On the Spectrum of the Surface Maryland Model *

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Abstract

We study spectral properties of the discrete Laplacian H on the half space $\mathbf{Z}_+^{d+1} = \mathbf{Z}^d \times \mathbf{Z}_+$ with a boundary condition $\psi(n,-1) = \lambda \tan(\pi\alpha \cdot n + \theta)\psi(n,0)$, where $\alpha \in [0,1]^d$. We denote by H_0 the Dirichlet Laplacian on $\mathbf{Z}_+^{d+1}$. Whenever α is independent over rationals $\sigma(H) = \mathbf{R}$. Khoruzenko and Pastur [KP] have shown that for a set of α's of Lebesgue measure 1, the spectrum of H on $\mathbf{R} \setminus \sigma(H_0)$ is pure point and that corresponding eigenfunctions decay exponentially. In this paper we show that if α is independent over rationals then the spectrum of H on the set $\sigma(H_0)$ is purely absolutely continuous.

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1 Introduction

Let $d \geq 1$ be given and let $\mathbf{Z}_+^{d+1} = \mathbf{Z}^d \times \mathbf{Z}_+$, where $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$. We denote the points in $\mathbf{Z}_+^{d+1}$ by (n, x), $n \in \mathbf{Z}^d$, $x \geq 0$. Let $V : \mathbf{Z}^d \mapsto \mathbf{R}$ be given function and let H be the discrete Laplacian on $l^2(\mathbf{Z}_+^{d+1})$ with the boundary condition $\psi(n, -1) = V(n)\psi(n)$. When $V \equiv 0$ this operator reduces to the Dirichlet Laplacian which we denote by H_0. We recall that $\sigma(H_0) = [-2(d+1), 2(d+1)]$ and that the spectrum of H_0 is purely absolutely continuous.

The operator H acts as

$$(H\psi)(n, x) = \begin{cases} \sum_{|n-n'|_+ + |x-x'|=1} \psi(n', x') & \text{if } x > 0, \\ \psi(n, 1) + \sum_{|n-n'|_+=1} \psi(n, 0) + V(n)\psi(n, 0) & \text{if } x = 0, \end{cases} \tag{1.1}$$

where $|n|_+ = \sum_{j=1}^d |n_j|$. Note that operator H can be viewed as the Schrödinger operator

$$H = H_0 + V, \tag{1.2}$$

where the potential V acts only along the boundary $\partial \mathbf{Z}_+^{d+1} = \mathbf{Z}^d$. More precisely, $(V\psi)(n, x) = 0$ if $x > 0$ and $(V\psi)(n, 0) = V(n)\psi(n, 0)$. For many purposes, it is convenient to adopt this point of view and we will do so in the sequel. Since H_0 is bounded, the operator H is properly defined as a self-adjoint operator on $l^2(\mathbf{Z}_+^{d+1})$.

The spectral theory of operators H in the cases where V is a random or almost periodic function has been studied in [AM], [BS], [G], [G1], [JL], [JM], [JMP], [KP], [P], [M]. The principal physical motivation is to understand the formation and propagation properties of surface waves in regions with corrugated boundaries. For additional information on this program we refer the reader to a review article [JMP].

The Maryland model is the family of operators on $l^2(\mathbf{Z}^d)$ of the form $h = h_0 + V(n)$, where

$$V(n) \equiv V_{\alpha,\theta,\lambda} = \lambda \tan(\pi\alpha \cdot n + \theta), \tag{1.3}$$

h_0 is the usual free Laplacian on $\mathbf{Z}^d$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in [0, 1]^d$ and $\theta \in [0, \pi]$. To avoid singular cases, we will assume that for a given α, θ is chosen so that $\forall n$,

$$\pi\alpha \cdot n + \theta \not\equiv 0 \mod \pi/2. \tag{1.4}$$

The Maryland model has been extensively studied in [FP], [FGP], [FGP1],[GFP], [PRG], [S1], [S2]. We recall that $\alpha = (\alpha_1, \ldots, \alpha_d)$ is independent over rationals if for any choice of rational numbers $r_1. \ldots, r_d \in \mathbf{Q}$,

$$\sum_k r_k \alpha_k \notin \mathbf{Q}.$$

It is not difficult to show that if α is independent over rationals then for any $\lambda \neq 0$ and a.e. $\theta \in [0, \pi]$, $\sigma(h) = \mathbf{R}$ (see e.g. [CFKS]). We say that α has typical Diophantine properties if there exists constants $C, k > 0$ such that

$$|n \cdot \alpha - m| > C|n|^{-k}, \tag{1.5}$$

for all $n \in \mathbf{Z}^d$ and $m \in \mathbf{Z}$. The set of α's in $[0,1]^d$ for which (1.5) holds has Lebesgue measure 1. If α has typical Diophantine properties then $\sigma(h) = \mathbf{R}$ for any θ which satisfies (1.4), the spectrum is pure point, the eigenvalues of h are simple and the corresponding eigenfunctions decay exponentially, see [FP] or [CFKS]. Thus, in any dimension and for typical α, the deterministic potential (1.3) is strongly localizing.

