A Wulff Shape from Constructive Field Theory *

François DUNLOP$^{(1)}$, **Jacques MAGNEN**$^{(2)}$

$^{(1)}$ *Laboratoire de Physique Théorique et Modélisation* (CNRS – UPRES-A)
Université de Cergy-Pontoise, 95302 Cergy-Pontoise, France
$^{(2)}$ *Centre de Physique Théorique* (CNRS – UMR 7644)
Ecole Polytechnique, 91128 Palaiseau, France

Abstract: We consider a sessile droplet as given by a height function $h(x)$, subject to a Hamiltonian of the form $|\nabla h|^2 + \lambda P(\nabla h)$, where P is a polynomial and λ is small. The corresponding Gibbs measure is conditioned on the value of the droplet volume $\mathrm{V} = \int_\Lambda dx\, h(x)$, where $\Lambda \subset \mathbb{R}^2$ is a bounded domain of the plane. The droplet shape, in a scaling limit $\mathrm{V} \approx |\Lambda|^{3/2} \to \infty$, is then a Wulff shape, with logarithmic fluctuations. The proof, outlined in these proceedings, is based on a phase space cluster expansion and renormalization of a varying slope chemical potential.

Keywords: Interfaces, Wulff construction, SOS model, Constructive field theory, Renormalization group.

1 Introduction

A Wulff shape [Wu] is the shape of a drop of given volume, with minimal surface free energy. In the case of a sessile drop, the Wulff construction becomes the Winterbottom construction [Wi]. The Winterbottom shape is a Wulff shape cut by the plane supporting the sessile drop. The amount of Wulff shape which is cut off depends upon the wettability of the supporting plane.

In the present paper we deal with a sessile drop whose basis is an arbitrary given connected open domain of the plane, $\Lambda \subset \mathbb{R}^2$, with $\partial\Lambda \in \mathcal{C}^2$. The given basis is thus in general different from the intersection of a plane and a Wulff shape. The drop of minimal surface free energy conditioned by the given volume and given basis still obeys the same partial differential equation, which we shall call the Wulff equation, only the boundary conditions is different.

We work with a continuum Solid-On-Solid representation, so that the interface bounding the drop is given by a height function $h(x) \in \mathbb{R}$, $x \in \Lambda$, with a volume constraint

$$\int_\Lambda dx\, h(x) = \mathrm{V}$$

and boundary conditions $h|_{\partial\Lambda} = 0$. The surface free energy or surface tension of solid materials and of many statistical mechanical models is a function of surface orientation, which is specified here by the local slope $\nabla h(x)$. Denoting $\sigma(\nabla h)$ the surface tension, the Wulff equation takes the form

$$\nabla \cdot \frac{\delta}{\delta \nabla h} \left(\sqrt{1 + |\nabla h|^2}\; \sigma(\nabla h) \right) = \text{constant}$$

Existence and unicity of solution of this partial differential equation, for suitable surface tension $\sigma(\cdot)$ and suitable boundary conditions, have been established by Taylor [T]; see also Fonseca [F].

Fluctuations on the atomic scale play an important role in the surface tension and in the resulting Wulff shape. Accordingly, in statistical mechanics, the surface tension is not given a priori, but can be obtained as a surface free energy from a Gibbs measure, with boundary conditions giving a plane interface. When the Gibbs measure is conditioned to produce a macroscopic drop, one expects this drop to take a Wulff shape in the thermodynamic limit.

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* Contribution to the proceedings of the colloquium “Mathematical Results in Statistical Mechanics” (Marseille, 1998), to be published by World Scientific.

Preliminary results in this direction were obtained by Minlos and Sinai [MS] for the Ising model. The 1+1 dimensional Solid-On-Solid droplet shape was established by De Coninck, Dunlop and Rivasseau [DDR]. Dobrushin, Kotecký and Shlosman [DKS] proved the Wulff shape for the two dimensional Ising model at low temperature. A simpler proof was given by Pfister [P]. The result was then extended up to the critical temperature by Ioffe [I]. The case of a two dimensional Ising sessile drop was treated by Kotecký and Pfister [KP].

The three-dimensional problem is more difficult, notably because coarse graining a random surface is difficult. For the Gaussian Solid-On-Solid model, the effect of conditioning the Gibbs measure by the constraint $h \geq 0$, together with the volume constraint for one drop, has been controlled by Ben Arous and Deuschel [BD]. The gas of Gaussian droplets on a wall with a total volume constraint was shown to obey the Winterbottom construction by Bolthausen and Ioffe [BI]. An extension of the result to non-Gaussian interactions has been announced [DGI].

The $|\nabla h|^2 + \lambda |\nabla h|^4$ model in two dimensions, with λ small, has been much studied as testing ground for rigorous renormalization group [GK,MS, BY]. The interest of proving a Wulff construction for this model was advocated and discussed in [D].

Let us now be more precise in order to formulate the results of the present paper. Let $\Lambda = \Lambda_L = L\Lambda_1 = \{x | \frac{x}{L} \in \Lambda_1\}$ and $\mathrm{V} = \mathrm{V}_L = L^3 \mathrm{V}_1$. Our starting point is the capillary wave model, or two-dimensional massless Euclidean field, and the corresponding Gaussian measure. We choose for covariance

$$C = \int_1^\infty d\alpha \, e^{\alpha \Delta_{\partial\Lambda}}$$

where $\Delta_{\partial\Lambda}$ is the Laplacian with zero Dirichlet data on $\partial\Lambda$. Dirichlet boundary conditions are required by $h|_{\partial\Lambda} = 0$ and also provide the needed infra-red cutoff. $\alpha \geq 1$ will give the needed short distance cutoff. The Gaussian measure $d\mu_C(h)$ is then a well defined measure on the space of $\mathcal{C}^\infty$ functions vanishing on $\partial\Lambda$.

Next we introduce a local interaction $V(\nabla h) = \lambda P(\nabla h)$, where $P(x)$ is a convex polynomial in the two components of $x \in \mathbb{R}^2$. The convexity hypothesis is not a serious restriction within the regime λ small, required for the convergence of the cluster expansion. Indeed given any polynomial $P(\nabla h)$ bounded from below, a small part $\mathcal{O}(\lambda)|\nabla h|^2$ could be taken from the Gaussian measure so that $\mathcal{O}(\lambda)|\nabla h|^2 + \lambda P(\nabla h)$ would be convex.

The interacting measure for the drop is then defined as

$$d\mu_{C,\lambda,\gamma}(h) = Z_{\lambda,\gamma}(\Lambda)^{-1} e^{-\int_\Lambda dx \, \left[V(\nabla h) - \gamma \frac{h(x) - \langle h(x) \rangle_{C,\lambda,\gamma}}{L} \right]} d\mu_C(h)$$

where

$$Z_{\lambda,\gamma}(\Lambda) = \int d\mu_C(h) \, e^{-\int_\Lambda dx \, \left[V(\nabla h) - \gamma \frac{h(x) - \langle h(x) \rangle_{C,\lambda,\gamma}}{L} \right]}$$

and

$$\langle \, \cdot \, \rangle_{C,\lambda,\gamma} = \int d\mu_{C,\lambda,\gamma}(h) \, \cdot$$

denotes the corresponding expectation. Rather than a sharp volume constraint, we work in an ensemble with a conjugate field γ. By monotonicity, there is no difficulty in choosing γ such that

$$\left\langle \int_\Lambda h(x) \right\rangle_{C,\lambda,\gamma} = \mathrm{V}$$

Our main result is as follows:

Theorem: *Let* $\langle \, \cdot \, \rangle = \langle \, \cdot \, \rangle_{C,\lambda,\gamma}$ *be as desribed above. Then* $\exists \lambda_0$ *so that for* $0 \leq \lambda < \lambda_0$*, as* $L \to \infty$

$$\left\langle \Big(h(x) - \langle h(x) \rangle \Big)^2 \right\rangle = \mathcal{O}(\ln L)$$

$$
\begin{aligned}
-\ln Z_{\lambda,\gamma}(\Lambda) &= \int_\Lambda dx\, \Big[\tilde\sigma_\lambda(\,\langle\,\nabla h(x)\,\rangle\,) - \tilde\sigma_0(0)\Big] + \mathcal{O}(L\ln L)\\
\nabla\cdot\ \frac{\delta}{\delta\nabla h}\,\tilde\sigma_\lambda(\,\langle\,\nabla h(x)\,\rangle\,) &= \mathcal{O}(\frac{1}{L})
\end{aligned}
$$

where $\tilde\sigma_\lambda(\nabla h)$ *is the projected surface tension, i.e.* $\tilde\sigma_\lambda(\nabla h) = \sqrt{1+|\nabla h|^2}\,\sigma_\lambda(\nabla h)$ *with* $\sigma_\lambda(\nabla h)$ *the surface tension at slope* ∇h*.*

The strategy of proof and the main ingredients are given in the following sections. The various algorithms defining the expansion and the renormalization group flow are described, but the proof of convergence is left to a separate paper [DM].

