CRITICAL SETS OF SMOOTH SOLUTIONS TO ELLIPTIC EQUATIONS*+

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August 13, 1997

Abstract. Let $u \not\equiv$ const satisfy an elliptic equation $L_0 u \equiv \Sigma a_{i,j} D_{ij} u + \Sigma b_j D_j u = 0$ with smooth coefficients in a domain in $\mathbb{R}^n$. It is shown that the critical set $|\nabla u|^{-1}\{0\}$ has locally finite $n-2$-dimensional Hausdorff measure. This implies in particular that for a solution $u \not\equiv 0$ of $(L_0 + c)u = 0$, with $c \in C^\infty$, the critical zero set $u^{-1}\{0\} \cap |\nabla u|^{-1}\{0\}$ has locally finite $n-2$-dimensional Hausdorff measure.

1. Introduction and Main Results

Let Ω be a domain in $\mathbb{R}^n, n \geq 3$ and let $u \not\equiv 0$ be a real valued classical solution of the elliptic partial differential equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{j=1}^{n} b_j D_j u + cu = 0 \text{ in } \Omega \tag{1.1}$$

where the real valued coefficients a_{ij}, b_j, c are C^∞ -functions in Ω.

Let Ω' be a compact subset of Ω; we denote

$$N(u,\Omega') \equiv \{x \in \Omega' | u(x) > 0\}, \Sigma(u,\Omega') \equiv \{x \in \Omega' || \nabla u|(x) = 0\}$$
$$\text{and } \Sigma_0(u,\Omega') \equiv N(u,\Omega') \cap \Sigma(u,\Omega').$$

In the following we shall show that locally the singular set Σ_0 of u has finite $n-2$-dimensional Hausdorff measure, i.e. $H^{n-2}(\Sigma_0(u,\Omega')) < \infty$.

Recently there is a rather rich literature describing the "size" of the zero sets and in particular the singular sets Σ_0 of solutions to elliptic equations in terms of

*Supported by Bundesministerium für Wissenschaft und Verkehr, Austria.
+Work partially supported by the European Union TMR grant FMRX-CT96-0001

the appropriate Hausdorff measure respectively Hausdorff dimension [Al, CF, CM, DF1, DF2, H, HS, L, N, R].

In [HOHON2] it was shown for the 3-dimensional case that locally the singular set Σ_0 of such a solution u has finite 1-dimensional Hausdorff measure. Here we generalize the result to $n \geq 3$ dimensions. The result is obtained by showing that the critical set Σ of a solution of (1.1) with $c \equiv 0$ has locally finite $n-2$-dimensional Hausdorff measure.

In contrast to the 2-dimensional case [Al,D,DF3,N] where Σ_0 consists of finitely many points the results on Σ_0 in dimension $n \geq 3$ are not as explicit [Y]. For solutions of equations with smooth enough coefficients it is known that Σ_0 is countably $n-2$-rectifiable [HS,H,RS].

Basic for all these investigations is the asymptotic behaviour of $u(x)$ for $x \to x_0$, where $u(x_0) = 0$. Let $\mathcal{O} \in \Omega$, and let u be a solution of (1.1), then it is well known (see e.g. [B]) that

(1.2)

$u(x) = p_M + \mathfrak{O}(|x|^{M+1})$ for $|x| \to 0$

with $p_M \not\equiv 0$

a homogeneous polynomial of degree M satisfying the osculating equation

$$\sum_{i,j} a_{ij}(\mathcal{O}) D_{ij} p_M = 0.$$

Assume without loss that $a_{ij}(\mathcal{O}) = \delta_{ij}$, then p_M is harmonic.

Therefore the investigations of the zero set respectively singular set of a solution of (1.1) are motivated by the desire to understand to which extent these sets can be described locally by the zero sets respectively critical sets of harmonic homogeneous polynomials. For a harmonic polynomial P_M of degree M in n variables it is known (see e.g. [HS]) that for some $C(n) < \infty$

$$H^{n-2}(\Sigma(P_M, B_1)) \leq C(n) M^2 \tag{1.3}$$

B_1 denoting a ball with radius 1. Lin [L] stated the corresponding conjecture for solutions of elliptic equations.

On the other hand there are examples (see [HOHON1]) showing that the singular set of a solution of a Schrödinger equation in $\mathbb{R}^3$ can locally be the segment of a straight line.

To state now our main results we define the elliptic operator $\mathcal{L}_0$ by

$$\mathcal{L}_0 = \mathcal{L} - c$$

with $\mathcal{L}$ and c given accordingly to (1.1).

Theorem 1.1. *Let $u \not\equiv$ const satisfy*

$$\mathcal{L}_0 u = 0 \ \textit{in} \ \Omega, \Omega \subset \mathbb{R}^n. \tag{1.1'}$$

Then for every compact subset Ω' of Ω

$$H^{n-2}(\Sigma(u, \Omega')) < \infty.$$

Corollary 1.1. *Let $u \not\equiv 0$ satisfy equation* (1.1). *Then for every compact subset Ω' of Ω*

$$H^{n-2}(\Sigma_0(u,\Omega')) < \infty.$$

The Corollary is a rather immediate consequence of Theorem 1.1:

Proof of the Corollary. Given $x_0 \in \Omega$ there is a neighbourhood $U(x_0)$ and a $u_0 \in C^\infty(U(x_0))$ with $u_0 > 0$ and $\mathcal{L}u_0 = 0$ in $U(x_0)$. (see e.g. [BJS], p. 228). It is easily seen that $\mu \equiv \frac{u}{u_0}$ satisfies in $U(x_0)$ an equation of type (1.1), so that by Theorem 1.1, $H^{n-2}(\Sigma(\mu,U')) < \infty$ for every compact subset U' of $U(x_0)$. Furthermore the singular set of u is a subset of the critical set of μ. $\square$

In Section 2 we give the proof of Theorem 1.1. The structure of the proof is in principle the same as for the 3-dimensional case, but due to the higher dimension new difficulties arise which require new ideas. Very roughly speaking the existence of so called "good" 2-dimensional subspaces L of $\mathbb{R}^n$ will be proven: They have the property that the number of critical points of the solution u in $L+\{y\}$, where $y \in L^\perp$, $L^\perp$ the orthogonal complement of L in $\mathbb{R}^n$, can be bounded from above uniformly in a neighbourhood B of the origin. For this we need to show that we can apply some results from singularity theory [AGV], for which in turn it is necessary to investigate the critical set of a harmonic homogeneous polynomial P in $\mathbb{C}^n$ with real coefficients. The uniform estimates mentioned above together with a result on the location of the critical set of u in a small neighbourhood of a critical point of u are then used to represent $\Sigma(u,B)$ as a union of finitely many sets, where each of them is in a unique correspondence to a "good" 2-plane. Then suitable δ-coverings of $\Sigma(u,B)$ are constructed, which for $\delta \downarrow 0$ yield that $H^{n-2}(\Sigma(u,B))$ is finite.

For the proof of Theorem 1.1 in the case $n = 3$ we used in the main steps the special geometry in the 3-dimensional space: In particular we used that the number of critical points of a harmonic homogeneous polynomial P on the unit sphere is finite. We also used the special structure of the critical set of such a polynomial in complex space. For $n \geq 3$ these arguments have to be replaced by new ones, which will essentially reduce to arguments concerning the dimension of the variety of "good" 2-dimensional subspaces in $\mathbb{R}^n$ and the dimension of the critical set of P in complex space.

