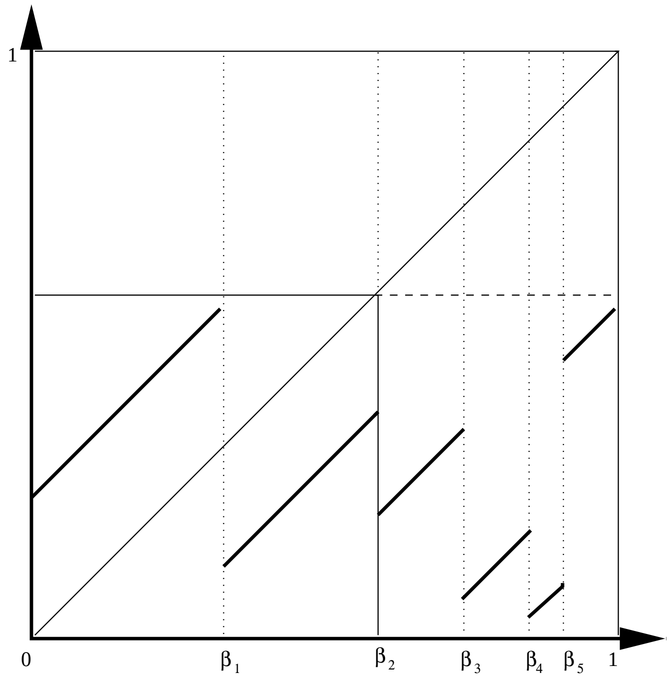

FIGURE 1.

Proof of theorem 2.8: Let T with r branches have the following properties (see figure 1):

1. $\overline{T[0,\beta_2)} \subset (0,\beta_2)$
2. $\overline{T[\beta_2,1)} \subset (0,\beta_2)$

Obviously the set of ITM's fulfilling 1. and 2. is open. Conditions 1. and 2. guarantee that $\Omega_1 \subset (0,\beta_2)$ and $T|_{\Omega_1}$ has only two branches. Thus corollary 2.2 tells us that T is of finite type. **q.e.d.**

Proof of theorem 2.10: Let T with r branches have the following property (see figure 2):

$$\overline{T[\beta_{2k-2},\beta_{2k})} \subset (\beta_{2k-2},\beta_{2k}) := J_k$$

for $1 \le k \le [r/2]$. For odd r we additionally demand

$$\overline{T[\beta_{r-1},\beta_r)} \subset (\beta_0,\beta_2).$$

Again these conditions are fulfilled for an open set of ITM's. They ensure that any minimal component is contained in one of the J_k's. This proves the assertion. **q.e.d.**

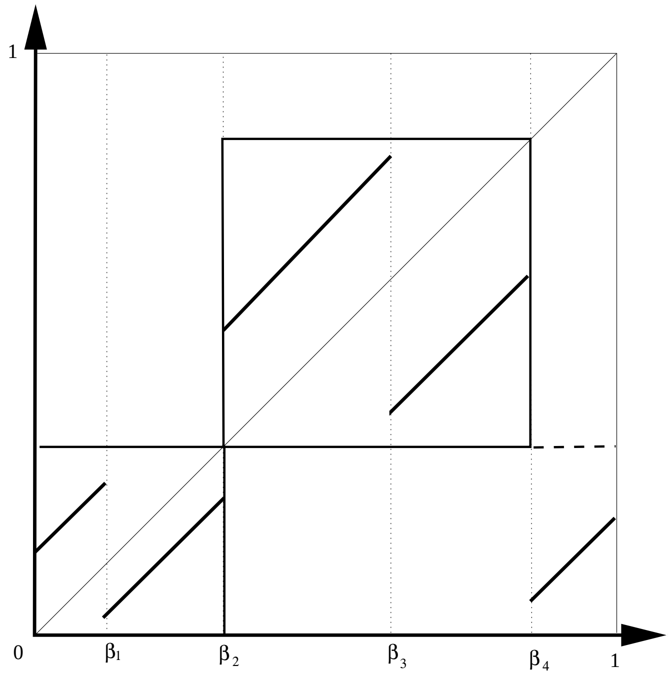

Figure 2.

