Wannier-Bloch oscillators

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Abstract. We consider a Wannier-Stark problem for small field f in the one-ladder case. We prove that a generical first band state is a metastable state (Wannier-Bloch oscillator) with the lifetime determined by the imaginary part of the Wannier-Stark ladder. The infinite resonances of the ladder cause Bloch oscillations as a global beating effect. For an adiabatic time $\tau = ft$ large enough, but much smaller than the resonance lifetime, we have a new version of the acceleration theorem and well specified Bloch oscillators. In the x representation and in the adiabatic scale: $x \to x(f) = \xi/f + y$ the state vanishes externally to a pulsating region of $|\xi|$ defined by $|\xi| < \xi^+(\tau)$ where $\xi^+(n) = 0$ and $\xi^+(n+1/2)$ is the maximum value equal to the first band width. For ξ and τ such that $|\xi|$ is in this region and for y in a fixed domain, the state approaches a finite combination of oscillating Bloch states.

1. Introduction

Let us consider the dynamics of an electron driven by a constant electric field in one-dimensional crystals. The Hamiltonian operator is formally given by:

$$H_f = -d^2/dx^2 + V(x) + fx \tag{1}$$

where $V(x + 2\pi) = V(x)$ and $f > 0$ is the electric field stregth. This very basic problem has attracted a lot of interest since the beginning of quantum mechanics. The names given to this problem are meaningful as long as the other possible, but anusual, names. For instance the "Stark effect for the Bloch problem" is not used for the absence of bound states at $f = 0$. The "Wannier-Stark problem" is the usual name because, as showed by Wannier [27], the bound states exist for $f > 0$ in the one-band approximation. The "Bloch oscillator" too is a name used for this problem, for the reason we are going to recall now. Bloch [7], by a semiclassical approximation, suggested the validity of a kind of acceleration theorem for one-band states. It means that if the state is restricted to the first band and represented in the crystal-momentum space by a function $a_1(k)$ at $t = 0$, at a later time t it will remain there and will be represented by $a_1(k + \tau)$, where $\tau = ft$ is the adiabatic time. Since k is a torus variable, i.e. an angle, the uniform translation means a periodic rotation. As a direct consequence we have bound states with a periodic behavior called "Bloch oscillators" (BO). This kind of acceleration theorem was later proved by Kittel [23], but in the limit case of a function $a_1(k)$ localized at a point (as a Dirac delta fuction) so that the Bloch oscillator obtained is an oscillating Bloch state but is not a quantum state. On the other hand, Zener [29] called in question the very existence of any kind of bound states, oscillating or not, because of the probability of transmission through the "tilted gap barrier". Nevertheless, as anticipated above, Wannier [27] was able to quantize the energy simply applying the one-band approximation: he obtained infinite levels arranged in a ladder (the Wannier-Stark ladder: WSL). Thus, combining the results of Zener and Wannier we espect resonances: the WSL of resonances.

For many years the research on the field was mostly devoted on the proof of the existence of such resonances (see [1],[6],[12],[14]). Actually, we have the prove of existence of the resonant states, for any field small enough, in the particular class of problems with a finite number of bands [15]. The prove was obtained by means of the single-band approximation of Wannier and the perturbation theory on the complex-translated problem [20]. Moreover we know that the resonances are arranged on regular ladders parallel to the real axis and that the imaginary part of the resonances is exponentially small [17] as predicted by Zener, the Fermi Golden Rule, Kane and Blount [21], numerical and analitical researches ([3], [5]) and the adiabatic approximation ([9], [10]). Hence, the dynamics of a generical one-band state is strongly affected by an infinite number of sharp resonances producing a global beating effect.

Thus, after many years of discussions on other aspects of the problem, we return to the original Bloch issue. The result is a synthesis of the different elements of the problem giving a kind of metastable Bloch oscillators, i.e. the Wannier-Bloch oscillators (WBO), explicitly related to the Wannier ladder of resonances and consequently to the Zener probability of decay. The time evolution of the first band component of a state $\mathbf{a}^t = (a_1^t, a_2^t)$ represented in the crystal momentum representation, given at $t = 0$ by $\mathbf{a} = (a_1, a_2 = 0)$ with a_1 smooth, has the asymptotic behavior:

$$a_1^t \overset{w}{\sim} \Omega_1^t T_\eta \overline{\Omega}_1^0 a_1, \tag{2}$$

as f goes to zero and for any $\tau > \tau_1(f) = (C + \epsilon)\log(1/f) + C_\epsilon$ for any $\epsilon > 0$, where C and C_ϵ are positive constants, (Ω_1^t, Ω_2^t) is any Wannier-Stark resonance state, T_η is the translation operator on the circle R/Z: $(T_\eta a_1)(k) = a_1(k + \eta)$, $\eta = \tau - [\tau] \in [0, 1)$ and $[\tau]$ is the integer part of τ.

The main differerences between a WBO and a Bloch oscillator are the slow vanishing and the existence of new factors given by the Wannier-Stark state function. For a time much smaller than the Wannier-Stark life-time the first difference is negligible but the second one is generically relevant. Thus, for a well defined $C_0 > 0$, $\tau < \tau_2(f) = \exp C/f$ with $0 < C < C_0$, we have:

$$U^{1/f} T_{-\eta} U^{-1/f} a_1^t = T_{-\eta} U_\eta^{-1/f} a_1^t \to a_1 \tag{3}$$

as $f \to 0$, where $U = e^{i\theta}$, θ is the function $\theta(k) = \int_0^k \mathcal{E}_1(p)dp$ and $\mathcal{E}_1$ is the first band energy function with mean value $\langle \mathcal{E}_1 \rangle = 0$. The unitary operator U_η is defined as the multiplication by the function: $U_\eta(k) = \exp[i\theta(k) - i\theta(k+\eta)]$. The last limit is the new version of the acceleration theorem and gives the new kind of Bloch oscillators. As a consequence of the asymptotic conservation of the norm of a_1^t as the one of $\mathbf{a}^t$, we also have:

$$a_2^t \to a_2 = 0 \tag{4}$$

and, taking integer values for τ,

$$\mathbf{a}^{n/f} \to \mathbf{a} \tag{5}$$

as $f \to 0$ with $\tau_1(f) < n < \tau_2(f)$. Actually, if the field f is small fixed and the original state $a_1(k)$ is concentrated about a given point k_0, it remains concentrated about the moving point $k_0 - \eta$ for many Bloch periods, as it was expected. Of course in the Kittel limit of total concentration (delta function) the WBO coincides with the Bloch oscillator. Since the peculiarity of this new kind of Bloch oscillator is the non trivial phase factor strongly oscillating for vanishing field, in this limit we have an effect of concentration at the stationary phase points. Let now consider the limit as above of vanishing f and diverging τ, but in the x-representation with wave function $\psi^t(x)$, or better in the adiabatic ξ-representation, for $x \to \xi/f + y$, and the wave function $\phi^t(\xi, y) = \psi^t(\xi/f + y)/\sqrt{f}$. The state is essentially restricted on a pulsating $|\xi|$-region defined by a function $\xi^+(\eta)$ whose maximum value $\xi^+(1/2) = |B_1|$ is equal to the first band width and the minimum is $\xi^+(0) = 0$. For a fixed $\xi < |B_1|$ and for y on a compact set, the stationary phase asymptotics is a finite combination of Bloch states with oscillating quasimomentum:

$$\phi^t(\xi, y) \sim \sum_{j=1}^{2N} C_\eta(k_j) U_\eta^{1/f}(k_j) a_1(\eta + k_j) \psi_1(\xi/f + y, k_j) \tag{6}$$

for η such that $|\xi| < \xi^+(\eta)$, where $N = N(\xi, \eta) \in \mathrm{N}$, $C_\eta(k) = \sqrt{i^{s_\eta(k)}/|g_\eta(k)|}$, $g_\eta(k) = \mathcal{E}_1'(k) - \mathcal{E}_1'(k+\eta)$, $s_\eta(k)$ is the signum of $g_\eta(k)$ and $\psi_1(x, k)$ is the Bloch state of the first band. The quasimomentum $k_j = k_j(\eta, \xi)$ is a solution of the equation $\mathcal{E}_1(k + \eta) - \mathcal{E}_1(k) = \xi$. In particular $k_1(\eta, 0) = \eta/2$ and $k_2(\eta, 0) =$

$\eta/2+1/2$, and in the limit of the free band function: $\mathcal{E}_1(k) \approx C(k^2-1/12)$ for $|k| < 1/2$, for $\eta \neq 0$, $k_1(\eta, \xi) \approx -\xi/2\eta - \eta/2$, $k_2(\eta, \xi) \approx -\xi/(2\eta-2)-(\eta-1)/2$ and $\xi^+(\eta) \approx C(\eta - \eta^2)$. In the other interesting limit: $\mathcal{E}_1(k) \approx -C\cos(2\pi k)$, we have $\xi^+(\eta) \approx \mathcal{E}_1(\eta/2 + 1/4) - \mathcal{E}_1(\eta/2 - 1/4)$.