The surface Maryland model is a family of operators on $l^2(\mathbf{Z}_+^{d+1})$ defined by (1.1)-(1.3),

$$H_{\alpha,\theta,\lambda} = H_0 + V_{\alpha,\theta,\lambda}.$$

Notation. In the sequel, whenever the meaning is clear within the context, we will drop subscripts α, θ, λ. Thus, we write H for $H_{\alpha,\theta,\lambda}$, etc. We will also use the shorthand $c_d = 2(d+1)$, so $\sigma(H_0) = [-c_d, c_d]$.
An easy Weyl's sequence argument yields that for any $\alpha \in [0,1]^d$, and any θ which satisfies (1.4), $\sigma(H_0) \subset \sigma(H)$. If α is independent over rationals, the standard argument (see e.g. [CFKS] or [JMP1]) yields that for a.e. $\theta \in [0,\pi]$, $\sigma(H) = \mathbf{R}$.

Since the potential V models strongly corrugated boundary, it is natural to expect that the spectrum of H outside $\sigma(H_0)$ is dense pure point for any α which satisfies (1.5). Indeed, Khoruzenko and Pastur [KP] have proven the following result.

Theorem 1.1 *Assume that α has typical Diophantine properties. Then for any $\lambda \neq 0$ and θ which satisfies (1.4), $\sigma(H) = \mathbf{R}$ and the spectrum of H on the set $\mathbf{R} \setminus (-c_d, c_d)$ is pure point. On this set, the eigenvalues are simple and the corresponding eigenfunctions decay exponentially.*

In this paper, we study the spectrum of H on the set $\sigma(H_0)$. One can show (see [JMP], [JMP1]) that for any $\alpha \in [0,1]^d$, the wave operators

$$\Omega^{\pm} = \mathrm{s} - \lim_{t \to \mp\infty} e^{itH} e^{-itH_0},$$

exist for a.e. $\theta \in [0,\pi]$, and consequently, that $[-c_d, c_d] \subset \sigma_{ac}(H)$. This is not a surprising result: Due to the free propagation along the x-axis, the operator H should have some absolutely continuous spectrum. There where various speculations that H might have some point spectrum on $(-c_d, c_d)$, and if α is "extremely well approximated" by rationals even some singular continuous spectrum. Thus, the following result comes perhaps as a surprise.

Theorem 1.2 *If $\alpha \in [0,1]^d$ is independent over rationals then for any $\lambda \neq 0$ and θ which satisfies (1.4), the spectrum of H on the set $(-c_d, c_d)$ is purely absolutely continuous.*

The relation between this result and the program of [JMP] will be discussed in [JMP1].

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2 Dimension reduction

One of the basic ideas used in practically all works on the spectral theory of operators (1.1) is to "integrate" the x-variable and to reduce the $d+1$-dimensional spectral problem to a d-dimensional problem, which will depend non-linearly on the spectral parameter E. In this section we carry out this dimension reduction for the surface Maryland model, and lay the ground for the proof of Theorem 1.3

The first steps follow closely Section 2 in [JM]. We give details for readers convenience. We recall that the points in $\mathbf{Z}_+^{d+1}$ are denoted by $\mathbf{n} = (n, x)$, $n \in \mathbf{Z}^d$, $x \in \mathbf{Z}_+$. Let $z \in \mathbf{C}$, $\mathrm{Im}(z) \neq 0$, be given and let

$$R(\mathbf{m}, \mathbf{n}, z) = (\delta_{\mathbf{m}}, (H - z)^{-1}\delta_{\mathbf{n}}).$$

