(Global and Local) Fluctuations of Phase Space Contraction in Deterministic Stationary Non-equilibrium

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Abstact: We studied numerically the validity of the fluctuation theorem introduced in [1][2] for a 2-dimensional system of particles maintained in a steady shear flow by Maxwell daemon boundary conditions [3]. The theorem was found to hold if one considers the total phase space contraction σ occuring at collisions with both walls: $\sigma = \sigma^{\uparrow} + \sigma^{\downarrow}$. An attempt to extend it to more local quantities $\sigma^{\uparrow}$ and $\sigma^{\downarrow}$, corresponding to the collisions with the top or bottom wall only, gave negative results. The time decay of the correlations in $\sigma^{\uparrow,\downarrow}$ was very slow compared to that of σ.

Keywords: dynamical system, time reversibility, fluctuation theorem, local phase space contraction.

1. Introduction

The microscopic structure of systems through which there is transport of energy or momentum, is a central problem in non-equilibrium statistical mechanics. For gases a partial answer to this question (on the mesoscopic/kinetic level) is provided by the stationary solution of the Boltzmann equation with suitable, *e.g.* Maxwellian, boundary conditions. Going beyond kinetic theory has proven to be very difficult and we still lack a full understanding of stationary non-equilibrium states (SNS) on the microscopic level.

This gap goes beyond that of computational complexity or even of technical difficulty in proving the validity of the formulas derived formally, something already present also in the kinetic theory and in equilibrium statistical mechanics. What is still missing, at the present time, are well defined formal procedure which would, at least in principle, provide answers to questions of physical relevance for SNS of macroscopic systems. Thus, we have no "statistical mechanical formula" for the decay of spatial or time displaced correlation functions in a SNS. In particular, we have no *a priori* formalism for deciding on the slow power law decay of the spatial correlations predicted by computer simulations, fluctuating hydrodynamics and confirmed by experiments [4]. We also have no formula for computing the stationary heat flux through a metal or plasma or the momentum flux

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through a fluid which goes beyond linear response theory. Such a formula should, like the Green-Kubo formula for linear transport, depend only on on the internal Hamiltonian of the system and the impressed macroscopic constraints driving the system, *e.g.* a specified temperature or velocity field on the boundaries of the system.

While the usual modeling of such systems has been via stochastic boundaries [5] there has been much interest recently in the study of SNS of particle systems evolving via entirely deterministic dynamics in the hope that dynamical systems theory will provide new insights into non-equilibrium behavior [6][7]. As is well known the existence of such SNS in finite systems is incompatible with the usual Hamiltonian evolution, believed to accurately describe the dynamics of systems in which quantum effects are unimportant. For such dynamics the only realizable stationary states are those which depend solely on the global constants of motion. In realistic systems these are just the total energy and (under suitable boundary conditions) the total momentum and angular momentum.

Adding external (globally non-gradient) forces to the dynamics, *e.g.* a uniform field in a system with periodic boundary conditions (such as would arise from a changing magnetic flux crossing the surface bounded by a conductor) will typically result in the system gaining energy continuously from the field, excluding the possibility of SNS. This necessitates the use of non phase space volume conserving forces for modeling SNS. Various models of such dynamics have been investigated through computer simulations and heuristic analysis [7], and there have also been some mathematical results motivated by physical considerations of such SNS [2]. There is however, as already mentioned, still a very wide gap between the mathematical and physical results obtained from these models and a full theory of macroscopic SNS.

While it is not clear at present whether and how the dynamical system approach will answer such questions it seems reasonable to explore the behavior of such SNS. This is particularly so for models in which the imposed, model dependent, driving and thermostating terms are confined to the boundaries of the system while the dynamics in the interior of the system remain realistically Hamiltonian. Several such models of shear flow were investigated via computer simulations and some heuristic analysis in [3]. Here we continue our investigation focusing on the behavior of the fluctuation in the phase space volume contraction σ which occurs at the boundaries. σ was found in [3] to be approximately equal, when the size of the system is sufficiently large for it to be in local thermal equilibrium, to the hydrodynamic entropy production inside the system. The same quantity, in a different model of shear flow, was studied in [1] where an interesting relation for its large deviations was experimentally found. This relation is now a rigorous result for large deviations of phase space contraction under suitable conditions on the dynamics [2]. Whether the results hold when the conditions are not satisfied exactly is a question of great relevance. The exploration of this and related quantities is the subject of the present work.

In the next section we describe the model and the check of the Gallavotti-Cohen result [2] for our system. In section 3 we investigate fluctuations and time dependent correlations of the volume contractions $\sigma^{\uparrow}$ and $\sigma^{\downarrow}$ produced by collisions with the top and bottom walls.

2. Numerical simulations: the fluctuation theorem

A: The model

We consider the 2-dimensional shear flow model first introduced in [3] and further studied in [8]: N identical disks of radius r evolve in the interior of the system according to Hamiltonian dynamics with hard core interaction among them while Maxwell deamons at the walls drive the system away from equilibrium. More specifically the particles move on the surface of a cylinder (*i.e.* the system has periodic boundary conditions on the vertical sides) with reflection rules at the top and bottom walls simulating macroscopic moving walls. The rules are as follows: when a particle collides with the upper wall, making an angle ϕ between the positive x-direction and the incoming velocity then the outgoing velocity angle ψ will be given by a (to be specified) reflection rule f:

$$\psi = f(\phi) \tag{2.1}$$

A similar rule applies on the lower wall with the only difference that ψ and ϕ refer to the angle between the incoming and outgoing velocity and the negative x-direction. Since the modulus of the velocity is preserved during a collision the total kinetic energy of the system is a constant of the motion.

We will assume that f is "time reversible", *i.e.* it satisfies $\phi = f(\pi - f(\pi - \phi))$. Observe that if this condition holds the system is time reversible: if one inverts the velocities of all the particle the system traces back its past trajectory. In the numerical simulations we will consider one of the reflection rules introduced in [3], namely:

$$\psi = (\pi + b) - \sqrt{(\pi + b)^2 - \phi(\phi + 2b)} \tag{2.2}$$

where b is used to modulate the intensity of the shear with $b = \infty$ representing elastic collisions (see Fig. 1).

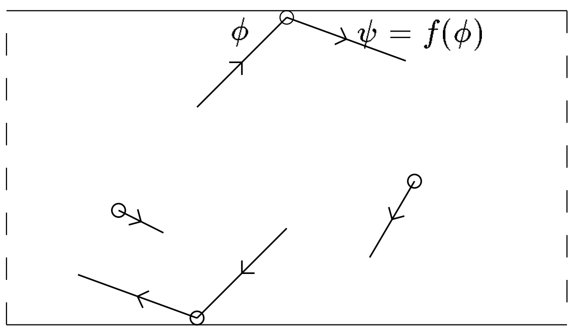

Fig. 1: Schematic representation of the dynamics of the system

During a collision with the walls the Liouville measure is not conserved. In fact at each "reflection" there is a phase space contraction α equal to [3]:

$$\alpha = -\log\left(\frac{\sin\psi}{\sin\phi} f'(\phi)\right) \tag{2.3}$$

It is natural to consider the collisions with the walls as timing events for the system, *i.e.* we can consider the function T that associates to any point X in the phase space of the system, *i.e.* the energy surface Σ, the first time at which a particle collides with a wall and define the map, $S : \Sigma \to \tilde{\Sigma}$, $S(X) = F(\Phi(X, T(X)))$, $\tilde{\Sigma}$ is the set of points at which at least one particle is colliding with the walls where $\Phi(X, t)$ is the point X_t obtained from X via the flow generated by the dynamics and $F : \tilde{\Sigma} \to \tilde{\Sigma}$ is the collision map, *i.e.* the map that change the velocity of the colliding particle according to the reflection rule f.

