Lifschitz tail in a magnetic field: the nonclassical regime

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Abstract

We estimate the Lifschitz tail of the two dimensional magnetic Schrödinger operator with a uniform magnetic field and random Poissonian impurities. The single site potential is compactly supported. We obtain the exact low energy asymptotics in the case of the hard-core potential at high density, and we prove lower and upper bounds for soft potential. Our methods involve magnetic localization techniques (both in space and energy) in addition to a modified version of the "enlargement of obstacles" method developed by A.-S. Sznitman.

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1 Introduction

Magnetic Schrödinger operators with random potentials have been intensively studied by physicists, in particular because of their relevance to the quantum Hall effect. Rigorous mathematical studies of these operators have appeared only recently (e.g. [3], [4], [14], [12]). For a wider background and many references (including physical ones) we refer to [3]. In this paper we focus on the integrated density of the states (IDS) in the low energy limit, i.e. on the so-called Lifschitz tail. We restrict ourselves to the two-dimensional situation which is the most relevant case for magnetic problems.

We shall consider both soft and hard-core obstacles. The asymptotic behaviour of the Lifschitz tail is not supposed to depend on the shape of the actual obstacle potential and our results are consistent with this expectation. For technical reasons, however, we are able to prove a stronger theorem for the hard-core case, but the crucial part of the proof is common for both cases. We mainly focus on the soft potential case (which is more general) and only indicate the necessary changes for the hard-core case. In particular we shall emphasize the reason why we have a stronger result for this latter case.

1.1 Definitions

Soft potential

Consider a nonnegative, measurable function $V^{(0)}$ on $\mathbf{R}^2$, which is positive on an open set of positive measure, i.e. $V^{(0)}(x) \geq v$ for $|x| \leq a$ for some $a, v > 0$. For technical simplicity, we assume that $V^{(0)}$ is continuous. Let

$$V(x) = V_\omega(x) = \sum_i V^{(0)}(x - x_i(\omega)) \tag{1.1}$$

be a random potential, where $x_i(\omega)$ is a Poisson point process on $\mathbf{R}^2$ with density ν (here ω denotes the randomness, but we shall usually omit it from the notations). The expectation with respect to the probability measure, $\mathcal{P}_\omega$, of the Poisson process is denoted by $\mathcal{E} = \mathcal{E}_\omega$. We consider the following magnetic Schrödinger operator with random potential V_ω

$$H(V) = H(V_\omega) = H_\omega = \frac{1}{2}(-i\nabla - A)^2 + V_\omega, \tag{1.2}$$

where $A : \mathbf{R}^2 \to \mathbf{R}^2$ is a deterministic vector potential generating the constant $B > 0$ magnetic field, i.e. $\operatorname{curl} A = B$. The properties we are interested in are independent of the actual gauge choice, so, conveniently, we choose $A(x) := (-Bx_2/2, Bx_1/2)$. Here $x = (x_1, x_2) \in \mathbf{R}^2$. We also define H_Λ as the restriction of H onto a domain $\Lambda \subset \mathbf{R}^2$ (with Dirichlet boundary conditions). In this paper by domain we mean an open, bounded subset of $\mathbf{R}^2$ with regular (C^1) boundary, which is not necessarily connected.

These operators are almost surely selfadjoint since $V_\omega \in L^1_{loc}$ with probability one. We always assume that ω belongs to this set of full measure. We define the integrated density of states (IDS) as

$$N(E) = \lim_{\Lambda \nearrow \mathbf{R}^2} \frac{1}{|\Lambda|} \mathcal{E}_\omega \mathrm{Tr} P_E(H_\Lambda), \tag{1.3}$$

where P_E is the spectral projection onto the half line $(-\infty, E]$, and $\Lambda \nearrow \mathbf{R}^2$ denotes an increasing sequence of nested regular domains, say cubes. In fact, the above limit equals to $\mathcal{E}_\omega[P_E(H)(x,x)]$, moreover it has the so called self-averaging property, i.e. the random quantity $|\Lambda|^{-1}\mathrm{Tr} P_E(H_\Lambda)$ becomes deterministic in the macroscopic limit. For details, we refer to [5] and references therein.

Hard-core potential

Let K be a compact set with non-empty interior and regular boundary, say $B(0,a) \subset K \subset B(0,\overline{a})$ for some $0 < a < \overline{a}$ and $\partial K \in C^1$. $B(x,r)$ denotes the ball of radius r centered at x. Consider the random set

$$\mathcal{T}_\omega = \mathcal{T} = \mathbf{R}^2 \setminus \bigcup_i \Big(K + x_i(\omega) \Big) \tag{1.4}$$

and let $H_\mathcal{T}$ be the operator $\frac{1}{2}(-i\nabla - A)^2$ with Dirichlet boundary conditions on $\mathcal{T}$. Since $\{x_i(\omega)\}_i$ is a.s. locally finite, $\mathcal{T}$ is open almost surely. Hence $H_\mathcal{T}$ and $H_{\mathcal{T}\cap\Lambda}$ are well defined as a selfadjoint operators. Let

$$N^{hc}(E) := \lim_{\Lambda \nearrow \mathbf{R}^2} \frac{1}{|\Lambda|} \mathcal{E}_\omega \mathrm{Tr} P_E(H_{\mathcal{T}\cap\Lambda}), \tag{1.5}$$

where P_E is the spectral projection as before. The existence of this limit is standard by the ergodic technique of superadditive processes. We refer to [11] for the general technique and to [12] for the magnetic case. This latter work states the theorem for the case of finite potential only, but all it uses is that the operator has compact resolvent, hence the hard-core case follows as well.

1.2 Results

Since the spectrum of the free magnetic operator starts at $B_0 := B/2$ (first Landau level), the low energy behaviour of the IDS amounts to studying the asymptotics of $N(B_0 + E)$ as $E \to 0 + 0$.

First, we recall the relevant results for the nonmagnetic ($B = 0$) case in any dimension d (see [15]). Assume that the single site potential, $V^{(0)}$, is in L^2_{loc} and it has a definite tail behaviour with exponent $\alpha > d$, i.e.

$$\lim_{|x|\to\infty} |x|^\alpha V^{(0)}(x) = \mu, \quad 0 < \mu < \infty. \tag{1.6}$$

For slowly decaying potentials, i.e. for $d < \alpha < d + 2$, the asymptotics of $N(E)$ is given by

$$\lim_{E\to 0+0} E^{d/(\alpha-d)} \log N(E) = -C(d, \alpha, \mu, \nu), \tag{1.7}$$

where the constant is explicit. This behaviour is completely governed by the potential energy, i.e. by classical effects. For $\alpha > d + 2$ (more precisely, if $V^{(0)}(x) = o(|x|^{-d-2})$ at infinity), one has (see [6] and [17])

$$\lim_{E\to 0+0} E^{d/2} \log N(E) = -C(d, \nu), \tag{1.8}$$

i.e. the tail is independent of the parameters of the single site potential. In fact, we obtain the same behaviour for hard obstacles (formally, it corresponds to $\alpha = \infty$), indicating that the localization properties of the kinetic energy plays the major role in contrast to the previous case of slowly decaying potential. We call this regime nonclassical. Note that the threshold for this transition is at the decay exponent $\alpha = d + 2$. Heuristically, it is obtained from the competition between the kinetic energy within the ball of radius R ($\sim R^{-2}$) and the potential energy within this ball originated from obstacles outside the ball ($\sim \int_{|x| \geq R} \frac{\mu}{|x|^\alpha} \nu \, \mathrm{d}x \sim R^{d-\alpha}$).

In the magnetic case, the situation is different. The magnetic field itself has a strong localization effect, i.e. the kinetic energy is expected to play less role. In fact, it has been proven in [3] that the classical effects dominate for any $d = 2 < \alpha < \infty$, and any $B > 0$, i.e. (compare with (1.7); even the constant is the same)

$$\lim_{E \to 0+0} E^{2/(\alpha-2)} \log N(B_0 + E) = -C(d = 2, \alpha, \mu, \nu) \tag{1.9}$$

under the condition (1.6). Note that the strength of the magnetic field does not appear in the asymptotic behaviour of $N(E + B_0)$, only the fact that $B > 0$ is used. Analogous result is valid in higher even dimensions as well.

One of the conclusion of [3] is that the low energy tail is always determined by the classical effect. In the present paper, we show that with a more careful analysis one can detect the nonclassical behaviour, but the threshold decay of the potential, where the transition occurs, has to be faster than polynomial. Therefore it is invisible in the regime investigated in [3]. Moreover, we show that the asymptotic behaviour of $N(B_0 + E)$ does depend on the strength of B in the nonclassical regime. Our theorem is the following [$\mathbf{1}(\cdot)$ denotes the characteristic function]:

Theorem 1.1 *In two dimensions let ν be the intensity of the Poisson point process and let $B = 2B_0$ be a constant magnetic field.*

(i) (Soft potential) Let $V^{(0)}$ be continuous, compactly supported and $V^{(0)} \geq v\mathbf{1}_{B(0,a)}$ for some $a, v > 0$. Then the integrated density of states of $H(V)$, defined in (1.3), satisfies

$$-\frac{2\pi\nu}{B} \leq \liminf_{E \to 0+0} \frac{\log N(E + B_0)}{|\log E|} \leq \limsup_{E \to 0+0} \frac{\log N(E + B_0)}{|\log E|} \leq -\frac{2\pi\nu}{B} + 2. \tag{1.10}$$

(ii) (Hard-core potential) Assume that the compact set K contains a ball $B(0,a)$ and has a regular boundary. Consider $H_{\mathcal{T}}$ as defined above, where $\mathcal{T}$ is given by (1.4). The integrated density of states $N^{hc}(E)$, defined in (1.5), satisfies

$$\lim_{E\to 0+0} \frac{\log N^{hc}(E+B_0)}{|\log E|} = -\frac{2\pi\nu}{B} \tag{1.11}$$

if the density is large enough, namely if

$$\nu a^2 \geq 27.97. \tag{1.12}$$

Remark 1. The gap between the lower and the upper bound in part (i) is due to the method of our proof. We are certain that the correct behavior is given by the lower bound. The upper bound is meaningful only for weak magnetic field: $B \leq \nu\pi$.