These matrix elements satisfy the equation

$$R(\mathbf{m}, (n, x+1); z) + R(\mathbf{m}, (n, x-1); z) + \sum_{|n-n'|_+=1} R(\mathbf{m}, (n', x); z) = \delta_{\mathbf{mn}} + zR(\mathbf{m}, (n, x); z) \quad (2.6)$$

if $x > 0$, and

$$R(\mathbf{m}, (n, 1); z) + \sum_{|n-n'|_+=1} R(\mathbf{m}, (n', 0); z) + (V(n) - z)R(\mathbf{m}, (n, 0); z) = \delta_{\mathbf{mn}}, \quad (2.7)$$

if $x = 0$. If $\mathbf{m} = (m, 0)$ is the point on the boundary, Equation (2.6) can be "integrated". This is most conveniently done in the Fourier representation associated to the variable n. Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be the circle and $\mathbf{T}^d$ the d-dimensional torus. We denote the points in $\mathbf{T}^d$ by $\phi = (\phi_1, \ldots, \phi_d)$, and by $d\phi$ the usual Lebesgue measure. In the sequel we will use the shorthand $\Phi(\phi) = \sum_{k=1}^d 2\cos\phi_k$. Let

$$\mathcal{F} : l^2(\mathbf{Z}_+^{d+1}) \mapsto L^2(\mathbf{T}^d) \otimes l^2(\mathbf{Z}_+)$$

be a unitary map defined by the formula

$$(\mathcal{F}\psi)(n, x) \equiv \hat{\psi}(\phi, x) = \frac{1}{(2\pi)^{d/2}} \sum_{n \in \mathbf{Z}^d} \psi(n, x) \exp(in \cdot \phi).$$

In the new representation, Equations (2.6) and (2.7) are (recall that $\mathbf{m} = (m, 0)$)

$$\hat{R}(\mathbf{m}, (\phi, x+1; z)) + \hat{R}(\mathbf{m}, (\phi, x-1); z) + (\Phi(\phi) - z)\hat{R}(\mathbf{m}, (\phi, x); z) = 0, \quad (2.8)$$

and

$$\hat{R}(\mathbf{m},(\phi,1);z)+\Phi(\phi)\hat{R}(\mathbf{m},(\phi,0);z)+\widehat{VR}(\mathbf{m},(\phi,0);z)-z\hat{R}(\mathbf{m},(\phi,0);z)=e_m(\phi), \tag{2.9}$$

where $e_m(\phi)=\mathcal{F}(\delta_{mn})=(2\pi)^{-d/2}\exp(im\cdot\phi)$. It follows from Equation (2.8) that if $x>0$,

$$\hat{R}(\mathbf{m},(\phi,x);z)=\hat{R}(\mathbf{m},(\phi,0);z)r(\phi,z)^x, \tag{2.10}$$

where $r(\phi,z)$ is the root of the quadratic equation

$$X+\frac{1}{X}+\Phi(\phi)=z, \tag{2.11}$$

such that $|r(\phi,z)|<1$. The other root is given by $\tilde{r}(\phi,z)=1/r(\phi,z)$. Substituting (2.10) into (2.9) and using that $r+\Phi-z=-\tilde{r}$, we get the equation

$$-\hat{R}(m,(\phi,0);z)\tilde{r}(\phi,z)+\widehat{VR}(m,(\phi,0);z)=e_m(\phi). \tag{2.12}$$

Let

$$\begin{aligned}\mathcal{R}(m,n;z) &\equiv R((m,0),(n,0);z)=(\delta_{(m,0)},(H-z)^{-1}\delta_{(n,0)}),\\ \mathcal{R}(m,\phi;z) &\equiv \hat{R}((m,0),(\phi,0);z),\end{aligned}$$

and

$$j(n,z)=\int_{\mathbf{T}^d}e^{-in\cdot\phi}\tilde{r}(\phi,z)d\phi.$$

Aplying $\mathcal{F}^{-1}$ to (2.12) we get that matrix elements $\mathcal{R}(m,n;z)$ satisfy the equation

$$-\sum_k j(n-k,z)\mathcal{R}(m,k;z)+V(n)\mathcal{R}(m,n;z)=\delta_{mn}. \tag{2.13}$$

The above construction is of course applicable to any V. To proceed we have to use the particular structure of the potential V. Note that

$$V(n)=\lambda\tan(\pi\alpha\cdot n+\theta)=-\lambda i\frac{1-\exp(-2\pi i\alpha\cdot n-2i\theta)}{1+\exp(-2\pi i\alpha\cdot n-2i\theta)}.$$