Thus we have discussed in a rigorous way some aspects of the WBO, as compared to the Bloch oscillators: the metastability, the phase factor, the adiabatic pulsations and the natural decomposition in Bloch states. Let us recall that the WBO$'^s$ have been observed experimentally and discussed in simple cases by an Aachen group [28]. Very recently we have received a paper by Lyssenko *et al.* [24] where the experimental observation of the spatial displacement of the WBO is reported. In the last paper analytical [13] and numerical [8] studies of the pulsations (or breathings) of Bloch oscillators in different models are quoted.

This paper is organized as follows: in Section 2 we state our main results (Theorem, Corollary 1 and Corollary 2); in Section 3 we give the proof of the Theorem following the general method of Herbst [19] (see also [26]) and using the peculiarity of our model; in Section 4 we consider the ξ-representation of the Bloch oscillators and we give the adiabatic limit by a Proposition. Our main results have been announced in [18].

2. Main results

We consider the time—dependent Schrödinger equation

$$i\frac{\partial\psi^t}{\partial t} = -\frac{\partial^2\psi^t}{\partial x^2} + V(x)\psi^t + fx\psi^t, \quad f > 0, \tag{7}$$

with the choice of units such that $\hbar^2 = 2m = 1$, where $V(x)$ is a periodic function and where m is the mass of the electron (actually, for superlattices, m denotes an effective mass, see [2]). Let H_f be the Wannier-Stark operator acting on $L^2(\mathrm{R}, dx)$ and formally defined as

$$H_f = H_B + fx, \quad H_B = -\frac{d^2}{dx^2} + V(x). \tag{8}$$

If $V(x)$ is regular enough, H_f is a self-adjoint operator and its spectrum is purely absolutely continuous: $\sigma(H_f) = \sigma_{\mathrm{ac}}(H_f) = \mathrm{R}$ (see [4] and [22]).

Let $\psi \in L^2(\mathrm{R}, dx)$ and let

$$\psi^t = e^{-iH_f t}\psi \tag{9}$$

be the solution of (7) satisfying to the initial condition $\psi^t(x)|_{t=0} = \psi(x)$. Let

$$\alpha(t) := \langle \gamma, \psi^t \rangle_{L^2(\mathrm{R},dx)} \tag{10}$$

where $\gamma \in L^2(\mathrm{R}, dx)$ is a test function. We are going to give the asymptotics of $\alpha(t)$ as f goes to zero when the initial state represented by ψ and the test state represented by γ are prepared in the first band.

To this end we assume

Hypotheses.

i) $V(x)$ is a real-valued even function with period 2π, i.e.

$$V(x) = V(-x) = V(x + 2\pi);$$

ii) The Bloch operator H_B has the first gap open and the others closed, i.e.:

$$\sigma(H_B) = [E_1^b, E_1^t] \cup [E_2^b, +\infty), \quad E_1^b < E_1^t < E_2^b;$$

iii) Let k_c and $\bar{k}_c$ be the Kohn branch points connecting the two band functions, i.e. $\mathcal{E}_1(k) = \mathcal{E}_2(k)$ at $k = k_c$ and $k = \bar{k}_c$ where $\mathcal{E}_\ell(k)$, $\ell = 1, 2$, are the two band functions. We assume that the two anti-Stokes lines, starting from the branch point $\bar{k}_c$ and such that $\Im \int_{\bar{k}_c}^{k} [\mathcal{E}_2(q) - \mathcal{E}_1(q)] dq = 0$, belong to the strip $|\Im k| < |\Im k_c|$ (but at the starting point).

iv) Let $\mathcal{B} = \mathrm{R}/\mathrm{Z}$, with representative values taken in $[0,1)$, be the Brillouin zone and let

$$\langle \mathcal{E}_1 \rangle = \int_{\mathcal{B}} \mathcal{E}_1(k) dk$$

be the mean value of the first band function. We fix the energy constant such that $\langle \mathcal{E}_1 \rangle = 0$.

In the crystal momentum representation (hereafter CMR, for details see [17]) the function $\alpha(t)$ takes the form

$$\alpha(t) := \langle \mathbf{b}, e^{-iH_f t} \mathbf{a} \rangle_{\mathcal{H}} \tag{11}$$

where $\mathbf{a} = (a_1, a_2) = \mathcal{U}\psi$, $\mathbf{b} = (b_1, b_2) = \mathcal{U}\gamma \in \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ represent the states associated to ψ and γ in the CMR; $\mathcal{H}_1 = L^2(\mathcal{B}, dk)$, with periodic boundary conditions, and $\mathcal{H}_2 = L^2(\mathrm{R}, dp)$. Since these states are prepared in the first band we have $a_2 = b_2 = 0$.

The unitary transformation $\mathcal{U}$ from $L^2(\mathrm{R}, dx)$ onto $\mathcal{H}$ is defined on $\psi \in \mathcal{S}(\mathrm{R})$, where $\mathcal{S}$ is the space of smooth functions of rapid decrease, as

$$a_1(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathrm{R}} \bar{\psi}_1(x,k) \psi(x) dx, \quad a_2(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathrm{R}} \bar{\psi}_2(x,p) \psi(x) dx,$$

with inverse

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \left[\int_{\mathcal{B}} a_1(k) \psi_1(x,k) dk + \int_{\mathrm{R}} a_2(p) \psi_2(x,p) dp \right], \tag{12}$$

and extended on the whole Hilbert space $L^2(\mathrm{R}, dx)$ by continuity, where $\psi_j(x,p) = e^{ipx} u_j(x,p)$, $u_j(x,p) = u_j(x+2\pi, p)$, are the two Bloch functions.

We recall that $V(z)$ is an analytic function in the strip $|\Im z| < \lambda_0$ for some $\lambda_0 > 0$ (see Theorem XIII.91(d) in [25]) and resonances of H_f are defined as complex eigenvalues of the non-self-adjoint operator H_f^λ, $-f\lambda_0 < \Im\lambda < 0$, obtained from H_f by means of an analytic translation [20]. This deformation leaves the space $\mathcal{H}_1$ unaffected and it is such that H_f^λ takes the form (still denoted H_f^λ)

$$H_f^\lambda = \mathrm{diag}(H_1, H_2^\lambda) + fX^\lambda,$$

where

$$H_1 = if\frac{d}{dk} + \mathcal{E}_1(k), \quad H_2^\lambda = if\frac{d}{dp} + \mathcal{E}_2(p) + f\lambda$$

and where $X^\lambda = (X_{\ell m}^\lambda)_{\ell,m=1,2}$ is a bounded operator, representing the coupling term between the two bands, with $X_{11}^\lambda \equiv 0$ and X_{12}^λ, X_{21}^λ, X_{22}^λ are defined in formulas (18), (19) and (26) in [17].