Remark 2. The lower bound in (1.12) comes from a robust numerical bound in percolation theory. It is not our goal to optimize it, especially since we believe that (1.11) is true without any condition on the density.

Remark 3. This theorem does not establish the optimal condition on the decay of $V^{(0)}$. In particular, we do not address the question how to improve our result to include potentials with unbounded support (the results of [3] show that the decay of the potential has to be faster than any polynomial). Similarly to the nonmagnetic case, a competition between the kinetic energy and the potential energy suggests that the threshold decay is probably Gaussian.

Remark 4. For Dirac delta potential $V^{(0)}$, the precise asymptotics was obtained in [2] using functional integrals (the asymptotic behaviour coincides with the lower bound in (1.10)). That method does not seem to apply to a wider class of potentials, though the result makes the conjecture in Remark 1 plausible since one expects that the Lifschitz tail behaviour in the nonclassical regime is independent of the actual shape of the potential.

Remark 5. Our method does not work to investigate Lifschitz tails above higher Landau levels. Note that statements analogous to (1.9) were proven in [3] for the restriction of H onto any fixed Landau band.

1.3 Idea of the proof

Similarly to the difference in complexity between the proofs of (1.7) and (1.8), one can expect that the proof of (1.10) requires more detailed analysis than that of (1.9). In particular, the lower bound on the Hamiltonian (upper bound on the IDS) is more complicated; Golden-Thompson and related inequalities used in [3] are insufficient.

First, we translate the problem into a heat kernel estimate. Then, there are two main ingredients in our work. Not unexpectedly, the proof of (1.10) must contain an argument similar to the proof of (1.8), i.e. a Donsker-Varadhan type upper bound on the heat kernel (see [6]). A.-S. Sznitman has worked out an alternative method for proving this upper bound (in fact, his technique is applicable to a much wider class of problems), and we found this approach suitable to the magnetic problem. This is the first major ingredient.

The second ingredient is an isoperimetric inequality. For the nonmagnetic case this boils down to the standard isoperimetric problem; i.e. minimization of the lowest Dirichlet eigenvalue of a domain with a fixed volume. Note that neither Donsker-Varadhan's nor Sznitman's method can avoid this step. Apparently the same is true for the magnetic problem; this was our main motivation to prove the corresponding isoperimetric inequality with a magnetic field in [8].

The present paper contains the heat kernel estimates and here we explain briefly the strategy.

Sznitman's method is not directly applicable, since it heavily uses that the transition kernels are positive, in contrast to the oscillatory character of the magnetic problem. It also requires a real valued "free" diffusion process, such that the Feynman-Kac formula for $e^{-tH(V)}$ could be viewed as its perturbation. We were not able to find a suitable diffusion process associated with the free magnetic problem. Diffusion with a drift comes as a natural candidate but we are not aware of any process with a real drift that mimics the free magnetic Schrödinger operator. We are able to replace the effect of the magnetic field by a potential via a variational principle (see Eqs. (6) and (8) in [8]), but this potential changes sign which makes it hard to estimate in the Feynman-Kac formula.

Instead, our key idea is to use the spectral gap in the free magnetic Hamiltonian to separate the lowest Landau band. It turns out, by a simple energy argument, that the contributions from the higher Landau bands are irrelevant as we are investigating unusually low lying eigenvalues. Hence almost the whole principal eigenstate lives in the lowest Landau band. By the Aharonov-Casher theorem [1], the projection of any state onto the lowest Landau band can be expressed as he^{Bg} where h is analytic, and g solves $\Delta g = -1$ on the domain. We write g in terms of the expected value of the boundary hitting time of a free Brownian motion. It is this stage where we use an argument similar to Sznitman's, applied to a process which is, unlike in Sznitman's papers, completely different from the one generated by the original Hamiltonian.

Before starting the proof, we wish to elucidate the origin of the gap between the two bounds in (1.10).

The IDS in infinite volume contains information about infinitely many (generalized) eigenvalues. However, the crucial part of our analysis (as well as in Sznitman's original work) proves that the lowest eigenvalue of $H(V)$ is comparable to the lowest eigenvalue of another Hamiltonian (the one with the so-called "enlarged obstacles"), whose potential configuration has a smaller entropy factor. This method is able to estimate only the lowest eigenvalue, hence, before applying it, we have to localize the problem onto a small enough box on which the lowest eigenvalue solely determines the low energy asymptotics of the averaged IDS. Fortunately, by (1.3), the infinite volume IDS can be approximated by a finite volume IDS. Hence, it will be enough to consider the Hamiltonian restricted to a finite box. We call this step the apriori localization. The choice of the size of this "apriori" box depends on the energy threshold E according to the following two requirements.

On one hand, it turns out that for each energy E the main contribution to the averaged IDS comes from very atypical configurations (where the Poisson cloud has a large clearing) which support an eigenfunction with a very low lying eigenvalue. This eigenfunction has a natural lengthscale, depending only on E. Obviously, the apriori box has to be at least as big as this natural lengthscale in order not to destroy these low lying states by the artificial apriori localization.

On the other hand, the actual number of eigenvalues below E should be irrelevant (on logarithmic scale) compared to the large deviation probability that there is any eigenvalue below E at all. This is the case, at least, if the size of the apriori box is not much bigger than the natural lengthscale of the typically contributing eigenfunction. The number of the relevant low lying eigenvalues is approximately equal to the volume ratio between the apriori box and the box with side lengths equal to the natural lengthscale.

The whole proof is actually done in the language of the heat kernel as we explain it in Section 2. We shall take the Laplace transform of $N(E)$ to obtain the heat kernel, whose time parameter t is the conjugate variable to E. The small energy asymptotics of $N(E)$ is related to the large time asymptotics of the heat kernel by a Tauberian type argument, which also establishes the relation between t and E. The apriori localization entails a large deviation estimate of the probability that the Brownian loop (in the Feynman-Kac-Ito representation of the heat kernel) goes extremely far. By the simplest Gaussian tail estimate on this probability, one obtains the necessary decay (exponential in t is good enough) if the linear size of the apriori box is of order t. This estimate does not use any extra localization due to the magnetic field (if any); the oscillatory factor in the Feynman-Kac-Ito formula, which supposedly enhances localization, is estimated by absolute value. We remark that it is not clear to what extent the strong Gaussian type off-diagonal decay of the free magnetic heat kernel survives under potential perturbation. Warning examples and related phenomena are discussed in [7] and [9].

In the nonmagnetic case the Tauberian argument sets $E \sim t^{-2/(d+2)}$, and the natural lengthscale is of order $E^{-1/2} \sim t^{1/(d+2)}$, d being the space dimension. Since the volume of the apriori box is of order t^d, the number of low lying eigenvalues is around $t^d/t^{d/(d+2)}$, i.e. a power of t. This overcounting is irrelevant compared to the subexponentially small size of the averaged heat kernel ($\sim \exp -t^{d/(d+2)}$).

In the magnetic case, the situation is much tighter. The natural lengthscale is proportional to $\sqrt{|\log E|} \sim \sqrt{\log t}$, and the overcounting is still a power of t. But the actual heat kernel decay is algebraic as well ($t^{-2\pi\nu/B}$, see (2.8) later). Hence the overcounting error is comparable to the main term and it influences the upper bound in (1.10). Even the proof of this weaker result is fairly involved, in fact we use a two step localization scheme and the special magnetic localization technique developed in [10].

It is only in the large density hard-core case where we are able to control the overcounting more effectively. If the density is large enough, then the obstacle configuration is in the percolating regime. The connected components of $\mathcal{T}$ are bounded, and, from percolation theory, one has an effective exponential estimate on the probability that a component is large. This gives an apriori localization for free on a much smaller lengthscale than the one given by Brownian loop large deviation.

