ON ANGLES WHOSE SQUARED TRIGONOMETRIC FUNCTIONS ARE RATIONAL

by

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Abstract

We consider the rational linear relations between real numbers whose squared trigonometric values are rational, angles we call "geodetic". We construct a convenient basis for the vector space over $\mathbb{Q}$ generated by these angles. Geodetic angles, and rational linear combinations of geodetic angles, appear naturally in Euclidean geometry; for illustration we apply our results to equidecomposability of polyhedra.

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0. Introduction

Many well known geometric objects involve angles that are irrational when measured in degrees or are irrational multiples of π in radian measure. For instance we might mention the dihedral angle α ($\approx 70°31'44''$) of the regular tetrahedron, whose supplement ($\approx 109°28'16''$) is known to chemists as the carbon valence bond angle. A goodly number of these angles have the property that their six trigonometric functions have rational squares. For instance,

$$\sin^2\alpha = \frac{8}{9}, \quad \cos^2\alpha = \frac{1}{9}, \quad \tan^2\alpha = 8, \quad \cot^2\alpha = \frac{1}{8}, \quad \sec^2\alpha = 9, \quad \csc^2\alpha = \frac{9}{8} \qquad 1)$$

and, for the dihedral angle β ($\approx 144°44'8''$) of the cuboctahedron,

$$\sin^2\beta = \frac{2}{3}, \quad \cos^2\beta = \frac{1}{3}, \quad \tan^2\beta = 2, \quad \cot^2\beta = \frac{1}{2}, \quad \sec^2\beta = 3, \quad \csc^2\beta = \frac{3}{2}. \qquad 2)$$

There are many additive relations between angles of this kind; for instance α and β satisfy $\beta + \alpha/2 = \pi$. In this paper we find all relations between these angles.

To be precise, we shall say that θ is a "pure geodetic angle" if any one (and therefore all) of its six squared trigonometric functions is rational (or infinite), and use the term "mixed geodetic angle" to mean a linear combination of pure geodetic angles with rational coefficients. The mixed geodetic angles form a vector space over the rationals and we shall find an explicit basis for this space.

Another aim of this paper is to introduce an elegant notation for these angles which we hope will find general acceptance. Namely, if $0 \le r \le 1$ is rational we define

$$\angle r = \sin^{-1}\sqrt{r}. \qquad 3)$$

(We feel free to write these angles in either degrees or radians.) The well-known particular cases are:

$$\angle 0 = 0° = 0, \ \angle\frac{1}{4} = 30° = \frac{\pi}{6}, \ \angle\frac{1}{2} = 45° = \frac{\pi}{4}, \ \angle\frac{3}{4} = 60° = \frac{\pi}{3}, \ \angle 1 = 90° = \frac{\pi}{2}. \quad 4)$$

We extend this notation by writing, for all integers n,

$$\angle n = 90n° = \frac{n\pi}{2}, \quad \angle(n + r) = \angle n + \angle r. \qquad 5)$$

Our basis contains certain angles $\langle p\rangle_d$ for prime p and square-free d. (Some exercise in the notation is provided in Tables 1 and 2, which show the first few elements in the basis.) $\langle p\rangle_d$ is defined just when $-d$ is congruent to a square modulo p and is found as follows. Express $4p^s$ as $a^2 + db^2$ for the smallest possible positive s. Then

$$\langle p\rangle_d = \frac{1}{s}\angle\frac{db^2}{4p^s} = \frac{1}{s}\sin^{-1}\sqrt{\frac{db^2}{4p^s}} = \frac{1}{s}\cos^{-1}\sqrt{\frac{a^2}{4p^s}} = \frac{1}{s}\tan^{-1}\sqrt{\frac{db^2}{a^2}}. \qquad 6)$$

The expression is unique except when $d = 1$ or 3, when we make it so by demanding that b be even (if $d = 3$) or divisible by four (if $d = 1$). Our main result is then

Theorem 1. Every pure geodetic angle is uniquely expressible as a rational multiple of π plus an integral linear combination of the angles $\langle p\rangle_d$. So the angles $\langle p\rangle_d$, supplemented by π (or $\angle 1$ or $1°$), form a basis for the space of mixed geodetic angles.

It is easy to find the representation of any pure geodetic angle θ in terms of the basis.

Theorem 2. If $\tan\theta = \frac{b}{a}\sqrt{d}$ for integers a, b, d, with d square-free and with a and b relatively prime, and if the prime factorization of $a^2 + db^2$ is $p_1p_2\cdots p_n$ (including multiplicity), then we have

$$\theta = t\pi \pm \langle p_1\rangle_d \pm \langle p_2\rangle_d \pm \cdots \pm \langle p_n\rangle_d \qquad 7)$$

for some rational t. (In fact the denominator of t will be a divisor of the class number of $\mathbb{Q}(\sqrt{-d})$.)

For example, for $\tan\theta = \frac{5}{4}\sqrt{3}$ we find $a^2 + db^2 = 16 + 75 = 91 = 7\cdot 13$ and indeed $\theta = n\pi - \langle 7\rangle_3 - \langle 13\rangle_3$.

Our results have many applications but we shall only mention one outstanding one here. In 1901 Dehn [Deh] solved Hilbert's 3^{rd} problem by giving a necessary condition for the mutual equidecomposability of polyhedra in terms of their dihedral angles. In 1965, Sydler proved [Syd] that Dehn's criterion was also sufficient. For polyhedra with geodetic dihedral angles our Theorem 1 makes the Dehn-Sydler criterion effective. At the end of this paper we shall apply our theory to the non-snub Archimedean polyhedra (whose dihedral angles are all geodetic.)

1. Angles with polyquadratic tangents and the Splitting Theorem.

The addition formula for tangents enables us to show that the tangent of any sum of pure geodetic angles is a "polyquadratic number", that is a number of the form $\sqrt{a}+\sqrt{b}+\sqrt{c}+\cdots$, with $a, b, c\cdots$ rational. For instance, if $\tan\alpha = \sqrt{2}/2$ and $\tan\beta = \sqrt{3}/3$, then

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sqrt{2}/2+\sqrt{3}/3}{1-\sqrt{2}\sqrt{3}/6} = \frac{4}{5}\sqrt{2}+\frac{3}{5}\sqrt{3};\\ \tan(\alpha-\beta) &= \frac{4}{5}\sqrt{2}-\frac{3}{5}\sqrt{3}.\end{aligned} \qquad 8)$$