2. Proof of Theorem 1.1

Let B denote a ball centered in $\mathcal{O} \in \mathbb{R}^n$ and without loss let $\overline{B} \subset \Omega$ and

$$a_{ij}(\mathcal{O}) = \delta_{ij} \quad 1 \leq i,j \leq n. \tag{2.1}$$

Further, we assume without loss that

$$\mathcal{O} \in \Sigma_0(u). \tag{2.2}$$

Since $u \in C^\infty$ and $\forall x_0 \in B$, $u - u(x_0)$ is a solution of (1.1') it follows (see e.g. [B]) that $u - u(x_0)$ vanishes in x_0 with some finite order $M(x_0)$, i.e. for some homogeneous polynomial $p^{(x_0)}_{M(x_0)} \not\equiv 0$ with degree $M(x_0) \geq 1$ we have for $x \to x_0$

$$u(x_0) - u(x_0) = p^{(x_0)}_{M(x_0)} + \mathfrak{o}(|x - x_0|^{M(x_0)}) \tag{2.3}$$

and for every multi index $\alpha, |\alpha| \leq M(x_0)$

$$D^\alpha u(x) = D^\alpha p^{(x_0)}_{M(x_0)}(x - x_0) + \mathfrak{o}(|x - x_0|^{M(x_0)-|\alpha|}). \tag{2.4}$$

Furthermore, $p^{(x_0)}_{M(x_0)}$ satisfies the corresponding osculating equation in the point x_0, namely

$$\sum_{i,j} a_{ij}(x_0) D_{ij} p^{(x_0)}_{M(x_0)}(x - x_0) = 0. \tag{2.5}$$

¿From the above it follows (via unique continuation) that

$$\overline{M} \equiv \sup\{M(x_0) | x_0 \in B\} < \infty. \tag{2.6}$$

$$\begin{aligned} &\text{For } x_0 = \mathcal{O} \text{ we denote } p^{(\mathcal{O})}_{M(\mathcal{O})} = P_M \\ &\text{and due to the foregoing (compare also (1.2))} \quad \Delta P_M = 0. \end{aligned} \tag{2.7}$$

□

In the first step of the proof we shall obtain information on the location of the critical set of u in the neighbourhood of a critical point of u. For these consideration and also for the following ones the 2-dimensional linear subspaces of V^n, where V stands for $\mathbb{R}$ or $\mathbb{C}$ will play an essential role. This algebraic variety, the Grassmannian, will be denoted by $G_{2,n}(V)$. In the following proposition $G_{2,n} = G_{2,n}(\mathbb{R})$.

Proposition 2.1. *Let $A(x) = (a_{i,j}(x))_{1 \leq i,j \leq n}$, $\forall x \in \overline{B}$ be a real valued symmetric matrix with $a_{ij} \in C(\overline{B})$, $\forall i, j$, where for some $\gamma > 0$, $A(x) \geq \gamma$, $\forall x \in \overline{B}$.*

Let $M \in \mathbb{N}, M \geq 2$ and let $\mathcal{P}_M$ be the set of real valued homogeneous polynomials in $\mathbb{R}^n$ of degree ≥ 2 and $\leq M$ with the property:

$$\begin{aligned} &\textit{Every } p \in \mathcal{P}_M \textit{ satisfies a differential equation} \\ &\sum_{i,j=1}^{n} a_{ij}(x_p) D_{ij} p = 0 \textit{ in } \mathbb{R}^n, \textit{ for some } x_p \in \overline{B}. \end{aligned} \tag{2.8}$$

Then

$$\begin{aligned} &\textit{there is a } c = c(M) > 0 \textit{ such that } \forall p \in \mathcal{P}_M, \\ &\exists L \in G_{2,n} \textit{ with } \textit{ dist } (\Sigma(p, S^{n-1}), \sigma) > c, \\ &\textit{where } \sigma = L \cap S^{n-1}. \end{aligned} \tag{2.9}$$

Remark 2.1.

1. We note that the set $\{p^{(x_0)}_{M(x_0)}, x_0 \in \Sigma(u, \overline{B})\} \equiv \mathcal{P}$ given accordingly to (2.3) and (2.5) obviously satisfies condition (2.8), with $A(x)$ being the matrix of the coefficients of the equation (1.1), and $\mathcal{P} \subset \mathcal{P}_{\overline{M}}$ with $\overline{M}$ given in (2.6).

2. Every $p \in \mathcal{P}_M$ can be transformed by a non singular linear transform T_p into a harmonic homogeneous polynomial P, where $P(y) = p(T_p y), y \in \mathbb{R}^n$, with $x = T_p y$. Furthermore $A(x_p) = T_p T_p^+$.

Proof of Proposition 2.1:. Suppose for contradiction that (2.10) does not hold, then

$$\forall \delta > 0, \exists p \in \mathcal{P}_M : \forall L \in G_{2,n}, \quad \text{dist } (\Sigma(p, S^{n-1}), \sigma) \leq \delta, \quad \sigma = L_\cap S^{n-1}. \tag{2.10}$$

Picking a sequence $\delta_m = \frac{1}{m}$, $m \in \mathbb{N}$ we obtain

$$\forall m \in \mathbb{N}, \quad \exists p_m \in \mathcal{P}_M : \forall L \in G_{2,n} \quad \text{dist } (\Sigma(p_m, S^{n-1}), \sigma) \leq \delta. \tag{2.11}$$

Because of the homogeneity of p_m we can normalize the coefficients of p_m and obtain (with $M(m)$ denoting the degree of p_m)

$$\tilde{p}_m(x) \equiv p_m(\lambda x) = \lambda^{M(m)} p_m(x), \text{ for some } \lambda > 0$$

where

$$\tilde{p}_m(x) = \sum_{|\alpha| = M(m)} a_\alpha^{(m)} x^\alpha, \sum_{|\alpha| = M(m)} |a_\alpha^{(m)}|^2 = 1$$

and

$$\Sigma(\tilde{p}_m) = \Sigma(p_m). \tag{2.12}$$

This obviously implies that there is a convergent subsequence of $\{\tilde{p}_m\}$, again denoted by $\{\tilde{p}_m\}$. Since $M(m) \leq M$, $\forall m$ we assume without loss that $M(m) = k$, $\forall m$ for some $k \in \{2, \ldots, M\}$. Furthermore we obtain that for some homogeneous polynomial of degree k with normalized coefficients

$$\begin{aligned} &D^\alpha \tilde{p}_m \to D^\alpha \tilde{p}, \text{ for } m \to \infty, \quad \forall |\alpha| \leq k, \text{ where the} \\ &\text{convergence is pointwise uniformly in every compact subset of } \mathbb{R}^n. \end{aligned} \tag{2.13}$$

Therefrom we conclude

$$\Sigma(\tilde{p}) \cap L \neq \{\mathcal{O}\}, \forall L \in G_{2,n}, \tag{2.14}$$

which can be seen as follows:

Suppose for contradiction that (2.14) is wrong, then

$$\exists \tilde{L} \in G_{2,n} : \Sigma(\tilde{p}, S^{n-1}) \cap \tilde{\sigma} = \emptyset \text{ where } \tilde{\sigma} = \tilde{L} \cap S^{n-1}. \tag{2.15}$$

(2.11) and (2.12) imply that

$$\forall m, \exists \tilde{x}_m \in \Sigma(\tilde{p}_m, S^{n-1}) \text{ with dist } (\tilde{x}_m, \tilde{\sigma}) \leq \frac{1}{m}. \tag{2.16}$$

Hence we can choose a convergent subsequence of $\{\tilde{x}_m\}$, again denoted by $\{\tilde{x}_m\}$, with $\tilde{x}_m \to \tilde{x}$, for $m \to \infty$, for some $\tilde{x} \in \tilde{\sigma}$. Due to (2.16)

$$|D_j \tilde{p}(\tilde{x})| \leq |D_j \tilde{p}(\tilde{x}) - D_j \tilde{p}(\tilde{x}_m)| + |D_j \tilde{p}(\tilde{x}_m) - D_j \tilde{p}_m(\tilde{x}_m)| \quad \forall j, 1 \leq j \leq n$$

and via (2.13) we obtain $\tilde{x} \in \Sigma(\tilde{p}) \cap \tilde{\sigma}$, which is a contradiction to (2.15). This verifies (2.14).