2 Heat kernel, Laplace transform

Soft potential:

Let $L(t)$ $(t > 0)$ be the shifted Laplace transform of the density of states $N(E)$:

$$L(t) := \int e^{-\lambda t}\, \mathrm{d}N(B_0 + \lambda) = e^{B_0 t} \int e^{-\lambda t}\, \mathrm{d}N(\lambda). \tag{2.1}$$

For all ω and $x \in \mathbf{R}^2$ we can define

$$L_\omega^x(t) := (2\pi t)^{-1} e^{B_0 t} \mathbf{E}_{x,0}^{x,t} e^{-i \int_0^t A(W_s) \mathrm{d}W_s - \int_0^t V_\omega(W_s) \mathrm{d}s} = e^{-tH_\omega}(x, x), \tag{2.2}$$

where $\mathbf{E}_{y,0}^{x,t}$ stands for the expectation with respect to the probability measure of the two dimensional Brownian bridge W_s $(0 \le s \le t)$ with $W_0 = x$, $W_t = y$. Notice that the heat kernel, $e^{-tH_\omega}(x, y)$ in general exists only for almost all x and y and almost surely. But by Theorem 3.1. from [5] it is continuous almost surely (if $V_\omega \in L^1_{loc}$), and the Feynman-Kac-Ito formula gives this continuous representation. In particular, the diagonal element is well defined.

Notice that $L_\omega^x(t)$ is independent of the gauge choice, in particular its distribution is ergodic with respect to the spatial translations in x, i.e. $\mathcal{E}_\omega L_\omega^x(t)$ is independent of the choice of x.

Using the approximation result in [3] (formula (A.10)) we get that

$$L(t) = \mathcal{E}_\omega L_\omega^x(t) \tag{2.3}$$

for any x (in particular this shows that $L(t)$ is finite).

Hard-core potential

Analogously to (2.1) and (2.2) we define

$$L^{hc}(t) := \int e^{-\lambda t}\, \mathrm{d}N^{hc}(B_0 + \lambda) = e^{B_0 t} \int e^{-\lambda t}\, \mathrm{d}N^{hc}(\lambda), \tag{2.4}$$

and for all $x \in \mathbf{R}^2$

$$L_\omega^{hc,x}(t) := (2\pi t)^{-1} e^{B_0 t} \mathbf{E}_{x,0}^{x,t} e^{-i\int_0^t A(W_s)\mathrm{d}W_s} \mathbf{1}(T_{\mathcal{T}_\omega} > t) = e^{-tH_{\mathcal{T}_\omega}}(x,x). \tag{2.5}$$

Here and in the sequel T_Ω denotes the exit time from the domain Ω.

The continuity of the heat kernel of $H_{\mathcal{T}}$ has not been explicitly proven in [5], but it follows immediately from the soft potential case by the following approximation. Since $\mathcal{T} = \mathcal{T}_\omega$ is open (almost surely), by dominated convergence

$$\mathbf{E}_{x,0}^{y,t} e^{-i\int_0^t A(W_s)\mathrm{d}W_s - \int_0^t V_n(W_s)\mathrm{d}s} \to \mathbf{E}_{x,0}^{y,t} e^{-i\int_0^t A(W_s)\mathrm{d}W_s} \mathbf{1}(T_{\mathcal{T}} > t) \tag{2.6}$$

locally uniformly in $x, y \in \mathcal{T}$ and in t, where V_n is an increasing sequence of continuous functions supported on $\mathcal{T}^c$ such that $V_n(x) \to \infty$ for all $x \in \mathcal{T}^c$. Such a sequence exists because $\mathrm{int}(K)$ is non-empty and ∂K is regular.

By continuity of the heat kernel, its diagonal element in (2.5) is well defined and for all x

$$L^{hc}(t) = \mathcal{E}_\omega L_\omega^{hc,x}(t). \tag{2.7}$$

Again, this relation is proven in [3] only for soft potential, but the proof of (A.10) goes through with obvious changes for the hard-core case as well.

Theorem 2.1 *(i) (Soft potential) Under the conditions of part (i) of Theorem 1.1, we have*

$$-\frac{2\pi\nu}{B} \le \liminf_{t\to\infty} \frac{\log L(t)}{\log t} \le \limsup_{t\to\infty} \frac{\log L(t)}{\log t} \le -\frac{2\pi\nu}{B} + 2. \tag{2.8}$$

(ii) (Hard-core potential) Under the conditions of (ii) of Theorem 1.1, we have

$$\lim_{t\to\infty} \frac{\log L^{hc}(t)}{\log t} = -\frac{2\pi\nu}{B}. \tag{2.9}$$

By a standard Tauberian argument, Theorem 1.1 immediately follows from Theorem 2.1.

The rest of the paper contains the proof of Theorem 2.1. We shall focus on the proof of part (i), and only indicate the changes required for the proof of part (ii).

3 Apriori localization

Let $Q = [-q(t), q(t)]^2$ be a large box with some $q(t) \gg 1$ to be determined later. For almost all configurations ω, we consider the self-adjoint operator $H_{Q,\omega} = \frac{1}{2}(-i\nabla - A)^2 - B_0 + V_\omega$ with Dirichlet boundary condition on Q. Let $\lambda_{Q,\omega}$ be its lowest eigenvalue. For any $x \in Q$, let

$$L^x_{Q,\omega}(t) := e^{-tH_{Q,\omega}}(x,x) = (2\pi t)^{-1} e^{B_0 t} \mathbf{E}^{x,t}_{x,0} e^{-i\int_0^t A(W_s)dW_s - \int_0^t V_\omega(W_s)\mathrm{d}s} \mathbf{1}(T_Q > t) \tag{3.1}$$

be the diagonal element of the heat kernel by the magnetic Feynman-Kac formula. Recall that for any domain $\Omega \subset \mathbf{R}^2$, T_Ω denotes the exit time from the set Ω.

The heat kernel of $H_{Q,\omega}$ is trace class, therefore

$$L_{Q,\omega}(t) := \frac{1}{|Q|}\int_Q L^x_{Q,\omega}(t)\mathrm{d}x = \frac{1}{|Q|}\mathrm{Tr}\, e^{-tH_{Q,\omega}} \tag{3.2}$$

$$= \frac{1}{|Q|}(2\pi t)^{-1} e^{B_0 t}\int_Q \mathrm{d}x\, \mathbf{E}^{x,t}_{x,0} e^{-i\int_0^t A(W_s)\mathrm{d}W_s - \int_0^t V_\omega(W_s)\mathrm{d}s}\mathbf{1}(T_Q > t)$$

exists and is finite ($|\cdot|$ denotes the Lebesgue measure of a set). Finally let

$$L_Q(t) := \mathcal{E}_\omega L_{Q,\omega}(t).$$

The quantities $\lambda^{hc}_{Q,\omega}$, $L^{hc,x}_{Q,\omega}(t)$, $L^{hc}_{Q,\omega}(t)$ and $L^{hc}_Q(t)$ for the hard-core case are defined analogously.

We need the following robust estimates.

Lemma 3.1 *(i) For $t > 2B^{-1}$, for any $x \in \mathbf{R}^2$ and for almost all ω*

$$L^x_\omega(t) = e^{-tH_\omega}(x,x) \le B, \tag{3.3}$$

and for any square Q

$$L^x_{Q,\omega}(t) = e^{-tH_{Q,\omega}}(x,x) \le B. \tag{3.4}$$

(ii) For any square Q, for any x, $t > 4B^{-1}$ and for almost all ω

$$L^x_{Q,\omega}(t) = e^{-tH_{Q,\omega}}(x,x) \le 10B\Big(e^{-t\lambda_{Q,\omega}} + e^{-B_0 t}\Big). \tag{3.5}$$

The same estimates hold for the hard-core case.

Proof of Lemma 3.1. (i) The robust diamagnetic estimate in the Feynman-Kac-Ito formula (2.2) gives

$$e^{-\tau H_\omega}(x,x) \le (2\pi\tau)^{-1} e^{B_0\tau}, \tag{3.6}$$

for any $\tau > 0$. Hence, using $H_\omega \ge 0$, we have

$$e^{-tH_\omega}(x,x) = \Big(e^{-B^{-1}H_\omega}(\cdot,x), e^{-(t-2B^{-1})H_\omega} e^{-B^{-1}H_\omega}(\cdot,x)\Big)_{L^2(\mathbf{R}^2)}$$

$$\le \Big\|e^{-B^{-1}H_\omega}(\cdot,x)\Big\|^2_{L^2(\mathbf{R}^2)} = e^{-B^{-1}H_\omega}(x,x) \le B.$$

The proof of (3.4) is identical.

(ii) Similarly to part (i),

$$e^{-tH_{Q,\omega}}(x,x) = \Big(e^{-B^{-1}H_{Q,\omega}}(\cdot,x), e^{-(t-2B^{-1})H_{Q,\omega}} e^{-B^{-1}H_{Q,\omega}}(\cdot,x)\Big)_{L^2(\mathbf{R}^2)} \tag{3.7}$$

$$\le e^{-(t-2B^{-1})\lambda_{Q,\omega}} \Big\|e^{-B^{-1}H_{Q,\omega}}(\cdot,x)\Big\|^2_{L^2(\mathbf{R}^2)} = e^{-(t-2B^{-1})\lambda_{Q,\omega}} e^{-B^{-1}H_{Q,\omega}}(x,x)$$

$$\le 10B\Big(e^{-t\lambda_{Q,\omega}} + e^{-B_0 t}\Big)$$

using that $H_{Q,\omega} \ge \lambda_{Q,\omega}$, $t > 4B^{-1}$ and that

$$e^{-\tau H_{Q,\omega}}(x,x) \le (2\pi\tau)^{-1} e^{B_0\tau}, \tag{3.8}$$

for any τ, again by a robust estimate in the Feynman-Kac formula (3.1). In the last line of (3.7) we distinguished the cases $\lambda_{Q,\omega} \le B$ and $\lambda_{Q,\omega} \ge B$. $\square$

3.1 Estimates on the lowest eigenvalue on a disk

Let $\lambda(B,R)$ be the lowest eigenvalue of $\frac{1}{2}(-i\nabla - A)^2 - B_0$ with Dirichlet boundary conditions on the disk $B(0,R)$. Recall that $B = 2B_0$ is the magnetic field. In the Appendix of [8] we gave a lower and an upper bound on $\lambda(B,R)$ (the notation was slightly different). The choice of the parameters in the proof given there is not well-suited for our problem, hence we slightly reformulate the statement and the proof.