2 Outline of proof

We introduce a vector function $u(x)$ such that $\nabla\cdot u = -\,\gamma/L$, so that, given $h|_{\partial\Lambda} = 0$,

$$
\int_\Lambda dx\,\gamma\frac{h(x)}{L} = -\int_\Lambda dx\,(\nabla\cdot u(x))\,h(x) = \int_\Lambda dx\,u(x)\cdot\nabla h(x)
$$

and

$$
d\mu_{C,\lambda,\gamma}(h) = Z_{\lambda,\gamma}(\Lambda)^{-1} e^{-\int_\Lambda dx\,\left[\,V(\nabla h) - u\cdot\left(\nabla h - \langle\nabla h\rangle_{C,\lambda,\gamma}\right)\,\right]}\, d\mu_C(h)
$$

The height function is then decomposed into a non-random part $\bar h$ and a fluctuation ϕ, that is $h = \bar h + \phi$. Eventually we'll have

$$
\bar h(x) = \langle h(x)\rangle_{C,\lambda,\gamma}
$$

but $\langle h(x)\rangle_{C,\lambda,\gamma}$ is unknown at this stage. In order to make the change of function in the Gaussian measure $d\mu_C(h)$, it is convenient to introduce an unknown function $\tilde h \in \mathcal{C}^2$ with $\tilde h|_{\partial\Lambda} = 0$ and let $\bar h = -C\Delta\tilde h = \tilde h + \mathcal{O}(1)$. Then, using

$$
\int_{\Lambda\times\Lambda} dxdy\,\bar h(x) C^{-1}(x,y)\phi(y) = -\int_\Lambda dx\,\big(\Delta\tilde h(x)\big)\phi(x) = \int_\Lambda dx\,\nabla\tilde h(x)\cdot\nabla\phi(x)
$$

we get, in law, $h = \bar h + \phi$ with ϕ distributed according to

$$
\begin{aligned}
d\mu_{C,\bar h,\tilde u}(\phi) &= Z(\Lambda)^{-1}\, d\mu_C(\phi)\ e^{-H(\Lambda;\phi)}\\
H(\Lambda;\phi) &= \int_\Lambda dx\,\Big[V(\nabla\bar h + \nabla\phi) - \tilde u\cdot\nabla\phi\Big]
\end{aligned}
$$

where $\tilde u = u - \nabla\tilde h$ and

$$
Z(\Lambda) = \int d\mu_C(\phi)\ e^{-H(\Lambda;\phi)}
$$

The corresponding expectation of functions of ϕ will be denoted $\langle\ \cdot\ \rangle_{C,\bar h,\tilde u}$. For the partition functions, the change of variables gives

$$
\begin{aligned}
-\ln Z_{\lambda,\gamma}(\Lambda) &= -\ln Z(\Lambda) + \int_\Lambda dx\,\frac{1}{2}\nabla\tilde h\cdot\nabla\bar h - \int_\Lambda dx\,u\cdot\left(\nabla\bar h - \langle\nabla h\rangle_{C,\lambda,\gamma}\right)\\
&= -\ln Z(\Lambda) + \int_\Lambda dx\,\Big[\tilde\sigma_0(\nabla\bar h(x)) - \tilde\sigma_0(0)\Big] + \mathcal{O}(L\ln L)
\end{aligned}
$$

where

$$
\tilde\sigma_0(\nabla\bar h) = \frac{1}{2}|\nabla\bar h|^2 + \tilde\sigma_0(0)
$$

is the projected surface tension for the Gaussian model, $\lambda = 0$, and $\tilde{\sigma}_0(0)$ is an arbitrary constant.

The last statement in the theorem will then follow from

$$-\ln Z(\Lambda) = \int_\Lambda dx \left[\tilde{\sigma}_\lambda(\nabla \bar{h}(x)) - \tilde{\sigma}_0(\nabla \bar{h}(x))\right] + \mathcal{O}(L \ln L)$$

which will be a result of the cluster expansion. The projected surface tension $\tilde{\sigma}_\lambda(\nabla \bar{h})$ can be defined from

$$\tilde{\sigma}_\lambda(\nabla \bar{h}) - \tilde{\sigma}_0(\nabla \bar{h}) = \lim_{\Lambda_L \nearrow \mathbb{R}^2} -\frac{1}{|\Lambda|} \ln \int d\mu_C(\phi)\, e^{-\int_\Lambda dx \left[V(\nabla \bar{h} + \nabla \phi) - \tilde{u} \cdot \nabla \phi \right]}$$

where the mean slope $\nabla \bar{h}$ and the slope chemical potential $\tilde{u}$ are uniform over Λ. The same cluster expansion as for the non-uniform case will show that the definition above is independent of the basis shape Λ_1. Of course

$$\int_\Lambda dx\, \tilde{u} \cdot \nabla \phi = 0$$

when ϕ obeys Dirichlet boundary conditions, so that the definition is also independent of $\tilde{u}$, and one may take $\tilde{u} = 0$ to recover the usual definition of surface tension. But $\tilde{u}$ can be chosen instead so that the expansion converges independently of boundary conditions.

Existence and properties of the projected surface tension have been established [NS,FS] for continuous SOS models, on $\mathbb{Z}^2$ rather than $\mathbb{R}^2$, with potentials which are strictly convex and of bounded convexity. For general properties of surface tension, see [MMR,M].

Let us now return to the Wulff problem. We have

$$\frac{\delta}{\delta \tilde{u}}\, \tilde{\sigma}_\lambda(\nabla \bar{h}) = -\frac{1}{|\Lambda|} \int dx \, \langle \nabla \phi(x) \rangle = 0$$

$$\frac{\delta}{\delta \nabla \bar{h}}\, \tilde{\sigma}_\lambda(\nabla \bar{h}) = \nabla \bar{h} + \left\langle \frac{\delta}{\delta \nabla \bar{h}}\, V(\nabla \bar{h} + \nabla \phi) \right\rangle$$

Integration by parts, with

$$\frac{\delta}{\delta \phi(x)}\, F(\nabla \phi) = -\nabla \cdot \frac{\delta}{\delta \nabla \phi(x)} F(\nabla \phi)$$

gives

$$\Big\langle \phi(x) \Big\rangle = \int_\Lambda dy\, C(x,y) \nabla \cdot \left[\left\langle \frac{\delta}{\delta \nabla \phi(y)}\, V\big(\, \nabla \bar{h}(y) + \nabla \phi(y) \,\big) \right\rangle - \tilde{u}(y) \right]$$

We aim at proving that the Wulff equation is satisfied by $\bar{h}$, with $\langle \phi \rangle = 0$. The Wulff equation here takes the form

$$\nabla \cdot \frac{\delta}{\delta \nabla \bar{h}}\, \tilde{\sigma}_\lambda(\nabla \bar{h}) = \text{constant} = -\frac{\gamma}{L}$$

Therefore both $\langle \phi(x) \rangle = 0$ and the Wulff equation follow if

$$\left\langle \frac{\delta}{\delta \nabla \phi(x)}\, V\big(\, \nabla \bar{h}(x) + \nabla \phi(x) \,\big) \right\rangle - \tilde{u}(x) \; = \; 0 \qquad \forall\, x$$

with $\tilde{u} = u - \nabla \bar{h}$ and

$$\nabla \cdot u = -\frac{\gamma}{L} \qquad \forall\, x$$

Let us now give some preliminaries to the phase space expansion. First let

$$C = C^1 + C^2 + \ldots + C^{n-1} + C_n$$

$$C^i = \int_{M^{2(i-1)}}^{M^{2i}} d\alpha\, e^{\alpha\Delta_{\partial\Lambda}}$$

$$C_i = \int_{M^{2(i-1)}}^{\infty} d\alpha\, e^{\alpha\Delta_{\partial\Lambda}}$$

where e.g. $M = 2$, and let $\phi = \phi^1 + \phi^2 + ... + \phi^{n-1} + \phi_n$ be the corresponding decomposition of the Gaussian field.