Multiplying both sides of Equation (2.13) by $1+\exp(-2\pi i\alpha\cdot n-2i\theta)$ and applying $\mathcal{F}$ again, we get after simple algebra

$$e^{-2i\theta}\mathcal{R}(m,\phi-2\pi\alpha;z)[\lambda i-\tilde{r}(\phi-2\pi\alpha,z)]-\mathcal{R}(m,\phi;z)[\tilde{r}(\phi,z)+\lambda i]=h_m(\phi)$$

where

$$h_m(\phi)=e_m(\phi)+\exp(-2i\theta)e_m(\phi-2\pi\alpha).$$

In the sequel we will assume that $\lambda > 0$. Similar analysis applies if $\lambda < 0$. It follows from Equation (2.11) that if Im$z > 0$ then Im $\tilde{r}(\phi, z) > 0$ (write $\tilde{r}$ in polar form). Thus, if Im$(z) > 0$ and

$$\tilde{\mathcal{R}}(m, \phi; z) = \mathcal{R}(m, \phi; z)[\lambda i + \tilde{r}(\phi, z)], \tag{2.14}$$

we arrive at the equation

$$e^{-2i\theta}\tilde{\mathcal{R}}(m, \phi - 2\pi\alpha; z)\gamma(\phi - 2\pi\alpha, z) - \tilde{\mathcal{R}}(m, \phi; z) = h_m(\phi), \tag{2.15}$$

where

$$\gamma(\phi, z) = \frac{\lambda i - \tilde{r}(\phi, z)}{\lambda i + \tilde{r}(\phi, z)}.$$

For latter applications, the following simple observation is critical. Recall that $\lambda > 0$. Let $\mathbf{R}^+ = [0, \infty)$.

Lemma 2.1 *1. If* Im$(z) > 0$ *then* $\forall \phi \in \mathbf{T}^d$, $|\gamma(\phi, z)| < 1$.
2. Let $|E| < 2(d+1)$ *and let* $\mathcal{O}$ *be an open set inside* $\{\phi \in \mathbf{T}^d : |\Phi(\phi) - E| < 2\}$. *Then* $t_E(\phi, \varepsilon) \equiv \gamma(\phi, E + i\varepsilon)$ *is a continuous function on* $\mathcal{O} \times \mathbf{R}^+$, *and for any* (ϕ, ε) *in this set,* $|t_E(\phi, \varepsilon)| < 1$.

Proof: Since Im$\tilde{r}(\phi, z) > 0$, an elementary computation yields Part 1. It follows from Equation (2.11) and basic complex analysis that $\tilde{r}(\phi, E + i\varepsilon)$ is continuous in variables ϕ and ε on the set $\mathcal{O} \times \mathbf{R}^+$, and that for $\phi \in \mathcal{O}$, Im $\tilde{r}(\phi, E) > 0$. This yields Part 2. □

Finally, the proof of Theorem 1.2 will be based on

Proposition 2.2 *Assume that for any* $E \in (-c_d, c_d)$, *and any* θ *which satisfies (1.4),*

$$\limsup_{\varepsilon \to 0} \int_{\mathbf{T}^d} |\tilde{\mathcal{R}}(0, \phi, E + i\varepsilon)| d\phi < \infty.$$

Then the spectrum of H *on* $(-c_d, c_d)$ *is purely absolutely continuous.*

Proof: Let $\mathcal{H}_n$ be the cyclic subspace generated by the vector $\delta_{(n,0)}$ and H. It is easy to show (see e.g. [JL]) that the linear span of $\mathcal{H}_n$'s is dense in $l^2(\mathbf{Z}^{d+1}_+)$. Let U_{n_0} be the unitary operator of translation by the vector $(n_0, 0)$. If $\theta' = \theta - 2\pi\alpha \cdot n_0$ then

$$\begin{aligned}
(\delta_{(n_0,0)}, (H_{\alpha,\theta,\lambda} - z)^{-1}\delta_{(n_0,0)}) &= (\delta_{(0,0)}, U_{n_0}^{-1}(H_{\alpha,\theta,\lambda} - z)^{-1}U_{n_0}\delta_{(0,0)}) \\
&= (\delta_{(0,0)}, (H_{\alpha,\theta',\lambda} - z)^{-1}\delta_{(0,0)}).
\end{aligned}$$

This argument and Fatou's theorem (see e.g. [RS] or [S3]) yield that H has purely absolutely continuous spectrum on $(-c_d, c_d)$ for all θ satisfying (1.4) if for such θ's and all $E \in (-c_d, c_d)$,

$$A(E) = \limsup_{\varepsilon \to 0} |(\delta_{(0,0)}, (H - E - i\varepsilon)^{-1}\delta_{(0,0)})| < \infty.$$

Note that

$$(\delta_{(0,0)}, (H - E - i\varepsilon)^{-1}\delta_{(0,0)}) = (2\pi)^{-d/2} \int_{\mathbf{T}^d} \mathcal{R}(0, \phi; E + i\varepsilon) d\phi.$$