Given a point X we define

$$\alpha_\tau(X) = \sum_{i=-[\tau/2]}^{[\tau/2]-1} \alpha(S^i(X)) \tag{2.4}$$

and

$$\sigma_\tau(X) = \frac{\alpha_\tau(X)}{\langle \alpha_\tau \rangle_+} \tag{2.5}$$

where $[t]$ is the biggest integer less that t, $\langle \cdot \rangle_+$ refer to the mean with respect to the forward invariant distribution. Call $\pi_\tau(p)$ the distribution function of σ_τ, and let

$$x_\tau(p) = \frac{1}{\langle \alpha_\tau \rangle_+} \log \frac{\pi_\tau(p)}{\pi_\tau(-p)} \tag{2.6}$$

then the chaotic hypothesis of [2] implies that

$$\lim_{\tau\to\infty} x_\tau(p) = p \tag{2.7}$$

To check numerically this relation numerical simulations were carried out on systems of $N = 10, 20, 30$ and 40 particles of radius $R = 1$ in a “square” box of size $L = \sqrt{dN}$, where $d = N/L^2$ is the number density kept constant to $d = 3.4 \times 10^{-2}$, and with the parameter $b = 25$. The initial configuration was chosen randomly in such a way that the energy per particle $e_0 = \frac{1}{2N}\sum_{i=0}^{N} v_i^2 = 0.5$. In the next table we give some of the interesting dynamical quantities associated with the system.

	10	20	30	40
d	3.4×10^{-2}	3.4×10^{-2}	3.4×10^{-2}	
t_0	20.6	30.6	38.1	44.4
ν	2.11	3.19	4.00	4.67
$\langle\alpha\rangle_+$	1.24×10^{-2}	1.68×10^{-2}	2.00×10^{-2}	2.26×10^{-2}
λ_{max}	0.92	0.74	0.63	0.57

Table 1: static and dynamic quantity for the simulation where p.s.c stands for phase space contraction.

Here t_0 is the mean time between successive collisions with the walls for a given particle, ν is the number of binary collisions (*i.e.* collisions between two particles) between two consecutive collisions with the walls for a given particle and $\langle\alpha\rangle_+$ in the mean phase space contraction rate. Finally we give the maximum Lyapunov exponent of the map S. The data, given without error estimates, are intended to give a rough image of the dynamics.

B: The fluctuation law

The check of the fluctuation theorem was done as in [9]. A long trajectory of 5×10^8 collisions with the walls was simulated and the phase space contraction was recorded for every 100 collisions. The main difference with [9] is that we did not attempt to decorrelate the adjacent data segment by leaving out a fixed number of collision between them. In fact in [9] this was possible because the self-correlation of the phase space contraction was decaying rapidly enough to leave out just few collisions. In the present system we find that while the total phase space contraction rate has rapidly decaying correlation the partial ones (to be defined precisely and studied in the next section) have a very slow decay. Hence we would have to discard too many collisions to decorrelate the adjacent data segments and we therefore decided to discard no collision also for the total phase space contraction to have a consistent analysis.

In Fig. 2 we show the graph of $x_\tau(p)$ for $N = 20$ and several value of τ.

As we can see already for $\tau = 400$ the agreement between the theoretical prediction and the experiment is very good.

To better follow the behavior of the fluctuation we have constructed the function $x_\tau(p)$ for $N = 10, 20, 30$ and 40 for several value of τ from 1 to 1000. We have then used a least square fit to fit the experimental data with a law of the form $x_\tau(p) = \chi_\tau p$. We chose a one parameter fit because $x_\tau(0) \equiv 0$. The result are shown in Fig. 3. As can be easily seen the evaluated χ_τ contain 1 within their error-bars starting from quite small value of τ. Moreover, one appears to be the asympotical value of χ_τ.

The analysis of the fluctuation law requires the construction of the whole distribution function $\pi_\tau(p)$. But as τ grows big fluctuations become more and more improbable and it is impossible to construct $x_\tau(p)$ for $\tau > 1000$. To go further with τ we can observe that, starting with $\tau \simeq 5$, the distribution $\pi_\tau(p)$ look very much like a Gaussian. Observe that if we assume that the system is Anosov the central limit theorem implies a Gaussian behavior near the maximum of the distribution but the observed agreement goes, in our opinion, beyond the prediction of the central limit theorem. Nevertheless it must be noted that the distribution can't be Gaussian because it is easy to observe that $|\alpha_\tau(X)|$ is bounded. Fig 4 show the comparison with a Gaussian for $N = 20$ and several values of τ.

Assuming that π_τ is a Gaussian immediately implies that $x_\tau(p)$ is linear in p and, calling C_τ the covariance of σ_τ, *i.e.* $C_\tau = \langle\sigma_\tau^2\rangle - 1$,we have that

$$\chi_\tau = \frac{2}{\langle\alpha_\tau\rangle C_\tau} \tag{2.8}$$

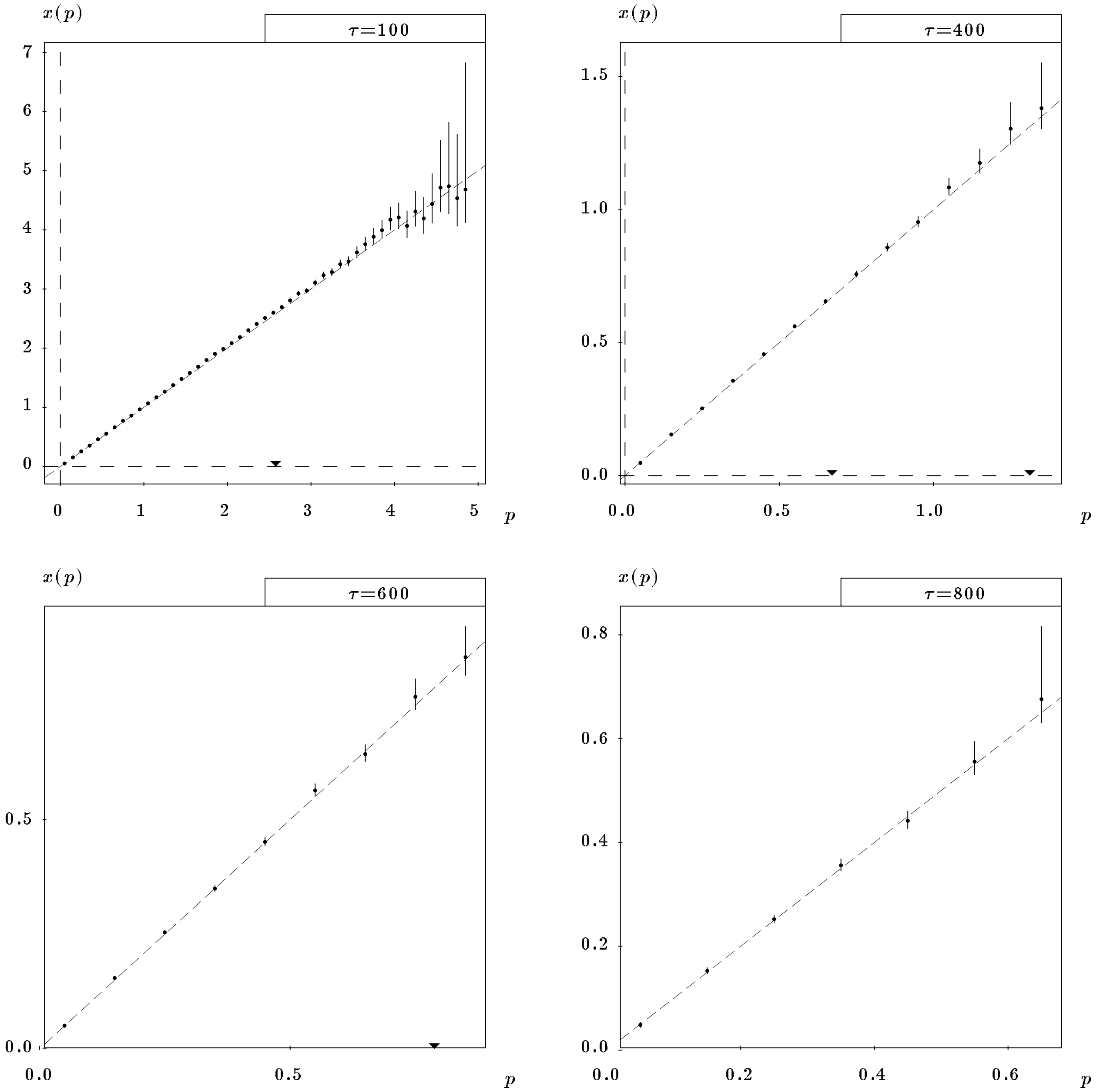

Fig. 2: Graphs of $x_\tau(p)$ for $\tau = 100, 400, 600, 800$ for $N = 20$. The arrows represents the value 1 plus or minus the standard deviation of σ_τ.