Proposition 3.2 *For any $\kappa > 0$, there exists $C(\kappa) > 0$ such that*

$$e^{-B_0R^2(1+\kappa)} \le \lambda(B,R) \le e^{-B_0R^2(1-\kappa)} \tag{3.9}$$

if $B_0R^2 \ge C(\kappa)$.

Proof. We slightly modify the proof given in [8]. Recall that it is sufficient to consider the case $B = 1$ by scaling. Then, we identified $\lambda(B=1,R)$ with the smallest eigenvalue of the one dimensional harmonic oscillator $-\partial_x^2 + x^2/4 - 1/2$ on the interval $[-R,R]$.

To construct a trial function for the upper bound in (3.9), we cut off the Gaussian eigenfunction $\phi(x) = e^{-x^2/4}$ of the unrestricted oscillator by straight line segments on $R-1 \le |x| \le R$. For large R this is more effective than the line segments on $R/2 \le |x| \le R$ used in [8].

For the lower bound we follow the same proof based on the Birman-Schwinger principle, but we choose the following parameters: $E := \exp(\kappa R^2)$, $N := \Big(\frac{1}{2} + 2\kappa\Big)R^2E$, $\eta := \exp\Big(-[\frac{1}{2}+\kappa]R^2\Big)$. $\square$.

3.2 Lower bound

Here we give the lower bound on $L(t)$ in Theorem 2.1 (the same argument works for $L^{hc}(t)$ as well). First we show that

$$L_Q(t) \le L(t). \tag{3.10}$$

Let $\Lambda_M := [-Mq(t), Mq(t)]$ with M being an integer, and let $M' < M$. We have

$$L(t) \geq \frac{1}{|\Lambda_M|} \int_{\Lambda_{M'}} \mathcal{E}_\omega e^{-tH_\omega}(x,x)\,\mathrm{d}x \tag{3.11}$$

$$= \frac{1}{|\Lambda_M|} \int_{\Lambda_{M'}} \mathcal{E}_\omega e^{-tH_{\Lambda_M,\omega}}(x,x)\,\mathrm{d}x + \frac{1}{|\Lambda_M|} \int_{\Lambda_{M'}} \mathcal{E}_\omega \Big[e^{-tH_\omega}(x,x) - e^{-tH_{\Lambda_M,\omega}}(x,x)\Big]\mathrm{d}x$$

$$\geq \frac{1}{|\Lambda_M|} \int_{\Lambda_M} \mathcal{E}_\omega e^{-tH_{\Lambda_M,\omega}}(x,x)\,\mathrm{d}x - \frac{B(|\Lambda_M| - |\Lambda_{M'}|)}{|\Lambda_M|} - \frac{4e^{B_0 t}|\Lambda_{M'}|}{2\pi t|\Lambda_M|} \exp\Big(-\frac{[(M-M')q(t)]^2}{8t}\Big).$$

Here we used (3.4) for $Q = \Lambda_M$ and we estimated

$$\Big|\mathcal{E}_\omega\Big[e^{-tH_\omega}(x,x) - e^{-tH_{\Lambda_M,\omega}}(x,x)\Big]\Big| = \Big|\mathcal{E}_\omega (2\pi t)^{-1} e^{B_0 t} \mathbf{E}_{x,0}^{x,t} e^{-i\int_0^t A(W_s)\mathrm{d}s - \int_0^t V_\omega(W_s)\mathrm{d}s} \mathbf{1}(T_{\Lambda_M} \leq t)\Big|$$

$$\leq 4(2\pi t)^{-1} e^{B_0 t} \exp\Big(-\frac{[(M-M')q(t)]^2}{8t}\Big)$$

for any $x \in \Lambda_{M'}$. The last estimate follows from the fact that x is at least at a distance $(M-M')q(t)$ from the boundary of Λ_M and from the standard large deviation estimate for the Brownian loop ([16])

$$\mathbf{P}_{0,0}^{0,t}(\sup_{0\leq s\leq t} |W_s| \geq L) \leq 4\exp\left(-\frac{L^2}{8t}\right) \tag{3.12}$$

for any $L > 0$.

Since M is an integer, there are squares $\{Q_i\}_{i=1}^{M^2}$, each being congruent to Q, which partition Λ_M. Obviously $H_{\Lambda_M,\omega} \leq \bigoplus_i H_{Q_i,\omega}$. Since the trace of the heat kernel is operator monoton, we have

$$\frac{1}{|\Lambda_M|} \int_{\Lambda_M} e^{-tH_{\Lambda_M,\omega}}(x,x)\,\mathrm{d}x = \frac{1}{|\Lambda_M|} \mathrm{Tr}\, e^{-tH_{\Lambda_M,\omega}} \geq \frac{1}{M^2} \sum_i \frac{1}{|Q_i|} \mathrm{Tr}\, e^{-tH_{Q_i,\omega}}.$$

Hence (3.11) yields

$$L(t) \geq L_Q(t) - \frac{B(|\Lambda_M| - |\Lambda_{M'}|)}{|\Lambda_M|} - \frac{4e^{B_0 t}|\Lambda_{M'}|}{2\pi t|\Lambda_M|} \exp\Big(-\frac{[(M-M')q(t)]^2}{8t}\Big).$$

Since this is true for any M, M', letting $M, M' \to \infty$ such that $M - M' \to \infty$ but $M - M' \ll M$, we obtain (3.10).

Now we give a lower bound on $L_Q(t)$. Since it corresponds to an upper bound on the Hamiltonian $H_{Q,\omega}$, it is enough to choose a convenient obstacle configuration and a trial function.

Let $\ell(t) := 10\Big(\frac{\log t}{B_0}\Big)^{1/2}$. This is the natural lengthscale, mentioned in the introduction, on which the eigenfunctions, with eigenvalues substantially contributing to the Lifschitz tail, are supported.

For the lower bound one can choose $q(t) := \ell(t)$ for the size of the apriori localization. Let $\operatorname{supp} V^{(0)} \subset B(0, \overline{a})$ for some $\overline{a}$. Hence

$$L_Q(t) = \frac{1}{|Q|}\mathcal{E}_\omega \operatorname{Tr} e^{-tH_{Q,\omega}} \geq \frac{1}{|Q|}\sup_{\Omega\subset Q}\Big\{e^{-t\lambda(\Omega)} \cdot \mathcal{P}_\omega\Big(\Omega \cap \bigcup_i B(x_i(\omega), \overline{a}) = \emptyset\Big)\Big\}$$

$$\geq \frac{1}{|Q|}\sup_{\Omega\subset Q}\Big\{e^{-t\lambda(\Omega)}e^{-\nu|\Omega + B(0,\overline{a})|}\Big\}, \tag{3.13}$$

where the supremum runs through all open sets $\Omega \subset Q := [-q(t), q(t)]^2$, and $\lambda(\Omega) = \lambda^{(B_0)}(\Omega)$ denotes the lowest eigenvalue of $H_\Omega := \frac{1}{2}(-i\nabla - A)^2 - B_0$ with Dirichlet boundary conditions on Ω. We usually omit the superscript B_0.

For Ω, one can choose a disk with radius $R := \Big(\frac{\log t}{B_0(1-\kappa)}\Big)^{1/2} \leq \ell(t)$ with some $0 < \kappa < 1/2$. Then $\lambda(\Omega) \leq e^{-B_0 R^2(1-\kappa)} = t^{-1}$ by Proposition 3.2 for large enough t. Hence, we obtain from (3.13) and (3.10)

$$L(t) \geq L_Q(t) \geq \frac{1}{\ell(t)^2} \cdot t^{-\frac{2\pi\nu}{B(1-2\kappa)}} \tag{3.14}$$

for large enough $t \geq \tilde{t}(B, \nu, \overline{a})$. Taking logarithm of (3.14), and considering $\liminf_{t\to\infty}$ first, then $\lim_{\kappa\to 0}$, we immediately obtain the lower bound in (2.8). The lower estimate in (2.9) is identical.

3.3 Upper bound

For the upper bound, we have to choose the lengthscale of the apriori localization $q(t)$ much bigger than the natural lengthscale $\ell(t)$.