We now suppose the sum of a number of pure geodetic angles is an integral multiple of π; let us say $\alpha_1 + \alpha_2 + \cdots + \beta_1 + \beta_2 + \cdots = m\pi$, where we have chosen the notation so that the tangents of $\alpha_1, \alpha_2, \cdots$ are in $\mathbb{Q}(\sqrt{d_1}, \cdots, \sqrt{d_n})$ and those of $\beta_1, \beta_2, \cdots$ are in $\mathbb{Q}(\sqrt{d_1}, \cdots, \sqrt{d_n})\sqrt{d}$, where $\sqrt{d} \notin \mathbb{Q}(\sqrt{d_1}, \cdots, \sqrt{d_n})$. Then $\tan(\alpha_1 + \alpha_2 + \cdots + \beta_1 + \beta_2 + \cdots)$ and $\tan(\alpha_1 + \alpha_2 + \cdots - \beta_1 - \beta_2 - \cdots)$ will be of the form $a + b\sqrt{d}$ and $a - b\sqrt{d}$ where $a, b \in \mathbb{Q}(\sqrt{d_1}, \cdots, \sqrt{d_n})$. But by assumption $a + b\sqrt{d} = 0$, so $a = b = 0$, from which it follows that $\alpha_1 + \alpha_2 + \cdots - \beta_1 - \beta_2 - \cdots$ is also an integral multiple of π. Adding and subtracting we deduce that the two subsums $\alpha_1 + \alpha_2 + \cdots$ and $\beta_1 + \beta_2 + \cdots$ are integral multiples of $\pi/2$. Repeating this argument we obtain

Theorem 3 (The Splitting Theorem). If the value of a rational linear combination of pure geodetic angles is a rational multiple of π then so is the value of its restriction to those angles whose tangents are rational multiples of any given square root.

Sketch of proof. We exhibit the argument in the instance that

$$\sum_i A_i\alpha_i + \sum_j B_j\beta_j + \sum_k C_k\gamma_k + \sum_\ell D_\ell\delta_\ell = E\pi, \tag{9}$$

where the tangents of the α's, β's, γ's and δ's are rational multiples of $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ respectively and the A's, B's, C's, D's and E are rational numbers. By clearing denominators, we may suppose that the A's, B's, C's, D's and E are all integers. Applying our argument with $d = 3$ we see first that $2(\sum_i A_i\alpha_i + \sum_j B_j\beta_j)$ and $2(\sum_k C_k\gamma_k + \sum_\ell D_\ell\delta_\ell)$ are integral multiples of π. Applying it again with $d = 2$ shows that $4\sum_i A_i\alpha_i$, $4\sum_j B_j\beta_j$, $4\sum_k C_k\gamma_k$ and $4\sum_\ell D_\ell\delta_\ell$ are integral multiples of π as well. ∎

We remark that the same method can be used to show that any angle whose tangent is polyquadratic is a mixed geodetic angle. If α is an angle whose tangent is polyquadratic, we express 2α as the sum of two angles each of whose tangents are simpler polyquadratic numbers; repeating the process we obtain α as a finite linear combination of pure geodetic angles. For example, suppose $\tan\alpha = \sqrt{6}+\sqrt{3}+\sqrt{2}+1$. Let α' be an angle with $\tan\alpha' = -\sqrt{6} + \sqrt{3} - \sqrt{2} + 1$. $\tan(\alpha)$ and $\tan(\alpha')$ are conjugates in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ under the operation $\sqrt{2} \to -\sqrt{2}$. Let $\gamma = \alpha + \alpha'$ and $\delta = \alpha - \alpha'$. By the addition formula for tangents,

$$\begin{aligned} \tan(\gamma) &= \frac{\tan(\alpha) + \tan(\alpha')}{1 - \tan(\alpha)\tan(\alpha')} = \frac{2 + 2\sqrt{3}}{5 + 2\sqrt{3}} = \frac{-2}{13} + \frac{6}{13}\sqrt{3}; \\ \tan(\delta) &= \frac{\tan(\alpha) - \tan(\alpha')}{1 + \tan(\alpha)\tan(\alpha')} = \frac{-2\sqrt{2}(1 + \sqrt{3})}{3 + 2\sqrt{3}} = \frac{2}{3}\sqrt{6} - 2\sqrt{2}. \end{aligned} \tag{10}$$

Thus $2\alpha = \gamma + \delta$ is the sum of two angles, each of whose tangents is the sum of two square roots. By a similar reduction 2γ and 2δ are each the sum of two angles whose tangents are a single square root, i.e. pure geodetic angles, so 4α is the sum of four pure geodetic angles. Specifically, $4\alpha = \angle(1 + \frac{441}{457}) + \angle\frac{432}{457} + \angle\frac{96}{457} + \angle\frac{2592}{4113}$.

2. Størmer theory and its generalization.

The Splitting Theorem reduces the study of the rational linear relationships between angles of the form $\tan^{-1}(\frac{b}{a}\sqrt{d})$ to those with a fixed d. These angles are the arguments of algebraic integers $a + b\sqrt{-d}$ and their theory is essentially the factorization theory of numbers in $\mathcal{O}_d$, the ring of algebraic integers of $\mathbb{Q}(\sqrt{-d})$. The method was first used by C. Størmer [Stø] (in the case $d = 1$) who found all the linear relations between the arctangents of rational numbers using the unique factorization of Gaussian integers. (See also [Con].) We recall Størmer's analysis of the case $d = 1$ and then generalize it to arbitrary d, which will prove Theorems 1 and 2.

It is known that the Gaussian integers have unique factorization up to multiplication by the 4 units: $1,\ -1,\ i,\ -i$. It is also known how each rational prime p factorizes in the Gaussian integers. Namely: 1) if $p \equiv -1 \pmod 4$ then p remains prime; 2) if $p \equiv +1 \pmod 4$, $p = a^2 + b^2$ is the product of the distinct Gaussian primes $a + ib$ and $a - ib$ (for uniqueness we choose a odd, b even, both positive); and 3) $2 = -i(1+i)^2$ "ramifies", that is to say it is (a unit times) the square of a Gaussian prime.

Now let $\kappa = \pi_1\pi_2\pi_3\cdots$ be the prime factorization of a Gaussian integer κ. Then plainly $\arg(\kappa) \equiv \arg(\pi_1)+\arg(\pi_2)+\arg(\pi_3)+\cdots \pmod{2\pi}$. So the arguments of Gaussian primes (together with π) span the subspace of mixed geodetic angles generated by the pure geodetic angles with rational tangent. However,

1) If $p = 4k - 1,\ \arg(p) = 0$ and can be ignored
2) If $p = 4k + 1 = a^2 + b^2$, then $\arg(a + bi) + \arg(a - bi) = 0$. We define $\langle p\rangle_1 = \arg(a + bi) = -\arg(a - bi)$.
3) $\arg(1 + i) = \pi/4$.
4) The arguments of the units are multiples of $\pi/2$, and so are multiples of $\arg(1 + i)$.

Thus the argument of any Gaussian integer is an integral linear combination of $\pi/4$ and the angles $\langle p\rangle_1$, with $p \equiv 1 \pmod 4$.

It is also easy to see that the numbers π and the $\langle p\rangle_1$ are rationally independent. Otherwise some integral linear combination of them would be an integral multiple of π. But suppose for instance that $2\langle p_1\rangle_1 - 3\langle p_2\rangle_1 + 5\langle p_3\rangle_1 = 0$. The left hand side is the argument of $\pi_1^2\bar{\pi}_2^3\pi_3^5$ which must therefore be a real number and hence equal to its conjugate $\bar{\pi}_1^2\pi_2^3\bar{\pi}_3^5$. But this contradicts unique factorization of Gaussian integers.