The covariance C_τ can be easily computed from the data and permit us to go beyond the limit of $\tau = 100$ met before. As a further check we show the value of χ_τ as computed from the best fit and from the Gaussian hypothesis for small value of τ in Fig. 5.

The agreement is very good and justifies to use of eq.(2.8) for large value of τ. The evaluated behavior of χ_τ for large value of τ from the Gaussian hypothesis is shown in Fig. 6.

In all cases the values of χ_τ are very close to 1 and we can therefore say that the prediction of the fluctuation law appears to be verified by our numerical simulations.

Finally observe that the approach to 1 of χ_τ can be connected to the decay property of the

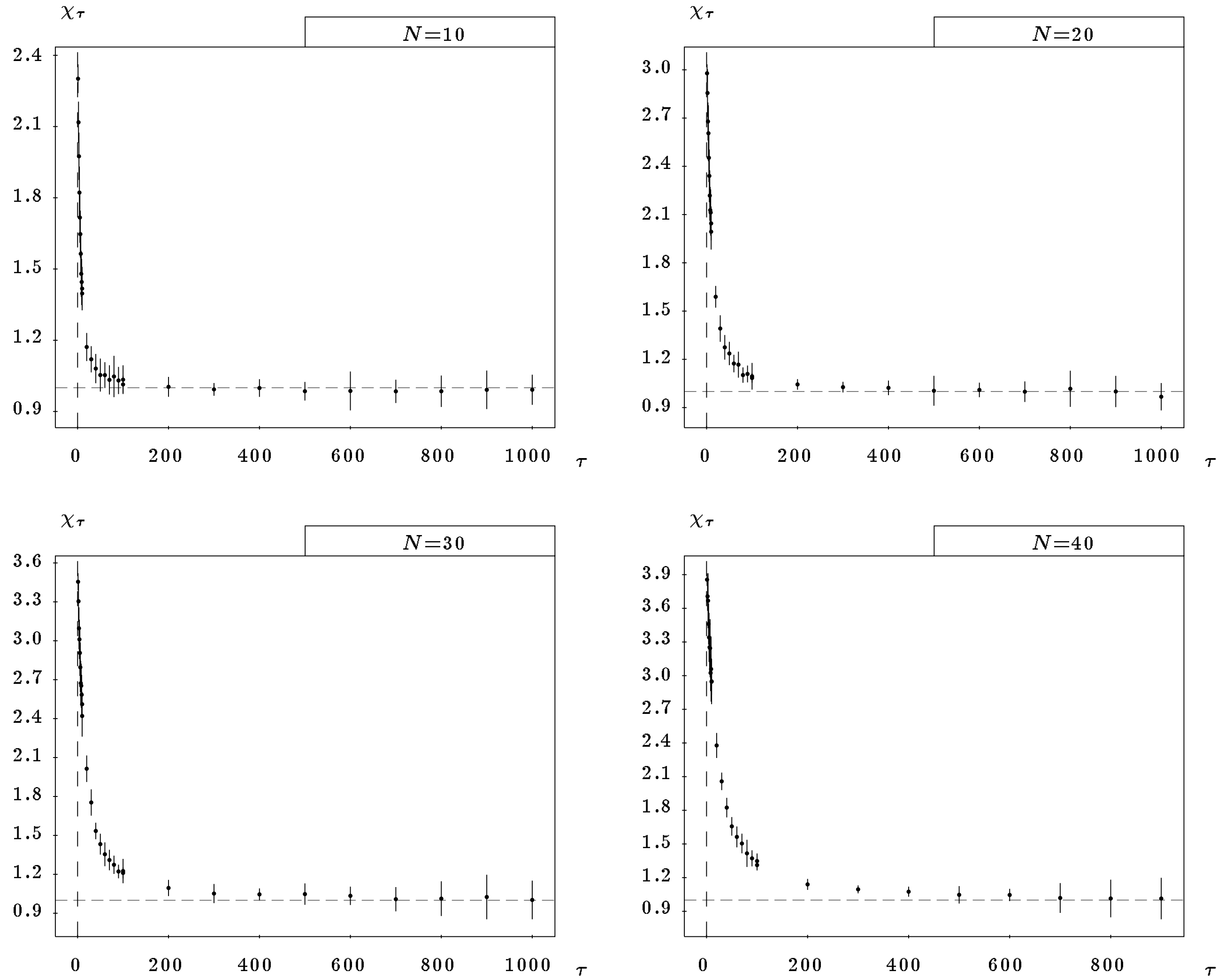

Fig. 3: Behavior of χ_τ as a function of τ for $N = 10, 20, 30$ and 40.

auto-correlation $D(t) = \langle \sigma(S^t(\cdot))\sigma(\cdot)\rangle - 1$. In fact we have:

$$C(\tau) = \frac{2}{\tau}\sum_{t=-\tau}^{\tau} D(t) - \frac{2}{\tau^2}\sum_{t=-\tau}^{\tau} |t| D(t) \tag{2.9}$$

The fast approach of χ_τ to its limit can thus be interpreted as a rapid decay of the correlation for the phase space contraction of the system. We will see that this behavior changes greatly when we consider partial phase space contractions.

3. Local fluctuations

An interesting question is whether one can give a local version of the fluctuation theorem. To