The decoupled band approximation, obtained neglecting the coupling term X^λ, gives a ladder of simple eigenvalues, with eigenvectors $\omega_j = (\omega_{1,j}, \omega_{2,j})$, $j \in \mathbb{Z}$, where:

$$\omega_{1,j}(k) = e_1^{-j}(k) \exp\left[\frac{i}{f}\int_0^k \mathcal{E}_1(q)dq\right], \quad \omega_{2,j}(p) \equiv 0.$$

and where $e_1(k) = \exp(i2\pi k)$. By means of the regular perturbation theory it was proved [15], [17] that, for f small enough, there exists exactly one ladder $E_j(f) = E_0(f) + 2\pi jf$, $j \in \mathbb{Z}$, of simple eigenvalues of H_f^λ in the strip $-f\lambda_0/2 < \Im z < 0$, that is of resonances of H_f, with

$$\Re E_0(f) = \mathcal{O}(f^2), \quad \Im E_0(f) = -\tfrac{1}{2} f e^{-2\rho_Z}\left(1 + o(1)\right), \tag{13}$$

as f goes to zero, where

$$\rho_Z = \frac{1}{8f}\int_0^{2\pi} [V(x) - \langle V\rangle]^2 dx.$$

Let $\Omega_j^\lambda = (\Omega_{1,j}, \Omega_{2,j}^\lambda)$ be the eigenvector associated to $E_j(f)$ with norm $\|\Omega_j^\lambda\|_{\mathcal{H}} = 1$. Let $\Omega_{1,0}^t = e^{-iE_0 t}\Omega_{1,0}$ be the time evolution of the resonant state Ω_0^λ projected on the first band.

We state now our main results:

Theorem. *Let Hypotheses i—iv) be satisfied, let* $\tau = ft$, $\eta = \tau - [\tau] \in [0,1)$ *and let* $\mathcal{T}_\eta$ *be the translation operator* $(\mathcal{T}_\eta a_1)(k) = a_1(k+\eta)$. *Then there exists* $f_0 > 0$ *such that for any fixed* $a_1, b_1 \in \mathcal{D}(H_1)$, $f \in (0, f_0)$ *and* $t \geq 0$ *we have*

$$\alpha(t) = A(t) + R(t) \tag{14}$$

where

$$A(t) = \langle b_1, \Omega_{1,0}^t \mathcal{T}_\eta a_1 \overline{\Omega_{1,0}} \rangle_{\mathcal{H}_1} = e^{-iE_0 t} h(\eta) \tag{15}$$

and $h(\eta)$ *is a periodic function with period* 1 *such that*

$$h(\eta) = \langle b_1, \omega_{1,0} \mathcal{T}_\eta a_1 / \omega_{1,0} \rangle_{\mathcal{H}_1} + \mathcal{O}(f). \tag{16}$$

as f *goes to zero. Moreover, there exists a positive constant* c_1, *independent of* f *and* t, *such that*

$$|R(t)| \leq c_1 f^{-3} e^{-\lambda_0 \tau/2}. \tag{17}$$

Remark 1. Hypotheses *i)* and *iv)* are taken only for the sake of definiteness. Hypothesis *iii)* is strictly necessary in order to obtain the asymptotics (13) of the imaginary part of the resonance; in absence of Hypothesis *iii)* the Theorem is still true but (13) must be replaced by the behavior $\Im E_0(f) = \mathcal{O}(f^\infty)$. Finally, Hypothesis *ii)* can be weakened by assuming that the first N gaps are open and the others are close. In such a case (14) is still true (with a different bound for $R(t)$) for any f small enough and such that the ladders of resonances do not cross. When the ladders cross a similar formula could be obtained following the analysis, based on the degenerate perturbation theory, performed in [16] for a three bands model.

Corollary 1. *Let* $\tau_1(f) := (6/\lambda_0 + \epsilon)\log(1/f) + C_\epsilon$*, for any fixed* $\epsilon > 0$ *and some* $C_\epsilon > 0$*. We have:*

$$R(t) \to 0$$

as f *goes to zero with* $\tau > \tau_1(f)$ *for every* f*. In particular, let* $\mathbf{a} = (a_1, 0) \in \mathcal{H}$*,* $a_1 \in \mathcal{D}(H_1)$*, and let* $a_1^t = \mathcal{P}_1 e^{-iH_f t}\mathbf{a}$*, where* $\mathcal{P}_1$ *is the projection on* $\mathcal{H}_1$*. Then, keeping* $\tau > \tau_1(f)$ *we have*

$$a_1^t(k) \stackrel{w}{\sim} \Omega_{1,0}^t \mathcal{T}_\eta \overline{\Omega_{1,0}} a_1 \tag{18}$$

as f *goes to zero.*

Proof From (11) and (15) it follows that $R(t) = \langle b_1, \tilde{R} a_1 \rangle_{\mathcal{H}_1}$ where $\tilde{R}$ is a bounded operator acting on $\mathcal{H}_1$ as $\tilde{R} a_1 = \mathcal{P}_1 e^{-iH_f t}\Pi_1 a_1 - \Omega_{1,0}^t \mathcal{T}_\eta \overline{\Omega_{1,0}} a_1$, where $\Pi_1 a_1 = \mathbf{a}$. Moreover, from (17) we have that

$$R(t) = \langle b_1, \tilde{R} a_1 \rangle_{\mathcal{H}_1} \to 0 \tag{19}$$

as f goes to zero with $\tau > \tau_1(f)$ and for any $b_1 \in \mathcal{D}(H_1)$. From this and from the fact that $\mathcal{D}(H_1)$ is dense in $\mathcal{H}_1$ it follows that $\tilde{R} a_1 \stackrel{w}{\to} 0$ as f goes to zero. $\sharp$

Remark 2. The physical meaning of Corollary 1 concerns the general behavior of regular states of the first band as metastable Bloch oscillators called Wannier-Bloch oscillators (BWO) because of their relation to the Wannier-Stark states. From the decay rate and the period of oscillation we compute the Zener probability of tunneling p coinciding with the value suggested by Kane and Blount [18]:

$$p = e^{-2\rho z(f)}(1 + o(1)) \quad \text{as } f \to 0. \tag{20}$$

Remark 3. In (18) the term $\Omega_{1,0}^t$ represents a Wannier-Stark resonant state restricted to the first band and the other term $\mathcal{T}_\eta \overline{\Omega_{1,0}} a_1$ represents the beating effect among the WS states, which is a translation on the torus. We should notice that the beating effect is not affected by the decay since the lifetime is much longer than the beating period, for f small enough.

Corollary 2. *Let f go to 0 and $\tau_1(f) < \tau < \tau_2(f)$, where $\tau_2(f) = \exp(C/f)$ and $0 < C < f\rho_Z$. Let also take a different gauge in order to eliminate the phase factor $\exp(-i\Re E_0(f)t)$. We have:*

$$T_{-\eta}U_\eta^{-1/f}a_1^t \to a_1 \tag{21}$$

as f goes to 0, where $\eta = \tau - [\tau] \in [0,1)$ and:

$$U_\eta^{1/f} = \omega_{1,0}(k)/\omega_{1,0}(k+\eta) = \exp\left\{\frac{i}{f}\int_{k+\eta}^{k}\mathcal{E}_1(q)dq\right\}$$

is a phase factor dependent on k.