Next we show that $\tilde{p}$ satisfies property (2.8): due to (2.8)

$$\sum_{i,j} a_{ij}(x_{p_m}) D_{ij} \tilde{p}_m = 0, \text{ for some } x_{p_m} \in \overline{B}, \forall m. \tag{2.17}$$

Clearly we can pick a convergent subsequence of $\{x_{p_m}\}$, denoted by $\{\tilde{x}_{p_m}\}$, with $\tilde{x}_{p_m} \to \tilde{x}$ for $m \to \infty$, for some $\tilde{x} \in \overline{B}$. Then due to the assumptions on $A(x)$, we obtain $a_{ij}(\tilde{x}_{p_m}) \to a_{ij}(\tilde{x})$, for $m \to \infty$, $\forall i, j$. Together with (2.13) this implies that for $m \to \infty$

$$\sum_{i,j} a_{ij}(\tilde{x}_{p_m}) D_{ij} \tilde{p}_m(x) \to \sum_{i,j} a_{ij}(\tilde{x}) D_{ij} \tilde{p}(x), \forall x \in \mathbb{R}^n$$

and because of (2.17)

$$\sum_{i,j} a_{ij}(\tilde{x}) D_{ij} \tilde{p} = 0 \tag{2.18}$$

follows, verifying that $\tilde{p} \in \mathcal{P}_m$.

Taking into account Remark 2.1.2 we note that $\tilde{p}$ can be transformed into a harmonic homogeneous polynomial $\tilde{P}(y) = \tilde{p}(Ty)$. From (2.14)

$$\Sigma(\tilde{p}) \cap L \neq \{\mathcal{O}\}, \forall L \in G_{2,n} \tag{2.19}$$

follows easily, namely:

Since $\det T \neq 0$ we obviously have

$$y \in \Sigma(\tilde{P}) \backslash \{\mathcal{O}\} \text{ iff } x = Ty \in \Sigma(\tilde{p}) \backslash \{\mathcal{O}\}. \tag{2.20}$$

Let $L \in G_{2,n}$, then for some $a, b \in \mathbb{R}^n$, a, b linear independent, $L = \{\lambda a + \mu b | \lambda, \mu \in \mathbb{R}\}$. Denote further $\tilde{L} = \{\lambda Ta + \mu Tb | \lambda, \mu \in \mathbb{R}\}$, then $\tilde{L} \in G_{2,n}$ and because of (2.14) there is an $\tilde{x} \neq \mathcal{O}$ with $\tilde{x} \in \Sigma(\tilde{p}) \cap \tilde{L}$. By (2.20) we obtain $\tilde{y} = T^{-1}\tilde{x} \in \Sigma(\tilde{P}) \backslash \{\mathcal{O}\}$. Since $\tilde{x} = \tilde{\lambda} Ta + \tilde{\mu} Tb$ for some $\tilde{\lambda}, \tilde{\mu} \in \mathbb{R}$ we conclude by the linearity of T that $\tilde{y} = \tilde{\lambda} a + \tilde{\mu} b \in L$, verifying (2.19).

(2.19) implies that

$$G = \{L \in G_{2,n} | \Sigma(\tilde{P}) \cap L \neq \{\mathcal{O}\}\} = G_{2,n}. \tag{2.21}$$

On the other hand since $\tilde{P}$ is harmonic we have (compare (1.3))

$$\dim \Sigma(\tilde{P}) \leq n - 2. \tag{2.22}$$

We shall derive a contradiction to (2.21) by taking into account (2.22) via a dimension argument for the subvariety G of $G_{2,n}$. For further use we state it for the complex and real case. We note first that $G_{2,n}(V)$ is a subvariety of the projective space $\mathbb{P}(\wedge^2 V)$with

$$\dim G_{2,n}(V) = 2(n - 2). \tag{2.23}$$

Here and in the following it will be clear from the context (whether V equals $\mathbb{R}$ or $\mathbb{C}$) when dim denotes the real respectively complex dimension.

Proposition 2.1.1. *Let p be a homogeneous polynomial on V^n (with real valued coefficients) with the property that the dimension of $\Sigma(p)$ as a projective variety in $\mathbb{P}^{n-1}(V)$ is at most $n-3$, then*

$$\dim G \leq 2n-5, \text{ where } G = \{L \in G_{2,n}(V) | \Sigma(p) \cap L \neq \{\mathcal{O}\}\}.$$

Proof of Proposition 2.1.1. : Following Harris [Ha,p. 142] G is a variety of incident planes, having dimension at most $2n-5$. To show the assertion directly: Let g be a 1-dimensional linear subspace of V^n with $g \subset \Sigma(p)$, then clearly with $\mathcal{V}_g \equiv \{L \in G_{2,n}(V) | g \subset L\}$, $\dim \mathcal{V}_g = n-2$. But $G = \cup\{\mathcal{V}_g | g \subset \Sigma(p)\}$ and $\dim\{g | g \subset \Sigma(p)\} \leq n-3$. Since G has locally the structure of a smooth product manifold dim $G \leq 2n-5$ follows. $\square$

Due to (2.22) we can apply Proposition 2.1.1 with $p = \tilde{P}$. But because of (2.21) and (2.23) $\dim G = 2n-4$ which is a contradiction. This finishes the proof of Proposition 2.1 $\square$

For the next considerations we identify every $L \in G_{2,n}(\mathbb{R})$ with the corresponding great circle $\sigma = L \cap S^{n-1}$. We endow $G_{2,n}$ with the Hausdorff metric by defining

$$d(L_1, L_2) \equiv d(\sigma_1, \sigma_2) = \max\{\sup_{x \in \sigma_1} \inf_{y \in \sigma_2} |x-y|, \sup_{y \in \sigma_2} \inf_{x \in \sigma_1} |x-y|\}$$

(2.24)

$$\forall L_1, L_2 \in G_{2,n}, \text{ where } \sigma_j = L_j \cap S^{n-1}, j = 1, 2.$$

Then $G_{2,n}$with the topology induced by this metric is a compact space.