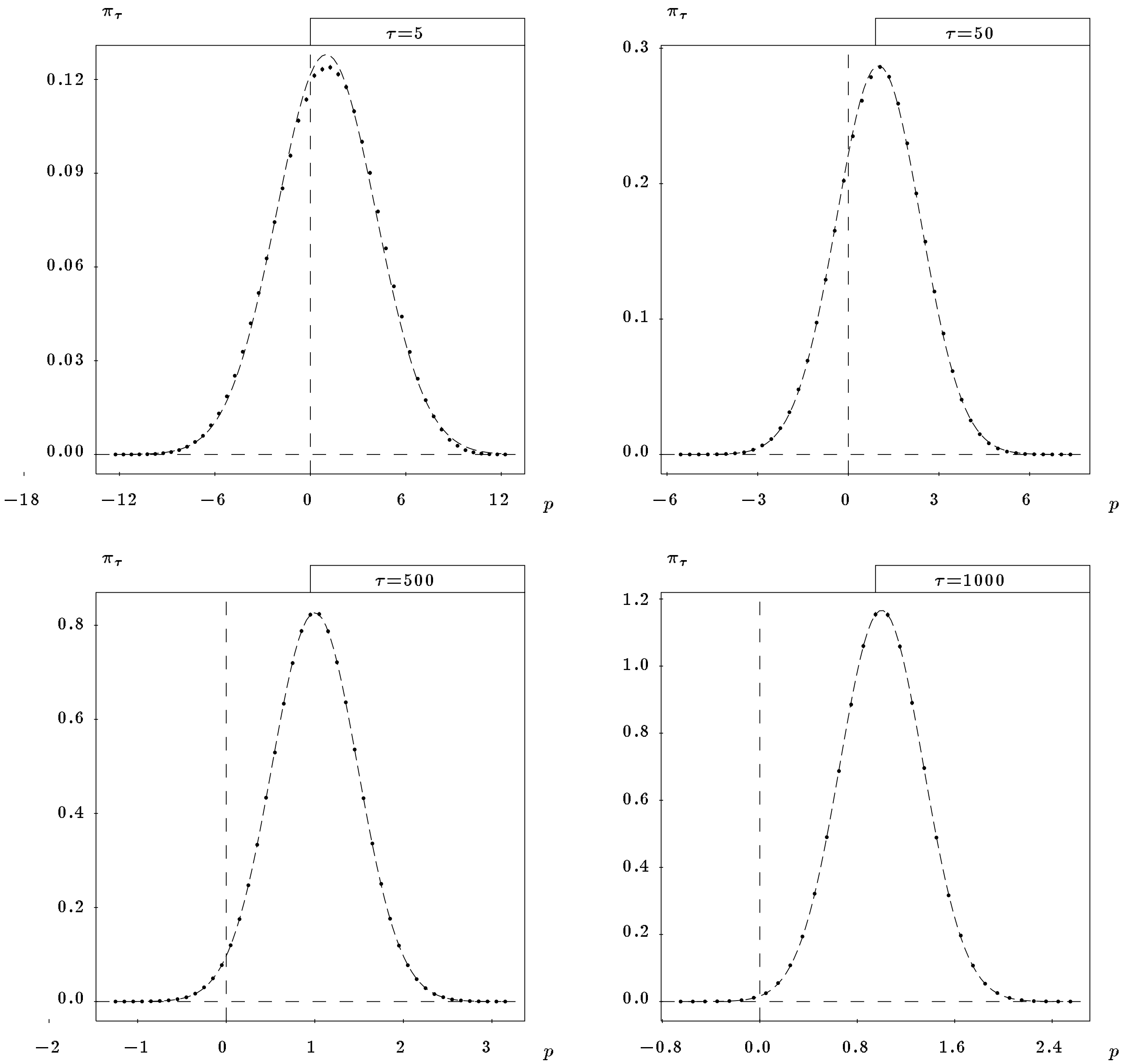

Fig. 4: Comparison between the distribution $\pi_\tau(p)$, for $N = 20$ and several values of τ, and a Gaussian with the same variance and mean. The error-bars are smaller than the dimension of the points.

this extent observe that we can write $\alpha_\tau(X) = \alpha^{\uparrow}_\tau(X) + \alpha^{\downarrow}_\tau(X)$ where

$$\alpha^{\uparrow}_\tau(X) = \sum_{i=-\tau/2}^{\tau/2} \alpha(S^i(X))\chi^{\uparrow}(S^i(X)) \tag{3.1}$$

and $\chi^{\uparrow}(x) = 1$ if X correspond to a collision of a particle with the upper wall and 0 otherwise. An analogous definition holds for $\alpha^{\downarrow}_\tau(X)$.

The generalized version of the fluctuation theorem of [10] gives

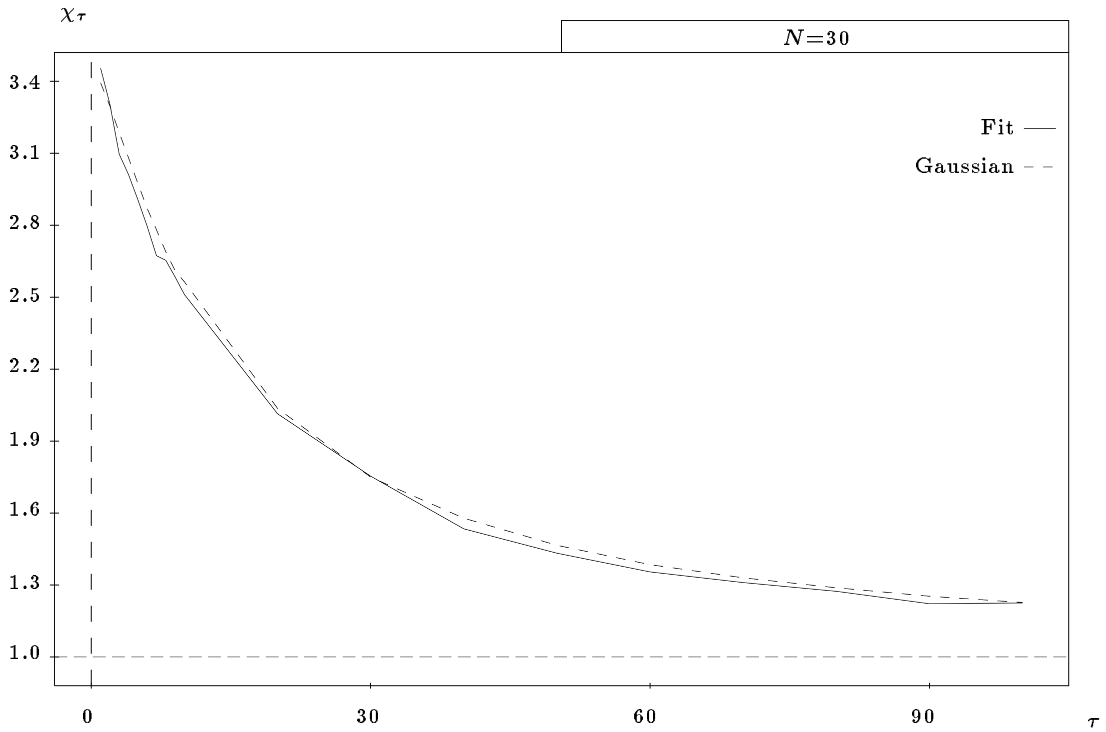

Fig. 5: Comparison between the value of χ_τ computed directly and using the Gaussian hypothesis for $N = 30$. Similar result are found for $N = 10, 20, 40$

$$\lim_{\tau\to\infty} \frac{1}{\langle\alpha_\tau^{\uparrow}\rangle_+} \log \frac{\pi_\tau^{\uparrow\downarrow}(p_1,p_2)}{\pi_\tau^{\uparrow\downarrow}(-p_1,-p_2)} = p_1 + p_2 \tag{3.2}$$

where $\pi_\tau^{\uparrow\downarrow}(p_1,p_2)$ is the joint distribution of $\sigma_\tau^{\uparrow}(X) = \alpha_\tau^{\uparrow}(X)/\langle\alpha_\tau^{\uparrow}\rangle_+$ and $\sigma_\tau^{\downarrow}(X) = \alpha_\tau^{\downarrow}(X)/\langle\alpha_\tau^{\downarrow}\rangle_+$. Due to the symmetry of the problem the two variables $\sigma_\tau^{\uparrow}(X)$ and $\sigma_\tau^{\downarrow}(X)$ can be assumed to be identically distributed[1]. It is easy to see that if we suppose them independent, both of them will separately satisfy the fluctuation theorem:

$$\lim_{\tau\to\infty} x_\tau^{\uparrow}(p) = p \tag{3.3}$$

where

$$x_\tau^{\uparrow}(p) = \frac{1}{\langle\alpha_\tau^{\uparrow}\rangle_+} \log \frac{\pi_\tau^{\uparrow}(p)}{\pi_\tau^{\uparrow}(-p)} \tag{3.4}$$

[1] Although this is well verified numerically it is not evident form a theoretical point of view. It is in fact easy to see that if the particles do not interact, *i.e.* they do not collide, the mean momentum of the center of mass is, generically, not zero and this will probably create asymmetry between the two walls. This phenomenon is probably destroyed by the "mixing" behavior generated by the binary collisions. It must be nevertheless noted that for small systems, like ours, the fluctuations of the center of mass momentum are quite big and its correlations decay slowly