Proof: From Corollary 1 and since $\Omega_{1,0} = \omega_{1,0}\left[1 + O(f)\right]$ (see formula (23) below) it immediately follows that $T_{-\eta}U_\eta^{-1/f}a_1^t \xrightarrow{w} a_1$ in the limit above. From this and from the following inequality:

$$\begin{aligned}
\|a_1^t\|_{\mathcal{H}_1} &= \|\mathcal{P}_1 e^{-iH_f t}\mathbf{a}\|_{\mathcal{H}_1} \leq \|e^{-iH_f t}\mathbf{a}\|_{\mathcal{H}} = \|\mathbf{a}\|_{\mathcal{H}} \\
&= \|a_1\|_{\mathcal{H}_1} = \|U_\eta^{1/f}T_\eta a_1\|_{\mathcal{H}_1}
\end{aligned}$$

we prove that the limit above holds in norm. $\sharp$

Remark 4. From the essential constancy of the norm of a_1^t it follows that the state, initially prepared in the first band, is still there for f vanishing and τ diverging as specified above. That is $a_2^t \to a_2 = 0$ in the same limit. Moreover, if we take only integer values for τ, in the same limit as above for f and τ, we have:

$$\mathbf{a}^t \to \mathbf{a}. \tag{22}$$

Remark 5. Formula (21) represents a new version of the acceleration Theorem and a new kind of Bloch oscillators. If we consider the function $a_1(k)$ in the limit case of a $\delta(k-k_0)$, so that in the x-representation we have a Bloch state, we recover the classical versions of the Bloch oscillator and of the acceleration Theorem as given by Kittel [23]. Actually in this limit the phase factor $U_\eta^{1/f}(k)$ can be taken as k-independent, i.e. pysically negligible, so that it can be eliminated in (21). In such a case, the translation of the mean value $\langle k\rangle_1^t + \eta \to \langle k\rangle_1^0$ as can be easly verified from the Corollary 2, strengthens to the translation of the exact value of the crystal momentum k_0^t: $k_0^t + \eta \to k_0$.

Remark 6. Let us consider the adiabatic behavior directly on a_1^t. For τ integer, i. e. $\eta = 0$, the behavior is given in Remark 4 and it is essentially stationary at the initial state value. For $\eta \neq 0$ the term a_1^t contains the phase factor $U_\eta^{1/f}$ which quickly oscillates as f goes to zero. There are two stationary phase points such that $\mathcal{E}_1(k) = \mathcal{E}_1(k-\eta)$, that is $k_1(\eta) = \eta/2$ and $k_2(\eta) = 1/2 + k_1(\eta)$. Hence, we have the following behavior in the distributional sense:

$$a_1^t(k) \sim \sqrt{i2\pi f/|g(k)|}(U_\eta^{1/f}T_\eta a_1)(k)\left[\delta\left(k-k_1(\eta)\right) - i\delta\left(k-k_2(\eta)\right)\right]$$

as f goes to zero, for non-integer values of τ going to $+\infty$ as above , and where $g(k) := \mathcal{E}_1'(k)$ and δ denotes the Dirac delta function. That is the WBO wavefunction is locally a linear combination of two Bloch states. Let us notice that this k-localization is naturally given by the vanishing of f so that it is definitely different from the Kittel one imposed by the initial conditions. The meaning of this distributional limit will be clear in the x-representation and extending the limit above to the adiabatic scale: $x \to \xi/f + y$ for any real ξ and y. So we will formally recover the conservation of the norm of the state.

3. Proof of the Theorem

The proof of the Theorem essentially follows the one given by Herbst [19], for a Stark potential superimposed to a Coulomb-type potential. We have adapted the method to our case where the Coulomb-type potential, vanishing at infinity, is replaced by a periodic potential, and there are infinite resonances periodically arranged in a strip parallel to the real axis. As a first step, we collect some results about the WS resonant states.

We recall that

$$\Omega_{1,0} = \hat{\omega}_0 \mu_0, \ \|\mu_0 - \mathbf{1}\|_{\mathcal{H}_1} = O(f) \ \text{ and } \ \|\Omega^\lambda_{2,j}\|_{\mathcal{H}_2} = O_\lambda(f) \tag{23}$$

as f goes to zero, where $\hat{\omega}_0$ is the multiplication operator defined on $\mathcal{H}_1$ by $\omega_{1,0}(k)$ and $\mathbf{1}(k) \equiv 1$ (see Lemma 1 in [17]) and that $\Omega_{1,j}(p)$ is an analytic function in the strip $|\Im p| < k_c$ containing the real axis. Moreover:

Lemma 1. *Let* ê *be the unitary operator defined on* $(a_1, a_2) \in \mathcal{H}$ *as*

$$\hat{\mathrm{e}}(a_1, a_2) = (\hat{\mathrm{e}}_1 a_1, \hat{\mathrm{e}}_2 a_2)$$

where $\hat{\mathrm{e}}_\ell$, $\ell = 1, 2$, *is the multiplication operator defined on* $\mathcal{H}_\ell$ *by* e_ℓ *where* $e_1(k) = e^{i2\pi k}$ *and* $e_2(p) = e^{i2\pi p}$. *Then*

$$\hat{\mathrm{e}}^j \Omega^\lambda_j = \Omega^\lambda_0.$$

Proof: The Lemma immediately follows from the simplicity of the ladder and because

$$\mathrm{diag}(H_1, H_2^\lambda)\hat{\mathrm{e}} = \hat{\mathrm{e}} \left[\mathrm{diag}(H_1 - 2\pi f, H_2^\lambda - 2\pi f)\right] \tag{24}$$

and

$$X^\lambda_{12}\hat{\mathrm{e}}_2 = \hat{\mathrm{e}}_1 X^\lambda_{12}, \ X^\lambda_{21}\hat{\mathrm{e}}_1 = \hat{\mathrm{e}}_2 X^\lambda_{21}, \ X^\lambda_{22}\hat{\mathrm{e}}_2 = \hat{\mathrm{e}}_2 X^\lambda_{22} \tag{25}$$

as one can directly verify from formulas (18), (19) and (26) in [17]. $\sharp$

Let $a_1, \ b_1 \in \mathcal{H}_1$ and absolutely continuous, and let $\mathbf{b} = (b_1, 0)$, $\mathbf{a} = (a_1, 0) \in \mathcal{H}$; let

$$F^+(z) := \langle \mathbf{b}, [H^\lambda_f - z]^{-1}\mathbf{a}\rangle_{\mathcal{H}} = \langle b_1, [H_1 - z - f\mathcal{F}^+(z)]^{-1} a_1\rangle_{\mathcal{H}_1}$$

be defined for $\Im z > 0$ where $\mathcal{F}^+(z)$ is defined as

$$\mathcal{F}^+(z) = X_{12} f [H_2 + f X_{22} - z]^{-1} X_{21} \quad \text{for } \Im z > 0.$$

$F^+(z)$ admits an analytic extension to $-f\lambda_0 < \Im z$ as

$$F^+(z) \equiv F^{+,\lambda}(z) := \langle b_1, [H_1 - z - f\mathcal{F}^{+,\lambda}(z)]^{-1} a_1 \rangle_{\mathcal{H}_1}$$

for any λ such that $-\lambda_0 < \Im\lambda < \Im z$, where

$$\mathcal{F}^{+,\lambda}(z) := X_{12}^\lambda f [H_2^\lambda + f X_{22}^\lambda - z]^{-1} X_{21}^\lambda.$$

Let us recall that (Lemma 10 in [15]) $\mathcal{F}^{+,\lambda}(z)$ is norm bounded and

$$\|\mathcal{F}^{+,\lambda}(z)[H_1 - z]^{-1}\|_{\mathcal{L}(\mathcal{H}_1)} \leq C \tag{26}$$

for any $z \in \partial S_0$ where $S_0 = [-\pi f, \pi f] \times i[-f\lambda_0/2, f\lambda_0/2]$ for f small enough and for some positive constant C independent of f and z. Similarly

$$F^-(z) = \langle \mathbf{b}, [H_f^\lambda - z]^{-1} \mathbf{a} \rangle_{\mathcal{H}} = \langle b_1, [H_1 - z - f\mathcal{F}^-(z)]^{-1} a_1 \rangle_{\mathcal{H}_1}$$

is defined for $\Im z < 0$ and it admits an analytic extension to $\Im z < f\lambda_0$ as

$$F^-(z) = \langle b_1, [H_1 - z - f\mathcal{F}^{-,\lambda}(z)]^{-1} a_1 \rangle_{\mathcal{H}_1}$$

and (26) holds for $\mathcal{F}^{-,\lambda}(z)$ too. Then, the function

$$G(z) = F^+(z) - F^-(z)$$

admits an analytic extension to the strip $-f\lambda_0 < \Im z < +f\lambda_0$.

In the following let us drop λ, if not necessary, and let us denote by C any positive constant.