Lemma 2.1. *Let $\tilde{G}$ be a countable dense subset of $G_{2,n}(\mathbb{R})$. Then there exists $L^{(j)} \in \tilde{G}$, $1 \leq j \leq N < \infty$ and there exists a $c > 0$ such that $\forall x_0 \in \Sigma(u, \overline{B})$ one can choose an integer j, $1 \leq j \leq N$ with*

$$(a) \quad \text{dist } (\Sigma(p^{(x_0)}, S^{n-1}), \sigma_j) > c$$

$$(b) \quad \lim_{\substack{x \to x_0 \\ x \in \Sigma(u, \overline{B})}} \text{dist } (\frac{x - x_0}{|x - x_0|}, \sigma_j) > c$$

where $\sigma_j = L^{(j)} \cap S^{n-1}$ and $p^{(x_0)}$ is the homogeneous polynomial of degree $M(x_0)$ corresponding to x_0 according to (2.3). *$N = N(\overline{M})$ with $\overline{M}$ given in* (2.6).

Proof of Lemma 2.1. :
ad(a): Let $U = \{U_\lambda | \lambda \in I\}$ be an open δ-covering of $G_{2,n}$, i.e.

$$\text{diam } U_\lambda = \sup_{L, K \in U_\lambda} d(L, K) \leq \delta, \quad \forall \lambda \in I. \tag{2.25}$$

Since $G_{2,n}$ is compact there is always a finite subcovering

$$G_{2,n} = \cup_{k=1}^N U_k, \quad U_k \in U, \quad \forall k \tag{2.26}$$

where N depends on δ.

Let $p^{(x_0)}$ denote the homogeneous polynomials given according to (2.3)-(2.6) and let

$$\mathcal{P} \equiv \{p^{(x_0)} | x_0 \in \Sigma(u, \overline{B})\},$$

then by Proposition 2.1.and Remark 2.1 we have:

$$\begin{aligned} &\text{There exists } c = c(\overline{M}) > 0 \text{ such that} \\ &\forall p \in \mathcal{P}, \quad \exists L \in G_{2,n} \text{ with} \\ &\text{dist } (\Sigma(p, S^{n-1}), \sigma) > c, \quad \sigma \equiv L \cap S^{n-1}. \end{aligned} \tag{2.27}$$

Now we choose $\delta = \frac{c(\overline{M})}{4}$, so that $N = N(\overline{M})$ and pick for every $j \in \{1, \ldots, N\}$ an $L_j \in U_j$. Let $\mathcal{L}_\delta = \{L_1, \ldots, L_N\}$, then (2.25) implies:

$$\forall L \in G_{2,n}, \quad \exists L_j \in \mathcal{L}_\delta \text{ with } d(L, L_j) \leq \delta. \tag{2.28}$$

Let $\tilde{G} = \{K_j, j \in \mathbb{N}\}$ be a dense subset of $G_{2,n}$.

Then

$$\forall L_j \in \mathcal{L}_\delta, \quad \exists \tilde{K}_j \in \tilde{G} \text{ with } d(L_j, \tilde{K}_j) \leq \delta \tag{. 2.29}$$

Denoting $\tilde{\mathcal{L}} = \{\tilde{K}_1, \ldots, \tilde{K}_N\}$ we obtain via (2.28) and (2.29) that

$$\begin{aligned} &\forall L \in G_{2,n}, \quad \exists j \in \{1, \ldots, N\} : \\ &d(L, \tilde{K}_j) \leq d(L, L_j) + d(L_j, \tilde{K}_j) \leq 2\delta = \frac{c(\overline{M})}{2}. \end{aligned} \tag{2.30}$$

By the triangle inequality and the definition of the Hausdorff metric we obtain easily via (2.27) and (2.30):

Given $\mathcal{P}$ and a countable dense subset $\tilde{G}$ of $G_{2,n}$ then there is a $c(\overline{M}) > 0$ and a finite subset $\tilde{L}$ of $\tilde{G}$ such that $\forall p \in \mathcal{P}$, $\exists \tilde{L} \in \tilde{\mathcal{L}}$ with

$$\text{dist } (\Sigma(p, S^{n-1}), \tilde{\sigma}) \geq \frac{c(M)}{(2)}.$$

This verifies (a).

ad (b): We first note that (a) is equivalent with

$$\begin{aligned} &\exists \mu > 0 : \forall x_0 \in \Sigma(u, \overline{B}), \quad \exists j, \quad 1 \leq j \leq N \text{ with} \\ &\sup_{\omega \in \sigma_j} | < \omega | y > | < 1 - \mu, \quad \forall y \in \Sigma(p^{(x_0)}, S^{n-1}) \end{aligned} \tag{a'}$$

and we show that (a') implies

$$\overline{\lim}_{x \to x_0} \sup_{\omega \in \sigma_j} | < \omega | \frac{x - x_0}{|x - x_0|} > | < 1 - \mu, \quad x \in \Sigma(u) \tag{b'}$$

This can be seen from the following: Let $x_m \to \overline{x}$, for $m \to \infty$, $x_m \in \Sigma(u, \overline{B})$ and denote in polar coordinates $x_m - \overline{x} = r_m y_m(r_m)$, $\forall m$ with $y_m \in S^{n-1}$. Without loss we assume that $y_m \to \overline{y}$, for $m \to \infty$. Since

$$0 = D_j u(x_m) = D_j p_{M(\overline{x})}^{(\overline{x})}(r_m y_m) + \mathfrak{o}(r_m^{M(\overline{x})-1}), \forall j,$$

$\overline{y} \in \Sigma(p_{M(\overline{x})}^{(\overline{x})}) \cap S^{n-1}$ follows.

(a') implies for $m \to \infty$

$$| < \omega | y_m > | \to < \omega | \overline{y} > \quad < 1 - \mu, \quad \forall \omega \in \sigma_j$$

which yields (b') and clearly (b') implies (b). This finishes the proof of Lemma 2.1. □

For the next step of the proof of Theorem 1.1 we need a property of harmonic homogeneous polynomials in $\mathbb{C}^n$ with real coefficients, which is interesting for itself (We thank H. Knörrer for crucial remarks):

Theorem 2.1. *: Let P be a harmonic homogeneous polynomial in $\mathbb{C}^n$ with real coefficients, $P \neq$ const , then dim $\Sigma(P) \leq n-2$.*

Proof of Theorem 2.1. For an irreducible polynomial in $\mathbb{C}^n$ the assertion is true. See e.g. [S, Chap. II, §1.4].

So let P be reducible. We assume for contradiction that $\dim \Sigma(P) = n-1$. then P can be represented as

(2.31) $\quad P = p^2 q$, where p, q are homogeneous polynomials and p is irreducible .

This can be seen as follows: Let $P = \prod_{j=1}^k q_j$, $k \geq 2$, $\quad q_j$ irreducible $\forall j$, and denote $N_j = N(q_j)$. If for $i \neq j$, dim $N_i \cap N_j < n-1$, then clearly $\dim \Sigma(P) < n-1$. So let without loss $\dim N_1 \cap N_2 = n-1$. Since q_1, q_2 are irreducible this implies (see e.g. [M] Lemma 2.5) $q_1 =$ const q_2, proving (2.31).

(2.31) implies

(2.32) $\quad P = \tilde{p}^2 \tilde{q}$, where $\tilde{p}, \tilde{q}$ are homogeneous polynomials with real coefficients.