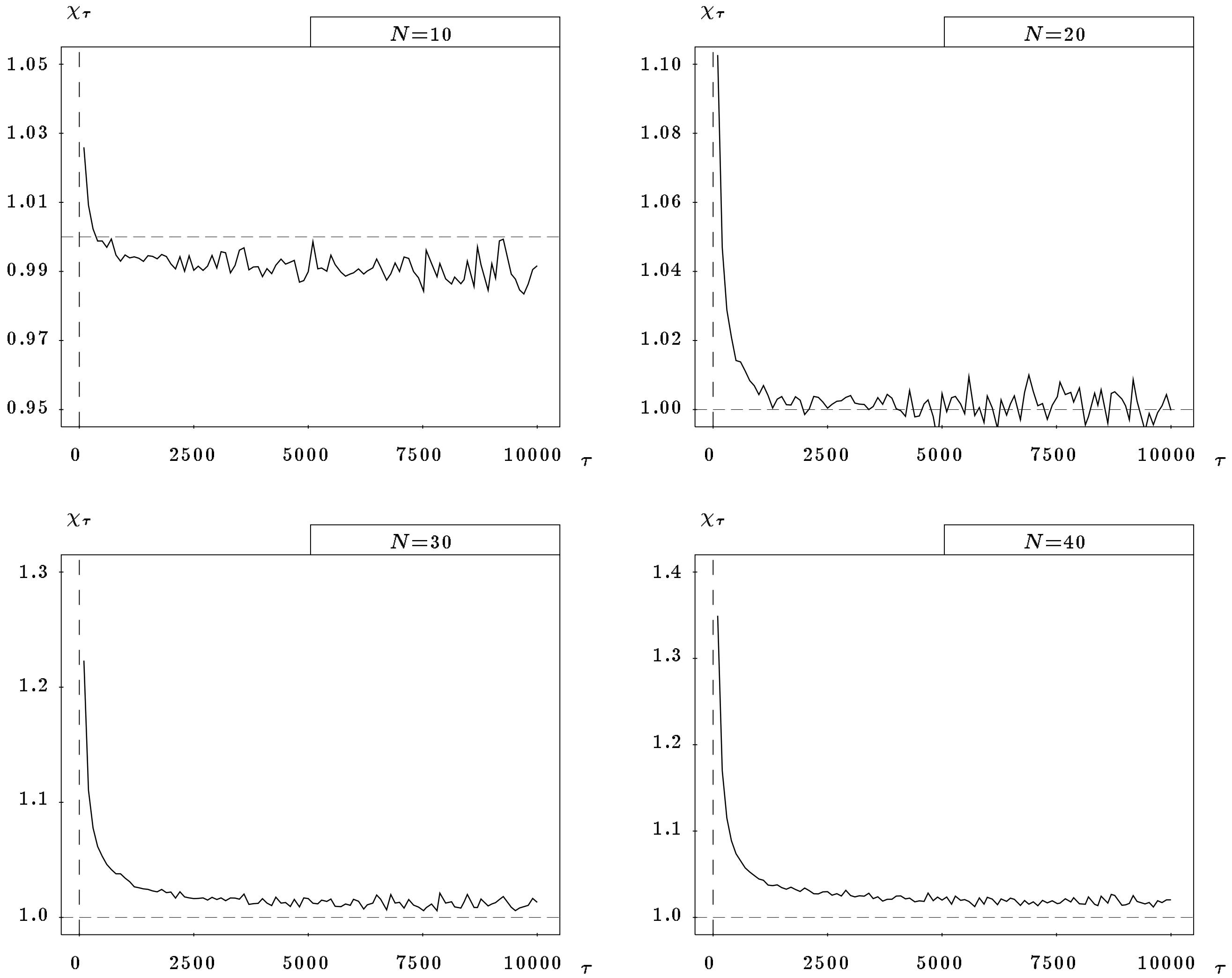

Fig. 6: Value of χ_τ obtained from the Gaussian hypothesis for large τ

This can be checked as we did for $\alpha_\tau(x)$. In Fig. 7 we show the graph of $x_\tau^\uparrow(p)$ for $N = 20$ and several values of τ.

Observe that here we are able to construct the distribution function π_τ up to $\tau = 2000$. This is not surprising considering that $\alpha_\tau^\uparrow(X)$ is, in the mean, the sum of $\tau/2$ non zero terms so that it can be expected to have fluctuation roughly similar to those of $\alpha_{\tau/2}(X)$.

As before we can look at the best fit of $x_\tau^\uparrow(p)$ of the form $x_\tau^\uparrow(p) = \chi_\tau^\uparrow p$. The results are shown in Fig. 8.

It is clear that for $N = 10$ and 20 the fit goes significatively below 1 at a value of τ for which χ_τ is very near its theoretical value of 1. Moreover $\chi_\tau^\uparrow$ does not seem to have reached a limiting value while χ_τ reaches its limiting value of one quite soon.

The cases $N = 30$ and 40 are less clear because we are unable to construct the distribution function for τ large enough to have a clear idea of the limiting value of χ_τ. But it seems reasonable

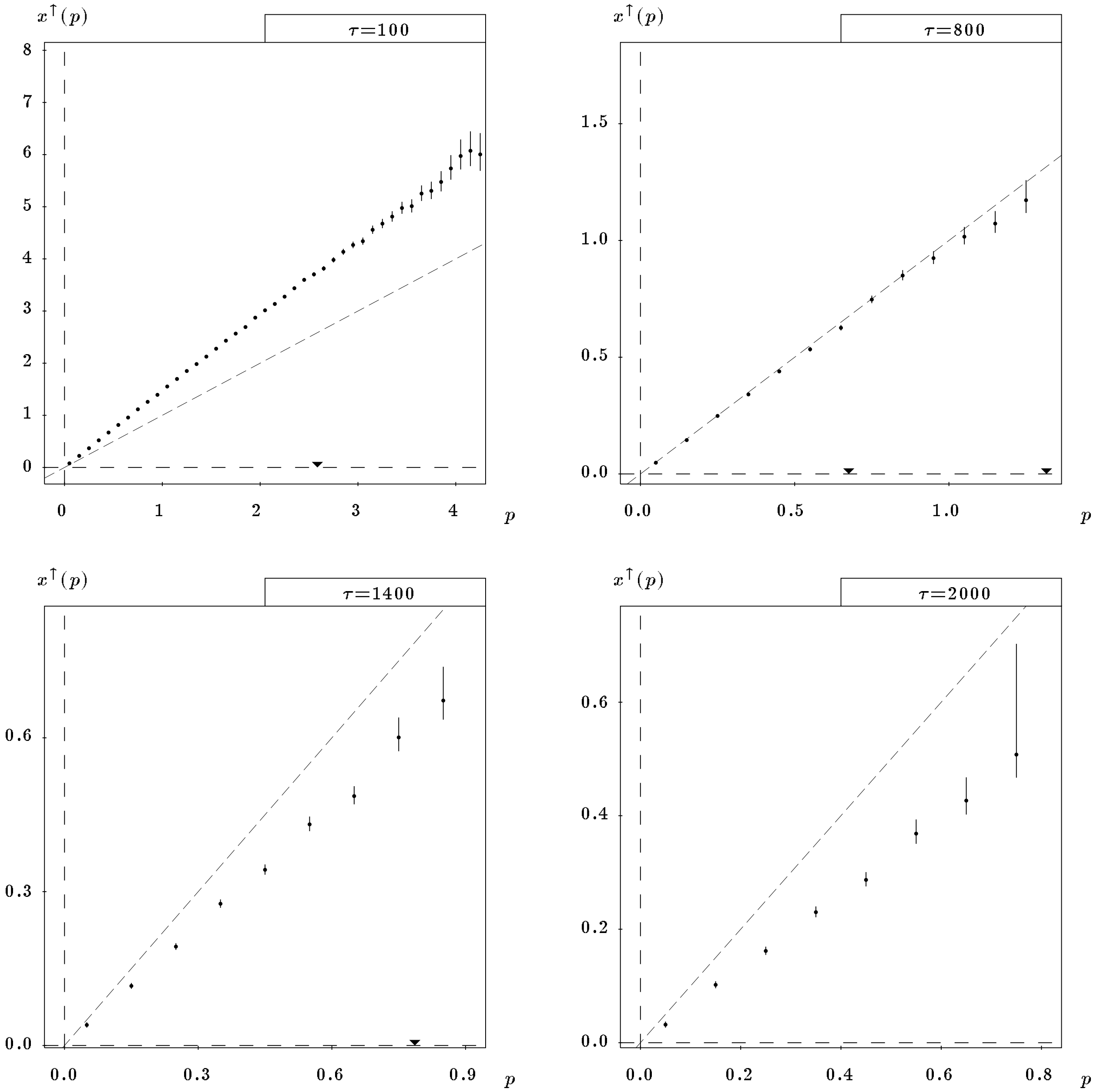

Fig. 7: Graphs of $x^{\uparrow}_{\tau}(p)$ for $\tau = 100, 800, 1400, 2000$ for $N = 20$. The arrows represents the value 1 plus o minus the standard deviation of $\sigma^{\uparrow}_{\tau}$.

to deduce from the graph that also in this case χ_{τ} will became smaller that 1.