For the horizontal cluster expansions, we use partitions of Λ, which are smooth except for the Dirichlet boundary conditions on $\partial\Lambda$. Let D^1 denote the paving of $\mathbb{R}^2$ by unit squares denoted $\triangle$, and let $\{\triangle(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^2) \,|\, \triangle \in D^1\}$ be a family of smooth characteristic functions such that

$$\begin{aligned}(\triangle + \mathbf{n})(x) =& \triangle(x - \mathbf{n}) \quad \forall\, \mathbf{n} \in \mathbb{Z}^2 \\ \triangle(x) =& 1 \quad \text{if} \quad x \in \triangle \\ \triangle(x) =& 0 \quad \text{if} \quad \operatorname{dist}(x, \triangle) \geq \frac{1}{2} \\ \sum_{\triangle \in D^1} \triangle(x) =& 1 \quad \forall\, x \in \mathbb{R}^2\end{aligned}$$

Such a partition will allow us to write e.g.

$$H(\Lambda; \phi) = \sum_{\triangle \in D^1} H(\triangle \cap \Lambda; \phi)$$

where

$$H(\triangle \cap \Lambda; \phi) = \int_\Lambda dx\, \triangle(x) \Big[V(\nabla\bar{h} + \nabla\phi) - \tilde{u} \cdot \nabla\phi \Big]$$

At scale M^i we shall use $D^i = M^{i-1}D^1$ and for $\triangle \in D^i$, $\triangle(x) = (M^{-i+1}\triangle)(M^{-i+1}x)$.

The phase space expansion and renormalization involve four operations at each scale: (1) Polymer expansion of ϕ^i ; (2) Coupling-decoupling expansion between ϕ^i and ϕ_{i+1} ; (3) Mayer expansion of the polymers produced by the two previous operations ; (4) Adjusting the mean slope, slope chemical potential, and quadratic terms. The next four sections describe in detail these four operations for the first momentum slice, $i = 1$. Next the essential formulae are given for an arbitrary momentum slice i. A final section discusses specific features of the last slice $i = n$ with $n \approx \log L / \log M$. General references to cluster expansions and renormalization are e.g. [B,R] and references therein.

3 Polymer expansion of ϕ^1

From $C = C^1 + C_2$, $\phi = \phi^1 + \phi_2$, we have

$$Z(\Lambda) = \int d\mu_{C_2}(\phi_2)\, Z^1(\Lambda; \phi_2)$$

$$Z^1(\Lambda; \phi_2) = \int d\mu_{C^1}(\phi^1)\, e^{-H_1(\Lambda; \phi^1 + \phi_2)}$$

where

$$H_1(\Lambda; \phi) = H(\Lambda; \phi) = \int_\Lambda dx \left[V(\nabla\bar{h} + \nabla\phi) - \tilde{u} \cdot \nabla\phi \right]$$

First step of expansion:

$$C^1(s_1) = s_1 C^1$$

$$\frac{d}{ds_1}C^1(s_1) = C^1 = \sum_{\triangle_{i_1}\triangle_{j_1}} \triangle_{i_1} C^1 \triangle_{j_1}$$

$$Z^1(\Lambda;\phi_2) = Z^1(\Lambda;\phi_2;s_1=1) = Z^1(\Lambda;\phi_2;s_1=0) + \sum_{\triangle_{i_1}\triangle_{j_1}} Z^1_{\triangle_{i_1}\triangle_{j_1}}(\Lambda;\phi_2)$$

with

$$Z^1(\Lambda;\phi_2;0) = e^{-H_1(\Lambda;\phi_2)}$$

$$\begin{aligned} Z^1_{\triangle_{i_1}\triangle_{j_1}}(\Lambda;\phi_2) = \int_0^1 ds_1 \int_\Lambda dx_1\, \triangle_{i_1}(x_1) \int_\Lambda dy_1\, \triangle_{j_1}(y_1)\; C^1(x_1,y_1)\,. \\ .\int d\mu_{C^1(s_1)}(\phi^1) \frac{\delta}{\delta\phi^1(x_1)} \frac{\delta}{\delta\phi^1(y_1)} e^{-H_1(\Lambda;\phi^1+\phi_2)} \end{aligned}$$

Note that $\triangle_{i_1} = \triangle_{j_1}$ is allowed. In any case let us denote $X_1 = \{\triangle_{i_1}, \triangle_{j_1}\}$ and

$$X_1(x) = \begin{cases} \triangle_{i_1}(x) + \triangle_{j_1}(x) & \text{if} \quad \triangle_{i_1} \neq \triangle_{j_1} \\ \triangle_{i_1}(x) & \text{if} \quad \triangle_{i_1} = \triangle_{j_1} \end{cases}$$

Second step of expansion: given X_1, let

$$\begin{aligned} C^1(s_1,s_2) =& s_2 C^1 + (s_1 - s_2) X_1 C^1 X_1 \\ =& s_2 C^1 + X_1\big(C^1(s_1) - s_2 C^1\big) X_1 \\ =& X_1 C^1(s_1) X_1 + s_2\big(C^1 - X_1 C^1 X_1\big) \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{ds_2}C^1(s_1,s_2) =& C^1 - X_1 C^1 X_1 \\ =& \sum_{\triangle_{i_2}\triangle_{j_2}} \triangle_{i_2} C^1 \triangle_{j_2} - X_1 C^1 X_1 \\ =& \sum_{\{\triangle_{i_2},\triangle_{j_2}\} \not\subset X_1} \triangle_{i_2} C^1 \triangle_{j_2} \end{aligned}$$

$$Z^1_{X_1}(\Lambda;\phi_2) = Z^1_{X_1}(\Lambda;\phi_2;s_2=s_1) = Z^1_{X_1}(\Lambda;\phi_2;s_2=0) + \sum_{\{\triangle_{i_2},\triangle_{j_2}\}\not\subset X_1} Z^1_{X_1\{\triangle_{i_2}\triangle_{j_2}\}}(\Lambda;\phi_2)$$

$$\begin{aligned} Z^1_{X_1\{\triangle_{i_2}\triangle_{j_2}\}}(\Lambda;\phi_2) = \int_0^1 ds_1 \int_0^{s_1} ds_2 \int d\mu_{C^1(s_1)}(\phi^1)\,. \\ .\Big(\int_\Lambda dx_1\, \triangle_{i_1}(x_1) \int_\Lambda dy_1\, \triangle_{j_1}(y_1)\; C^1(x_1,y_1) \frac{\delta}{\delta\phi^1(x_1)} \frac{\delta}{\delta\phi^1(y_1)}\Big)\,. \\ .\Big(\int_\Lambda dx_2\, \triangle_{i_2}(x_2) \int_\Lambda dy_2\, \triangle_{j_2}(y_2)\; C^1(x_2,y_2) \frac{\delta}{\delta\phi^1(x_2)} \frac{\delta}{\delta\phi^1(y_2)}\Big)\, e^{-H_1(\Lambda;\phi^1+\phi_2)} \end{aligned}$$

Let us denote $\mathcal{X}_2 = \{X_1, \{\triangle_{i_2}\triangle_{j_2}\}\}$ if $\text{dist}(X_1, \{\triangle_{i_2}, \triangle_{j_2}\}) \geq 2$ and $\mathcal{X}_2 = \{X_1 \cup \{\triangle_{i_2}, \triangle_{j_2}\}\}$ otherwise. Thus $\mathcal{X}_2$ is the set of connected components produced after two steps of expansion, connectedness being defined with allowed jumps up to next nearest neighbors, or jumps of length strictly less than 2. Third step of expansion:

$$\begin{aligned} C^1(s_1,s_2,s_3) =& s_3 C^1 + (s_1 - s_2) \sum_{X\in\mathcal{X}_1} X C^1 X + (s_2 - s_3) \sum_{X\in\mathcal{X}_2} X C^1 X \\ =& s_3 C^1 + \sum_{X\in\mathcal{X}_2} X\big(C^1(s_1,s_2) - s_3 C^1\big) X \\ =& \sum_{X\in\mathcal{X}_2} X C^1(s_1,s_2) X + s_3\big(C^1 - \sum_{X\in\mathcal{X}_2} X C^1 X\big) \end{aligned}$$

Induction: let $\mathcal{X}_p$ be the set of connected components of the family of $\triangle$'s produced by the s_k derivatives up to $k = p$. We denote $C^1(s_{\leq p}) = C^1(s_1 ... s_p)$ and define

$$\begin{aligned} C^1(s_{\leq p+1}) =& s_{p+1} C^1 + \sum_{i=1}^{p} (s_i - s_{i+1}) \sum_{X \in \mathcal{X}_i} X C^1 X \\ =& s_{p+1} C^1 + \sum_{X \in \mathcal{X}_p} X \big(C^1(s_{\leq p}) - s_{p+1} C^1 \big) X \\ =& \sum_{X \in \mathcal{X}_p} X C^1(s_{\leq p}) X + s_{p+1} \big(C^1 - \sum_{X \in \mathcal{X}_p} X C^1 X \big) \end{aligned}$$

Then

$$\frac{d}{ds_p} C^1(s_{\leq p}) = C^1 - \sum_{X \in \mathcal{X}_{p-1}} X C^1 X = \sum_{\substack{\{\triangle_{i_p}, \triangle_{j_p}\} \not\subset X \\ \forall X \in \mathcal{X}_{p-1}}} \triangle_{i_p} C^1 \triangle_{j_p}$$

This inductive procedure builds up trees of $\triangle$'s, and ends when $s_p = 0$ or ultimately when $\mathcal{X}_p$ covers Λ. The interpolating parameters are totally ordered, $s_1 < s_2 < \ldots$, but the order between interpolating parameters belonging to different trees, and the order in the building up of different trees are irrelevant and can be forgotten. Then the covariance between two points is

$$C^1(\mathbf{s})(x, y) = \sum_{\triangle \triangle'} \triangle(x) \triangle'(y) s(\triangle, \triangle') C^1(x, y)$$

where $s(\triangle, \triangle') = 0$ if $\triangle$ and $\triangle'$ don't belong to the same tree, and otherwise is the smallest interpolating parameter occurring on the tree-path between $\triangle$ and $\triangle'$.