The analogue of the Størmer theory for the general case is complicated by the fact that elements of $\mathcal{O}_d$ may not have unique factorization. However the ideals do. Instead of assigning arguments to numbers, we simply assign an angle to each ideal

$\mathbb{I}$ by the rule

$$\arg(\mathbb{I}) = \begin{cases} \arg(\kappa) & \text{if } \mathbb{I} = (\kappa) \text{ is principal;} \\ \frac{1}{s}\arg(\mathbb{I}^s) & \text{if } \mathbb{I} \text{ is not principal,} \end{cases} \tag{11}$$

where s is the smallest exponent for which $\mathbb{I}^s$ is a principal ideal, and (κ) denotes the principal ideal generated by κ. Recall that for every d the ideal class group is finite, so such an s exists for every ideal, and s divides the class number of $\mathcal{O}_d$. Since the generator of a principal ideal is defined up to multiplication by a unit, the argument of an ideal must be understood to be modulo the argument of a unit divided by the class number of $\mathcal{O}_d$.

It is not hard to check that, for general ideals $\mathbb{I}$ and $\mathbb{J}$,

$$\arg(\mathbb{I}\mathbb{J}) = \arg(\mathbb{I}) + \arg(\mathbb{J}), \tag{12}$$

modulo the ambiguity, by a rational multiple of π, in the definition of arg. Thus the argument of any ideal (and in particular the principal ideal generated by any algebraic integer) is an integral linear combination of the arguments of the prime ideals. What remains is to determine the nontrivial arguments of prime ideals. As in the case $d = 1$, there will be one such angle for each rational prime p for which (p) splits as the product of distinct ideals.

We illustrate the procedure by working in $\mathcal{O}_5$, for which the ideal factorizations of the first few rational primes are:

$$\begin{aligned}
(2) =& (2, 1+\sqrt{-5})^2 \\
(3) =& (3, 1+\sqrt{-5})(3, 1-\sqrt{-5}) \\
(5) =& (\sqrt{-5})^2 \\
(7) =& (7, 3+\sqrt{-5})(7, 3-\sqrt{-5}) \\
(11) =& (11) \\
(13) =& (13) \\
(17) =& (17) \\
(19) =& (19) \\
(23) =& (23, 22+3\sqrt{-5})(23, 22-3\sqrt{-5}) \\
(29) =& (3+2\sqrt{-5})(3-2\sqrt{-5}).
\end{aligned} \tag{13}$$

Here (x, y) denotes the ideal generated by x and y. The reader will see that (2) ramifies as the square of a non-principal ideal and (5) as the square of a principal ideal, (3), (7) and (23) split into products of non-principal ideals, (29) splits as the product of distinct principal ideals, while (11), (13), (17) and (19) remain prime.

As in the Størmer case the principal ideals generated by rational primes that remain prime have argument zero and can be ignored. We also ignore those that ramify, since their angles will be rational multiples of π. Otherwise we define $\langle p\rangle_d$ to be the argument of one of the two ideal factors of (p), making it unique by requiring $0 < \langle p\rangle_d < \pi/2$ if the factors of (p) are principal and $0 < \langle p\rangle_d < \pi/4$ if not.

To illustrate this we determine $\langle 3\rangle_5$. The ideal factors of (3) are non-principal so we square them:

$$\begin{aligned}(3,1+\sqrt{-5})^2 &= (3^2, 3(1+\sqrt{-5}), (1+\sqrt{-5})^2)\\ &= (9, 3+3\sqrt{-5}, -4+2\sqrt{-5}),\end{aligned} \tag{14}$$

which reduces to $(2-\sqrt{-5})$. Similarly, $(3,1-\sqrt{-5})^2 = (2+\sqrt{-5})$. So $\langle 3\rangle_5 = \frac{1}{2}\arg(2+\sqrt{-5}) = \frac{1}{2}\tan^{-1}(\frac{1}{2}\sqrt{5}) = \frac{1}{2}\angle\frac{5}{9}$.

In the general case (d an arbitrary square-free positive integer) we assign an angle $\langle p\rangle_d$ to every rational prime p for which (p) splits as the product of two distinct prime ideals $\mathbb{I}$ and $\mathbb{J}$. Let s be the smallest integer for which $\mathbb{I}^s$ (and therefore $\mathbb{J}^s$) is principal. Recall that the elements of $\mathcal{O}_d$ are of the form $\frac{a}{2}+\frac{b}{2}\sqrt{-d}$, where a and b are rational integers. If $d \neq 3 \pmod 4$, then a and b must be even; if $d = 3$ (mod 4) then a and b are either both even or both odd. We can therefore write

$$\mathbb{I}^s = \left(\frac{a}{2}+\frac{b}{2}\sqrt{-d}\right), \quad \mathbb{J}^s = \left(\frac{a}{2}-\frac{b}{2}\sqrt{-d}\right), \tag{15}$$

where we can distinguish between $\mathbb{I}$ and $\mathbb{J}$ by supposing that a and b are positive. We take $\langle p\rangle_d = \frac{1}{s}\tan^{-1}(\frac{b}{a}\sqrt{d}) = \frac{1}{s}\angle\frac{b^2d}{a^2+b^2d}$.

The above defines $\langle p\rangle_d$ uniquely for all d other than 1 and 3, because then the only units are ± 1, so that the only generators of $\mathbb{I}^s$ and $\mathbb{J}^s$ are the four numbers $\pm\frac{a}{2}\pm\frac{b}{2}\sqrt{d}$. When $d=1$ we have the additional unit i which effectively allows us to interchange a and b: we then achieve uniqueness by demanding that the generators of $\mathbb{I}$ be $a+bi$ with a and b positive integers with b even. In the case $d=3$ the field has six units and the corresponding condition is that the generators of $\mathbb{I}$ should have the form $a+b\sqrt{-3}$ where a and b are positive integers.

Theorem 4. For fixed square-free d, consider the subspace of mixed geodetic angles generated by arctangents of rational multiples of $\sqrt{d}$. This subspace is spanned by the nonzero angles $\langle p\rangle_d$ and π, where p ranges over the rational primes for which (p) splits as a product of distinct ideals in $\mathcal{O}_d$.

Proof. The proof is essentially that of the Størmer decomposition, only substituting the arguments of ideals for the arguments of algebraic integers. ■

Combining Theorem 4 with the Splitting Theorem, we obtain Theorem 1.

Proof of Theorem 2. Recall that, if $\mathbb{I}$ is any ideal (principal or not) in $\mathcal{O}_d$, then $\mathbb{I}\bar{\mathbb{I}}$ is a principal ideal with a positive integer generator that we call the norm of $\mathbb{I}$, and that norms are multiplicative. From this it follows that the prime ideals are precisely the factors of (p), with p ranging over the rational primes, and that every prime ideal that is not generated by a rational prime has rational prime norm.