Considering the argument at the beginning of this section it is interesting to look at the cross correlation between $\sigma^{\uparrow}_{\tau}$ and $\sigma^{\downarrow}_{\tau}$ given by:

$$C^{\uparrow\downarrow}_{\tau} = \langle \sigma^{\uparrow}_{\tau} \sigma^{\downarrow}_{\tau} \rangle_{+} - 1$$

whose graph is plotted in Fig. 9.

It is interesting to note that, if we disregard the value of $C^{\uparrow\downarrow}_{\tau}$ for small value of τ, we have behavior

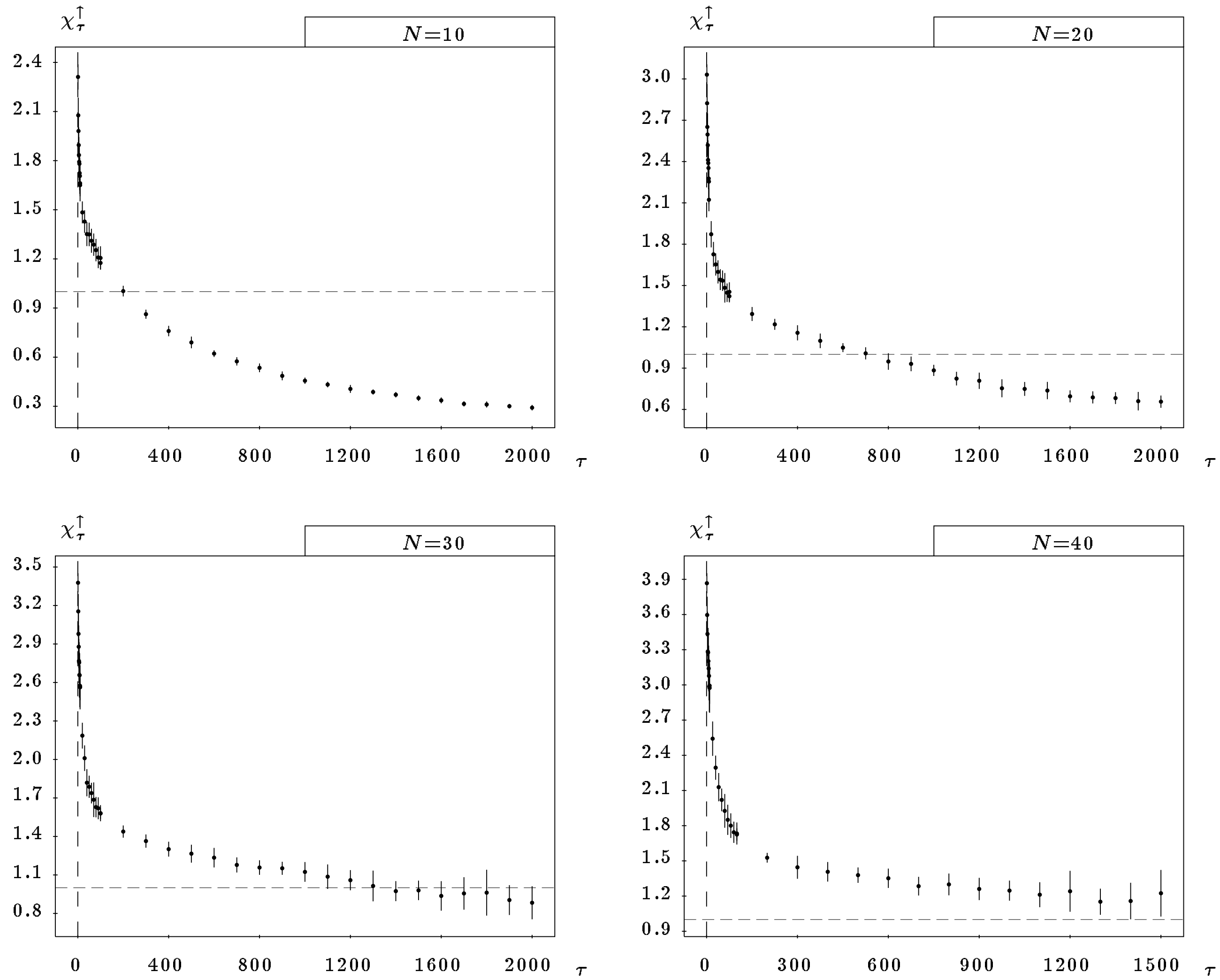

Fig. 8: Behavior of $\chi_\tau^{\uparrow}$ as a function of τ for $N = 10, 20, 30$ and 40.

of $C_\tau^{\uparrow\downarrow}$ is linked to that of χ_τ. In particular we can see that when $C_\tau^{\uparrow\downarrow} = 0$ we have $\chi_\tau = 1$. This is particularly clear in the case $N = 10$ and 20.

This suggests that the higher order correlations of $\sigma^{\uparrow}$ and $\sigma^{\downarrow}$ are small compared to the self-correlation. Moreover the distribution function $\pi_\tau^{\updownarrow}$ is again very well approximated by a Gaussian as can be seen in Fig. 10.

As in the previous section we can assume that the pair $(\sigma^{\uparrow}, \sigma^{\downarrow})$ represents a bidimensial Gaussian variable with mean $(1,1)$ and covariance matrix:

$$\mathcal{C}_\tau = \begin{pmatrix} C_\tau^{\uparrow} & C_\tau^{\uparrow\downarrow} \\ C_\tau^{\uparrow\downarrow} & C_\tau^{\downarrow} \end{pmatrix} \tag{3.5}$$