Functional derivatives:

$$\frac{\delta}{\delta \phi^1(x)} F(\nabla \phi^1) = -\nabla \cdot \frac{\delta}{\delta \nabla \phi^1(x)} F(\nabla \phi^1)$$

where typically $F(\nabla \phi^1) = e^{-H_1(\Lambda; \phi^1 + \phi_2)}$ or a derivative of it. The gradient ∇ can hit the relatively smooth external field $\bar{h}_1$, giving a factor $\mathcal{O}(1/L)$, or it can hit $\nabla \phi^1$, in which case we integrate by parts. This will produce a term with $\nabla C^1(x, y)$ but also a term $\nabla \triangle(x)$.

The induction results in

$$Z^1(\Lambda; \phi_2) = \sum_{Y_1^1, \ldots Y_q^1} A(Y_1^1; \phi_2) \ldots A(Y_q^1; \phi_2) e^{-H_1(\Lambda \setminus \cup \bar{Y}_k^1; \phi_2)}$$

where $Y_1^1, \ldots Y_q^1$ are polymers with $\mathrm{dist}(Y_k^1, Y_{k'}^1) \geq 2$ for $k \neq k'$, $\bar{Y} = (Y \cup \partial Y) \cap \Lambda$ with ∂Y the set of squares $\triangle$ neighbors of Y and

$$\begin{aligned} A(Y; \phi_2) = \sum_{T'} \int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{p-1}} ds_p \int d\mu_{C^1(s)}(\phi^1). \\ . \Big(\prod_{k=1}^{p} \int_\Lambda dx_k \, \triangle_{i_k} \int_\Lambda dy_k \, \triangle_{j_k} \; C^1(x_k, y_k) \frac{\delta}{\delta \phi^1(x_k)} \frac{\delta}{\delta \phi^1(y_k)} \Big) e^{-H_1(\bar{Y}; \phi^1 + \phi_2)} \end{aligned}$$

where the sum over T' is over ordered decorated trees, making the polymer connected with allowed jumps of length strictly less than 2, and $p \geq 1$. A decoration is a link between a square and itself, $\triangle_{i_k} = \triangle_{j_k}$, which is allowed only if the square $\triangle_{i_k}$ was isolated up to the k'th step of expansion. A simple case is a decorated monomer.

It will be convenient to denote the contribution of squares with $\phi^1 = 0$ or "empty squares" as

$$A(\triangle^{\emptyset}; \phi_2) = e^{-H_1(\triangle \cap \Lambda; \phi_2)}$$

so that

$$Z^1(\Lambda; \phi_2) = \sum_{Y_1^1,\ldots Y_q^1} A(Y_1^1; \phi_2)\ldots A(Y_q^1; \phi_2) \prod_{\triangle \notin \cup \bar{Y}_k^1} A(\triangle^{\emptyset}; \phi_2)$$

4 Coupling-decoupling expansion between ϕ^1 and ϕ_2

For each $A(Y_k^1; \phi_2)$, and for each $\triangle \notin \cup \bar{Y}_k^1$, an interpolating parameter t is introduced. $t = 1$ corresponds to the original measure with expected mean $\bar{h}_1 = \bar{h}$ and slope chemical potential $\tilde{u}_1 = \tilde{u}$, whereas $t = 0$ corresponds to decoupling $\nabla\phi^1$ from $\nabla\phi_2$, and giving a new expected mean slope $\bar{h}_2$ and slope chemical potential $\tilde{u}_2$ to $\nabla\phi_2$. Namely:

$$V(\phi^1; \phi_2; t) = V\big(\nabla\bar{h}_1 + \nabla\phi^1 + t\nabla\phi_2\big) - V\big(\nabla\bar{h}_2 + t\nabla\phi_2\big) + V\big(\nabla\bar{h}_2 + \nabla\phi_2\big)$$

and

$$\tilde{u}(\phi^1; \phi_2; t) = \tilde{u}_1 \cdot (\nabla\phi^1 + t\nabla\phi_2) - t\,\tilde{u}_2 \cdot \nabla\phi_2 + \tilde{u}_2 \cdot \nabla\phi_2$$

so that

$$\frac{d}{dt} V(\phi^1; \phi_2; t)\,\Big|_{t=0} = \nabla\phi_2 \cdot \Big[\bigtriangledown V\big(\nabla\bar{h}_1 + \nabla\phi^1\big) - \bigtriangledown V\big(\nabla\bar{h}_2\big)\Big]$$

$$\frac{d}{dt} \tilde{u}(\phi^1; \phi_2; t)\,\Big|_{t=0} = \nabla\phi_2 \cdot \Big[\tilde{u}_1 - \tilde{u}_2\Big]$$

where the notation $\bigtriangledown V(\nabla h)$ is a shorthand for

$$\bigtriangledown V(\nabla h) = \frac{\delta}{\delta\nabla h} V(\nabla h) = \left(\frac{\partial V}{\partial(\partial h/\partial x_1)}, \frac{\partial V}{\partial(\partial h/\partial x_2)}\right)$$

e.g. for $V(\nabla h) = \lambda|\nabla h|^4$ we have $\bigtriangledown V(\nabla h) = 4\lambda\nabla h|\nabla h|^2$.

Quadratic terms are less dangerous than linear terms, but the expansion will be easier to control if they are also renormalized. This is done by a counter-term

$$Q(\phi_2; t) = \frac{1}{2}(1 - t^2) q_2^{\alpha\beta} \partial_\alpha\phi_2 \partial_\beta\phi_2$$

where $q_2^{\alpha\beta}$ with $\alpha, \beta \in \{1, 2\}$ are functions of $x \in \Lambda$, and $\partial_\alpha\phi_2 = \partial\phi_2/\partial x_\alpha$. The resulting interpolating Hamiltonian is

$$H(\Lambda; \phi^1; \phi_2; t) = \int_\Lambda dx\,\Big[V(\phi^1; \phi_2; t) - \tilde{u}(\phi^1; \phi_2; t) + Q(\phi_2; t)\Big]$$

This interpolation and the integration over ϕ^1 is the basis of the first renormalization group step, associated to the flow $(\bar{h}_1, \tilde{u}_1, 0) \to (\bar{h}_2, \tilde{u}_2, q_2^{\alpha\beta})$.

It works with an expansion of each polymer amplitude $A(Y_k^1; \phi_2; t = 1)$ as

$$A(Y_k^1; \phi_2; 1) = A(Y_k^1; \phi_2; 0) + \frac{dA(Y_k^1; \phi_2; t)}{dt}\bigg|_{t=0} + \frac{1}{2}\frac{d^2A(Y_k^1; \phi_2; t)}{dt^2}\bigg|_{t=0} + \frac{1}{2}\int_0^1 dt\,(1-t)^2 \frac{d^3A(Y_k^1; \phi_2; t)}{dt^3}$$

or for easier reference

$$A(Y; \phi_2) = A^{(0)}(Y; \phi_2) + A^{(1)}(Y; \phi_2) + A^{(2)}(Y; \phi_2) + A^{(3)}(Y; \phi_2)$$