It follows from Relation (2.14) that

$$|\mathcal{R}(0, \phi, E + i\varepsilon)| \leq \frac{1}{\lambda} |\tilde{\mathcal{R}}(0, \phi; E + i\varepsilon)|,$$

and we have that

$$\begin{aligned} A(E) &\leq \limsup_{\varepsilon \to 0} (2\pi)^{-d/2} \int_{\mathbf{T}^d} |\mathcal{R}(0, \phi; E + i\varepsilon)| d\phi \\ &\leq \limsup_{\varepsilon \to 0} (2\pi)^{-d/2} \lambda^{-1} \int_{\mathbf{T}^d} |\tilde{\mathcal{R}}(0, \phi; E + i\varepsilon)| d\phi. \end{aligned}$$

The statement follows. □

3 Proof of Theorem 1.2

We are now ready to finish the proof of Theorem 1.2. We will use the shorthand $\tilde{\mathcal{R}}(\phi, z) \equiv \tilde{\mathcal{R}}(0, \phi; z)$.

Let $E \in (-c_d, c_d)$ be given and let $\varepsilon > 0$. It follows from Equation (2.15) that

$$|\tilde{\mathcal{R}}(\phi, E + i\varepsilon)| \leq 2(2\pi)^{-d/2} + |\gamma(\phi - 2\pi\alpha, E + i\varepsilon)| \cdot |\tilde{\mathcal{R}}(\phi - 2\pi\alpha, E + i\varepsilon)|. \tag{3.16}$$

Integrating over $\mathbf{T}^d$ we get that

$$\int_{\mathbf{T}^d} |\tilde{\mathcal{R}}(\phi, E + i\varepsilon)| d\phi \leq 2(2\pi)^{d/2} + \int_{\mathbf{T}^d} |\gamma(\phi, E + i\varepsilon)| \cdot |\tilde{\mathcal{R}}(\phi, E + i\varepsilon)| d\phi.$$

Let

$$\mathcal{C}_E^{(1)} = \{\phi \in \mathbf{T}^d : |\Phi(\phi) - E| \leq 2\}, \qquad \mathcal{C}_E^{(2)} = \{\phi \in \mathbf{T}^d : |\Phi(\phi) - E| \geq 2\}.$$

We will use the shorthand $C_0 = 2(2\pi)^{d/2}$. Splitting the integrals we get that

$$\int_{\mathcal{C}_E^{(1)}} |\tilde{\mathcal{R}}(\phi, E + i\varepsilon)| \cdot (1 - |\gamma(\phi, E + i\varepsilon)|) d\phi \leq C_0 + \int_{\mathcal{C}_E^{(2)}} |\tilde{\mathcal{R}}(\phi, E + i\varepsilon)| \cdot (|\gamma(\phi, E + i\varepsilon)| - 1) d\phi.$$

Since $|\gamma(\phi, E + i\varepsilon)| < 1$,

$$\int_{\mathcal{C}_E^{(1)}} |\tilde{R}(\phi, E + i\varepsilon)| \cdot (1 - |\gamma(\phi, E + i\varepsilon)|) d\phi < C_0. \tag{3.17}$$

Let $\mathcal{O}$ be an open set such that its closure is properly contained in $\mathcal{C}_E^{(1)}$. Then it follows from Part 2 of Lemma 2.1 that there exists constant $C > 0$ such that for any $\phi \in \mathcal{O}$ and $0 \leq \varepsilon < 1$,

$$1 - |\gamma(\phi, E + i\varepsilon)| \geq C > 0,$$

and from Relation (3.17) that

$$\int_{\mathcal{O}} |\tilde{\mathcal{R}}(\phi, E + i\varepsilon)| d\phi \leq C_0/C.$$

Let $T_\gamma : \mathbf{T}^d \mapsto \mathbf{T}^d$ be the translation map, $T_\gamma(\phi) = \phi + 2\pi\gamma$. For any positive integer m let $\mathcal{O}_m \equiv T_{m\alpha}(\mathcal{O})$. It follows from Equation (3.16) that

$$\int_{\mathcal{O}_1} |\tilde{\mathcal{R}}(\phi, E + i\varepsilon)| \leq C_0 + C_0/C,$$

for all $0 < \varepsilon < 1$, and arguing inductively, that

$$\int_{\mathcal{O}_m} |\tilde{\mathcal{R}}(\phi, E + i\varepsilon)| d\phi \leq mC_0 + C_0/C.$$

Since α is independent over rationals, T_α is ergodic, and the open sets $\mathcal{O}_m$ cover $\mathbf{T}^d$. Picking a finite subcover, we derive that

$$\limsup_{\varepsilon \to 0} \int_{\mathbf{T}^d} |\tilde{\mathcal{R}}(\phi, E + i\varepsilon)| d\phi < \infty.$$

Theorem 1.2 now follows from Proposition 2.2.

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