where

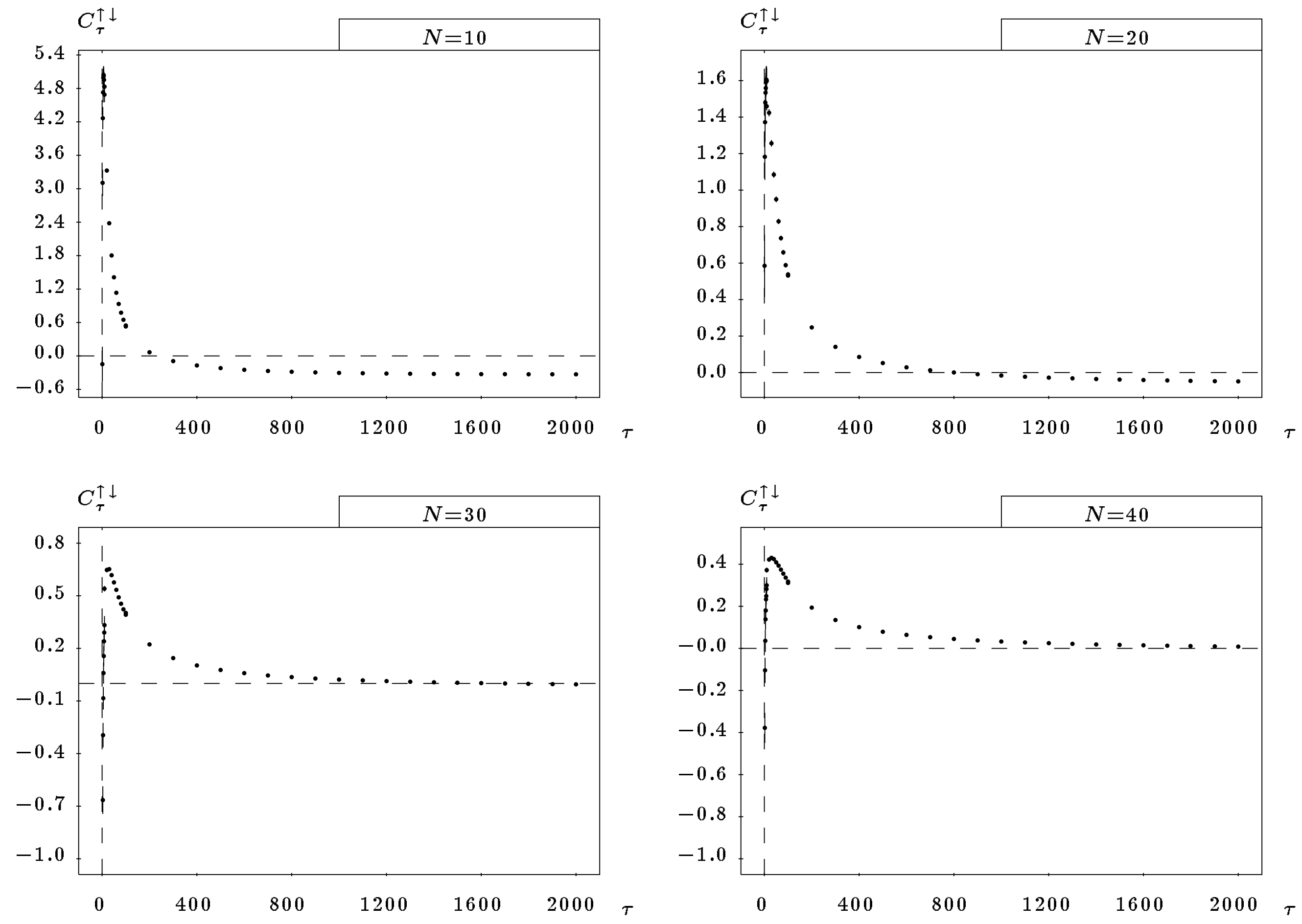

Fig. 9: Behavior of $C_\tau^{\uparrow\downarrow}$ as a function of τ for $N = 10, 20, 30$ and 40.

$$C_\tau^{\uparrow} = \langle(\sigma_\tau^{\uparrow})^2\rangle - 1 \qquad C_\tau^{\downarrow} = \langle(\sigma_\tau^{\downarrow})^2\rangle - 1 \tag{3.6}$$

Form now on we will suppose that $C_\tau^{\uparrow} = C_\tau^{\downarrow}$. This is well verified in the experiment and seems natural from the symmetry properties of the system (see however footnote 1). With the above Gaussian hypothesis we can again try to follow the behavior of $\chi_\tau^{\uparrow}$ for large value of τ using the expression:

$$\chi_\tau^{\uparrow} = \frac{2}{\langle\alpha^{\uparrow}\rangle\tau C_\tau^{\uparrow}} \tag{3.7}$$

Fig. 11 shows the results of this evaluation.

Finally we observe that we can write:

$$C_\tau^{\uparrow\downarrow} = 2C_\tau - C_\tau^{\uparrow} \tag{3.8}$$

Hence, if we assume gaussianity and take τ large enough that we can consider $\chi_\tau = 1$, gives

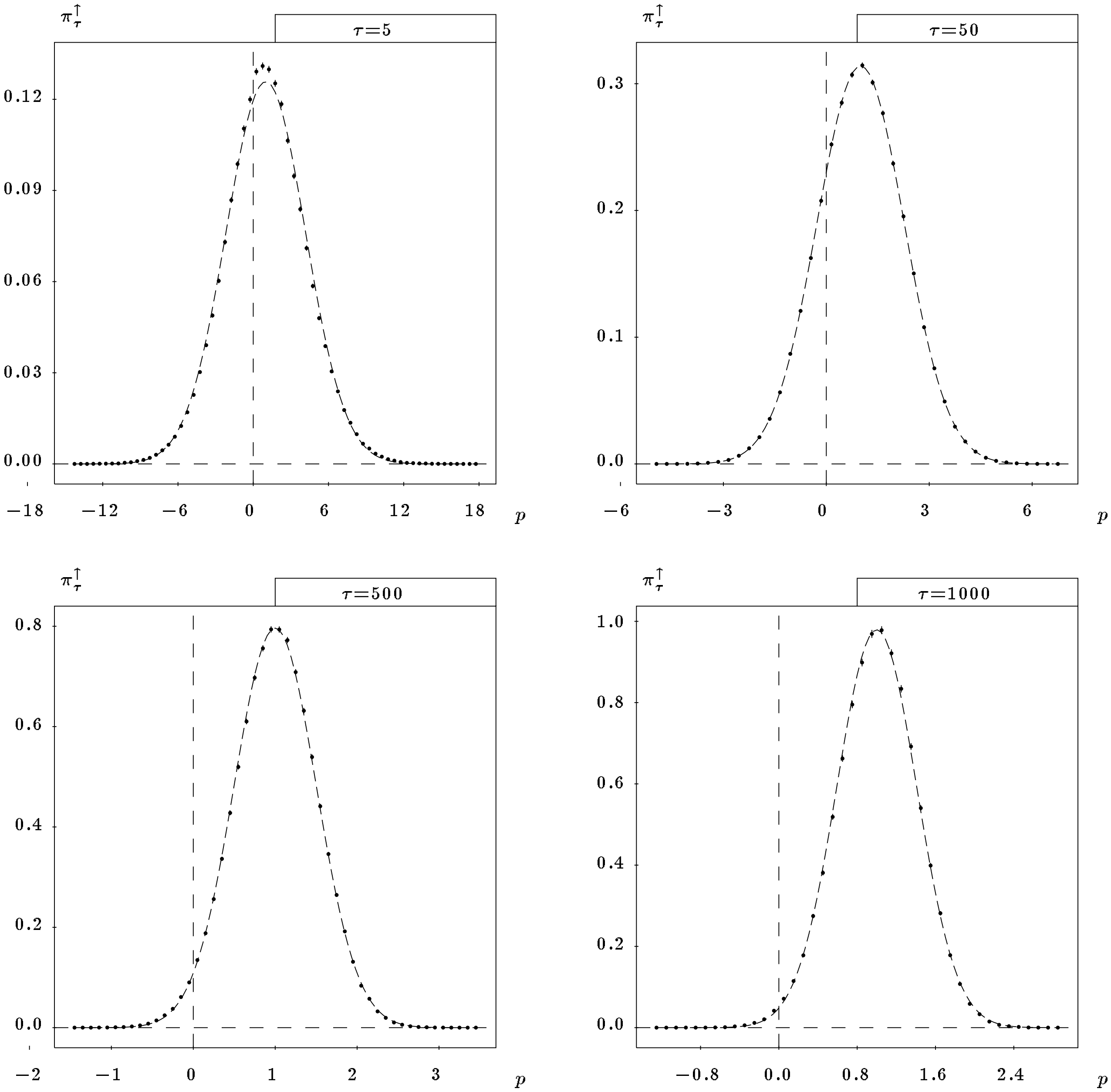

Fig. 10: comparison between the distribution $\pi_\tau^\uparrow(p)$ and a Gaussian with the same variance and mean. The errorbar are smaller than the dimension of the points.

$$K_\tau = \chi_\tau^\uparrow - 1 = \frac{C_\tau^{\uparrow\downarrow}}{C_\tau^\uparrow} \tag{3.9}$$

We can conclude this discussion by observing that, for finite N, the fluctuation law seems to be invalid if we take in consideration only half of the entropy production. Nevertheless it is interesting to note that $x_\tau^{\uparrow,\downarrow}(p)$ still look linear and that the slope $\chi_\tau^{\uparrow,\downarrow}$ seems to reach a finite limit as τ goes to infinity. Moreover this limit is very small for small N and seems to increase with N. It would