The formula for $A^{(0)}(Y;\phi_2)$ is obtained from $A(Y;\phi_2)$ by changing the Hamiltonian $H_1(Y;\phi^1+\phi_2)$ into $H(Y;\phi^1;\phi_2;0)$:

$$
\begin{aligned}
A^{(0)}(Y;\phi_2) = \sum_{T'} \int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{p-1}} ds_p \int d\mu_{C^1(s)}(\phi^1). \\
.\Big(\prod_{k=1}^{p} \int_\Lambda dx_k \, \triangle_{i_k} \int_\Lambda dy_k \, \triangle_{j_k} \, C^1(x_k,y_k) \frac{\delta}{\delta\phi^1(x_k)} \frac{\delta}{\delta\phi^1(y_k)} \Big) e^{-H(\bar{Y};\phi^1;\phi_2;0)}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
A^{(1)}(Y;\phi_2) = - \sum_{T'} \int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{p-1}} ds_p \int d\mu_{C^1(s)}(\phi^1). \\
.\Big(\prod_{k=1}^{p} \int_\Lambda dx_k \, \triangle_{i_k} \int_\Lambda dy_k \, \triangle_{j_k} \, C^1(x_k,y_k) \frac{\delta}{\delta\phi^1(x_k)} \frac{\delta}{\delta\phi^1(y_k)} \Big) e^{-H(\bar{Y};\phi^1;\phi_2;0)}. \\
. \sum_{\triangle\in\bar{Y}} \int_\Lambda dx \, \triangle \, \nabla\phi_2 \cdot \Big[\bigtriangledown V(\nabla\bar{h}_1 + \nabla\phi^1) - \bigtriangledown V(\nabla\bar{h}_2) - \tilde{u}_1 + \tilde{u}_2 \Big]
\end{aligned}
$$

$$
\begin{aligned}
A^{(2)}(Y;\phi_2) = \frac{1}{2} \sum_{T'} \int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{p-1}} ds_p \int d\mu_{C^1(s)}(\phi^1). \\
.\Big(\prod_{k=1}^{p} \int_\Lambda dx_k \, \triangle_{i_k} \int_\Lambda dy_k \, \triangle_{j_k} \, C^1(x_k,y_k) \frac{\delta}{\delta\phi^1(x_k)} \frac{\delta}{\delta\phi^1(y_k)} \Big) e^{-H(\bar{Y};\phi^1;\phi_2;0)}. \\
.\Bigg\{ \Big(\sum_{\triangle\in\bar{Y}} \int_\Lambda dx \, \triangle \, \nabla\phi_2 \cdot \Big[\bigtriangledown V(\nabla\bar{h}_1 + \nabla\phi^1) - \bigtriangledown V(\nabla\bar{h}_2) - \tilde{u}_1 + \tilde{u}_2 \Big] \Big). \\
.\Big(\sum_{\triangle'\in\bar{Y}} \int_\Lambda dx' \, \triangle' \, \nabla\phi_2 \cdot \Big[\bigtriangledown V(\nabla\bar{h}_1 + \nabla\phi^1) - \bigtriangledown V(\nabla\bar{h}_2) - \tilde{u}_1 + \tilde{u}_2 \Big] \Big) \\
- \sum_{\triangle\in\bar{Y}} \int_\Lambda dx \, \triangle \, \nabla\phi_2 \cdot \Big[\nabla\phi_2 \cdot \Big[\bigtriangledown\bigtriangledown V(\nabla\bar{h}_1 + \nabla\phi^1) - \bigtriangledown\bigtriangledown V(\nabla\bar{h}_2) \Big] \Big] \\
- \sum_{\triangle\in\bar{Y}} \int_\Lambda dx \, \triangle \, q_2^{\alpha\beta} \partial_\alpha\phi_2 \partial_\beta\phi_2 \Bigg\}
\end{aligned}
$$

For each $\triangle \notin \cup\bar{Y}_k^1$, the same expansion is applied to $A(\triangle^\emptyset;\phi^2) = e^{-H_1(\triangle\cap\Lambda;\phi_2)}$. Inserting $A(Y;\phi_2) = A^{(0)}(Y;\phi_2) + A^{(1)}(Y;\phi_2) + A^{(2)}(Y;\phi_2) + A^{(3)}(Y;\phi_2)$, with an appropriate polymer index to t, for each $A(Y_k^1;\phi_2)$ in the polymer expansion, and the similar expression for each $\triangle \notin \cup\bar{Y}_k^1$, completes this step of vertical expansion.

5 Mayer expansion with ϕ_2 as external field

Let

$$
H_2(\Lambda;\phi_2) = \int_\Lambda dx \Big[V(\nabla\bar{h}_2 + \nabla\phi_2) - \tilde{u}_2\nabla\phi_2 + \frac{1}{2} q_2^{\alpha\beta} \partial_\alpha\phi_2\partial_\beta\phi_2 \Big]
$$

$$
V(\Lambda;\bar{h}) = \int_\Lambda dx \, V(\nabla\bar{h})
$$

$$
V(\bar{Y};\bar{h}) = \sum_{\triangle\in\bar{Y}} \int_\Lambda dx \, \triangle(x) V(\nabla\bar{h}(x))
$$

$$\tilde{Z}^1(\Lambda;\phi_2) = e^{H_2(\Lambda;\phi_2) + V(\Lambda;\bar{h}_1) - V(\Lambda;\bar{h}_2)} Z^1(\Lambda;\phi_2)$$

$$\tilde{A}^{(n)}(Y;\phi_2) = e^{H_2(\bar{Y};\phi_2) + V(\bar{Y};\bar{h}_1) - V(\bar{Y};\bar{h}_2)} A^{(n)}(Y;\phi_2)$$

Note that $\tilde{A}^{(0)}(Y;\phi_2) = \tilde{A}^{(0)}(Y)$ is independent of ϕ_2 and $\tilde{A}^{(1)}(Y;\phi_2)$ is linear in ϕ_2 and $\tilde{A}^{(2)}(Y;\phi_2)$ is quadratic in ϕ_2. Also $\tilde{A}^0(\triangle^\emptyset;\phi^2) = 1$. Then

$$\tilde{Z}^1(\Lambda;\phi_2) = \sum_{q\geq 0} \sum_{\substack{n_1,\dots n_q \\ \in\{0,1,2,3\}}} \sum_{Y_1,\dots Y_q} \prod_{k=1}^{q} \tilde{A}^{(n_k)}(Y_k;\phi_2)$$

where $Y_1, \dots Y_q$ are non-overlapping polymers (distance at least two between support squares), in Λ. Factors $\tilde{A}^{(n_k)}(\triangle^\emptyset;\phi^2)$ with $n_k \geq 1$ are allowed.

Uniform absolute convergence of the expansion requires the cancellation of terms linear in any $\nabla\phi_2(x)$. The corresponding sum over polymers must be evaluated by a Mayer expansion. It will help also to cancel the dominant part of terms quadratic in $\nabla\phi_2$. For this purpose we decompose

$$\Big(\sum_{\triangle\in\bar{Y}} \int_\Lambda dx\, \triangle(x)\, \nabla\phi_2(x) \cdot \Big[\bigtriangledown V\big(\nabla\bar{h}_1 + \nabla\phi^1\big) - \bigtriangledown V\big(\nabla\bar{h}_2\big) - \tilde{u}_1 + \tilde{u}_2 \Big]_x \Big) \cdot$$
$$\cdot \Big(\sum_{\triangle'\in\bar{Y}} \int_\Lambda dx'\, \triangle'(x')\, \nabla\phi_2(x') \cdot \Big[\bigtriangledown V\big(\nabla\bar{h}_1 + \nabla\phi^1\big) - \bigtriangledown V\big(\nabla\bar{h}_2\big) - \tilde{u}_1 + \tilde{u}_2 \Big]_{x'} \Big) =$$
$$\Big(\sum_{\triangle\in\bar{Y}} \int_\Lambda dx\, \triangle(x)\, \nabla\phi_2(x) \cdot \Big[\bigtriangledown V\big(\nabla\bar{h}_1 + \nabla\phi^1\big) - \bigtriangledown V\big(\nabla\bar{h}_2\big) - \tilde{u}_1 + \tilde{u}_2 \Big]_x \Big) \cdot$$
$$.\nabla\phi_2(x) \cdot \Big(\sum_{\triangle'\in\bar{Y}} \int_\Lambda dx'\, \triangle'(x') \Big[\bigtriangledown V\big(\nabla\bar{h}_1 + \nabla\phi^1\big) - \bigtriangledown V\big(\nabla\bar{h}_2\big) - \tilde{u}_1 + \tilde{u}_2 \Big]_{x'} \Big) +$$
$$\Big(\sum_{\triangle\in\bar{Y}} \int_\Lambda dx\, \triangle(x)\, \nabla\phi_2(x) \cdot \Big[\bigtriangledown V\big(\nabla\bar{h}_1 + \nabla\phi^1\big) - \bigtriangledown V\big(\nabla\bar{h}_2\big) - \tilde{u}_1 + \tilde{u}_2 \Big]_x \Big) \cdot$$
$$\cdot \Big(\sum_{\triangle'\in\bar{Y}} \int_\Lambda dx'\, \triangle'(x') \Big(\nabla\phi_2(x') - \nabla\phi_2(x)\Big) \cdot \Big[\bigtriangledown V\big(\nabla\bar{h}_1 + \nabla\phi^1\big) - \bigtriangledown V\big(\nabla\bar{h}_2\big) - \tilde{u}_1 + \tilde{u}_2 \Big]_{x'} \Big)$$

The second term in this decomposition may be considered as having one extra (spatial) derivative. This motivates the notation

$$\tilde{A}^{(2)}(Y;\phi_2) = \tilde{A}^{(2)}(W;\phi_2) + \tilde{A}^{(3)}(Y;\phi_2)$$

where $\tilde{A}^{(3)}(Y;\phi_2)$ is the contribution of the second term in the decomposition formula above.