We give now the following preliminary results:

Lemma 2. *Let S_j be the complex box $S_j = [s(j), s(j+1)] \times i[-f\lambda_0/2, f\lambda_0/2]$ where $s(j) = (2j-1)f\pi$. Let $\mathbf{S} = \cup_{j\in Z}\partial S_j$, where ∂S_j denotes the boundary of S_j. For f small enough, it follows that*

$$|G(z)| \leq Cf^{-3}(|z|^2+1)^{-1}, \tag{27}$$

for any z belonging to $\mathbf{S}$, for some positive constant C independent of f and z.

Proof. We set

$$F_j^{\pm}(z) = F^{\pm}(z + 2\pi jf)$$

and we restrict our analysis to $z \in \partial S_0$ and large j. To this end we observe that

$$\begin{aligned} F^{\pm}(z+2\pi f) &= \langle b_1, [H_1 - z - 2\pi f - f\mathcal{F}^{\pm}(z+2\pi f)]^{-1} a_1\rangle_{\mathcal{H}_1} \\ &= \langle \hat{\mathrm{e}}_1 b_1, [H_1 - z - f\mathcal{F}^{\pm}(z)]^{-1}\hat{\mathrm{e}}_1 a_1\rangle_{\mathcal{H}_1} \end{aligned}$$

from (24) and (25). Hence, it follows that

$$F_j^{\pm}(z) = \langle \hat{\mathrm{e}}_1^j b_1, [H_1 - z - f\mathcal{F}^{\pm}(z)]^{-1}\hat{\mathrm{e}}_1^j a_1\rangle_{\mathcal{H}_1}. \tag{28}$$

From the first resolvent formula we obtain

$$F_j^{\pm}(z) = A_j^{\pm}(z) + B_j^{\pm}(z) + C_j^{\pm}(z)$$

where

$$A_j^{+}(z) = A_j^{-}(z) = \langle \hat{\mathrm{e}}_1^j b_1, [H_1 - z]^{-1}\hat{\mathrm{e}}_1^j a_1\rangle_{\mathcal{H}_1},$$

$$\begin{aligned} B_j^{\pm}(z) &= \langle \hat{\mathrm{e}}_1^j b_1, [H_1 - z]^{-1} f\mathcal{F}^{\pm}(z)[H_1 - z]^{-1}\hat{\mathrm{e}}_1^j a_1\rangle_{\mathcal{H}_1} \\ &= f\langle [H_1 - \bar{z}]^{-1}\hat{\mathrm{e}}_1^j b_1, \mathcal{F}^{\pm}(z)[H_1 - z]^{-1}\hat{\mathrm{e}}_1^j a_1\rangle_{\mathcal{H}_1} \end{aligned}$$

and

$$
\begin{aligned}
C_j^{\pm}(z) &= \langle \hat{e}_1^j b_1, [H_1 - z]^{-1} f \mathcal{Q}^{\pm}(z) f [H_1 - z]^{-1} \hat{e}_1^j a_1 \rangle_{\mathcal{H}_1} \\
&= f^2 \langle [H_1 - \bar{z}]^{-1} \hat{e}_1^j b_1, \mathcal{Q}^{\pm}(z) [H_1 - z]^{-1} \hat{e}_1^j a_1 \rangle_{\mathcal{H}_1}
\end{aligned}
$$

where we set

$$
\mathcal{Q}^{\pm}(z) = \mathcal{F}^{\pm}(z)[H_1 - z - f\mathcal{F}^{\pm}(z)]^{-1}\mathcal{F}^{\pm}(z).
$$

Since

$$
\begin{aligned}
&\left([H_1 - z]^{-1} \hat{e}_1^j a_1\right)(k) = \\
&= -\frac{i}{f}\phi(k)\left\{\frac{1}{e^{iz/f} - 1}\int_{\mathcal{B}} \phi^{-1}(q) a_1(q) e^{i2\pi qj} dq + \int_0^k \phi^{-1}(q) a_1(q) e^{i2\pi qj} dq\right\},
\end{aligned}
$$

where

$$
\phi(k) = \exp\left[\frac{i}{f}\int_0^k (\mathcal{E}_1(q) - z) dq\right],
$$

and since a_1 is absolutely continuous, then by integrating by parts we obtain

$$
\left|\left([H_1 - z]^{-1} \hat{e}_1^j a_1\right)(k)\right| \leq C f^{-2}(|j| + 1)^{-1} \tag{29}
$$

for some positive constant C independent of j, f and k; a similar bound holds for $[H_1 - z]^{-1}\hat{e}_1^j b_1$ too. From this and from (26) it follows that

$$
\begin{aligned}
|B_j^{\pm}| &\leq f\left\|[H_1 - z]^{-1}\hat{e}_1^j a_1\right\|_{\mathcal{H}_1}\left\|[H_1 - \bar{z}]^{-1}\hat{e}_1^j b_1\right\|_{\mathcal{H}_1}\|\mathcal{F}^{\pm}(z)\| \\
&\leq C f^{-3}(j^2 + 1)^{-1}
\end{aligned}
$$

for any $z \in \partial S_0$, for some positive constant C independent of f, j and z. A similar bound for $C_j^{\pm}$ follows. From this and from the fact that $A_j^+ - A_j^- = 0$ then (27) follows. $\sharp$

Lemma 3. *Let $E_j(f) = E_0(f) + 2\pi j f$ be the ladder of resonances of H_f contained in the strip $-f\lambda_0/2 < \Im z < 0$. It follows that the series*

$$A(t) = \sum_{j\in \mathrm{Z}} Res\ F^+(z)|_{z=E_j} e^{-iE_j t} = e^{-iE_0 t} h(\eta), \tag{30}$$

is absolutely convergent for any t, where $Res\ F^+(z)|_{z=E_j}$ denotes the residue of $F^+(z)$ at $z = E_j$. The function h is a periodic function with period 1.

Proof. Let P_j^λ be the eigenprojection on the space spanned by Ω_j^λ. We observe that

$$\begin{aligned}
q_j &:= Res\ F^+(E_j) = \langle \mathbf{b}, P_j^\lambda \mathbf{a}\rangle_{\mathcal{H}} = \langle \mathbf{b}, \Omega_j^\lambda\rangle_{\mathcal{H}} \langle \Omega_j^\lambda, \mathbf{a}\rangle_{\mathcal{H}} \\
&= \langle b_1, \Omega_{1,j}\rangle_{\mathcal{H}_1} \langle \Omega_{1,j}, a_1\rangle_{\mathcal{H}_1} \\
&= \langle b_1, \hat{\mathrm{e}}_1^{-j} \Omega_{1,0}\rangle_{\mathcal{H}_1} \langle \hat{\mathrm{e}}_1^{-j} \Omega_{1,0}, a_1\rangle_{\mathcal{H}_1} = \mathcal{O}(j^{-2})
\end{aligned}$$

for $|j|$ large, as can be proven by integration by parts since a_1 and b_1 are absolutely continuous functions. Hence, the Fourier series

$$h(\eta) = \sum_{j\in \mathrm{Z}} q_j e^{-i2\pi j\eta}$$

is absolutely convergent for any η. $\sharp$

Now, we are ready to conclude the proof of the Theorem.