Now suppose that $\tan(\theta) = \frac{b}{a}\sqrt{d}$ with d square-free and with a and b relatively prime. Consider the factorization of the principal ideal $\mathbb{I} = (a+b\sqrt{-d})$. If $\mathbb{I} = \pi_1\pi_2\cdots\pi_n$, where each ideal π_i is prime, then none of the π_i's are generated by rational primes, insofar as a and b are relatively prime. Thus each π_i satisfies $\pi_i\bar{\pi}_i = (p_i)$ for some rational prime p_i. We then have

$$(p_1)(p_2)\cdots(p_n) = (a^2 + b^2 d) = \mathbb{I}\bar{\mathbb{I}} = \pi_1\bar{\pi}_1\cdots\pi_n\bar{\pi}_n. \qquad 16)$$

So, up to permutation of the indices $1,\ldots,n$, we must have $\pi_i\bar{\pi}_i = (p_i)$, and so the argument of π_i is $\pm\langle p_i\rangle_d$. But $\theta = \arg(a + b\sqrt{-d}) = \arg(\mathbb{I})$ is the sum of the arguments of the π_i's, up to a rational multiple of π that comes from the ambiguity in the definition of the argument of an ideal. ■

All that remains is to identify the pairs (p, q) for which $\langle p\rangle_d$ is nonzero. The following theorems give the criteria. These may be easily checked, by hand for small d and p, and by computer for larger values. The theorems themselves are standard results, and we abbreviate the proofs.

Theorem 5. Let p be an odd rational prime. The ideal (p) of $\mathcal{O}_d$ splits as a product of distinct ideals if and only if we can write

$$4p^s = a^2 + b^2 d \qquad 17)$$

for integers a and b (neither a multiple of p) and for an exponent s that divides the class number of $\mathcal{O}_d$. If $d \not\equiv 3 \pmod 4$, or if $d = 3$, then the factor of 4 is unnecessary, and the criterion for splitting reduces to

$$p^s = a^2 + b^2 d, \qquad 18)$$

for a and b nonzero (mod p). The ideal (p) is prime in $\mathcal{O}_d$ if and only if $-d$ is not equal to a square modulo p. If (p) is not prime and does not split, then (p) ramifies.

Proof. Assume that d is not 1 or 3, so that the only units in $\mathcal{O}_d$ are ± 1. If $(p) = \mathbb{I}\mathbb{J}$, let s be the order of $\mathbb{I}$ in the ideal class group. Then $(p^s) = \mathbb{I}^s\mathbb{J}^s$, where $\mathbb{I}^s$ is a principal ideal. Since the elements of $\mathcal{O}_d$ are all of the form $\frac{a}{2} + \frac{b}{2}\sqrt{-d}$, with a and b rational integers, we can write

$$(p^s) = (p)^s = \mathbb{I}^s\mathbb{J}^s = \left(\frac{a}{2} + \frac{b}{2}\sqrt{-d}\right)\left(\frac{a}{2} - \frac{b}{2}\sqrt{-d}\right) = \left(\frac{a^2}{4} + \frac{b^2}{4}d\right), \qquad 19)$$

which implies that $4p^s = a^2 + b^2 d$. Now, if $d \not\equiv 3 \pmod 4$, then the only way to make $a^2 + b^2 d$ divisible by 4 is to have both a and b even, in which case we cancel the factors of 2 and write p^s in the form $a^2 + b^2 d$. Conversely, suppose $4p^s = a^2 + b^2 d = [a + b\sqrt{-d}][a - b\sqrt{-d}]$. The left hand side is divisible by p, while each factor in the right hand side is not, so p cannot be prime. Since the only units are ± 1, $(a + b\sqrt{-d})$ and $(a - b\sqrt{-d})$ are distinct ideals, so the factors $\mathbb{I}$ and $\mathbb{J}$ of (p) must be distinct, so (p) splits.

If $-d = x^2 \pmod p$, where x is a rational integer, then $x^2 + d = [x + \sqrt{-d}][x - \sqrt{-d}]$ is divisible by p. However, neither $x \pm \sqrt{-d}$ is divisible by p, so (p) cannot be prime. Conversely, if (p) is not prime, then there are elements $\frac{x}{2} + \frac{y}{2}\sqrt{-d}$ and

$\frac{a}{2} + \frac{b}{2}\sqrt{-d}$ of $\mathcal{O}_d$, neither divisible by p, whose product is divisible by p. But then the norms $[x^2 + y^2 d]/4$ and $[a^2 + b^2 d]/4$ multiply to a multiple of p^2, so at least one of these norms is divisible by p. Suppose $[x^2 + y^2 d]/4$ is divisible by p, so $x^2 = -y^2 d \pmod p$. If $y = 0 \pmod p$, then $x = 0 \pmod p$, contradicting the fact that $\frac{x}{2} + \frac{y}{2}\sqrt{-d}$ is not in (p). Since $y \neq 0 \pmod p$, we can find an integer z such that $yz = 1 \pmod p$, and then $-d = (xz)^2 \pmod p$.

The cases $d = 1$ and $d = 3$ are complicated slightly by the existence of units other than ± 1, but the analysis is essentially the same. ∎

Theorem 5 gives criteria for all odd primes. The prime $p = 2$ is somewhat different. Since both 0 and 1 are squares, every $-d$ is congruent to a square modulo 2. However, there are values of d for which (2) is prime.

Theorem 6. If $d \neq 3 \pmod 4$, then (2) ramifies in $\mathcal{O}_d$. If $d = 3 \pmod 8$, then (2) is prime. If $d = 7 \pmod 8$, then (2) splits and we can write a power of 2 as $a^2 + bd^2$.

Proof. If $d \neq 3 \pmod 4$, then the algebraic integers all have the form $a + b\sqrt{-d}$, with a and b rational integers. Either $-d = (\sqrt{-d})^2$ or $1+d = [1+\sqrt{-d}][1-\sqrt{-d}]$ is divisible by 2, but none of the factors are divisible by 2, so 2 is not prime. However, (2) does not split, for if $4 \cdot 2^s = a^2 + db^2$, with s minimal, then the left hand side is divisible by 4, while the right hand side is only divisible by 4 if both a and b are even, in which case s is not minimal. Thus (2) ramifies. (This argument breaks down if $d = 1$, but then $(2) = ([1 + i]/2)^2$, so the conclusion still holds).

If $d = 3 \pmod 8$, then the algebraic integers that are not divisible by 2 are precisely those with odd norm. Since the product of any two such numbers is another such number, (2) is prime.

If $d = 7 \pmod 8$, then $[d + 1]/4 = [\frac{1}{2} + \frac{1}{2}\sqrt{-d}][\frac{1}{2} - \frac{1}{2}\sqrt{-d}]$ is divisible by 2, while neither factor on the right hand side is, so (2) is not prime. The fact that (2) splits, as opposed to ramifying, is more difficult, and is not proven here. ∎

3. Applications to the Dehn-Sydler criterion of Archimedean polyhedra.

The Dehn invariant of a polyhedron whose i^{th} edge has length ℓ_i and dihedral angle θ_i is the formal expression $\sum_i \ell_i V[\theta_i]$ where the "vectors" $V[\theta_i]$ are subject to the relations

$$V[r\theta + s\phi] = rV[\theta] + sV[\phi], \quad V[r\pi] = 0, \tag{20}$$

for all rational numbers r and s. The $V[\theta]$ satisfy the same rational linear relations satisfied by the angles θ in the rational vector space they generate, together with the additional relation $V[\pi] = 0$; however we allow their coefficients to be arbitrary real numbers.