This can be seen as follows: Let $\underline{f}$ denote the polynomial which is obtained from a polynomial f by complex conjugation of its coefficients. Since P has real coefficients we conclude from (2.31) that $P = \underline{p}^2 \underline{q} = p^2 q$. Since p is irreducible $q = \underline{p}^2 \tilde{q}$ follows, for some homogeneous polynomial $\tilde{q}$. Hence $P = (p\underline{p})^2 \tilde{q}$ and $p\underline{p}$ has real coefficients.

Finally we use:

Proposition 2.2. *: Let $P \neq$ const be a harmonic homogeneous polynomial in $\mathbb{R}^n$ given by $P = p^2 q$, p, q polynomials with real coefficients and $p \neq$ const, then $P = 0$.*

Proof of Proposition 2.2. Let M denote the degree of P, then for some spherical harmonic $Y_M(\frac{x}{|x|})$, $P(x) = |x|^M Y_M$ in polar coordinates. Further we have

$$q|_{S^{n-1}} = \sum_{j=1}^{M-1} a_j Y_j,$$

where the $Y_j's$ are spherical harmonics of degree $\leq M-1$, which can be taken orthonormal on S^{n-1} and $a_j \in \mathbb{R}$, $\forall j$. Hence $\int_{S^{n-1}} Y_j Y_M d\omega = 0$ for $j \neq M$,

$$\text{and } \int_{S^{n-1}} pq d\omega = \int_{S^{n-1}} p^2 q^2 d\omega = 0$$

implying $p = q = 0$ $\square$

Together with (2.32) this gives $P \equiv 0$ which is a contradiction to $P \not\equiv$ const and finishes the proof of Theorem 2.1. $\square$

For the next step of the proof of Theorem 1.1 we need

Definition 2.1:. Let $P(z)$, $z \in \mathbb{C}^n$ denote a harmonic homogeneous polynomial, $\not\equiv$ const, with real coefficients. Let $W \in G_{2,n}(\mathbb{C})$ such that W is spanned by two vectors of $\mathbb{R}^n$. The subspace $L = W \cap \mathbb{R}^n$ of $\mathbb{R}^n$ will be called good with respect to $P(x)$, if $P|_W(z)$ has an isolated critical zero in the origin of $\mathbb{C}^2$.

Lemma 2.2. *: Let P be as in Definition 2.1, then the set*

$$G \equiv \{L \in G_{2,n}(\mathbb{R}) | L \textit{ is good } \}$$

is dense in $G_{2,n}(\mathbb{R})$.

Proof of Lemma 2.2. Let $G^c = G_{2,n}(\mathbb{R}) \backslash G$. Obviously it suffices to show that
(2.33)
$$\forall L \in G^c, \quad \exists L_n = [a_n, b_n] \in G \text{ with } (a_n, b_n) \to (a,b) \text{ for } n \to \infty, [a,b] = L.$$

Thereby [a,b] denotes the 2-dimensional subspace of $\mathbb{R}^n$ spanned by linear independent $a, b \in \mathbb{R}^n$.

Suppose now for contradiction that (2.33) does not hold. Then for some $L_0 = [a_0, b_0] \in G^c$ there is a neighbourhood U_0 of (a_0, b_0) in $\mathbb{R}^{2n}$ such that $\forall (a,b) \in U_0$, $L = [a,b] \in G^c$. Denote by $L_{a,b}$ the 2-dimensional subspace of $\mathbb{C}^n$, which is spanned by a, b. Then

(2.34)
$$\forall (a,b) \in U_0 \subset \mathbb{R}^{2n}, P|_{L_{a,b}} \text{ contains a critical line of zeros through the origin } \mathcal{O} \in \mathbb{C}^2.$$

Proposition 2.3. *Let*

$$G_\Sigma = \{L \in G_{2,n}(\mathbb{C}) | \quad P|_L \textit{ has a critical line of zeros through } \mathcal{O} \in \mathbb{C}^2\}.$$

Suppose (2.33) *does not hold, then* G_Σ *contains a non empty open subset of* $G_{2,n}(\mathbb{C})$.

Proof of Proposition 2.3. Due to (2.34), $P|_{L_{a,b}} \in G_\Sigma$, $\forall (a,b) \in U_0$. Let

$$P|_{L_{a,b}}(\alpha, \beta) = P(\alpha a + \beta b), \quad \forall (\alpha, \beta) \in \mathbb{C}^2$$

then the homogeneity of $P|_{L_{a,b}}$ in $(\alpha.\beta)$ implies that there is at least one $\tilde{\alpha} \in \mathbb{C}$ such that

$$P(\tilde{\alpha} a + b) = \frac{\partial}{\partial \alpha} P(\tilde{\alpha} a + b) = 0.$$

Define $f(\alpha, a, b) = P(\alpha a + b)$, then for fixed (a, b) f is a polynomial in α having at least one critical zero of order $k \leq M$, where M denotes the degree of P. Suppose now that $f(\alpha, a_0, b_0)$ has a zero of order $k \geq 2$ in α_0, where the order of the zero is stable in the sense that for a sufficiently small neighbourhood U_1 of (a_0, b_0), $f(\alpha, a, b)$ has a zero of order $\geq k$, $\forall (a, b) \in U_1$. We assume this without loss, since instead of (a_0, b_0) we could pick $(a', b') \in U$, U an open subset of U_0, such that k is the minimal order of critical zeros of $f(\alpha, a, b)$, $\forall (a, b) \in U$.

The foregoing now implies that for some continuous $\tilde{\alpha} : U_0 \to \mathbb{C}$, $f(\alpha, a, b)$ has $\forall (a, b) \in U_0$ a zero of order k in $\tilde{\alpha}(a, b)$, i.e.

$$\begin{aligned} &\frac{\partial^j}{\partial \alpha^j} f(\tilde{\alpha}(a,b), a, b) = 0, \quad 0 \leq j \leq k, \\ &\frac{\partial^{k+1}}{\partial \alpha^{k+1}} f(\tilde{\alpha}(a,b), a, b) \neq 0, \quad \forall (a,b) \in U_0 \end{aligned} \tag{2.35}$$

where $U_0 \subset \mathbb{R}^{2n}$ is some neighbourhood of (a_0, b_0). In particular we have

$$\frac{\partial^k}{\partial \alpha^k} f(\alpha_0, a_0, b_0) = 0, \frac{\partial^{k+1}}{\partial \alpha^{k+1}} f(\alpha_0, a_0, b_0) \neq 0. \tag{2.36}$$

Furthermore $\frac{\partial^k}{\partial \alpha^k} f\ (\alpha, a, b) = 0$ is an analytic function in $(\alpha, a, b) \in \mathbb{C} \times \mathbb{C}^{2n}$. This and (2.36) leads via the Weierstraß-Preparation Theorem to: For some neighbourhood $W_0 \subset \mathbb{C} \times \mathbb{C}^{2n}$ of (α_0, a_0, b_0) all the zeros of $\frac{\partial^k}{\partial \alpha^k} f\ (\alpha, a, b)$ are given by

$$\frac{\partial^k}{\partial \alpha^k} f(\alpha(a,b), a, b) = 0,$$

where $\alpha(a, b)$ is analytic.