From the spectral Theorem and since the spectrum of H_f is purely absolutely continuous, we can write

$$\begin{aligned}
\alpha(t) &= \langle \mathbf{b}, e^{-iH_f t}\mathbf{a}\rangle_{\mathcal{H}} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} G(z) e^{-izt} dz \\
&= \lim_{N_1, N_2 \to +\infty} I_{N_1,N_2}
\end{aligned} \tag{31}$$

where $N_1, N_2 \in \mathrm{N}$ and

$$I_{N_1,N_2} = \frac{1}{2\pi i}\int_{s(-N_1)}^{s(N_2+1)} G(z) e^{-izt} dz.$$

From the Cauchy Theorem we can replace the path of integration with the line $\gamma_{N_1,N_2} = \partial S_{N_1,N_2} \cap \mathrm{C}^-$, where $\mathrm{C}^- = \{z \in \mathrm{C} \ : \ \Im z \leq 0\}$ and where $S_{N_1,N_2} = \cup_{j=-N_1}^{N_2} S_j$, obtaining

$$I_{N_1,N_2} = \sum_{j=-N_1}^{N_2} Res\ F^+(z)|_{z=E_j} e^{-iE_j t} + \frac{1}{2\pi i}\int_{\gamma_{N_1,N_2}} G(z) e^{-izt} dz \tag{32}$$

In the limit $N_1, N_2 \to +\infty$ we have that the series converges (see Lemma 3) and the integral converges (see Lemma 2). Hence we obtain that

$$\alpha(t) = A(t) + R(t) \tag{33}$$

where

$$R(t) = e^{-\lambda_0 t f/2} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} G(z - i\lambda_0 f/2) e^{-izt} dz.$$

From Lemma 2 it follows that

$$|R(t)| \leq e^{-\lambda_0 \tau/2} \frac{C}{2\pi f^3} \int_{-\infty}^{+\infty} \frac{dz}{z^2+1} \leq c_1 f^{-3} e^{-\lambda_0 \tau/2}$$

for some positive constant c_1 independent of f and t. Moreover, it follows that

$$\begin{aligned}
A(t) &= \sum_{j\in \mathrm{Z}} e^{-iE_0 t - i2\pi j\eta} \langle b_1, \hat{\mathrm{e}}_1^{-j} \Omega_{1,0}\rangle_{\mathcal{H}_1} \langle \hat{\mathrm{e}}_1^{-j}\Omega_{1,0}, a_1\rangle_{\mathcal{H}_1} \\
&= \sum_{j\in \mathrm{Z}} \langle \overline{\Omega_{1,0}^t} b_1, e_1^{-j}\rangle_{\mathcal{H}_1} \langle e_1^{-j} e^{i2\pi j\eta}, \overline{\Omega_{1,0}} a_1\rangle_{\mathcal{H}_1} \\
&= \sum_{j\in \mathrm{Z}} \langle \overline{\Omega_{1,0}^t} b_1, e_1^{-j}\rangle_{\mathcal{H}_1} \langle \mathcal{T}_{-\eta}(e_1^{-j}), \overline{\Omega_{1,0}} a_1\rangle_{\mathcal{H}_1} \\
&= \sum_{j\in \mathrm{Z}} \langle \overline{\Omega_{1,0}^t} b_1, e_1^{-j}\rangle_{\mathcal{H}_1} \langle e_1^{-j}, \mathcal{T}_{\eta}\overline{\Omega_{1,0}} a_1\rangle_{\mathcal{H}_1} \\
&= \langle b_1, \Omega_{1,0}^t \mathcal{T}_\eta \overline{\Omega_{1,0}} a_1\rangle_{\mathcal{H}_1}
\end{aligned}$$

because $\{e_1^j\}_{j\in Z}$ is a complete orthonormal basis of $\mathcal{H}_1$. Finally, if we set

$$A(t) = e^{-iE_0 t} h(\eta)$$

then

$$h(\eta) = \langle b_1, \omega_{1,0} \mathcal{T}_\eta \overline{\omega_{1,0}} a_1 \rangle_{\mathcal{H}_1} + \mathcal{O}(f)$$

since (23). Thus the Theorem is completely proved.

4. The Wannier-Bloch Oscillators in the x representation and the adiabatic behavior

Let $\psi^t(x)$ be the time dependent wavefunction in the x representation (see (12)). From the Theorem (see Remark 4) we can study the asymptotics of the WBO wavefunction in the x representation. The asymptotic behavior of $\psi^t(x)$ comes directly from Remark 6 and the unitary trasform relating the two representations. From the Theorem follows that $a_2 \to 0$, hence the WBO wavefunction has the weak behavior:

$$\psi^t(x) \sim \frac{1}{\sqrt{2\pi}} \int_{\mathcal{B}} \psi_1(x,k) (U_\eta^{1/f} T_\eta a_1)(k) dk \tag{34}$$

$$\sim \sqrt{f} \sum_{j=1}^{2} C_\eta(k_j) U_\eta^{1/f}(k_j) a_1(k_j+\eta) \psi_1(x,k_j) \tag{35}$$

as f goes to zero. Where C_η and $k_j = k_j(\eta)$ are defined in Remark 6. The asymptotics in (35) is given by the stationary phase theorem, it is meant in the punctual sense and it is uniform in a compact domain of the x-axis. Let us notice that the wave function vanishes as $\sqrt{f}$. In order to extend the study to a larger region and in order to have a non-vanishing function, it is useful to make the same limit in an adiabatic scale: we consider

$$\phi^t(\xi) = \left(D_{1/f} \psi^t \right)(\xi) = \sqrt{1/f}\, \psi^t(\xi/f)$$

where $\xi = fx$ is the adiabatic variable. Moreover we can consider a fixed translation:

$$\phi^t(\xi, y) = \left(T_y D_{1/f} \psi^t\right)(\xi) = (1/\sqrt{f})\psi^t(\xi/f + y).$$

From this it follows

$$\phi^t(\xi, y) \sim \frac{1}{\sqrt{2\pi f}} \int_{\mathcal{B}} \psi_1(\xi/f + y, k)(U_\eta^{1/f} T_\eta a_1)(k) dk, \tag{36}$$

as f goes to zero with $\tau \in (\tau_1(f), \tau_2(f))$, where $\psi_1(x, k)$ is the Bloch function. In order to apply the stationary phase theorem, we state it in a suitable form:

Lemma 4. *Let*

$$I(f) = \int_a^b u(x(f), k) e^{i\omega(k)/f} g(k) dk, \quad -\infty < a < b < +\infty, \ f > 0, \tag{37}$$

where ω is a real-valued function, $x(f)$ is any function with values in $\mathrm{R}/2\pi\mathrm{Z}$, u is a periodic function with respect to x and u, g and ω are regular enough, i.e. $g(k) \in C^1([a,b])$, $u(\cdot, k) \in C^2([a,b])$, $\omega(k) \in C^2([a,b])$, $u(x, \cdot) \in C^1(\mathrm{R}/2\pi\mathrm{Z})$. Let $k_0 \in [a,b]$ be such that $\omega'(k_0) = 0$ and $\omega''(k_0) \neq 0$ and let $\omega'(k) \neq 0$ for any $k \neq k_0$. Then, for any x

$$\begin{aligned} I(f) &= u(x(f), k_0) \int_a^b e^{i\omega(k)/f} g(k) dk + O(f) & (38) \\ &= \sqrt{2\pi f} u(x(f), k_0) e^{i\omega(k_0)/f} g(k_0) \sqrt{i^s / |\omega''(k_0)|} + O(f) & (39) \end{aligned}$$

as f goes to zero and where s is the signum of $\omega''(k_0)$.

Proof: From the Lagrange theorem we have

$$u(x, k) = u(x, k_0) + (k - k_0)\tilde{u}(x, k)$$

where $\tilde{u}(x, k)$ and its derivative with respect to k are continuous functions in $x \in \mathrm{R}/2\pi\mathrm{Z}$ and bounded. Hence

$$I(f) \;=\; u(x(f),k_0)\int_a^b e^{i\omega(k)/f} g(k)dk + R(f)$$

where

$$R(f) = \int_a^b K(k,f)dk$$

and

$$K(k,f) = (k-k_0)\tilde{u}(x(f),k)g(k)e^{i\omega(k)/f}.$$

Let us assume, for the sake of definitness, that $k_0 \in (a,b)$ and let, for f small enough,

$$\begin{aligned} & R(f) = R_1(f) + R_2(f) + R_3(f) \\ = \; & \int_a^{k_0-\sqrt{f}} K(k,f)dk + \int_{k_0-\sqrt{f}}^{k_0+\sqrt{f}} K(k,f)dk + \int_{k_0+\sqrt{f}}^{b} K(k,f)dk. \end{aligned}$$

Since

$$|K(k,f)| \le C_2|k-k_0|/2$$

for some positive constant C_2 then $|R_2| \le C_2 f$. In order to estimate R_1 we integrate by parts obtaining

$$R_1(f) = -if\left[\frac{K(k,f)}{\omega'(k)}\right]_a^{k_0-\sqrt{f}} + if\int_a^{k_0-\sqrt{f}} e^{i\omega(k)/f}\frac{d}{dk}\left[\frac{\tilde{u}(x(f),k)g(k)}{\omega'(k)/(k-k_0)}\right]dk$$

where $\omega'(k)/(k-k_0)$ is a $C^1([a,b])$ function and is different from zero for any $k \in [a,b]$. Hence $K(k,f)/\omega_k(k)$ is bounded and $(k-k_0)\tilde{u}(x,k)g(k)/\omega'(\xi,k) \in C^1([a,b])$ and so $|R_1| \le C_1 f$ for some positive constant C_1; a similar bound holds for R_3 too. The proof of (38) is so completed. From this and from the stationary phase method (39) follows. $\sharp$

Remark 7. This result (with a different estimate of the remainder) can be easily extended to the different cases: $\omega(k) = \omega(k_0) + (k - k_0)^\alpha \tilde{\omega}(k)$ with $\alpha > 2$.