We now distinguish vacuum polymers ($n_k = 0$) denoted V's, polymers with one $\nabla\phi_2$ ($n_k = 1$) denoted $W^{(1)}$'s or two $\nabla\phi_2$ at coinciding points ($n_k = 2$) denoted $W^{(2)}$'s, and the other polymers ($n_k = 3$) which have at least three $\nabla\phi_2$ fields pulled down from the exponential, or two with an extra spatial derivative, denoted Y's. A W without superscript will denote a generic $W^{(1)}$ or $W^{(2)}$.

The Mayer expansion is worked out inductively, and pieces which get hooked to a Y are not expanded further. Also two W's which hook together with a total of three or four $\nabla\phi_2$ are also put with the Y's. When two $W^{(1)}$'s hook together, we apply the same decomposition as we did for $\tilde{A}^{(2)}(Y;\phi_2)$, so that the coinciding points part becomes a new $W^{(2)}$, whereas the gradient part becomes a new Y.

The W polymers are essentially of the same form as the vacuum polymers, except for a factor $\nabla\phi_2(x)$ or $|\nabla\phi_2(x)|^2$ or $(\nabla\phi_2(x)\cdot)(\nabla\phi_2(x)\cdot)$ which is localized in a particular square $\triangle_W$ of the polymer. The non-overlap condition between polymers is written as follows:

$$\tilde{Z}^1(\Lambda;\phi_2) = \Big(\sum_{V_1\ldots V_q}\prod_{k=1}^{q}\tilde{A}^{(0)}(V_k)\Big)\Big(\sum_{W_1\ldots W_{q'}}\prod_{k'=1}^{q'}\tilde{A}^{(n_{k'})}(W_{k'};\phi_2)\Big)\cdot$$
$$\cdot\Big(\sum_{Y_1\ldots Y_{q''}}\prod_{k''=1}^{q''}\tilde{A}^{(3)}(Y_{k''};\phi_2)\Big)P_1P_2$$

where

$$P_1 = \prod_{V\neq V'} 1_{\bar{V}\cap\bar{V}'=\emptyset}\prod_{V,W} 1_{\bar{V}\cap\bar{W}=\emptyset}\prod_{V,Y} 1_{\bar{V}\cap\bar{Y}=\emptyset}\prod_{W\neq W'} 1_{\bar{W}\cap\bar{W}'=\emptyset}\prod_{W,Y} 1_{\bar{W}\cap\bar{Y}=\emptyset}$$

$$P_2 = \prod_{W^{(1)},W^{(2)}} 1_{\triangle_{W^{(1)}}\neq\triangle_{W^{(2)}}}\prod_{W^{(2)}\neq W'^{2}} 1_{\triangle_{W^{(2)}}\neq\triangle_{W'^{2}}}\prod_{W,Y} 1_{\triangle_W\notin Y}\prod_{Y\neq Y'} 1_{\bar{Y}\cap\bar{Y}'=\emptyset}$$

with 1_A the indicator function of event A. We shall also use the notation

$$\bar{1}_A = 1_A - 1 = \begin{cases} 0 & \text{if} \quad A \\ -1 & \text{if} \quad (\text{not}\, A)\end{cases}$$

We do the Mayer expansion with interpolating parameters in P_1 only. For the first step of this expansion, we define $P_1(m_1)$, with $m_1\in[0,1]$ by replacing each 1_A factor in P_1 by

$$1_A(m_1) = 1 + m_1\bar{1}_A$$

Then

$$P_1 = P_1(1) = P_1(0) + \sum_{A_1}\bar{1}_{A_1}\int_0^1 dm_1 P_1^{A_1}(m_1)$$

where $P_1^{A_1}(m_1)$ is obtained from $P_1(m_1)$ by omitting the factor $1_A(m_1)$. If A_1 happens to be a $\{W^{(1)},W'^{(1)}\}$, the corresponding product of the two polymers is decomposed into a $W^{(2)}$ and a Y as explained, and $P_1^{A_1}(m_1)$ and P_2 are modified accordingly. For the second step of the Mayer expansion, each $1_A(m_1)$ in the new $P_1(m_1)$ is replaced by

$$1_A(m_2) = 1 + m_2\bar{1}_A$$

and

$$P_1^{A_1}(m_1) = P_1^{A_1}(0) + \sum_{A_2}\bar{1}_{A_2}\int_0^{m_1} dm_2\, P_1^{A_1A_2}(m_2)$$

We then proceed with the aim to have all polymers linked on trees, and no interactions between trees. In order not to make loops, at each step, factors 1_A in $P_1^{A_1A_2\cdots}(\mathbf{m})$ where A is a tree connected pair, are taken from P_1 into P_2.

In order to formulate the results of the Mayer expansion, let us introduce

$$\sum_{\substack{V_1\cdots V_q\\ q\geq 0}}\prod_{k=1}^{q}\tilde{A}^{(0)}(V_k)\prod_{k_1\neq k_2} 1_{V_{k_1}\cap V_{k_2}=\emptyset} = \frac{\int d\mu_{C^1}(\phi^1)\, e^{-H_1(\Lambda;\phi^1)}}{e^{-V(\Lambda;\bar{h}_1)}} = e^{-\tilde{F}_1(\Lambda)}$$

Then

$$\Big(\sum_{V_1\dots V_q}\prod_{k=1}^{q}\tilde A^{(0)}(V_k)\Big)\Big(\sum_{W\ni\triangle}\tilde A^{(1)}(W;\phi_2)\Big)\prod_{k_1\neq k_2}1_{V_{k_1}\cap V_{k_2}=\emptyset}\prod_{k}1_{V_k\cap W=\emptyset}=$$

$$-e^{-\tilde F_1(\Lambda)}\int_\Lambda dx\,\triangle(x)\,\nabla\phi_2\cdot\Big[\Big\langle\bigtriangledown V(\nabla\bar h_1+\nabla\phi^1)\Big\rangle_1-\bigtriangledown V(\nabla\bar h_2)-\tilde u_1+\tilde u_2\Big]$$

where the fields in the square brackets are at $x\in\triangle$, and

$$\Big\langle\cdot\Big\rangle_1=\frac{\int d\mu_{C^1}(\phi^1)\,e^{-H_1(\Lambda;\,\phi^1)}\,\cdot}{\int d\mu_{C^1}(\phi^1)\,e^{-H_1(\Lambda;\,\phi^1)}}$$

Then

$$\tilde Z^1(\Lambda;\phi_2)=e^{-\tilde F_1(\Lambda)}\sum_{\substack{\mathcal Y_1\dots\mathcal Y_{q_3}\\ q_3\geq 0}}\Big(\prod_{k=1}^{q_3}\mathcal A^{(3)}(\mathcal Y_k;\phi_2)\Big)\sum_{\substack{\triangle_1\dots\triangle_{q'}\\ q'\geq 0}}\ \sum_{\substack{\triangle_{q'+1}\dots\triangle_{q'+q''}\\ q''\geq 0}}P_2\,\cdot$$

$$\cdot\prod_{k'=1}^{q'}\Big\{-\int_\Lambda dx_{k'}\,\triangle_{k'}(x_{k'})\,\nabla\phi_2(x_{k'})\cdot\Big[\Big\langle\bigtriangledown V(\nabla\bar h_1+\nabla\phi^1)\Big\rangle_1-\bigtriangledown V(\nabla\bar h_2)-\tilde u_1+\tilde u_2\Big]_{x_{k'}}\Big\}$$

$$\cdot\prod_{k''=q'+1}^{q'+q''}\frac{1}{2}\int_\Lambda dx_{k''}\,\triangle_{k''}(x_{k''})\Big\{$$

$$\Big\langle\nabla\phi_2(x_{k''})\cdot\Big[\bigtriangledown V(\nabla\bar h_1+\nabla\phi^1)\Big]_{x_{k''}};\nabla\phi_2(x_{k''})\cdot\int_\Lambda dy_{k''}\Big[\bigtriangledown V(\nabla\bar h_1+\nabla\phi^1)\Big]_{y_{k''}}\Big\rangle_1$$