7. Infinite type is G_δ

Proof of theorem 2.5: Let r be fixed and $\mathfrak{P} \subset \mathbb{R}^{2r-1}$ be the set of parameters which define an ITM. We define a sequence of continuous functions $f_n \colon \mathfrak{P} \to \mathbb{R}$ by $f(p) = |\Omega_n(p)|$ where $\Omega_n(p)$ is the set $T^n(p)I$ for the map $T(p)$ defined by $p \in \mathfrak{P}$. These functions have the property that $f_n \leq f_{n+1} \leq 1$ for all n. Moreover, by theorem 2.1 p gives rise to an ITM of infinite type if and only if $f_n(p) < f_{n+1}(p)$ for all n. Also $f_m(p) = f_{m+1}(p)$ implies $f_l(p) = f_m(p)$ for all $l > p$. Therefore the set of $p \in \mathfrak{P}$ having infinite type ITM is a G_δ. **q.e.d.**

Proof of corollary 2.6: The above proof implies that the limit function $f = \lim_{n\to\infty} f_n$ exists and is lower semi–continuous. Hence, the set of points achieving 1 – the maximum value of f – is a G_δ. This means that the set of ITM's for which X has zero Lebesgue measure is a G_δ. **q.e.d.**

8. Cutting parks

In this section we are going to prove theorem 2.9. We introduce a coding of ITM's which we call *"cutting parks"*.

Let T be an ITM with r branches. The cutting park at time n consists of r *trees*, some *fertile paths* and some *blossoms*. The trees are defined as follows. The root of a tree is i, $1 \leq i \leq r$. The vertex at level n is a string $(i, j_1, \cdots, j_n)$, $1 \leq j_l \leq r$, $1 \leq l \leq n$. There is an directed arrow from $(i, j_1, \cdots, j_{k-1})$ to $(i, j_1, \cdots, j_{k-1}, j_k)$ for all k. These are universal objects for all ITM's with r branches.

A fertile path is specific to a given ITM. It is defined as a directed path along marked arrows and does not depend on the time n. An arrow from $(i, j_1, \cdots, j_{k-1})$ to $(i, j_1, \cdots, j_{k-1}, j_k)$ is marked iff $T^k\beta_i = \beta_i + \sum_{n=1}^k \gamma_{j_n}$. Finally, we define blossoms. These are objects which are specific to T but in contrary to fertile paths change with time n. A blossom at time n is a marked vertex. We mark a vertex $(i, j_1, \cdots, j_k)$ at time n iff it lies on a fertile path and $T^k\beta_i = \beta_i + \sum_{n=1}^k \gamma_{j_n}$ is an endpoint of an interval in Ω_n. Here we consider the singularities β_i as endpoints too. We want to remark that for blossoms the distance k to the root of the tree is always less than or equal to the time of marking n. So the evolution of the cutting park characterizes the combinatorial structure of an ITM. It is clear that the evolution of a cutting park's blossoms stops at a finite time n if and only if T is of finite type. In fact it is of finite type if and only if the blossoms at two consecutive time steps are the same. A very important observation is that both marking the arrows or marking the vertices is defined at a finite time (level) by not more than $(n+1)^2/2$ inequalities of the form

$$\beta_i + \sum_{l=1}^{k_1} \gamma_{i_l} \leq \beta_j + \sum_{l=1}^{k_2} \gamma_{j_l} \tag{1}$$

with $k_1, k_2 \leq n$, $1 \leq i, j \leq r$ and $1 \leq i_l, j_l \leq r$. These inequalities are called the defining inequalities at time n.

Lemma 8.1. *Let T have maximal rank. Then for each n there is a neighborhood of T such that all ITM's form this neighborhood have the same cutting park up to time n as T.*

Proof of the lemma 8.1: If all the inequalities in 1 are strict then the assertion follows immediately since there are only finitely many of these conditions. If some of these conditions in 1 are actually equalities we conclude as follows. Let

$$\beta_i + \sum_{l=1}^{k_1} \gamma_{i_l} = \beta_j + \sum_{l=1}^{k_2} \gamma_{j_l}. \tag{2}$$

Since T has maximal rank this is only possible if $k_1 = k_2$, $i = j$ and there is a permutation σ of $1, \cdots, k_1$ with $i_{\sigma(l)} = j_l$. Now if we change β_i

to β_i' and γ_j to γ_j' both sums in 2 change for the same amount and hence, equality is preserved. Therefore the system of defining inequalities is preserved in a neighborhood of T.**q.e.d.**

Proof of theorem 2.9: Let T be of maximal rank and of finite type. This means that the cutting park does not change any more after some time n. By lemma 8.1 there is a neighborhood of T such that all the ITM's in the neighborhood have the same cutting park as T up to time $n+1$. Thus since the blossoms of the cutting parks do not change from time n to $n+1$ all the ITM's in this neighborhood are of finite type.**q.e.d.**

References

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