Let us consider the asymptotic behavior of $\phi^t(\xi, y)$ for ξ fixed and y in a compact set. We use Lemma 4 with with u_1 for u, U_η for $e^{i\omega}$ and $e^{iky}(T_\eta a_1)(k)$ for $g(k)$. If τ takes only integer values, $\phi^t(\xi, y)$ approches the initial function $\phi^0(\xi, y)$.

For any fixed ξ the integral is asymptotically given by the stationary phase approximation as f goes to zero. The stationary phase points are the solutions of the equation

$$-\xi = \mathcal{E}_1(k) - \mathcal{E}_1(k + \eta). \tag{40}$$

This equation has solutions $k_j = k_j(\xi, \eta) \in \mathcal{B}$, $j = 1, 2, \ldots, 2N$ and $N = N(\xi, \eta) \geq 1$, provided that ξ belongs to the interval $[-\xi^+(\eta), \xi^+(\eta)]$ where $0 \leq \xi^+(\eta) = \max_{k \in \mathcal{B}} [\mathcal{E}_1(k) - \mathcal{E}_1(k + \eta)]$. In particular, $\xi^+(\eta) = 0$ at η integer and $\xi^+(\eta)$ has maximum value $E_1^t - E_1^b$ at $\eta = 1/2$. From this and from Lemma 4, we can conclude that, for any $\eta \neq 0$ such that

$$g_\eta(k_j) := \mathcal{E}_1'(k_j) - \mathcal{E}_1'(k_j + \eta) \neq 0, \tag{41}$$

$\phi^t(\xi, y)$ has an oscillating behavior for any fixed ξ , and y taken in a compact set, and for η such that $\xi \in [-\xi^+(\eta)\xi^+(\eta)]$ is given by

$$\phi^t(\xi, y) \sim \sum_{j=1}^{2N} C_\eta(k_j) U_\eta^{1/f}(k_j) a_1(k_j + \eta) \psi_1(\xi/f + y, k_j) \tag{42}$$

as f goes to zero, where $C_\eta = \sqrt{i^{s_\eta}/|g_\eta|}$, $s_\eta(k)$ denotes the signum of $g_\eta(k)$. At $\xi = \pm\xi^+(\eta)$ equation (40) has a degenerate solution and so $|\phi^t(\xi^\pm(\eta), y)| = O(f^{-n/(4+2n)})$ as f goes to zero, for some $n \geq 1$. A similar behavior could appear also at the points $\xi \in (-\xi^+(\eta), \xi^+(\eta))$ such that the solution $k_j(\xi, \eta)$

has multiplicity larger than one. The number of such points is finite because $\mathcal{E}_1(k)$ is an analytic function and it is exactly zero in the limit $\mathcal{E}_1(k) \approx -C\cos(2\pi k)$ or $C(k^2 - 1/12)$, for some $C > 0$. On the other hand, $\phi^t(\xi, y)$ as a function of ξ has an exponentially decreasing behavior (modulo the error of the asymptotic formula) outside the interval $[-\xi^+(\eta), \xi^+(\eta)]$ because the solutions of equation (40) have imaginary part different from zero.

We can summarize this result in the following proposition (see also Figure 1):

Proposition. *Let us consider ξ fixed. In the limit $f \to 0^+$ keeping $\tau \in (\tau_1(f), \tau_2(f))$, we obtain the adiabatic behavior of the WBO:*

i) If we take for τ only integer values, the WBO approaches the initial state.

ii) For any τ not integer and fixed the WBO wavefunction $\phi^t(\xi, y)$ is localized in an interval $\xi \in [-\xi^+(\eta), \xi^+(\eta)]$ where

$$\xi^+(\eta) = \max_{k \in \mathcal{B}} [\mathcal{E}_1(k) - \mathcal{E}_1(k + \eta)] \tag{43}$$

is a function with minimum value 0 at $\eta = 0$ and maximum value $E_1^t - E_1^b$ at $\eta = 1/2$. In particular the amplitude of the wavefunction is generically of order $O(1)$ at any $\xi \in (-\xi^+(\eta), \xi^+(\eta))$ in agreement with the normalization of $\phi^t(\xi, y)$. The amplitude of the wavefunction is generically of order $O(f^{-1/6})$ at the boundary points $\xi = \pm\xi^+(\eta)$. In the limit case of a locally free band: $\mathcal{E}_1(k) \approx C(k^2 - 1/12)$ for $-1/2 < k < 1/2$ we have $\xi^+(\eta) \approx C(\eta - \eta^2)$.

iii) For $\eta \neq 0$, the WBO wavefunction $\phi^t(\xi, y)$ is locally given by a linear combination (42) of a finite number of Bloch states. In particular (Remark 6) at $\xi = 0$ we observe two Bloch states with two opposite quasimomentum values $k_1(\eta) = \eta/2$ and $k_2(\eta) = k_1(\eta) + 1/2$ moving on the torus with a velocity equal to one half. In the limit of the band function locally equal to the free one $\mathcal{E}_1(k) \approx C(k^2 - 1/12)$ for $-1/2 < k < 1/2$, we have only two Bloch states with quasimomentum values: $k_1(\eta, \xi) \approx -\xi/2\eta - \eta/2$ and $k_2(\eta, \xi) \approx -\xi/2(\eta - 1) - (\eta - 1)/2$.

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Bibliography

1. Agler J, Froese R 1985 *Existence of Stark ladder resonances,* Commun. Math. Phys. **100** 161-171.

2. Altarelli M 1988 *Envelope function approach to electronic states in heterostructures,* in *Interfaces, Quantum well and superlattice* ed. by R.Leavens and R.Taylor, Plenum Publ. Corp., 43-66.

3. Avron J 1982 *The lifetime of Wannier ladder states,* Ann. Phys. **143**, 33-53.

4. Bentosela F, Carmona R, Duclos P, Simon B, Souillard B, Weder R 1983 *Schrödinger operators with an electric field and random or deterministic potentials,* Commun. Math. Phys. **88**, 387-397.

5. Bentosela F, Grecchi V, Zironi F 1983 *Oscillations of Wannier resonances,* Phys. Rev. Lett. **50**, 84-86.

6. Bentosela F, Grecchi V 1991 *Stark-Wannier ladders,* Commun. Math. Phys. **142**, 169-192.

7. Bloch F 1928 *Über die quanten mechanik der elektronen in kristallgittern*, Z. Phys. **52** 555-600.

8. Bouchard A M, Luban M 1995 *Bloch oscillations and other dynamical phenomena of electrons in semiconductor superlattices*, Phys. Rev. B **52** 5105-5123.

9. Buslaev V, Dmitrieva L 1990 *A Bloch electron in an external field,* Leningrad Math. J. **1**, 287-320.

10. Buslaev V, Grigis A 1997 *Imaginary parts of Stark-Wannier resonances*, Prépublications Mathématiques de l'Université Paris-Nord, 97-10.