If every dihedral angle θ of a polyhedron is geodetic we can write

$$\theta = r\pi + r_1\langle p_1\rangle_{d_1} + \cdots + r_j\langle p_j\rangle_{d_j} \tag{21}$$

for rational numbers $r, r_1, \cdots, r_j$, so

$$V[\theta] = r_1 V[\langle p_1\rangle_{d_1}] + \cdots + r_j V[\langle p_j\rangle_{d_j}]. \tag{22}$$

Its Dehn invariant will then be a rational linear combination of the $V[\langle p\rangle_d]$.

Now each face of an Archimedean polyhedron (other than the snub cube and snub dodecahedron) is orthogonal to a rotation axis of one of the Platonic solids, and the rotation groups of all the Platonic solids are contained in the icosahedral group $\mathcal{I}$. It follows that the dihedral angles of all the polyhedra are found among the angles between the rotation axes of $\mathcal{I}$.

Now let $\tau = (1+\sqrt{5})/2$ and $\sigma = \tau^{-1} = \tau - 1$. The 12 vectors whose coordinates are cyclic permutations of $0, \pm 1, \pm\tau$ lie along the pentad axes. Similarly the 20 vectors obtained by cyclicly permutating $\pm 1, \pm 1, \pm 1$ and $0, \pm\tau, \pm\sigma$ lie along the triad axes, and the 30 cyclic permutations of $\pm 2, 0, 0$ and $\pm 1, \pm\sigma, \pm\tau$ lie along the dyad ones. The cosines of the angles between the axes have the form $v \cdot w/|v|\,|w|$ where v and w are chosen from these vectors. These cosines are enumerated in Fig. 1. The angles that correspond to them are those shown in Fig. 2, together with their supplements.

Table 3 gives the components of the Dehn invariants for the non-snub Archimedean polyhedra of edge lengths 1. For instance the dihedral angles of the truncated tetrahedron are $\pi - 2\langle 3\rangle_2$ at six edges and $2\langle 3\rangle_2$ at the remaining 12, so that its Dehn invariant is

$$6V[\pi - 2\langle 3\rangle_2] + 12V[2\langle 3\rangle_2] = 12V[\langle 3\rangle_2], \qquad 23)$$

since $V[\pi] = 0$. In the values we abbreviate $V[\langle p\rangle_d]$ to $\langle p\rangle_d$.

We note that the Dehn invariants of the icosahedron, dodecahedron and icosidodecahedron with unit edge lengths, namely $60\langle 3\rangle_5$, $-30\langle 5\rangle_1$, and $30\langle 5\rangle_1 - 60\langle 3\rangle_5$, respectively, have zero sum, so Sydler's theorem shows that it is possible to dissect them into finitely many pieces that can be reassembled to form a large cube. This might make an intriguing wooden puzzle if an explicit dissection could be found. (We have no idea how to do this.)

4. Angles with algebraic trigonometric functions.

It is natural to consider a generalization of our theory that gives a basis for the rational vector space generated by all the angles whose six trigonometric functions are algebraic. What is missing here is the analogue of our Splitting Theorem. If such an analogue were found, the ideal theory would probably go through quite easily.

We ask a precise question: Does there exist an algorithm that finds all the rational linear relations between a finite number of such angles? The nicest answer would be one giving an explicit basis, analogous to our $\langle p\rangle_d$.

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$d\backslash p$	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
1	$*$	0	$\angle\frac{4}{5}$	0	0	$\angle\frac{4}{13}$	$\angle\frac{16}{17}$	0	0	$\angle\frac{4}{29}$	0	$\angle\frac{36}{37}$	$\angle\frac{16}{41}$	0	0
2	$*$	$\angle\frac{2}{3}$	0	0	$\angle\frac{2}{11}$	0	$\angle\frac{8}{17}$	$\angle\frac{18}{19}$	0	0	0	0	$\angle\frac{32}{41}$	$\angle\frac{18}{43}$	0
3	0	$*$	0	$\angle\frac{3}{7}$	0	$\angle\frac{12}{13}$	0	$\angle\frac{3}{19}$	0	0	$\angle\frac{27}{31}$	$\angle\frac{12}{37}$	0	$\angle\frac{27}{43}$	0
5	$*$	$\frac{1}{2}\angle\frac{5}{9}$	$*$	$\frac{1}{2}\angle\frac{45}{49}$	0	0	0	0	$\frac{1}{2}\angle\frac{45}{529}$	$\angle\frac{20}{29}$	0	0	$\angle\frac{5}{41}$	$\frac{1}{2}\angle\frac{405}{1849}$	$\frac{1}{2}\angle\frac{2205}{2209}$
6	$*$	$*$	$\frac{1}{2}\angle\frac{24}{25}$	$\angle\frac{6}{7}$	$\frac{1}{2}\angle\frac{96}{121}$	0	0	0	0	$\frac{1}{2}\angle\frac{216}{841}$	$\angle\frac{6}{31}$	0	0	0	0
7	$\angle\frac{7}{8}$	0	0	$*$	$\angle\frac{7}{11}$	0	0	0	$\angle\frac{7}{23}$	$\angle\frac{28}{29}$	0	$\angle\frac{28}{37}$	0	$\angle\frac{7}{43}$	0
10	$*$	0	$*$	$\frac{1}{2}\angle\frac{40}{49}$	$\angle\frac{10}{11}$	$\frac{1}{2}\angle\frac{160}{169}$	0	$\angle\frac{10}{19}$	$\frac{1}{2}\angle\frac{360}{529}$	0	0	$\frac{1}{2}\angle\frac{640}{1369}$	$\angle\frac{40}{41}$	0	$\frac{1}{2}\angle\frac{360}{2209}$
11	0	$\angle\frac{11}{12}$	$\angle\frac{11}{20}$	0	$*$	0	0	0	$\angle\frac{11}{92}$	0	$\angle\frac{99}{124}$	$\angle\frac{99}{148}$	0	0	$\angle\frac{11}{47}$
13	$*$	0	0	$\frac{1}{2}\angle\frac{13}{49}$	$\frac{1}{2}\angle\frac{117}{121}$	$*$	$\angle\frac{13}{17}$	$\frac{1}{2}\angle\frac{325}{361}$	0	$\angle\frac{13}{29}$	$\frac{1}{2}\angle\frac{637}{961}$	0	0	0	$\frac{1}{2}\angle\frac{1053}{2209}$
14	$*$	$\frac{1}{4}\angle\frac{56}{81}$	$\frac{1}{4}\angle\frac{504}{625}$	$*$	0	$\frac{1}{4}\angle\frac{4536}{28561}$	0	$\frac{1}{4}\angle\frac{24696}{130321}$	$\angle\frac{14}{23}$	0	0	0	0	0	0
15	$\frac{1}{2}\angle\frac{15}{16}$	$*$	$*$	0	0	0	$\frac{1}{2}\angle\frac{240}{289}$	$\angle\frac{15}{19}$	$\frac{1}{2}\angle\frac{240}{529}$	0	$\angle\frac{15}{31}$	0	0	0	$\frac{1}{2}\angle\frac{2160}{2209}$
17	$*$	$\frac{1}{4}\angle\frac{17}{81}$	0	$\frac{1}{4}\angle\frac{1377}{2401}$	$\frac{1}{4}\angle\frac{3825}{14641}$	$\frac{1}{2}\angle\frac{153}{169}$	$*$	0	$\frac{1}{4}\angle\frac{187425}{279841}$	0	$\frac{1}{4}\angle\frac{907137}{923521}$	0	0	0	0
19	0	0	$\angle\frac{19}{20}$	$\angle\frac{19}{28}$	$\angle\frac{19}{44}$	0	$\angle\frac{19}{68}$	$*$	$\angle\frac{19}{23}$	0	0	0	0	$\angle\frac{171}{172}$	$\angle\frac{19}{188}$
21	$*$	$*$	$\frac{1}{2}\angle\frac{21}{25}$	$*$	$\frac{1}{2}\angle\frac{21}{121}$	0	$\frac{1}{2}\angle\frac{189}{289}$	$\frac{1}{2}\angle\frac{336}{361}$	$\frac{1}{2}\angle\frac{525}{529}$	0	$\frac{1}{2}\angle\frac{336}{961}$	$\angle\frac{21}{37}$	$\frac{1}{2}\angle\frac{525}{1681}$	0	0
22	$*$	0	0	0	$*$	$\frac{1}{2}\angle\frac{88}{169}$	0	$\frac{1}{2}\angle\frac{352}{361}$	$\angle\frac{22}{23}$	$\frac{1}{2}\angle\frac{792}{841}$	$\angle\frac{22}{31}$	0	0	$\frac{1}{2}\angle\frac{1408}{1849}$	$\angle\frac{22}{47}$
23	$\frac{1}{3}\angle\frac{23}{32}$	$\frac{1}{3}\angle\frac{23}{27}$	0	0	0	$\frac{1}{3}\angle\frac{828}{2197}$	0	0	$*$	$\frac{1}{3}\angle\frac{4508}{24389}$	$\frac{1}{3}\angle\frac{207}{29791}$	0	$\frac{1}{3}\angle\frac{23552}{68921}$	0	$\frac{1}{3}\angle\frac{103247}{103823}$