Proposition 3.3 *Let $t > 4/B$.*
(i) (Soft potential) Assume that $q(t) \geq 4C_1 t$ with some $C_1 \geq (24B_0)^{1/2}$. Then

$$L(t) \leq \Big(1 + \frac{4C_1 t}{q(t)}\Big) L_Q(t) + \frac{160 C_1 t}{q(t)} \mathcal{E}_\omega e^{-t\lambda_{Q,\omega}} + 50 B e^{-B_0 t}. \tag{3.15}$$

This estimate is valid for the hard-core case as well. Furthermore, we have

$$L(t) \leq 160\Big(B + \frac{C_1 t}{q(t)}\Big) \mathcal{E}_\omega e^{-t\lambda_{Q,\omega}} + 70 B e^{-B_0 t}. \tag{3.16}$$

(ii) (Hard-core potential) Assume (1.12) holds, then there exist universal constants C_2, C_3, C_4 such that

$$L^{hc}(t) \leq 4 L_Q^{hc}(t) + 50 B\Big(C_2 e^{-C_3 q(t)} + e^{-B_0 t}\Big), \tag{3.17}$$

$$L^{hc}(t) \leq 100 B\Big(\mathcal{E}_\omega e^{-t\lambda_{Q,\omega}^{hc}} + C_2 e^{-C_3 q(t)} + e^{-B_0 t}\Big) \tag{3.18}$$

hold if $q(t) \geq C_4$.

Proof of Proposition 3.3. Lemma 3.1 implies (3.16) once (3.15) is proved using

$$L_Q(t) = \frac{1}{|Q|} \int_Q \mathcal{E}_\omega L_{Q,\omega}^x(t)\, \mathrm{d}x.$$

Similar argument is valid for the hard-core case to show that (3.17) implies (3.18).

For the proof of (3.15), we consider the cube $\tilde{Q} = [-(q(t) - C_1 t), q(t) - C_1 t]^2$. Then

$$L_{Q,\omega}(t) = \frac{1}{|Q|} \int_{\tilde{Q}} L_\omega^x(t)\, \mathrm{d}x + \frac{1}{|Q|} \int_{Q\setminus\tilde{Q}} L_{Q,\omega}^x(t)\, \mathrm{d}x + \frac{1}{|Q|} \int_{\tilde{Q}} \Big(L_{Q,\omega}^x(t) - L_\omega^x(t)\Big)\, \mathrm{d}x,$$

and by taking expectation

$$L_Q(t) := \mathcal{E}_\omega L_{Q,\omega}(t) = \frac{|\tilde{Q}|}{|Q|} L(t) + \frac{1}{|Q|} \int_{Q\setminus\tilde{Q}} \mathcal{E}_\omega L_{Q,\omega}^x(t)\, \mathrm{d}x + \frac{1}{|Q|} \int_{\tilde{Q}} \mathcal{E}_\omega \Big(L_{Q,\omega}^x(t) - L_\omega^x(t)\Big)\, \mathrm{d}x. \tag{3.19}$$

In the main term use that $|\tilde{Q}|/|Q| \geq 1 - 2C_1 t/q(t)$. The first error term is estimated by (3.5)

$$\frac{1}{|Q|}\int_{Q\setminus\tilde{Q}} \mathrm{d}x\, \mathcal{E}_\omega L^x_{Q,\omega}(t) \leq \frac{80C_1Bt}{q(t)}\Big(\mathcal{E}_\omega e^{-t\lambda_{Q,\omega}} + e^{-B_0 t}\Big) \tag{3.20}$$

since $|Q \setminus \tilde{Q}| \leq 8C_1 t q(t)$.

On $\tilde{Q}$ we can write

$$\begin{aligned} &\frac{1}{|Q|}\Big|\int_{\tilde{Q}} \mathrm{d}x\, \mathcal{E}_\omega\Big(L^x_{Q,\omega}(t) - L^x_\omega(t)\Big)\Big| \\ \leq\ &\frac{1}{|Q|}\Big|\int_{\tilde{Q}} \mathrm{d}x\, \mathcal{E}_\omega (2\pi t)^{-1} e^{B_0 t}\mathbf{E}^{x,t}_{x,0} e^{-i\int_0^t A(W_s)\mathrm{d}W(s) - \int_0^t V_\omega(W_s)\mathrm{d}s}\mathbf{1}(T_Q \leq t)\Big| \\ &\leq 4(2\pi t)^{-1} e^{B_0 t} e^{-(C_1 t)^2/(8t)} \leq Be^{-Bt}. \end{aligned} \tag{3.21}$$

In the last line we used that x is at least at a distance $C_1 t \geq (24B_0)^{1/2} t$ from the boundary of Q and the large deviation estimate (3.12). Hence (3.15) follows from (3.19), (3.20) and (3.21).

For the proof of part (ii) of Proposition 3.3, we recall a basic result from the theory of continuum percolation in the Boolean Poisson model of random disks centered at the points of a Poisson process (for background, see [13]). The radius a of the disks is deterministic in our application. Let $\mathcal{R}$ be the rectangle $[0,a] \times [0,3a] \subset \mathbf{R}^2$. Let $\sigma(\mathcal{R},\nu)$ be the probability, under $\mathcal{P}_\omega$, that there exists a vacant crossing in the short direction of $\mathcal{R}$. It means that there is a Jordan arc in $\mathcal{R}$, connecting the segments $\{0\} \times [0,3a]$ and $\{a\} \times [0,3a]$ which avoids the set $\cup_i B(x_i(\omega), a)$. For any $x \in \mathbf{R}^2$, let $d_\omega(x)$ be the diameter of the component of $\mathbf{R}^2 \setminus \cup_i B(x_i(\omega), a)$ that contains the point x. Then Lemma 4.1. of [13] states that

$$\mathcal{P}_\omega\{d_\omega(x) \geq L\} \leq C_2 e^{-C_3 L} \tag{3.22}$$

for any $L \geq C_4$, with universal constants C_2, C_3, C_4 if

$$\sigma(\mathcal{R},\nu) \leq (9e)^{-122} \tag{3.23}$$

(by stationarity, the probability in (3.22) is independent of x). By a simple construction one can show that

$$\sigma(\mathcal{R},\nu) \leq 1 - \left(1 - e^{-\nu a^2}\right)^3,$$

hence (1.12) guarantees (3.23).

Now for the proof of (3.17) consider again a huge box $\Lambda_M = [-Mq(t), Mq(t)]$, where $M \geq 2$ is an integer. Apply the estimate (3.15) for Λ_M instead of Q, with $M \geq 4C_1 t/q(t)$, to obtain

$$\limsup_{M\to\infty} \left| L^{hc}_{\Lambda_M}(t) - L^{hc}(t) \right| \leq 50 B e^{-B_0 t}. \tag{3.24}$$

Let $\tilde{\Lambda}_M := [-(M-1)q(t), (M-1)q(t)]$. Now partition $\tilde{\Lambda}_M$ into open boxes $\{\tilde{Q}_i\}_{i=1}^{(M-1)^2}$ with edge length $2q(t)$ (boundaries neglected) and let $\{Q_i\}_{i=1}^{(M-1)^2}$ be the concentric boxes with edge length $4q(t)$.

Then, using the hard-core analogue of (3.3), the positivity of the diagonal element of the heat kernel and the independence of $\int_{Q_i} \mathcal{E}_\omega L^{hc,x}_{Q_i,\omega}(t)\,\mathrm{d}x$ on i by stationarity, we obtain

$$\begin{aligned}
L^{hc}_{\Lambda_M}(t) &= \frac{1}{|\Lambda_M|}\int_{\Lambda_M\setminus\tilde{\Lambda}_M} \mathcal{E}_\omega L^{hc,x}_{\Lambda_M,\omega}(t)\,\mathrm{d}x + \frac{1}{|\Lambda_M|}\sum_{i=1}^{(M-1)^2}\int_{\tilde{Q}_i} \mathcal{E}_\omega L^{hc,x}_{\Lambda_M,\omega}(t)\,\mathrm{d}x \\
&\leq \frac{2B}{M} + \frac{1}{|\Lambda_M|}\sum_{i=1}^{(M-1)^2}\int_{\tilde{Q}_i}\Big[\mathcal{E}_\omega L^{hc,x}_{Q_i,\omega}(t) + \mathcal{E}_\omega\Big(L^{hc,x}_{\Lambda_M,\omega}(t) - L^{hc,x}_{Q_i,\omega}(t)\Big)\Big]\,\mathrm{d}x \\
&\leq \frac{2B}{M} + \frac{1}{|\Lambda_M|}\sum_{i=1}^{(M-1)^2}\int_{Q_i} \mathcal{E}_\omega L^{hc,x}_{Q_i,\omega}(t)\,\mathrm{d}x + \sup_i \sup_{x\in\tilde{Q}_i} \mathcal{E}_\omega \Big| L^{hc,x}_{\Lambda_M,\omega}(t) - L^{hc,x}_{Q_i,\omega}(t)\Big| \\
&\leq \frac{2B}{M} + 4L^{hc}_Q(t) + \sup_i \sup_{x\in\tilde{Q}_i} \mathcal{E}_\omega\Big[\mathbf{1}(d_\omega(x) \geq q(t))\Big|L^{hc,x}_{\Lambda_M,\omega}(t) - L^{hc,x}_{Q_i,\omega}(t)\Big|\Big] \\
&\leq \frac{2B}{M} + 4L^{hc}_Q(t) + 2BC_2 e^{-C_3 q(t)}.
\end{aligned} \tag{3.25}$$