11. Callaway J 1974 *Quantum theory of the solid state* (Academic Press, New-York).

12. Combes J M, Hislop P 1991 *Stark ladder resonances for small electric field*, Commun. Math. Phys. **140**, 291-320.

13. Dignam M, Sipe J E, Shah J 1994 *Coherent excitations in the Stark ladder: excitonic Bloch oscillations*, Phys. Rev. B **49** 10502-10513.

14. Grecchi V, Maioli M, Sacchetti A 1992 *Stark resonances in disordered systems,* Commun. Math. Phys. **146**, 231-240.

15. Grecchi V, Maioli M, Sacchetti A 1994 *Stark ladder of resonances: Wannier ladders and perturbation theory,* Commun. Math. Phys. **159**, 605-618.

16. Grecchi V, Sacchetti A 1995 *Crossing and anticrossing of resonances: the Wannier-Stark ladders,* Ann. Phys. **241**, 258-284.

17. Grecchi V, Sacchetti A 1997 *Lifetime of the Wannier-Stark resonances and perturbation theory,* Commun. Math. Phys. **185**, 359-378.

18. Grecchi V, Sacchetti A 1997 *Metastable Bloch oscillators,* Phys. Rev. Lett. **78**, 4474-4477.

19. Herbst I 1980 *Exponential decay in the Stark effect,* Commun. Math. Phys. **75**, 197-205.

20. Herbst I, Howland J 1981 *The Stark ladder and the other one-dimensional external electric field problems,* Commun. Math. Phys. **80**, 23-42.

21. Kane E O, Blount E 1969 *Interband tunneling* in *Tunneling phenomena in solids*, ed. Burstein E and Lundqvist S (Plenum Press - New York), 79-91.

22. Kieselev A 1997 *Absolutely continuous spectrum of perturbed Stark operators*, preprint mp-arc 97-201.

23. Kittel C 1987 *Quantum theory of solids,* John Wiley and Sons (second edition).

24. Lyssenko V G, Valusis G, Löser F, Hasche T, Leo K, Dignam M M, Köhler K 1997 *Direct measurement of the spatial displacement of Bloch-oscillating electrons in semiconductor superlattices*, to appear on Phys. Rev. Lett..

25. Reed M, Simon B 1978 *Methods of Modern Mathematical Physics: IV Analysis of operators,* (Academic, New-York).

26. Simon B 1973 *Resonances in n–body quantum systems with dilation analytic potentials and the foundations of time–dependent perturbation theory,* Ann. Math. **97**, 247-274.

27. Wannier G H 1960 *Wave functions and effective Hamiltonian for Bloch electrons in an electric field,* Phys. Rev. **117**, 432-439.

28. Waschke C, Roskos H G, Schwedler R, Leo K, Kurz H, Köhler K 1993 *Coherent submillimeter-wave emission from Bloch oscillations in a semiconductor superlattice*, Phys. Rev. Let. **70**, 3318.

29. Zener C 1934 *A theory of electric breakdown of solid dielectrics,* Proc. R. Soc. A **145**, 523-529.

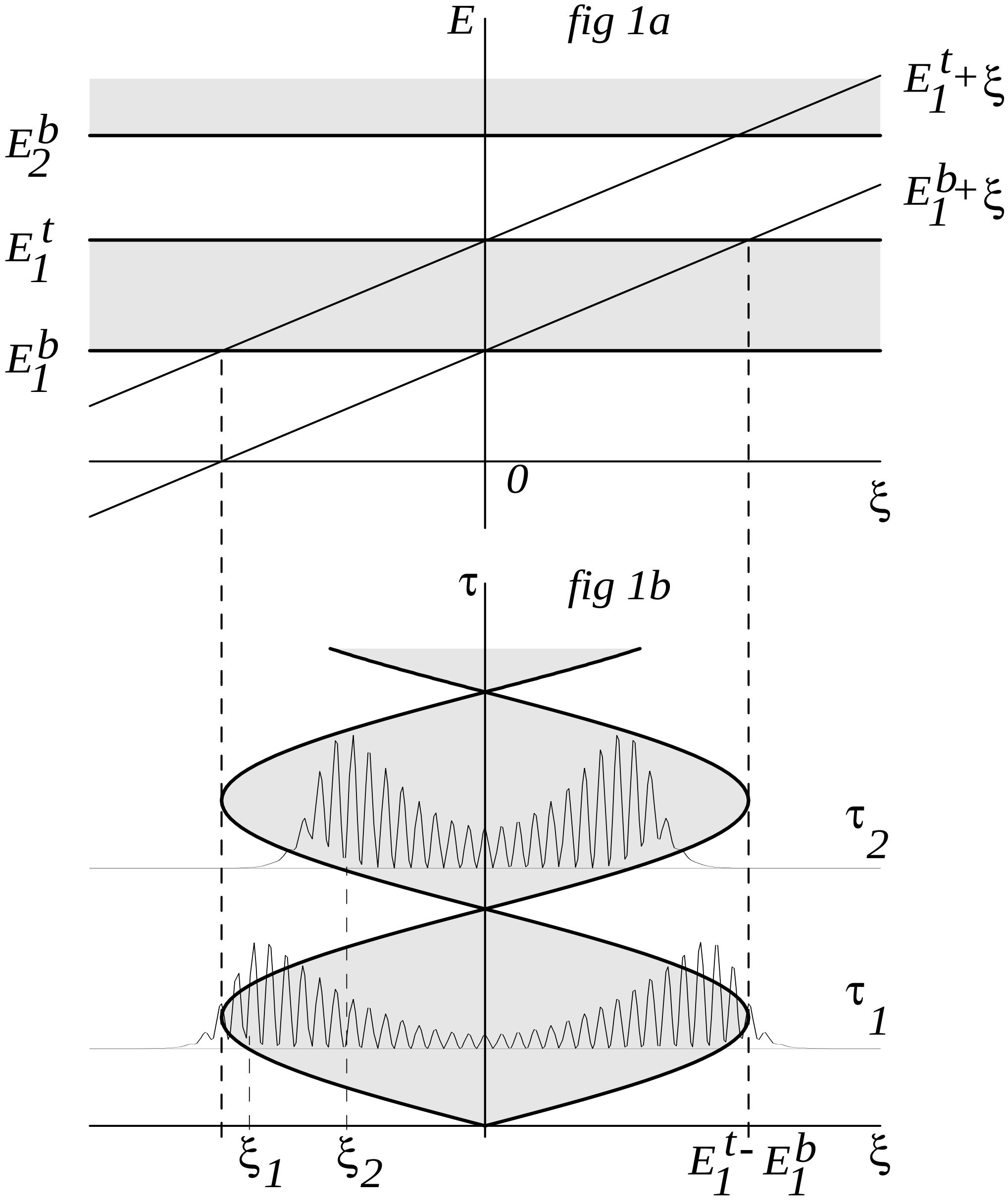

Figure 1: In the picture 1b the gray zone is the domain, in the (ξ, τ) plane, of localization of the Wannier-Bloch oscillator wavefunction (where we set $\mathcal{E}_1(k) \approx -C\cos(2\pi k)$, for some $C > 0$). For any τ (not integer) fixed the WBO wavefunction is localized in an interval $[-\xi^+(\tau), \xi^+(\tau)]$ and its amplitude attains a maximum value of order $\mathcal{O}(f^{-1/6})$ at the endpoints of the interval, inside the interval the amplitude is of order $\mathcal{O}(1)$. The interval $[-\xi^+(\tau), \xi^+(\tau)]$ is periodic with period 1 and it has a maximum amplitude given by $2(E_1^t - E_1^b)$ as suggested by the tilted band Zener picture (see picture 1a, where gray zone denotes the bands). In the picture 1b we draw the graph of the functions $\phi^t(\xi, 0)$ for $\tau_1 \approx 2/5$ and $\tau_2 \approx 6/5$, where $\xi_1 = -\xi^+(\tau_1)$ and $\xi_2 = -\xi^+(\tau_2)$.