$$-\nabla\phi_2(x_{k''})\cdot\Big[\nabla\phi_2(x_{k''})\cdot\Big\langle\bigtriangledown\bigtriangledown V(\nabla\bar h_1+\nabla\phi^1)-\bigtriangledown\bigtriangledown V(\nabla\bar h_2)\Big\rangle_1\Big]$$

$$-q_2^{\alpha\beta}(x_{k''})\,\partial_\alpha\phi_2(x_{k''})\,\partial_\beta\phi_2(x_{k''})\Big\}$$

where $\mathcal Y_k=\{Y_k;\triangle_{k,1}\dots\triangle_{k,q_k}\}$ with $q_k\geq 0$ or $\mathcal Y_k=\{\triangle_{k,1}\dots\triangle_{k,q_k}\}$ with $q_k\geq 2$ and at least three $\nabla\phi_2$ fields or two $\nabla\phi_2$ fields with an extra gradient. $\langle\ ;\ \rangle$ is the truncated expectation. For a general $\mathcal Y_k=\{Y;\triangle_1\dots\triangle_q\}$, the amplitude is given by

$$\mathcal A^{(3)}(\mathcal Y;\phi_2)=\tilde A^{(3)}(Y;\phi_2)\sum_{\substack{W_1\dots W_q\\ W_k\supset\triangle_k}}\prod_{k=1}^{q}\tilde A^{(n_k)}(W_k;\phi_2)\sum_{\substack{V_1\dots V_{q'}\\ q'\geq 0}}\prod_{k'=1}^{q'}\tilde A^{(0)}(V_{k'})\,.$$

$$\cdot\sum_{\substack{T\text{ over}\\ Y\,W_1\dots W_q\\ V_1\dots V_{q'}}}\Big(\prod_{\ell\in T}\int_0^1 dm_\ell\,\bar 1_\ell\Big)\prod_{\ell'\in\bar T}(1+m_{\ell'}\,\bar 1_{\ell'})$$

where T and ℓ are Mayer trees and Mayer links, $\bar T$ is the set of pairs of vertices of the tree which are not links of the tree, and $m_{\ell'}$ for $\ell'\in\bar T$ is the smallest m_ℓ on the path along the tree between the two vertices. The formula is the same without $\tilde A^{(3)}(Y;\phi_2)$ when $\mathcal Y$ has no Y.

The renormalization equations discussed in the next section will fix $\bar h_1-\bar h_2$, $\tilde u_2-\tilde u_1$, a_2 and b_2 so that only the terms with $q'=0$ and $q''=0$ in the formula for $\tilde Z^1(\Lambda;\phi_2)$ are non-zero.

6 Renormalization

The first renormalization equation is

$$\Big\langle \bigtriangledown V\big(\nabla \bar{h}_1 + \nabla \phi^1\big) \Big\rangle_1 - \bigtriangledown V(\nabla \bar{h}_2) - \tilde{u}_1 + \tilde{u}_2 = 0$$

In accordance with $\tilde{u}_1 = u_1 - \nabla \tilde{h}_1$ and $\nabla \cdot u_1 = -\gamma/\lambda = -\gamma_1/L$, we also choose $\tilde{u}_2$ of the form $\tilde{u}_2 = u_2 - \nabla \tilde{h}_2$ with $\nabla \cdot u_2 = -\gamma_2/L$. Applying $\nabla\cdot$ to the renormalization equation then gives

$$\nabla \Big\langle \bigtriangledown V\big(\nabla \bar{h}_1 + \nabla \phi^1\big) \Big\rangle_1 - \nabla \bigtriangledown V(\nabla \bar{h}_2) + \Delta \tilde{h}_1 - \Delta \tilde{h}_2 + \frac{\gamma_1 - \gamma_2}{L} = 0$$

Defining

$$\tilde{g}_i = \big(-\Delta_{\partial\Lambda}\big)^{-1} \frac{\gamma_i}{L}$$

we get

$$\tilde{h}_1 - \tilde{h}_2 = \tilde{g}_1 - \tilde{g}_2 + \big(-\Delta_{\partial\Lambda}\big)^{-1} \Big(\nabla \Big\langle \bigtriangledown V\big(\nabla \bar{h}_1 + \nabla \phi^1\big) \Big\rangle_1 - \nabla \bigtriangledown V(\nabla \bar{h}_2)\Big)$$

together with $\bar{h}_1 = -C\Delta \tilde{h}_1$ and the constraint

$$\int_\Lambda dx\, \bar{h}_1(x) = \mathrm{V}$$

given that $\bar{h}_2$ should satisfy the same constraint. Having obtained $\tilde{h}_1$ in terms of $\tilde{h}_2$ and $\gamma_1 - \gamma_2$, one may choose γ_1 to satisfy the volume constraint. Finally $\tilde{u}_1$ is obtained from the renormalization equation in its original form. Note that $\nabla \cdot u_1 = -\gamma_1/L$ alone doesn't determine u_1, because u_1 doesn't obey simple boundary conditions.

Let us now discuss the solution of the equation, taking for simplicity $\tilde{g}_2 = \tilde{g}_1$. Then

$$\tilde{h}_1 - \tilde{h}_2 = \big(-\Delta_{\partial\Lambda}\big)^{-1} \Big(\nabla \Big\langle \bigtriangledown V\big(\nabla \bar{h}_2 + \nabla \bar{h}_1 - \nabla \bar{h}_2 + \nabla \phi^1\big) \Big\rangle_1 - \nabla \bigtriangledown V(\nabla \bar{h}_2)\Big)$$

which is of the form

$$f = F(\tilde{h}_2, f)$$

with $f = \tilde{h}_1 - \tilde{h}_2$.

Lemma: *Assume $\tilde{h}_2$, f, g are in $\mathcal{C}^2(\Lambda) \cap \mathcal{C}^1(\bar{\Lambda})$ with $\nabla \tilde{h}_2$, ∇f, ∇g uniformly bounded by a universal constant. Let $F(\cdot,\cdot)$ be as above. Then*

$$|F(\tilde{h}_2, f) - F(\tilde{h}_2, g)| < \mathcal{O}(\lambda)|f - g|$$

Therefore

$$f = F(\tilde{h}_2, f)$$

has a unique solution, which can be obtained as the limit of the sequence $f_0 = 0$, $f_{n+1} = F(\tilde{h}_2, f_n)$. This solution can be assigned to $\tilde{h}_1 - \tilde{h}_2$. Then

$$\tilde{h}_1 - \tilde{h}_2 = \big(-\Delta_{\partial\Lambda}\big)^{-1} \Big(\nabla \Big\langle \bigtriangledown V\big(\nabla \bar{h}_2 + \nabla \phi^1\big) \Big\rangle_1 - \nabla \bigtriangledown V(\nabla \bar{h}_2)\Big) \Big(1 + \mathcal{O}(\lambda)\Big)$$

In the case $V(x) = \lambda x^4$, the result takes the form

$$\tilde{h}_1 - \tilde{h}_2 = 8\lambda \Big\langle \,|\, \nabla \phi^1 \,|^2 \Big\rangle_1 \tilde{h}_2 \Big(1 + \mathcal{O}(\lambda)\Big)$$

and the contractivity estimate takes the form

$$|F(\tilde{h}_2,f)-F(\tilde{h}_2,g)| \simeq 4\lambda(f-g)\Big(|\nabla\tilde{h}_2|^2+\Big\langle\,|\nabla\phi^1|^2\Big\rangle_1\Big)+4\lambda\big(-\Delta_{\partial\Lambda}\big)^{-1}\Big(\partial_\alpha\partial_\beta(f-g)\partial_\alpha\tilde{h}_2\partial_\beta\tilde{h}_2\Big)$$

The second renormalization equation is simply

$$\begin{aligned} q_2^{\alpha\beta}(x) = \Big\langle\, \partial_\alpha V\Big(\nabla\bar{h}_1(x)+\nabla\phi^1(x)\Big)\,;\,\int_\Lambda dy\,\partial_\beta V\Big(\nabla\bar{h}_1(y)+\nabla\phi^1(y)\Big)\Big\rangle_1 \\ -\Big\langle\,\partial_\alpha\partial_\beta V\Big(\nabla\bar{h}_1(x)+\nabla\phi^1(x)\Big)\Big\rangle_1+\partial_\alpha\partial_\beta V\Big(\nabla\bar{h}_2(x)\Big) \end{aligned}$$

where $\partial_\alpha V(\nabla h)=\partial V/\partial(\partial_\alpha h)$.