The key point is that one can insert $\mathbf{1}(d_\omega(x) \geq q(t))$ for free in the last but one line. For, by the Feynman-Kac-Ito formula and $Q_i \subset \Lambda_M$

$$L^{hc,x}_{\Lambda_M,\omega}(t) - L^{hc,x}_{Q_i,\omega}(t) = (2\pi t)^{-1}\mathbf{E}^{x,t}_{x,0} e^{B_0 t} e^{-i\int_0^t A(W_s)\mathrm{d}W_s}\mathbf{1}(T_{\mathcal{T}_\omega} > t)\mathbf{1}(T_{\Lambda_M} > t)\mathbf{1}(T_{Q_i} \leq t).$$

But the intersection of the events $\{T_{\mathcal{T}_\omega} > t\}$ and $\{T_{Q_i} \le t\}$ is clearly included in the event $\mathbf{1}(d_\omega(x) \ge q(t))$ since $x \in \tilde{Q}_i$ is at least at a distance $q(t)$ from the boundary of Q_i. In the last step of the estimate (3.25) we used the hard-core analogue of (3.4), for both $Q = Q_i$ and $Q = \Lambda_M$, and (3.22). Combining (3.24) and (3.25), taking $M \to \infty$, we obtain the upper bound in (3.17). $\square$

3.4 Second localization in the soft potential case

For the upper bound in the soft potential case we need a further localization from lengthscale $q = q(t) \sim t$ to scale $s = s(t) \sim (\log t)^2$ (the precise definition is given later). Here we use the magnetic localization technique developed in [10]. As it is explained there, the usual IMS formula makes a localization error of order L^{-2} in the energy, where L is the localization lengthscale. The magnetic kinetic energy can be localized at a price which is of order $\exp[-(const)L^2]$ if one is willing to change the magnetic field by a constant amount.

Fix a parameter $0 < \beta < B_0$ and suppose that $q(t) = \beta^{-1/2}N(t)$ with some positive integer $N = N(t)$. Let $\mathbf{L} := [-N(t), N(t)]^2 \cap \mathbf{Z}^2$, $S = [-s(t), s(t)]^2$ and $\tilde{S} = [-\frac{s(t)}{2}, \frac{s(t)}{2}]^2$, where $s = s(t) \ge 1$ is to be chosen later. Let $S_j := S + j$ and $\tilde{S}_j := \tilde{S} + j$ for any $j \in \mathbf{R}^2$. Let $\lambda^{(B_0+\beta)}_{S_j,\omega}$ be the lowest eigenvalue of $\frac{1}{2}(-i\nabla - (1 + \beta B_0^{-1})A)^2 + V_\omega$ with Dirichlet boundary conditions on S_j. Recall that $A(x) = (A_1(x), A_2(x)) = B_0(-x_2, x_1)$.

Proposition 3.4 *Assume that $N = \beta^{1/2}q$ is a positive integer and $s \ge 1$ satisfies*

$$s^{-2}e^{-\frac{\beta}{4}s^2}\beta q^2 \le \frac{1}{180}. \tag{3.26}$$

Then

$$\lambda_{Q,\omega} \ge \frac{1}{100} \min_{j \in \beta^{-1/2}\mathbf{L}} \lambda^{(B_0+\beta)}_{S_j,\omega} - 60 s^{-2} e^{-\frac{\beta}{4}s^2}\beta q^2. \tag{3.27}$$

Proof. Define

$$\zeta(x) = \zeta_\beta(x) := \sum_{j \in \beta^{-1/2}\mathbf{L}} e^{-\beta(x-j)^2}. \tag{3.28}$$

Easy calculation shows that

$$\frac{1}{5} \le \zeta_\beta(x) \le 5 \tag{3.29}$$

for all $x \in Q := [-q, q]^2$.

Let $\varphi = \varphi_{Q,\omega}$ be the normalized eigenfunction of $H_{Q,\omega}$ belonging to the lowest eigenvalue $\lambda_{Q,\omega}$. We can think of φ to be extended to the whole $\mathbf{R}^2$ by defining it to be zero outside of Q, and let $\int$ denote $\int_{\mathbf{R}^2}$ in this section. Then, using integration by parts,

$$\begin{aligned} \lambda_{Q,\omega} &= \int_Q \Big(\frac{1}{2}\Big|(-i\nabla - A)\varphi\Big|^2 - B_0|\varphi|^2 + V_\omega|\varphi|^2\Big) \qquad (3.30)\\ &= \int_Q \Big(\frac{1}{2}\Big|(-i\partial_1 + \partial_2 - A_1 - iA_2)\varphi\Big|^2 + V_\omega|\varphi|^2\Big) \\ &\ge \frac{1}{5}\int \zeta \cdot \Big(\frac{1}{2}\Big|(-i\partial_1 + \partial_2 - A_1 - iA_2)\varphi\Big|^2 + V_\omega|\varphi|^2\Big) \\ &= \frac{1}{5}\sum_{j\in\beta^{-1/2}\mathbf{L}} \int \Big(\frac{1}{2}e^{-\beta(x-j)^2}\Big|(-i\partial_1 + \partial_2 - A_1 - iA_2)\varphi\Big|^2 + V_\omega e^{-\beta(x-j)^2}|\varphi|^2\Big). \end{aligned}$$

Straightforward calculation shows that

$$\Big(-i\partial_1 + \partial_2 + (B_0+\beta)x_2 - i(B_0+\beta)x_1\Big)\varphi_j = e^{-\frac{\beta}{2}(x-j)^2} e^{i\beta(x_2 j_1 - x_1 j_2)}(-i\partial_1 + \partial_2 + B_0 x_2 - iB_0 x_1)\varphi,$$

where $j = (j_1, j_2) \in \beta^{-1/2}\mathbf{L}$ and

$$\varphi_j(x) := e^{-\frac{\beta}{2}(x-j)^2} e^{i\beta(x_2 j_1 - x_1 j_2)}\varphi.$$

Let

$$T_\beta := -i\partial_1 + \partial_2 + (B_0+\beta)x_2 - i(B_0+\beta)x_1 \tag{3.31}$$

for simplicity. Hence from (3.30), using $A(x) = B_0(-x_2, x_1)$

$$\lambda_{Q,\omega} \ge \frac{1}{5}\sum_{j\in\beta^{-1/2}\mathbf{L}} \int \Big\{\frac{1}{2}|T_\beta\varphi_j|^2 + V_\omega|\varphi_j|^2\Big\}. \tag{3.32}$$

Fix a smooth function $\theta(x)$ such that $\theta(x) \equiv 0$ for $x \in \mathbf{R}^2 \setminus S$, $\theta(x) \equiv 1$ on $\tilde{S}$, $0 \le \theta \le 1$ and $\|\nabla\theta\|_\infty \le 4s^{-1}$. Let $\theta_j(x) := \theta(x-j)$ for any $j \in \mathbf{R}^2$. Obviously

$$\Big|(1-\theta_j)\varphi_j\Big|^2 \le e^{-\frac{\beta}{4}s^2}|\varphi|^2 \quad \text{and} \quad |\varphi_j|^2 \cdot \mathbf{1}(\operatorname{supp}\nabla\theta_j) \le e^{-\frac{\beta}{4}s^2}|\varphi|^2, \tag{3.33}$$

where $\mathbf{1}(\cdot)$ is the characteristic function of a set as before. Hence we have the following pointwise bound:

$$|\varphi_j|^2 \le 2|\theta_j\varphi_j|^2 + 2|(1-\theta_j)\varphi_j|^2 \le 2|\theta_j\varphi_j|^2 + 2e^{-\frac{\beta}{4}s^2}|\varphi|^2, \tag{3.34}$$

and by the usual IMS estimate, recalling the definition of T_β

$$\theta_j^2|T_\beta\varphi_j|^2 \ge \frac{1}{2}|T_\beta(\theta_j\varphi_j)|^2 - 2\|\nabla\theta\|_\infty^2 \cdot \mathbf{1}(\operatorname{supp}\nabla\theta_j)|\varphi_j|^2 \ge \frac{1}{2}|T_\beta(\theta_j\varphi_j)|^2 - 32s^{-2}e^{-\frac{\beta}{4}s^2}|\varphi|^2. \tag{3.35}$$

Hence, using $\theta_j^2 \le 1$, $V_\omega \ge 0$ and $\int|\varphi|^2 = 1$, we have

$$\lambda_{Q,\omega} \ge \frac{1}{5}\sum_{j\in\beta^{-1/2}\mathbf{L}}\Big\{\int \frac{1}{2}|T_\beta(\theta_j\varphi_j)|^2 + \int V_\omega|\theta_j\varphi_j|^2\Big\} - \frac{32}{5}s^{-2}e^{-\frac{\beta}{4}s^2}(2N+1)^2. \tag{3.36}$$

From (3.34) we obtain

$$\sum_{j\in\beta^{-1/2}\mathbf{L}}\int|\theta_j\varphi_j|^2 \ge \frac{1}{2}\sum_{j\in\beta^{-1/2}\mathbf{L}}\int|\varphi_j|^2 - s^{-2}e^{-\frac{\beta}{4}s^2}(2N+1)^2 \tag{3.37}$$

$$\ge \frac{1}{10}\int|\varphi|^2 - 9s^{-2}e^{-\frac{\beta}{4}s^2}N^2 \ge \frac{1}{20}$$

using the definition of φ_j, (3.29), $\int|\varphi|^2 = 1$, $N \ge 1$ and the assumption (3.26).