Together with (2.35) the above implies that for some neighbourhood $U_1 \subset \mathbb{R}^{2n}$ of (a_0, b_0) there is a real analytic function $\alpha : U_1 \to \mathbb{C}$ such that $\alpha(a, b)$ is a zero of order k of $f(\alpha, a, b)$ and (2.35) holds with $\tilde{\alpha} = \alpha$ in U_1. Analytic continuation of $\frac{\partial^j}{\partial \alpha^j} f\ (\alpha(a, b), a, b)$ to some complex neighbourhood V_0 of (a_0, b_0) and (2.34) implies that $\forall (a, b) \in V_0 \subset \mathbb{C}^{2n}$, $P|_{L_{\mathbb{C}}(a,b)}$ has a critical line of zeros through $\mathcal{O} \in \mathbb{C}^2$. This verifies Proposition 2.3 $\square$

Proposition 2.3 implies that

$$\dim G_\Sigma = \dim G_{2,n} = 2n - 4 \tag{2.37}$$

But on the other hand we shall obtain that

$$\dim G_\Sigma \leq 2n - 5 \tag{2.38}$$

which is a contradiction, and will finish the proof of Lemma 2.2.

(2.38) can be seen as follows: Because of the homogeneity of P we have

$$\begin{aligned} &G_\Sigma = G_1 \cup G_2 \\ &\quad \text{with} \\ &G_1 = \{L \in G_{2,n}(\mathbb{C}) | \Sigma(P) \cap L \neq \{\mathcal{O}\}\}, \\ &G_2 = \{L \in G_{2,n}(\mathbb{C}) | \Sigma(P) \cap L = \{\mathcal{O}\}, \\ &\quad \exists \text{ a critical line of zeros of } P|_L \text{ through } \mathcal{O} \in \mathbb{C}^2\}. \end{aligned}$$

Due to Theorem 2.1, P satisfies the assumptions of Proposition 2.1.1 and thus

$$\dim\, G_1 \leq 2n-5. \tag{2.39}$$

Now let g be a 1-dimensional subspace of $\mathbb{C}^n$ with $g \subset N(P)\backslash\Sigma(P)$ and let E_g denote the corresponding tangent hyperplane to $P=0$ containing g. Suppose g is a critical line of zeros of $P|_L$. Then it is straight forward to see that $L \subset E_g$. Therefore we can represent G_2 as follows:

$$G_2 = \cup\{W_g | g \subset N(P)\backslash\Sigma(P)\}$$

where

$$W_g = \{L \in G_{2,n} | g \subset L \subset E_g\}.$$

Obviously $\dim\, W_g = n-3$. Further the dimension of $N(P)$ as a projective variety is at most $n-2$. Therefrom we conclude $\dim\, G_2 \leq 2n-5$ since G_2 is locally a product manifold. This together with (2.39) yields (2.38). $\square$

Now we combine Lemma 2.1 and 2.2 and obtain

Lemma 2.3. *Given P_M according to* (2.7)*, then for some $N=N(M)>0$ there exists a subset $G \equiv \{L^{(1)},\ldots,L^{(N)}\}$ of $G_{2,n}(\mathbb{R})$, where $\forall j, 1\leq j\leq N$, $L^{(j)}$ is good with respect to P_M and for some $c(M)>0$ the assertions* (a) *and* (b) *of Lemma 2.1 hold.*

Lemma 2.3 allows us to classify the critical points of u as follows:

$$\Sigma(u,\overline{B}) = \cup_{j=1}^N E_j,$$

where

$$E_j = \{x_o \in \Sigma(\overline{u},\overline{B}) | (a) \text{ and } (b) \text{ hold with } L^{(j)} \in G\}. \tag{2.42}$$

Note that the sets $E_j, 1\leq j\leq N$ are not necessarily pairwise disjoint.

The existence of "good" 2-dimensional subspaces of $\mathbb{R}^n$ is now used for the essential step of the proof of Theorem 1.1.

Lemma 2.4. *: Let $L \in G_{2,n}(\mathbb{R})$ be good with respect to P_M given in* (2.7)*. Then for some $d=d(M)>0$ there exists $\tilde{R}>0$:*

$$\#\Sigma(u, L(\tilde{R})+\{y\}) \leq d, \quad \forall y \in L^\perp \textit{ with } |y| \leq \tilde{t}$$

for $\tilde{t}$ small enough, where $L(\tilde{R}) = \{x \in L | |x| < \tilde{R}\}$ and $L^\perp$ is the orthogonal complement of L in $\mathbb{R}^n$.

Proof of Lemma 2.4. : The proof is similar to that of Lemma 2.3. We use Proposition 2.2 of [HOHON2], namely:

Proposition 2.4.1. *: Let $p(z_1,z_2)$ be a homogeneous polynomial in $\mathbb{C}^2$ of degree k with real coefficients, and assume that p has an isolated critical point in the origin in $\mathbb{C}^2$. Let further $\varphi \in C^\infty(D_r(\mathcal{O}))$, $D_r(\mathcal{O}) = \{y \in \mathbb{R}^2 | |y| < r\}$, $r>0$, with*

$$\varphi(y) = p(y) + \mathfrak{o}(|y|^k) \quad \textit{for } |y| \to 0$$

and let $\varphi_t(y) \in C^\infty(D(\mathcal{O}) \times I)$ for $t \in I, I = [-t_0, t_0]$ for some $t_0 > 0$, with $\varphi_0 = \varphi$. Then there exists $\tilde{r}$, $0 < \tilde{r} < r$ such that for $|t| \le t_0$, t_0 small enough, the number of critical points of $\varphi_t(.)$ in $D_{\tilde{r}}(\mathcal{O})$ is uniformly bounded by a constant $d(k)$. Thereby $d(k)$ is the algebraic multiplicity of ∇p in $\mathcal{O} \in \mathbb{C}^2$.

For the proof we used results of [AGV], namely: for a homogeneous polynomial $p(z)$, $z \in \mathbb{C}^2$ of degree k, with an isolated critical point in the origin $\mathcal{O}$ the algebraic multiplicity $d(k)$ of the gradient map of p in $\mathcal{O}$ is finite. This together with the subadditivity of the algebraic multiplicity yields the result, which can be stated as

$$\#\Sigma(\varphi_t, D_{\tilde{r}}(\mathcal{O})) \le d(k) \quad \forall t, |t| \le t_0.$$

We apply this to our case: let $L \in G_{2,n}(\mathbb{R})$ be a good plane with respect to P_M, which we represent as $L = \{y_1\nu_1 + y_2\nu_2 | y_1, y_2 \in \mathbb{R}\}$ where $\nu_1, \nu_2 \in S^{n-1}$ and $< \nu_1 | \nu_2 >= 0$.