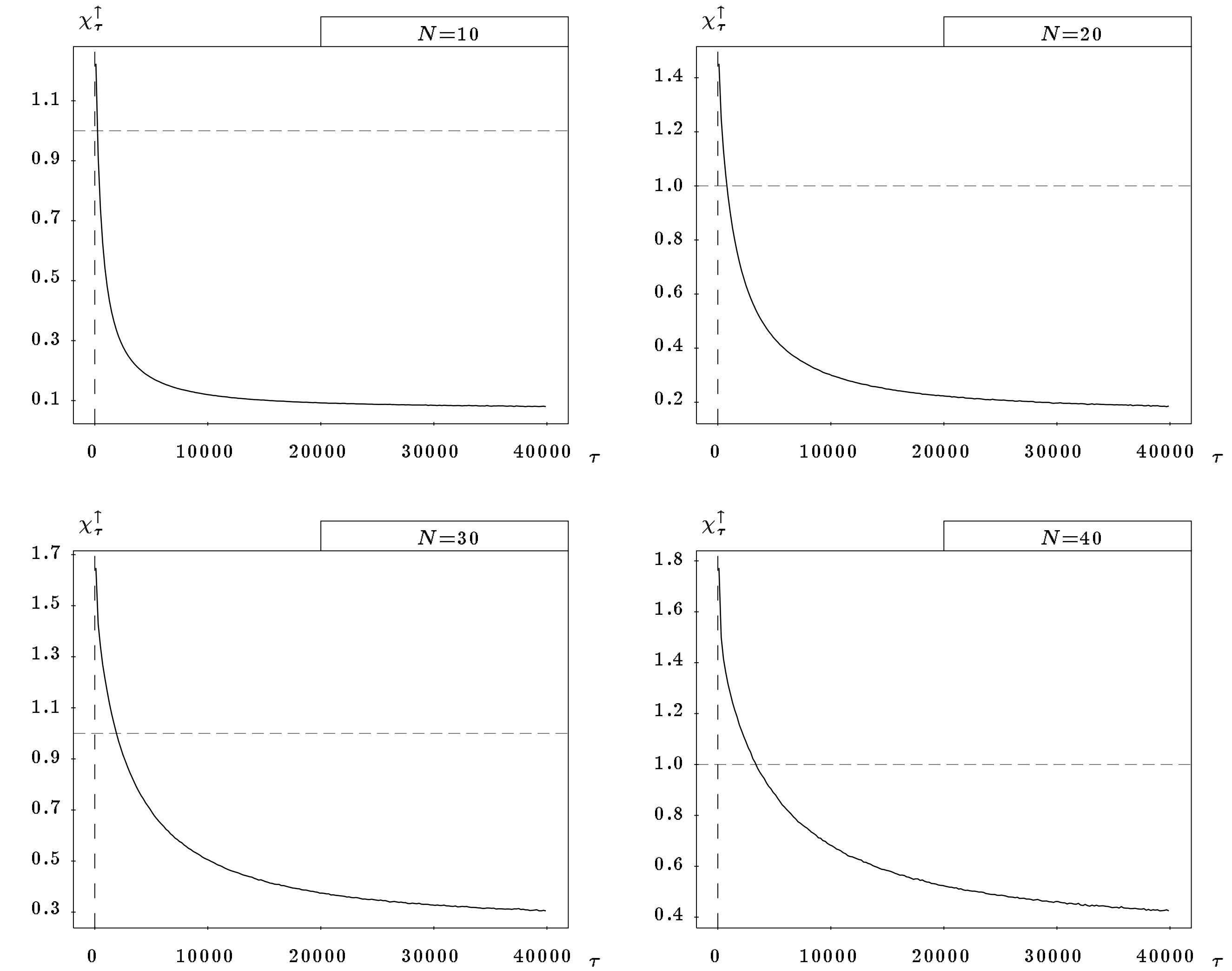

Fig. 11: Behavior of $\chi^{\uparrow}_{\tau}$ as a function of τ for $N = 10, 20, 30, 40$.

be intersting to fit the curve in Fig. 11 to give an estimate of the limiting value of $\chi^{\uparrow}_{\tau}$ as $\tau \to \infty$. We observe that, as for χ_{τ}, we can relate the approach to a limiting value of $\chi^{\uparrow}_{\tau}$ with the decay properties of $D^{\uparrow}(t) = \langle \sigma^{\uparrow}(S^t(\cdot))\sigma^{\uparrow}(\cdot)\rangle - 1$. In fact an equation identical to (2.9) holds for $C^{\uparrow}_{\tau}$ with $D^{\uparrow}(t)$ in the place of $D(t)$. A comparison of Fig. 6 and Fig. 11 immediately shows that the decay of correlations of $\sigma^{\uparrow}$ is much slower that that of σ. If we assume that $D^{\uparrow}(t) \simeq t^{-\beta}$, with $\beta > 0$, we get that:

$$\chi^{\uparrow}_{\tau} = \chi^{\uparrow}_{\infty} + \frac{\chi^{\uparrow\prime}}{\tau} + \frac{\chi^{\uparrow\prime\prime}}{\tau^{\beta-1}}$$

If β is smaller than 2 the third term dominate, in the asymptotic behavior on the second one. One can try to fit β looking at the log-log plot of $\chi^{\uparrow}_{\tau}$ and then using a least square fit to find the other coefficient. Unfortunately the log-log plot does not give a precise answer, so we can only say that β is probably less than 2. This implies that the approach to the limit of $\chi^{\uparrow,\downarrow}_{\tau}$ is very slow and its limiting value change considerably depending which β one uses. At the end we are unable to give a quantitative estimate of $\chi_\infty^{\uparrow,\downarrow}$ and we can just say that it is non zero and increases with N.

4. Conclusion

Our numerical results show that the Fluctuation Theorem, eq.(2.5), is well verified if we consider the total phase space contraction σ of the model described in sec. 1 sub A. For τ small enough we were able to construct the distribution $\pi_\tau(p)$ and check directly the validity of eq. (2.5). For higher τ we used a Gaussian hypothesis to compute the slope χ_τ of $x_\tau(p)$ obtaining again a very good agrrement with the theoretical prediction. We observe that if we interpret the Gaussianity of $\pi_\tau(p)$ as a Central Limit Theorem effect we get that $x_\tau(p)$ is linear but no conclusion con be drawn on the slope χ_τ. The validity of the fluctuation theorem predictions, toghether with the gaussian hypothesis, imply a Green-Kubo formula out of equilibrium as can be seen by combining eq. (2.8) and eq. (2.9) (see also [9]). In this sense we can think that our results depend in part on the fact that we are not very far from equilibrium. Increasing the shear make the experiment much harder becouse the probability of observing negative fluctuations decreases very rapidly with the shear. Moreover we expect that the gaussianity of $\pi_\tau(p)$ will be destroyed. We think, however, that if the shear is not too big, *i.e.* if the attractor can still be considered dense in phase space (see [11]), then the fluctuation theorem will holds (see [12] for a situation in wich $\pi_\tau(p)$ is clearly not Gaussian).

We also checked the validity of the fluctuation theorem for the partial phase space contraction $\sigma^\uparrow$ and $\sigma^\downarrow$. In this situation we see that $\pi^{\uparrow,\downarrow}$ is still well aproximated by a Gaussian so that $x_\tau^\uparrow(p)$ appears to be linear. The slope $\chi_\tau^\uparrow$ however behave in a different way from what one would expect. Two interesting features of this behaviour are:

1) It saturates very slowly. The fact that we considered, in the definition of $\sigma_\tau^{\uparrow,\downarrow}$ a fixed number of collisions τ with both walls clearly create a negative correlation between $\sigma_\tau^\uparrow$ and $\sigma_\tau^\downarrow$. This correlation can be roughly estimated as τ^{-1} so that it probably cannot account for the slow behavior found in sec. 3. That behavior shows a strong correlation between the two wall probably due to the slow decay of fluctuation the density near the walls. We do not have a clear understanding of this phenomenon and we hope to come back on it in a forthcaming work.

2) The limiting value $\chi_\infty^{\uparrow,\downarrow}$ in not easy to estimate but is clearly smaller than 1. Although it seems to increase with the number of particle we do not have enough data to draw any conclusion on its behaviour with N. Nevertheless the existence of a limiting non-zero value suggests the validity of a "local fluctuation theorem" in wich the slope 1 is replaced by some other value. This behavior may have relation to the recent experimental result on the local entropy production (supposed to be the equivalent of a phase space contraction) in a fluid moving in a convective cell [13]. They find there a clear linear behaviour but the slope seems to be different from one. The rusults are, however, still unclear and no real comparison can be made.

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