Combining (3.36) and (3.37) we have

$$\lambda_{Q,\omega} \ge \frac{1}{100}\frac{\sum_{j\in\beta^{-1/2}\mathbf{L}}\Big\{\int\frac{1}{2}|T_\beta(\theta_j\varphi_j)|^2 + \int V_\omega|\theta_j\varphi_j|^2\Big\}}{\sum_{j\in\beta^{-1/2}\mathbf{L}}\int|\theta_j\varphi_j|^2} - 60s^{-2}e^{-\frac{\beta}{4}s^2}N^2 \tag{3.38}$$

$$\ge \frac{1}{100}\min_{j\in\beta^{-1/2}\mathbf{L}}\lambda_{S_j,\omega}^{(B_0+\beta)} - 60s^{-2}e^{-\frac{\beta}{4}s^2}N^2.$$

In the last step we used the variational principle

$$\lambda_{S_j,\omega}^{(B_0+\beta)} \le \frac{\int \left\{ \frac{1}{2}|T_\beta(\theta_j\varphi_j)|^2 + V_\omega|\theta_j\varphi_j|^2 \right\}}{\int |\theta_j\varphi_j|^2},$$

and that $\tilde{S}_j \cap Q$ has positive measure for $j \in \beta^{-1/2}\mathbf{L}$, and the fact that φ (hence φ_j) does not vanish on an open subset of Q by general elliptic theory. We also used that the weighted average (with positive weights) is not smaller than the smallest term. This finishes the proof of Proposition 3.4. $\square$

4 Enlargement of obstacles

Estimating the IDS of $H_{Q,\omega}$ amounts to considering all eigenvalues below E (if any). But, as we mentioned in the introduction, for the Lifschitz tail in a finite box of appropriate size, only a much rougher information is needed, namely the location of the bottom of the spectrum. The main contribution to $L_Q(t)$, hence to $N(E)$, comes from few configurations with a large clearing of size of order $\ell(t)$, and the actual number of the eigenvalues below E is irrelevant compared to the large deviation probability that there is any eigenvalue below E at all. We shall see that focusing on the lowest eigenvalue is enough.

As before, let $\lambda_{Q,\omega}$ be the lowest eigenvalue of $H_{Q,\omega}$. Following Sznitman's original idea, we shall estimate this eigenvalue from below by the lowest eigenvalue of another magnetic Hamiltonian which has hard-core potential on the "enlarged obstacles", i.e. on a set which consists of larger balls centered at the points of the Poisson configuration. This part of the analysis yields estimates uniformly in the point configuration, hence the Poisson randomness plays no role.

The enlarged obstacles typically occupy a much larger portion of Q than the support of V_ω (in particular their union has a smaller combinatorial complexity), but the enlargement does not substantially influence the largest clearing on which the lowest eigenstate lives if this clearing is really large. In other words, the enlargement does not influence the lowest eigenvalue if it is really small.

The first part of the following construction follows Sznitman's work [18].

Consider a fixed locally finite set of points $\omega = \{x_i\}_{i=1,2,\ldots}$ in $S = [-s, s]^2$. Let C_m stand for the cube (square)

$$C_m := \{z \in \mathbf{R}^2 \,:\, m_i\ell \le z_i < (m_i + 1)\ell,\, i = 1, 2\},$$

where $\ell \le s$.

Fix two parameters $b > 10a$ and $\varepsilon > 0$ (recall that a is the radius of a disk located fully within the support of $V^{(0)}$). We say that a point $x_i \in C_m$ is good if for all closed balls $C = \overline{B}(x_i, 10^{j+1}b)$, $0 \le j$, and $10^{j+1}b \le \ell/2$

$$\Big|C_m \cap C \cap \Big(\bigcup_{x_j \in C_m} \overline{B}(x_j, b)\Big)\Big| \ge \frac{\varepsilon}{9}|C_m \cap C|, \tag{4.1}$$

in particular

$$\Big|C \cap \Big(\bigcup_{x_j \in C_m} \overline{B}(x_j, b)\Big)\Big| \ge \frac{\varepsilon}{36}|C|.$$

Let Good(m) be the set of good points in C_m, the rest is Bad(m), and let $\mathcal{G} := \cup_m \mathrm{Good}(m)$. By a covering argument we know that

$$\Big|C_m \cap \Big(\bigcup_{x_j \in \mathrm{Bad}(m)} \overline{B}(x_j, b)\Big)\Big| \le \varepsilon|C_m| = \varepsilon\ell^2. \tag{4.2}$$

Chop each segment $[k\ell, (k+1)\ell]$ into at most $[\sqrt{2}\ell/b] + 1$ intervals of length $b/\sqrt{2}$ (except the last one). This yields closed boxes with diameter less than b, with union $\overline{C}_m$.

Introduce a number $r > 0$. We define the event Cl_m that "there is a clearing of size r in C_m", i.e.

$$Cl_m := \{\omega \,:\, |\tilde{U}_m(\omega)| \ge 9^{-2}\pi r^2\ell^2\},$$

where $\tilde{U}_m(\omega)$ is the open subset of $\text{int}(C_m)$ obtained by taking the complement in the interior of C_m of the closed boxes where a good point of C_m falls. Let $A(\omega)$ be the union of all closed cubes $\overline{C}_m$ where there is clearing of size r:

$$\mathbf{1}_{A(\omega)}(z) = \sum_m \mathbf{1}_{\overline{C}_m}(z) \cdot \mathbf{1}_{Cl_m}(\omega).$$

Let $A^1 = A^1(\omega)$ be the open set of points at distance less than ℓ from $A(\omega)$. If $A(\omega)$ is empty, so is $A^1(\omega)$. Let S_-^ϱ be the open square $(-s-\varrho, s+\varrho)^2$ and $S_+^\varrho := (-s+\varrho, s-\varrho)^2$ for $s > \varrho > 0$. Finally, let

$$\Omega := S \setminus \bigcup_{i\in\mathcal{G}} \overline{B}(x_i, a),$$

and for any $b > a$, $s > b$

$$\Omega_+^b := \left(S_+^b \cap A^1\right) \setminus \bigcup_{i\in\mathcal{G}} \overline{B}(x_i, b).$$

Let $\delta := \min\{\frac{1}{200}, \frac{a}{6}\}$ and let

$$\Omega_-^\delta := B(0,\delta) + \Omega \quad \text{and} \quad \Omega_-^{2\delta} := B(0, 2\delta) + \Omega.$$

Notice that $\Omega_+^{2b} \subset \Omega_+^b \subset \Omega \subset \Omega_-^\delta \subset \Omega_-^{2\delta}$ and that there is no A^1 in the definition of Ω, Ω_-^δ and $\Omega_-^{2\delta}$.

Let $\lambda_{S,\omega}$ be the lowest eigenvalue of $\frac{1}{2}(-i\nabla - A)^2 - B_0 + V_\omega$ with Dirichlet boundary conditions on S. For a lower bound on $\lambda_{S,\omega}$ one can replace the true potential V by $\tilde{V} = \tilde{V}_\omega$ which is defined as

$$\tilde{V}_\omega(x) := v \cdot \mathbf{1}\Big(x \in \bigcup_{i:x_i(\omega)\in\mathcal{G}} \overline{B}(x_i, a)\Big), \tag{4.3}$$

and we can assume that $v \leq B$ and $0 < a < 1$.

Let $\lambda = \lambda_\omega$ be the lowest eigenvalue of $\frac{1}{2}(-i\nabla - A)^2 - B_0 + \tilde{V}_\omega$ with Dirichlet boundary condition on S. Certainly $\lambda_{S,\omega} \geq \lambda$, and similarly $\lambda_{S,\omega}^{hc} \geq \lambda$ in the hard-core case. Let $\lambda_b = \lambda_b(\omega)$ be the lowest eigenvalue of $\frac{1}{2}(-i\nabla - A)^2 - B_0$ with Dirichlet boundary condition on Ω_+^b.

For any bounded domain Ω with piecewise C^1 boundary we define its Green's function g_Ω as the solution to the following boundary value problem:

$$\Delta g_\Omega = -1 \quad \text{on } \Omega$$

$$g_\Omega \equiv 0 \quad \text{on } \partial\Omega.$$

By standard elliptic theory, $g_\Omega \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ uniquely exists and is positive and bounded. Let

$$G_\Omega := \max_{x\in\overline{\Omega}} g_\Omega(x). \tag{4.4}$$

Let us consider the standard Brownian motion W_s in Ω, starting from $x \in \Omega$. The probability with respect to this Brownian motion is denoted by $\mathbf{P}_x$, the expectation value by $\mathbf{E}_x$. Let T_Ω be the exit time from Ω. By Ito's formula

$$g_\Omega(W_s) - g_\Omega(x) - \frac{1}{2}\int_0^s \Delta g_\Omega(W_\tau)d\tau$$

is a martingale, hence

$$g_\Omega(x) = \frac{1}{2}\mathbf{E}_x T_\Omega. \tag{4.5}$$

Using these notations, the goal of this section is to prove the following theorem.