We identify $p(y_1, y_2) = P_M(y_1\nu_1 + y_2\nu_2)$ and $\varphi_0(y_1, y_2) = u(y_1\nu_1 + y_2\nu_2)$. Let $\gamma \in C^\infty$ denote a curve in $L^\perp$ passing through the origin, parameterized such that $\gamma(0) = \mathcal{O}$. we define

$$\varphi_t(y_1, y_2) = u(y_1\nu_1 + y_2\nu_2 + \gamma(t)), t \in I.$$

Then we obtain due to Proposition 2.2 that for some $\overline{R} > 0$, and $\bar{t}$ small enough

$$\#\Sigma(u, L(\overline{R}) + \{\gamma(t)\}) \le d(M), \quad \forall t, \quad |t| \le \bar{t}. \tag{2.43}$$

We suppose now for contradiction that Lemma 2.4 is false, then there are sequences $\{R_k\}$ and $\{y^{(k)}\}$, with $y^{(k)} \in L^\perp$, $R_k \to 0$, $|y^{(k)}| \to 0$ for $k \to \infty$ such that

$$\#\Sigma(u, L(R_k) + \{y^{(k)}\}) > d(M) \quad \forall k.$$

Proposition 2.4.2. *Let $\{y^{(k)}\}$ denote a sequence in $\mathbb{R}^n$ convergent to some $\overline{y}$. Then there is a subsequence which is a subset of a C^∞-curve in $\mathbb{R}^n$.*

Proof of Proposition 2.4.2. : We use a result of A. Kriegl [Kr] (see also Lemma 4.2.15 in [FK]):

Let $x_m \in \mathbb{R}^n$, $x_m \to \overline{x}$ for $m \to \infty$ and let $t_m \in \mathbb{R}$, $t_m \downarrow 0$ for $m \to \infty$. If $\forall k$, $k \in \mathbb{N}$, $\{(x_m - x_{m+1})(t_m - t_{m+1})^{-k}\}$ is bounded, then for some C^∞-curve γ, $\gamma(t_m) = x_m$, $\forall m$ and $\gamma^{(j)}(t_m) = 0$, $\forall j \in \mathbb{N}$.

Given a convergent sequence $\{y^{(k)}\}$ it is easily seen that we can pick a subsequence converging fast enough so that the above assumptions are fulfilled. □

Therefore we conclude: We can pick a subsequence of $\{y^{(k)}\}$ (again denoted by $\{y^{(k)}\}$) such that for some $\gamma \in C^\infty$, $\gamma(t_k) = y^{(k)}$, $\forall k$ and

$$\#\Sigma(u, L(R_k)) + \{\gamma(t_k)\} > d(M), \quad \forall k. \tag{2.44}$$

On the other hand given γ, there is $\overline{R}, \bar{t} > 0$ such that (2.43) holds which is a contradiction to (2.44). This verifies Lemma 2.4. □

In the following step of the proof we shall need Lemma 2.3 and 2.4: According to (2.42) we have $\Sigma(u, \overline{B}) = \cup_{j=1}^N E_j$ and to each E_j corresponds a good $L^{(j)} \in G_{2,n}$ with respect to P_M such that (a) and (b) of Lemma 2.3 hold with this L. Let $R > 0$ and denote $\Sigma_j = E_j \cap \overline{B}_R$, $1 \le j \le N$, where $B_R = \{x \in \mathbb{R}^n | |x| < R\}$.

Lemma 2.5. *: For $R > 0$ small enough there is a $C < \infty$ such that $H^{n-2}(A) \le C$ for every closed subset A of $\Sigma_j = E_j \cap \overline{B}_R$, $1 \le j \le N$.*

Proof of Lemma 2.5. For simplicity we suppress in the following the index j in Σ_j, E_j and the corresponding $L^{(j)}$. So let A denote a closed subset of Σ and let $\pi : A \to L^\perp$, $L^\perp$ the orthogonal complement of L in $\mathbb{R}^n$, denote the orthogonal projection of $x \in A$ to $L^\perp$. We have

Proposition 2.5.1. *: There exists $R, \tilde{t}, c_0 > 0$, $d < \infty$ such that*

$$\begin{aligned} &\text{for every closed subset} \quad A \text{ of } \Sigma \cap \overline{B}_R \text{ and} \\ &\forall \overline{y} \in L^\perp \text{ with } |\overline{y}| \le \overline{t} \\ &\pi^{-1}(\overline{y}) = \{\overline{x}^{(1)}, \ldots, \overline{x}^{(\overline{k})}\} \text{ with } \overline{k} \le d \end{aligned} \tag{2.45}$$

and $\forall \delta$, $\delta \le \delta_0(\overline{y})$

$$\begin{aligned} &\pi^{-1}(D_\delta(\overline{y})) \subset \cup_{j=1}^{\overline{k}} \overline{B}_\rho(\overline{x}^{(j)}), \text{ with } \rho = \frac{\delta}{c_0}, \\ &\text{where } D_\delta(\overline{y}) = \{y \in L^\perp | |y - \overline{y}| < \delta\}. \end{aligned} \tag{2.46}$$

Proof of Proposition 2.5.1. : Due to Lemma 2.3 $L \in G_{2,n}$ is good with respect to P_M, hence by Lemma 2.4 we obtain for $R, \overline{t}$ small enough that (2.45) holds with $d = d(\overline{M})$. Suppose now for contradiction that (2.46) does not hold, then

$$\begin{aligned} &\forall c_0 > 0 \text{ there is a sequence } \delta_m \downarrow 0 \text{ and} \\ &x^{(m)} \in \pi^{-1}(D_{\delta_m}(\overline{y})) \setminus \cup_{j=1}^{\overline{k}} \overline{B}_{\rho_m}(\overline{x}^{(j)}), \\ &\text{with } \rho_m = \frac{\delta_m}{c_0}. \end{aligned} \tag{2.47}$$

Without loss we assume that $x^{(m)} \to \overline{x}$, and since $x^{(m)} \in A$, $\forall m$ and A is compact, $\overline{x} \in A \cap \pi^{-1}(\overline{y})$. Because of (2.45) $\overline{x} = \overline{x}^{(j)}$ for some $1 \le j \le \overline{k}$.

Denoting by α_m the angle between $\omega_m = \frac{x^{(m)} - \overline{x}}{|x^m - \overline{x}|}$ and the 2-dimensional plane L, it follows easily from (2.47) that

$$|\sin \alpha_m| \le c_0, \quad \forall m. \tag{2.48}$$

On the other hand we know from Lemma 2.1 (b) (respectively from (b') in the proof) that for some $\mu < 1$

$$\sup_{\omega \in L \cap S^{n-1}} | < \omega | \omega_m > | < 1 - \mu \text{ for } m \ge m_0,$$

m_0 large enough.

This implies in particular that

$$|\sin \alpha_m| > \mu \quad \forall m, \quad m \ge m_0. \tag{2.49}$$

Picking $c_0 = \frac{\mu}{2}$ in (2.47), (2.49) is in contradiction to (2.48) $\square$

We proceed now with the proof of the Lemma: Let $L^\perp(\bar{t}) = \{y \in L^\perp | |y| \leq \bar{t}\}$, pick $\delta_0 > 0$ small, and $\forall y, y \in L^\perp(\bar{t})$, let $\delta(y)$, with $\delta(y) \leq \delta_0$, be given according to (2.46). Then $\mathcal{D} \equiv \{D_{\frac{\delta(y)}{5}}(y) | y \in L^\perp(\bar{t})\}$ is a covering of $L^\perp(\bar{t})$. Application of a covering theorem (see e.g. Lemma 1.9, p. 10[F]) yields: There exists a countable disjoint subcollection $\mathcal{D}'$ of $\mathcal{D}$, $\mathcal{D}' = \{D_{\frac{\delta(y^{(j)})}{5}}(y^{(j)}) | j \in \mathbb{N}\}$, such that

$$
\begin{aligned}
& L^\perp(\bar{t}) \subseteq \cup_{j\in\mathbb{N}} D_{\delta(y^{(j)})}(y^{(j)}) \\
(2.50) \qquad & \text{and } \sum_{j\in\mathbb{N}} (\frac{\delta(y^{(j)})}{5})^{n-2} \leq \tilde{t}^{n-2}.
\end{aligned}
$$