Table 1. Basis elements $\langle p\rangle_d$ for some p and d. 0 indicates that (p) is prime in $\mathcal{O}_d$, while $*$ indicates that (p) ramifies.

$\langle 5\rangle_{1} = \angle\frac{4}{5} \approx 63^\circ 26' 6''$
$\langle 13\rangle_{1} = \angle\frac{4}{13} \approx 33^\circ 41' 24''$
$\langle 17\rangle_{1} = \angle\frac{16}{17} \approx 75^\circ 57' 50''$
$\langle 29\rangle_{1} = \angle\frac{4}{29} \approx 21^\circ 48' 5''$
$\langle 37\rangle_{1} = \angle\frac{36}{37} \approx 80^\circ 32' 16''$
$\langle 41\rangle_{1} = \angle\frac{16}{41} \approx 38^\circ 39' 35''$

$\langle 3\rangle_{2} = \angle\frac{2}{3} \approx 54^\circ 44' 8''$
$\langle 11\rangle_{2} = \angle\frac{2}{11} \approx 25^\circ 14' 22''$
$\langle 17\rangle_{2} = \angle\frac{8}{17} \approx 43^\circ 18' 50''$
$\langle 19\rangle_{2} = \angle\frac{18}{19} \approx 76^\circ 44' 14''$
$\langle 41\rangle_{2} = \angle\frac{32}{41} \approx 62^\circ 3' 42''$
$\langle 43\rangle_{2} = \angle\frac{18}{43} \approx 40^\circ 18' 55''$

$\langle 7\rangle_{3} = \angle\frac{3}{7} \approx 40^\circ 53' 36''$
$\langle 13\rangle_{3} = \angle\frac{12}{13} \approx 73^\circ 53' 52''$
$\langle 19\rangle_{3} = \angle\frac{3}{19} \approx 23^\circ 24' 48''$
$\langle 31\rangle_{3} = \angle\frac{27}{31} \approx 68^\circ 56' 54''$
$\langle 37\rangle_{3} = \angle\frac{12}{37} \approx 34^\circ 42' 54''$
$\langle 43\rangle_{3} = \angle\frac{27}{43} \approx 52^\circ 24' 39''$

$\langle 3\rangle_{5} = \frac{1}{2}\angle\frac{5}{9} \approx 24^\circ 5' 41''$
$\langle 7\rangle_{5} = \frac{1}{2}\angle\frac{45}{49} \approx 36^\circ 41' 57''$
$\langle 23\rangle_{5} = \frac{1}{2}\angle\frac{45}{529} \approx 8^\circ 28' 43''$
$\langle 29\rangle_{5} = \angle\frac{20}{29} \approx 56^\circ 8' 44''$
$\langle 41\rangle_{5} = \angle\frac{5}{41} \approx 20^\circ 26' 22''$
$\langle 43\rangle_{5} = \frac{1}{2}\angle\frac{405}{1849} \approx 13^\circ 57' 10''$
$\langle 47\rangle_{5} = \frac{1}{2}\angle\frac{2205}{2209} \approx 43^\circ 46' 50''$

$\langle 5\rangle_{6} = \frac{1}{2}\angle\frac{24}{25} \approx 39^\circ 13' 53''$
$\langle 7\rangle_{6} = \angle\frac{6}{7} \approx 67^\circ 47' 32''$
$\langle 11\rangle_{6} = \frac{1}{2}\angle\frac{96}{121} \approx 31^\circ 28' 56''$
$\langle 29\rangle_{6} = \frac{1}{2}\angle\frac{216}{841} \approx 15^\circ 13' 31''$
$\langle 31\rangle_{6} = \angle\frac{6}{31} \approx 26^\circ 6' 0''$

$\langle 2\rangle_{7} = \angle\frac{7}{8} \approx 69^\circ 17' 43''$
$\langle 11\rangle_{7} = \angle\frac{7}{11} \approx 52^\circ 54' 48''$
$\langle 23\rangle_{7} = \angle\frac{7}{23} \approx 33^\circ 28' 56''$
$\langle 29\rangle_{7} = \angle\frac{28}{29} \approx 79^\circ 17' 54''$
$\langle 37\rangle_{7} = \angle\frac{28}{37} \approx 60^\circ 26' 57''$
$\langle 43\rangle_{7} = \angle\frac{7}{43} \approx 23^\circ 47' 44''$