7 Formulae for a generic momentum slice i

The essential formulae defining the expansion algorithm at a generic scale i are the coupling-decoupling inductive formula and the renormalization equations. The coupling-decoupling formula is based on replacing $\phi_i=\phi^i+\phi_{i+1}$ by $\phi^i+t_i\phi_{i+1}$ in the interaction. More precisely we define inductively

$$\begin{aligned} V\big(\phi^1\ldots\phi^i\,;\,\phi_{i+1}\,;\,t_1\ldots t_i\big) =& V\big(\phi^1\ldots\phi^{i-1}\,;\,\phi^i+t_i\phi_{i+1}\,;\,t_1\ldots t_{i-1}\big) \\ & -V\big(\nabla\bar{h}_{i+1}+t_i\nabla\phi_{i+1}\big)+V\big(\nabla\bar{h}_{i+1}+\nabla\phi_{i+1}\big) \end{aligned}$$

$$\tilde{u}\big(\phi^1\ldots\phi^i\,;\,\phi_{i+1}\,;\,t_1\ldots t_i\big)=\tilde{u}\big(\phi^1\ldots\phi^{i-1}\,;\,\phi^i+t_i\phi_{i+1}\,;\,t_1\ldots t_{i-1}\big)+(1-t_i)\tilde{u}_{i+1}\cdot\nabla\phi_{i+1}$$

$$Q\big(\phi^1\ldots\phi^i\,;\,\phi_{i+1}\,;\,t_1\ldots t_i\big)=Q\big(\phi^1\ldots\phi^{i-1}\,;\,\phi^i+t_i\phi_{i+1}\,;\,t_1\ldots t_{i-1}\big)+\frac{1}{2}(1-t_i^2)q_{i+1}^{\alpha\beta}\,\partial_\alpha\phi_{i+1}\partial_\beta\phi_{i+1}$$

starting from

$$V(\phi^1;\phi_2;t_1)=V\big(\nabla\bar{h}_1+\nabla\phi^1+t_1\nabla\phi_2\big)-V\big(\nabla\bar{h}_2+t_1\nabla\phi_2\big)+V(\nabla\bar{h}_2+\nabla\phi_2)$$

$$\tilde{u}(\phi^1;\phi_2;t_1)=\tilde{u}_1\cdot(\nabla\phi^1+t_1\nabla\phi_2)+(1-t_1)\tilde{u}_2\cdot\nabla\phi_2$$

$$Q(\phi^1;\phi_2;t_1)=Q(\phi_2;t_1)=\frac{1}{2}(1-t_1^2)q_2^{\alpha\beta}\,\partial_\alpha\phi_2\partial_\beta\phi_2$$

The resulting interpolating Hamiltonian is

$$\begin{aligned} H(\Lambda;\phi^1\ldots\phi^i;\phi_{i+1};t_1\ldots t_i)=\int_\Lambda dx\,\Big[\\ V(\phi^1\ldots\phi^i;\phi_{i+1};t_1\ldots t_i)-\tilde{u}(\phi^1\ldots\phi^i;\phi_{i+1};t_1\ldots t_i)+Q(\phi^1\ldots\phi^i;\phi_{i+1};t_1\ldots t_i)\Big] \end{aligned}$$

The first renormalization equation at slice i may be written as

$$\Big\langle\,\frac{d}{dt_i}H(\Lambda;\phi^1\ldots\phi^i;\phi_{i+1};1\ldots 1,t_i)\Big|_{t_i=0}\Big\rangle_{\le i}\equiv 0$$

where $\Big\langle\,\cdot\,\Big\rangle_{\le i}$ is the average over $\phi^{\le i}=\phi^1+\ldots+\phi^i$ with covariance $C^{\le i}=C^1+\ldots+C^i$ and interaction

$$V\Big(\nabla\bar{h}_1+\nabla\phi^{\le i}\Big)-\tilde{u}_1\cdot\nabla\phi^{\le i}$$

The identity should hold for every ϕ_{i+1} and is therefore equivalent to

$$\Big\langle \bigtriangledown V\big(\nabla\bar{h}_1 + \nabla\phi^{\le i}\big) \Big\rangle_{\le i} - \bigtriangledown V(\nabla\bar{h}_{i+1}) - \tilde{u}_1 + \tilde{u}_{i+1} = 0 \qquad \forall x \in \Lambda$$

It fixes $\bar{h}_1$ and $\tilde{u}_1$ relative to $\bar{h}_{i+1}$ and $\tilde{u}_{i+1}$. As $\bar{h}_1$ and $\tilde{u}_1$ have already been fixed relative to $\bar{h}_2$ and $\tilde{u}_2$, $\bar{h}_2$ and $\tilde{u}_2$ relative to $\bar{h}_3$ and $\tilde{u}_3$, etc..., the i'th renormalization equation can be solved by a fixed point theorem for the small functions $\bar{h}_i - \bar{h}_{i+1}$ and $\tilde{u}_i - \tilde{u}_{i+1}$. The renormalization of quadratic terms is useful for book-keeping but does not affect the renormalization group flow of the mean slope $\bar{h}_i$ and slope chemical potential $\tilde{u}_i$. The quadratic counter-terms introduced at slice i are given by

$$\begin{aligned} q^{\alpha\beta}_{i+1}(x) = {} & \Big\langle\, \partial_\alpha V\Big(\nabla\bar{h}_1(x) + \nabla\phi^{\le i}(x)\Big)\,;\, \int_\Lambda dy\, \partial_\beta V\Big(\nabla\bar{h}_1(y) + \nabla\phi^{\le i}(y)\Big)\,\Big\rangle_{\le i} \\ & - \Big\langle\, \partial_\alpha\partial_\beta V\Big(\nabla\bar{h}_1(x) + \nabla\phi^{\le i}(x)\Big)\Big\rangle_{\le i} + \partial_\alpha\partial_\beta V\Big(\nabla\bar{h}_{i+1}(x)\Big) \end{aligned}$$

8 Last momentum slice $n \approx \ln L/\ln M$

At the last slice, indexed by n with $M^n = \mathcal{O}(L)$, the Mayer expansion follows directly the polymer expansion for ϕ_n with C_n. The result of this last Mayer expansion takes the form

$$\int d\mu_{C_n}(\phi_n)\; e^{-H_n(\Lambda;\phi_n) + V(\Lambda;\bar{h}_n)} \sum_{\substack{\mathcal{Y}_1\cdots\mathcal{Y}_{q_3}\\ q_3\ge 0}} \Big(\prod_{k=1}^{q_3} \mathcal{A}^{(3)}(\mathcal{Y}_k;\phi_n)\Big) = e^{-\tilde{F}_n(\Lambda)}$$

so that altogether

$$Z(\Lambda) = e^{-V(\Lambda;\bar{h}_1) - \sum_{i=1}^{n} \tilde{F}_i(\Lambda)}$$

In order that the last Mayer expansion converge uniformly in the thermodynamic limit, $\langle\nabla\phi_n\rangle$ should be small, which gives the final renormalization equation. More precisely we impose

$$\Big\langle \bigtriangledown V\big(\nabla\bar{h}_1 + \nabla\phi^1 + \ldots + \nabla\phi^{n-1} + \nabla\phi_n\big) \Big\rangle - \tilde{u}_1 = \Big\langle \bigtriangledown V\big(\nabla\bar{h} + \nabla\phi\big) \Big\rangle - \tilde{u} = 0 \qquad \forall x$$

which implies $\langle\nabla\phi\rangle = 0$ and that $\bar{h}$ obeys the Wulff equation.

Since $\tilde{h}_i$, $\tilde{u}_i$ for $i = 1, \ldots, n-1$ have already been fixed as functions of $\tilde{h}_{i+1}$, $\tilde{u}_{i+1}$, the final renormalization equation is an equation for $\tilde{h}_n$ and $\tilde{u}_n$ only. Taking profit of the work done in previous steps and of

$$\langle\, |\nabla\phi_n|^2 \,\rangle = \mathcal{O}(M^{-2n}) = \mathcal{O}(L^{-2})$$

it can be worked into

$$\bigtriangledown V(\nabla\bar{h}_n) - \tilde{u}_n = \mathcal{O}(L^{-1})$$

which shows that $\bar{h}_n$ is close to a classical field solution of the variational Wulff problem on a fixed basis.

Convergence of the expansion cannot be discussed in a few lines. Let us comment only on the necessary "domination", which allows to bound some of the products of fields in polymers by the exponential of the interaction, rather than integrate them with the Gaussian measure. this requires $V(\nabla\bar{h} + \nabla\phi) - \tilde{u}\cdot\nabla\phi$ to be bounded from below for every $\bar{h}_i$, $\tilde{u}_i$, whence the convexity hypothesis.

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