Now we conclude via Proposition 2.5.1 and (2.50) that for every closed subset A of $\Sigma = E \cap \overline{B}_R$

$$
A \subset \pi^{-1}(L^\perp(\bar{t})) \subset \pi^{-1}(\cup_{j\in\mathbb{N}} D_{\delta(y^{(j)})}(y^{(j)})) = \cup_{j\in\mathbb{N}} \cup_{i=1}^{\overline{k}_j} \overline{B}_{\frac{\delta(y^{(j)})}{5}}(x_j^{(i)})
$$

where $\sum_{j=1}^{\infty} (\frac{\delta(y^{(j)})}{c_0})^{n-2} \leq (\frac{5}{c_0}\bar{t})^{n-2}$ and $\overline{k}_j \leq d(\overline{M})$, $\forall j$. Thereby $\delta(y^{(j)}) \leq \delta_0$ and δ_0 can be taken arbitrarily small. Hence we obtain for a δ-cover $\{U_i, i \in \mathbb{N}\}$ of A (i.e. the diameters $d(U_i)$ of U_i are $\leq \delta$)

$$
H_\delta^{n-2}(A) := \inf_{\{U_i\}} \sum_{i=1}^{\infty} d(U_i)^{n-2} < d(\overline{M})(\frac{10}{c_0}\bar{t})^{n-2}
$$

and since $H^{n-2}(A) = \sup_{\delta>0} H_\delta^{n-2}(A) = \lim_{\delta\downarrow 0} H_\delta^{n-2}(A)$, we have for some $C < \infty$ $H^{n-2}(A) \leq C$ which finishes the proof of Lemma 2.5. $\square$

Finally we proceed as in [HOHON2]. Let $R > 0$ small enough such that Lemma 2.5 is applicable. We define for $k \in \mathbb{N}$

$$
(2.51) \qquad \Gamma_k = \{\bar{x} \in \Sigma(u, \overline{B}_\rho(\mathcal{O})) | u - u(\bar{x}) \text{ vanishes of order } k \text{ in } \bar{x}\}
$$

and for $n \in \mathbb{N}$ let

$$
(2.52) \qquad \Gamma_{k,n} = \{x \in \Gamma_k | \text{ dist } (x, \cup_{j\geq k+1}\Gamma_j) \geq \frac{1}{n}\}.
$$

Note that since $k \leq \overline{M}$, $\Gamma_j = \emptyset$ for $j > \overline{M}$. $\Gamma_{k,n}$ is a closed subset of $\Sigma(u, \overline{B}_\rho(\mathcal{O}))$, which follows easily from the smoothness of u and from (2.3). Further it is easily seen that

$$
(2.53) \qquad \Gamma_k = \cup_{n\in\mathbb{N}} \Gamma_{k,n}.
$$

Therefore

$$
(2.54) \qquad \Sigma(u, \overline{B}_\rho(\mathcal{O})) = \cup_{k=2}^{\overline{M}} \cup_{n\in\mathbb{N}} \Gamma_{k,n}.
$$

To show that $H^1(\Gamma_{k,n})$ can be bounded uniformly in n we use

Proposition 2.5.2. *Let $\overline{x} \in \Gamma_{k,n} \cap \Sigma_j$, then for some $\overline{\rho} = \overline{\rho}(\overline{x}) > 0$, $\Gamma_{k,n} \cap \overline{B}_{\overline{\rho}}(\overline{x}) \subset \Sigma_j$.*

Proof of Proposition 2.5.2. : Suppose for contradiction that there is a sequence $\{x^{(m)}\}$, $x^{(m)} \in \Gamma_{k,n} \setminus \Sigma_j$, $\forall m$ with $x^{(m)} \to \overline{x}$, $\overline{x} \in \Gamma_{k,n} \cap \Sigma_j$. Due to Lemma 2.1. (a) and the definition of Σ_j we have

$$\text{dist } (\Sigma(p_k^{(x^{(m)})}, S^{n-1}), \sigma^{(j)}) \le c, \quad \forall m.$$

This implies that $\forall m$ there is a $y^{(m)} \in \Sigma(p_k^{(x^{(m)})}, S^{n-1})$ with $\text{dist}(y^{(m)}, \sigma^{(j)}) \le c$. Without loss let $y^{(m)} \to \overline{y}$ for $m \to \infty$, then

$$\text{dist } (\overline{y}, \sigma^{(j)}) \le c \tag{2.55}$$

follows. By the smoothness of u and (2.4) we obtain $|\nabla p_k^{(x^{(m)})} - \nabla p_k^{(\overline{x})}| \to 0$ for $m \to \infty$ pointwise uniformly, which implies further $\overline{y} \in \Sigma(p_k^{(\overline{x})}, S^{n-1})$. Since $\overline{x} \in \Sigma_j$, dist $(\overline{y}, \sigma^{(j)}) > c$, which is a contradiction to (2.55). $\square$

Clearly we have $\Gamma_{k,n} \subset \bigcup_{\bar{x} \in \Gamma_{k,n}} B_{\rho(\bar{x})}(\bar{x})$, with $\rho(\bar{x})$ given according to Proposition 2.5.2. Since $\Gamma_{k,n}$ is a compact set, there is due to Heine Borel a finite cover of balls centered in $\bar{x}_i$, $1 \le i \le l_{k,n}$ for some $l_{k,n} < \infty$ such that

$$\Gamma_{k,n} \subseteq \bigcup_{i=1}^{l_{k,n}} A_i, \text{ where } A_i = \overline{B_{\rho(\bar{x}_i)}}(\bar{x}_i) \cap \Gamma_{k,n}, \tag{2.56}$$

and because of Proposition 2.4 A_i is a closed subset of some $\Sigma_{j(i)}$. Since this is a finite union of closed sets we rewrite (2.25) as

$$\Gamma_{k,n} \subseteq \bigcup_{j=1}^{N} \mathcal{A}_j^{(k,n)}, \tag{2.57}$$

where $\mathcal{A}_j^{(k,n)}$ are closed subsets of Σ_j.

¿From Lemma 2.5 we have

$$H^{n-2}(\mathcal{A}_j^{(k,n)}) \le C, \quad 1 \le j \le N$$

which together with (2.57) leads to

$$H^{n-2}(\Gamma_{k,n}) \le \sum_{j=1}^{N} H^{n-2}(\mathcal{A}_j^{(k,n)}) \le CN. \tag{2.58}$$

Noting that obviously $\Gamma_{k,n} \subset \Gamma_{k,n+1}$ $\forall n$, $\{\Gamma_{k,n}\}_{n \in \mathbb{N}}$ is an increasing sequence of sets and therefore (see e.g. Lemma 1.3 [F])

$$H^{n-2}(\bigcup_{n \in \mathbb{N}} \Gamma_{k,n}) = \lim_{n \to \infty} H^{n-2}(\Gamma_{k,n}).$$

This together with (2.58) yields

$$H^{n-2}(\bigcup_{n\in\mathbb{N}} \Gamma_{k,n}) \leq CN, \quad 1 \leq k \leq \overline{M}.$$

Finally using (2.54) we obtain

$$H^{n-2}(\Sigma(u, \overline{B_R}(\mathcal{O}))) \leq \overline{M}CN.$$

This finishes the proof of Theorem 1.1. $\square$

Acknowledgement: It is a pleasure to thank H. Knörrer and P. Michor for helpful discussions.

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