$\langle 7\rangle_{10} = \frac{1}{2}\angle\frac{40}{49} \approx 32^\circ 18' 42''$
$\langle 11\rangle_{10} = \angle\frac{10}{11} \approx 72^\circ 27' 6''$
$\langle 13\rangle_{10} = \frac{1}{2}\angle\frac{160}{169} \approx 38^\circ 19' 44''$
$\langle 19\rangle_{10} = \angle\frac{10}{19} \approx 46^\circ 30' 31''$
$\langle 23\rangle_{10} = \frac{1}{2}\angle\frac{360}{529} \approx 27^\circ 47' 29''$
$\langle 37\rangle_{10} = \frac{1}{2}\angle\frac{640}{1369} \approx 21^\circ 34' 5''$
$\langle 41\rangle_{10} = \angle\frac{40}{41} \approx 81^\circ 0' 54''$
$\langle 47\rangle_{10} = \frac{1}{2}\angle\frac{360}{2209} \approx 11^\circ 54' 17''$

$\langle 3\rangle_{11} = \angle\frac{11}{12} \approx 73^\circ 13' 17''$
$\langle 5\rangle_{11} = \angle\frac{11}{20} \approx 47^\circ 52' 11''$
$\langle 23\rangle_{11} = \angle\frac{11}{92} \approx 20^\circ 13' 46''$
$\langle 31\rangle_{11} = \angle\frac{99}{124} \approx 63^\circ 19' 11''$
$\langle 37\rangle_{11} = \angle\frac{99}{148} \approx 54^\circ 52' 21''$
$\langle 47\rangle_{11} = \angle\frac{11}{47} \approx 28^\circ 55' 57''$

$\langle 7\rangle_{13} = \frac{1}{2}\angle\frac{13}{49} \approx 15^\circ 30' 5''$
$\langle 11\rangle_{13} = \frac{1}{2}\angle\frac{117}{121} \approx 39^\circ 45' 44''$
$\langle 17\rangle_{13} = \angle\frac{13}{17} \approx 60^\circ 58' 58''$
$\langle 19\rangle_{13} = \frac{1}{2}\angle\frac{325}{361} \approx 35^\circ 47' 45''$
$\langle 29\rangle_{13} = \angle\frac{13}{29} \approx 42^\circ 1' 52''$
$\langle 31\rangle_{13} = \frac{1}{2}\angle\frac{637}{961} \approx 27^\circ 15' 7''$
$\langle 47\rangle_{13} = \frac{1}{2}\angle\frac{1053}{2209} \approx 21^\circ 49' 55''$

$\langle 3\rangle_{14} = \frac{1}{4}\angle\frac{56}{81} \approx 14^\circ 3' 46''$
$\langle 5\rangle_{14} = \frac{1}{4}\angle\frac{504}{625} \approx 15^\circ 58' 27''$
$\langle 13\rangle_{14} = \frac{1}{4}\angle\frac{4536}{28561} \approx 5^\circ 52' 17''$
$\langle 19\rangle_{14} = \frac{1}{4}\angle\frac{24696}{130321} \approx 6^\circ 27' 5''$
$\langle 23\rangle_{14} = \angle\frac{14}{23} \approx 51^\circ 16' 41''$

$\langle 2\rangle_{15} = \frac{1}{2}\angle\frac{15}{16} \approx 37^\circ 45' 40''$
$\langle 17\rangle_{15} = \frac{1}{2}\angle\frac{240}{289} \approx 32^\circ 50' 32''$
$\langle 19\rangle_{15} = \angle\frac{15}{19} \approx 62^\circ 41' 18''$
$\langle 23\rangle_{15} = \frac{1}{2}\angle\frac{240}{529} \approx 21^\circ 10' 17''$
$\langle 31\rangle_{15} = \angle\frac{15}{31} \approx 44^\circ 4' 33''$
$\langle 47\rangle_{15} = \frac{1}{2}\angle\frac{2160}{2209} \approx 40^\circ 43' 2''$

$\langle 3\rangle_{17} = \frac{1}{4}\angle\frac{17}{81} \approx 6^\circ 48' 59''$
$\langle 7\rangle_{17} = \frac{1}{4}\angle\frac{1377}{2401} \approx 12^\circ 18' 24''$
$\langle 11\rangle_{17} = \frac{1}{4}\angle\frac{3825}{14641} \approx 7^\circ 41' 5''$
$\langle 13\rangle_{17} = \frac{1}{2}\angle\frac{153}{169} \approx 36^\circ 2' 24''$
$\langle 23\rangle_{17} = \frac{1}{4}\angle\frac{187425}{279841} \approx 13^\circ 43' 51''$
$\langle 31\rangle_{17} = \frac{1}{4}\angle\frac{907137}{923521} \approx 20^\circ 35' 11''$

$\langle 5\rangle_{19} = \angle\frac{19}{20} \approx 77^\circ 4' 45''$
$\langle 7\rangle_{19} = \angle\frac{19}{28} \approx 55^\circ 27' 45''$
$\langle 11\rangle_{19} = \angle\frac{19}{44} \approx 41^\circ 4' 53''$
$\langle 17\rangle_{19} = \angle\frac{19}{68} \approx 31^\circ 54' 38''$
$\langle 23\rangle_{19} = \angle\frac{19}{23} \approx 65^\circ 21' 10''$
$\langle 43\rangle_{19} = \angle\frac{171}{172} \approx 85^\circ 37' 37''$
$\langle 47\rangle_{19} = \angle\frac{19}{188} \approx 18^\circ 32' 11''$

$\langle 5\rangle_{21} = \frac{1}{2}\angle\frac{21}{25} \approx 33^\circ 12' 39''$
$\langle 11\rangle_{21} = \frac{1}{2}\angle\frac{21}{121} \approx 12^\circ 18' 36''$
$\langle 17\rangle_{21} = \frac{1}{2}\angle\frac{189}{289} \approx 26^\circ 59' 3''$
$\langle 19\rangle_{21} = \frac{1}{2}\angle\frac{336}{361} \approx 37^\circ 22' 16''$
$\langle 23\rangle_{21} = \frac{1}{2}\angle\frac{525}{529} \approx 42^\circ 30' 21''$
$\langle 31\rangle_{21} = \frac{1}{2}\angle\frac{336}{961} \approx 18^\circ 7' 29''$
$\langle 37\rangle_{21} = \angle\frac{21}{37} \approx 48^\circ 53' 0''$
$\langle 41\rangle_{21} = \frac{1}{2}\angle\frac{525}{1681} \approx 16^\circ 59' 17''$

$\langle 13\rangle_{22} = \frac{1}{2}\angle\frac{88}{169} \approx 23^\circ 5' 36''$
$\langle 19\rangle_{22} = \frac{1}{2}\angle\frac{352}{361} \approx 40^\circ 27' 28''$
$\langle 23\rangle_{22} = \angle\frac{22}{23} \approx 77^\circ 57' 53''$
$\langle 29\rangle_{22} = \frac{1}{2}\angle\frac{792}{841} \approx 38^\circ 0' 58''$
$\langle 31\rangle_{22} = \angle\frac{22}{31} \approx 57^\circ 23' 49''$
$\langle 43\rangle_{22} = \frac{1}{2}\angle\frac{1408}{1849} \approx 30^\circ 22' 59''$
$\langle 47\rangle_{22} = \angle\frac{22}{47} \approx 43^\circ 10' 11''$

$\langle 2\rangle_{23} = \frac{1}{3}\angle\frac{23}{32} \approx 19^\circ 19' 27''$
$\langle 3\rangle_{23} = \frac{1}{3}\angle\frac{23}{27} \approx 22^\circ 27' 15''$
$\langle 13\rangle_{23} = \frac{1}{3}\angle\frac{828}{2197} \approx 12^\circ 37' 28''$
$\langle 29\rangle_{23} = \frac{1}{3}\angle\frac{4508}{24389} \approx 8^\circ 29' 16''$
$\langle 31\rangle_{23} = \frac{1}{3}\angle\frac{207}{29791} \approx 1^\circ 35' 38''$
$\langle 41\rangle_{23} = \frac{1}{3}\angle\frac{23552}{68921} \approx 11^\circ 55' 27''$
$\langle 47\rangle_{23} = \frac{1}{3}\angle\frac{103247}{103823} \approx 28^\circ 34' 34''$

Table 2. The values of some basis elements $\langle p\rangle_d$.

Tetrahedron	$6\pi - 12\langle 3\rangle_2$
Truncated tetrahedron	$6\pi + 12\langle 3\rangle_2$
Cube	6π
Truncated cube	$30\pi - 24\langle 3\rangle_2$
Octahedron	$24\langle 3\rangle_2$
Truncated octahedron	24π
Rhombicuboctahedron	$9\pi - 24\langle 3\rangle_2$
Cuboctahedron	$24\pi - 24\langle 3\rangle_2$
Truncated cuboctahedron	54π
Icosahedron	$15\pi + 60\langle 3\rangle_5$
Truncated icosahedron	$60\pi + 30\langle 5\rangle_1$
Dodecahedron	$-30\langle 5\rangle_1$
Truncated dodecahedron	$75\pi - 60\langle 3\rangle_5$
Rhombicosidodecahedron	$105\pi + 60\langle 3\rangle_5 - 30\langle 5\rangle_1$
Icosidodecahedron	$45\pi - 60\langle 3\rangle_5 + 30\langle 5\rangle_1$
Truncated icosidodecahedron	150π

Table 3. The Dehn invariant for the non-snub unit edge Archimedean polyhedra.

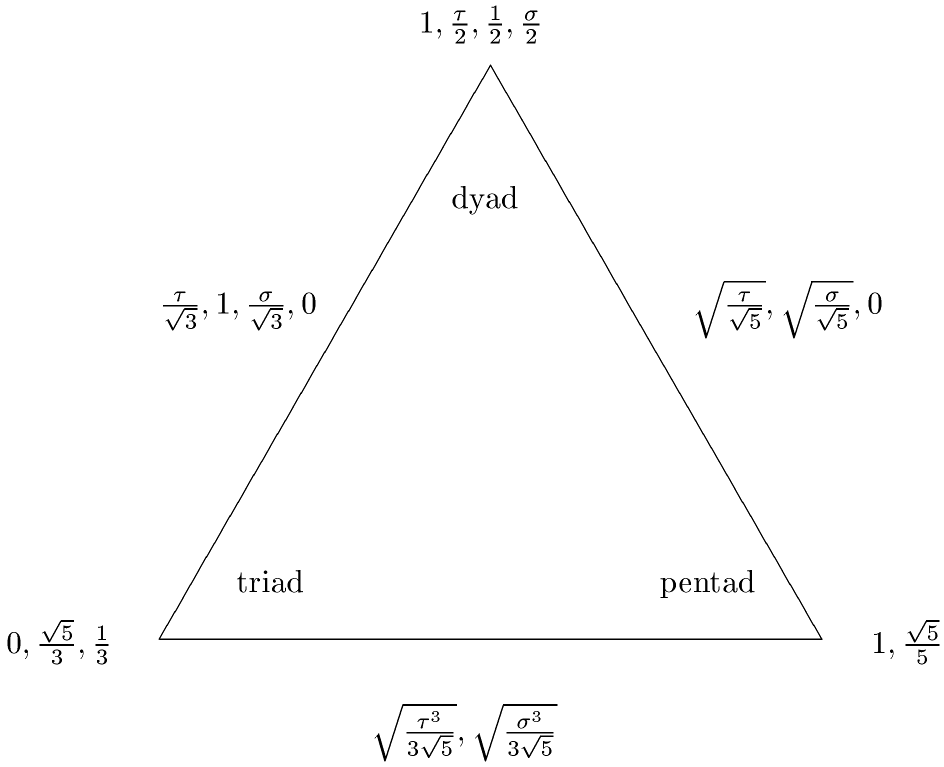

Figure 1. Cosines of angles between axes of fixed rotational symmetry (shown at corners), and between axes of different rotational symmetry (shown at edges).

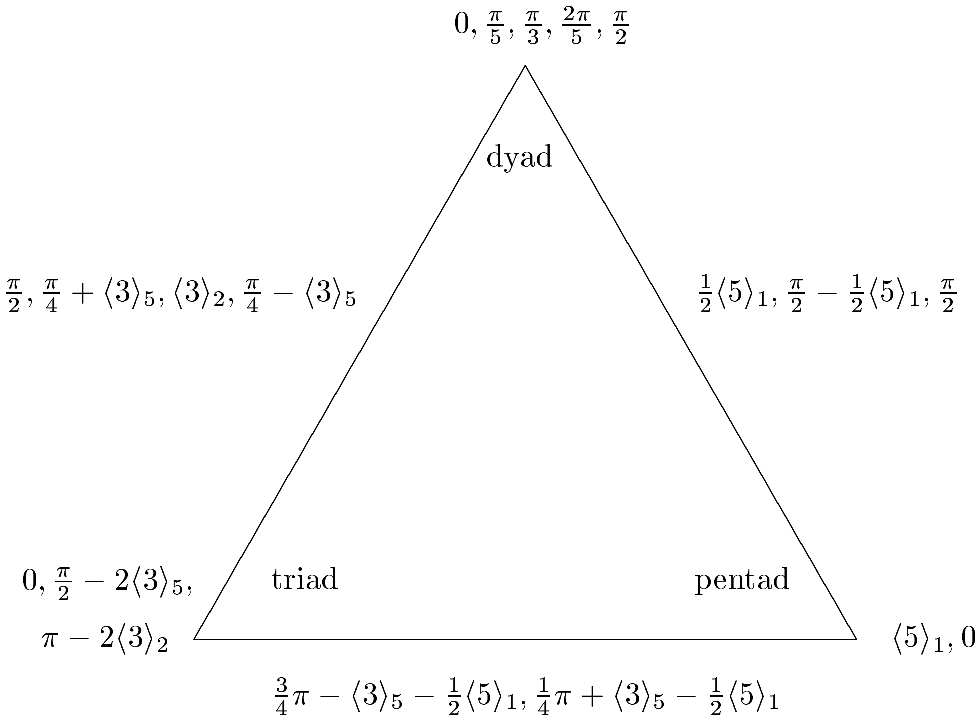

Figure 2. Angles between axes of fixed rotational symmetry (shown at corners), and between axes of different rotational symmetry (shown at edges).