Theorem 4.1 *Given a finite point configuration ω in S. Let λ be the lowest eigenvalue of $\frac{1}{2}(-i\nabla - A)^2 - B_0 + \tilde{V}$ with Dirichlet boundary condition on S, and λ_b be the lowest eigevalue of $\frac{1}{2}(-i\nabla - A)^2 - B_0$ with Dirichlet boundary condition on Ω_+^b. There exist two universal constants ε_0, $c_0 > 0$ and a function $\tilde{\ell}(\varepsilon, b, B, r) > 0$ for all $\varepsilon \leq \varepsilon_0$, $b \geq 1$ such that for all $\ell \geq \tilde{\ell}(\varepsilon, b, B, r)$*

$$\lambda_b \leq \Big(\frac{\lambda}{K}\Big)^{\frac{1}{1+w(r)}} \tag{4.6}$$

with

$$K = K(b, B, v, \delta, r, \ell, s) := \Big(80b^{-2}v^{-1}\delta^{-2}s^{2+20(1-c_0)^{1/r}}e^{20Bc_0^{-1}r^2\ell^2}e^{30B\ell}\Big)^{-(1+w(r))} \tag{4.7}$$

and

$$w(r) := \frac{1}{1 - 20(1-c_0)^{1/r}} - 1 \tag{4.8}$$

under the following conditions: $\delta := \min\{\frac{1}{200}, \frac{a}{6}\}$, $a < 1 < b$, $r \le 1/4$, $r < \log\left(\frac{1}{1-c_0}\right)/(\log 30)$, $v \le B$, *and*

$$\left(\frac{\lambda}{K}\right)^{\frac{1}{1+w(r)}} \le \frac{1}{2}. \tag{4.9}$$

All the estimates are uniform in the point configuration ω.

Since the relevant size of λ is of order $\exp[-(const)B\ell^2]$, we see that $\log\lambda \ll \log K < 0$ for large ℓ and small r. Hence (4.6) tells us that $\log\lambda_b \le (1 - o(1))\log\lambda < 0$.

The proof requires several lemmas. The intuitive idea is that the magnetic Dirichlet eigenvalue $\lambda(\Omega)$ of a domain Ω is essentially e^{-BG_Ω} and the corresponding eigenfunction is roughly $e^{B(g_\Omega(x) - G_\Omega)}$ if Ω is large. The increase of the eigenvalue due to the enlargement is determined by the size of the eigenfunction, hence of g_Ω, within a distance of order b from the boundary of Ω (Lemma 4.2). Then, by applying the method of enlargement of obstacles, we show that $g_\Omega(x) \ll G_\Omega$ if Ω is large and x is close to the boundary of Ω (Lemma 4.3). Here we use the probabilistic representation of g_Ω (4.5). For technical reasons, we have to work on a slightly bigger domain, $\Omega_-^{2\delta}$, and we have to compare G_Ω and $G_{\Omega_-^{2\delta}}$ (Lemma 4.4). Finally we have to go back and estimate G_Ω in terms of $\lambda(\Omega)$ (Lemma 4.5). An additional difficulty arises from the fact that λ in the theorem is not exactly $\lambda(\Omega)$ because the obstacles are soft.

Lemma 4.2 *With the notations above, if* $\delta \le s$, $v \le B$, *and* $\lambda v^{-1}s^2e^{2B\eta}\delta^{-2} < 1/16$, *then*

$$\lambda \le \lambda_b \le 16\lambda \cdot \frac{1 + 8b^{-2}v^{-1}s^2e^{2B\eta}\delta^{-2}}{1 - 8\lambda v^{-1}s^2e^{2B\eta}\delta^{-2}}, \tag{4.10}$$

where

$$\eta := \max\left\{g_{\Omega_-^{2\delta}}(z) \;:\; z \in \overline{\Omega_-^{2\delta}} \setminus \Omega_+^{2b}\right\}. \tag{4.11}$$

Proof of Lemma 4.2. The first inequality is trivial by variational principle. For the second one, let φ be a normalized eigenfunction corresponding to λ. Again by variational principle

$$\lambda_b = \inf\Big\{ \frac{\int_{\Omega_+^b} \left(\frac{1}{2}|(-i\nabla - A)\psi|^2 - B_0|\psi|^2\right)}{\int_{\Omega_+^b} |\psi|^2} \ : \ \psi \in H_0^1(\Omega_+^b) \Big\}$$

$$= \inf\Big\{ \frac{\int_{\Omega_+^b} \frac{1}{2}|(-i\partial_1 + \partial_2 - A_1 - iA_2)\psi|^2}{\int_{\Omega_+^b} |\psi|^2} \ : \ \psi \in H_0^1(\Omega_+^b) \Big\}$$

by integration by parts.

Let θ be a cutoff function such that $\theta \equiv 1$ on Ω_+^{2b}, $\theta \equiv 0$ on $\mathbf{R}^2 \setminus \Omega_+^b$, $0 \le \theta \le 1$ and $|\nabla\theta| \le 4b^{-1}$. Since

$$\lambda = \frac{1}{2}\int_S |(-i\partial_1 + \partial_2 - A_1 - iA_2)\varphi|^2 + \int_S \tilde{V}|\varphi|^2$$

$$\ge \frac{1}{2}\int_S \theta^2|(-i\partial_1 + \partial_2 - A_1 - iA_2)\varphi|^2 + \int_S \tilde{V}|\varphi|^2$$

$$\ge \frac{1}{4}\int_S |(-i\partial_1 + \partial_2 - A_1 - iA_2)(\theta\varphi)|^2 - \frac{1}{2}\|\nabla\theta\|_\infty^2 \int_{S\cap supp\,\nabla\theta} |\varphi|^2$$

by the IMS localization and $\tilde{V} \ge 0$; we can use $\psi := \theta\varphi$ as a trial function to obtain

$$\lambda_b \le \frac{\int_{\Omega_+^b} \frac{1}{2}|(-i\partial_1 + \partial_2 - A_1 - iA_2)\psi|^2}{\int_{\Omega_+^b} |\psi|^2} \tag{4.12}$$

$$\le \frac{2\lambda + 16b^{-2}\int_{S\cap\, supp\nabla\theta} |\varphi|^2}{\int_{\Omega_+^b} \theta^2|\varphi|^2} \le \frac{2\lambda + 16b^{-2}\int_{S\setminus\Omega_+^{2b}} |\varphi|^2}{1 - \int_{S\setminus\Omega_+^{2b}} |\varphi|^2}$$

using $\int_S |\varphi|^2 = 1$ and $\theta \equiv 1$ on Ω_+^{2b}. Hence we have to estimate $\int_{S\setminus\Omega_+^{2b}} |\varphi|^2$ from above.

Recall that we use the standard gauge $A := B_0(-x_2, x_1)$. Let Π be the spectral projection onto the lowest (zero energy) Landau level of the unperturbed operator $\frac{1}{2}(-i\nabla - A)^2 - B_0$ defined on $\mathbf{R}^2$. Extend φ on the whole plane to be zero outside S. Then $\varphi = \Pi\varphi + (1-\Pi)\varphi$. Since $(-i\partial_1 + \partial_2 - A_1 - iA_2)\Pi = 0$, we have

$$\lambda = \frac{1}{2}\int_{\mathbf{R}^2} |(-i\partial_1 + \partial_2 - A_1 - iA_2)\varphi|^2 + \int_S \tilde{V}|\varphi|^2$$

$$= \frac{1}{2}\int_{\mathbf{R}^2} |(-i\partial_1 + \partial_2 - A_1 - iA_2)(I-\Pi)\varphi|^2 + \int_S \tilde{V}|\varphi|^2 \geq B\int_{\mathbf{R}^2} |(I-\Pi)\varphi|^2 + \int_S \tilde{V}|\varphi|^2$$

by the gap in the spectrum of the free magnetic operator $\frac{1}{2}(-i\nabla - A)^2 - B_0$ defined on $\mathbf{R}^2$. Hence

$$\int_{\mathbf{R}^2} |(I-\Pi)\varphi|^2 \leq \frac{\lambda}{B} \tag{4.13}$$

and

$$\int_{\bigcup_{i\in\mathcal{G}} \overline{B}(x_i,a)} |\varphi|^2 \leq \frac{\lambda}{v} \tag{4.14}$$

using the explicit form of $\tilde{V}$.

Notice that

$$S \setminus \Omega_+^{2b} \subset \Big(\Omega \setminus \Omega_+^{2b}\Big) \cup \bigcup_{i\in\mathcal{G}} \overline{B}(x_i, a).$$

Hence

$$\int_{S\setminus\Omega_+^{2b}} |\varphi|^2 \leq \int_{\Omega\setminus\Omega_+^{2b}} |\varphi|^2 + \frac{\lambda}{v} \tag{4.15}$$

$$\leq 2\int_{\Omega\setminus\Omega_+^{2b}} |\Pi\varphi|^2 + 2\int_{\Omega\setminus\Omega_+^{2b}} |(I-\Pi)\varphi|^2 + \frac{\lambda}{v} \leq 2\int_{\Omega\setminus\Omega_+^{2b}} |\Pi\varphi|^2 + \frac{2\lambda}{B} + \frac{\lambda}{v}.$$

Let $g = g_{\Omega_-^{2\delta}}$ for simplicity (recall that $\Delta g = -1$ on $\Omega_-^{2\delta}$ and $g \equiv 0$ on $\partial\Omega_-^{2\delta}$). Let $\xi : \Omega_-^{2\delta} \to \mathbf{R}^2$ be a (real) solution to

$$\nabla\xi = A + B(-\partial_2 g, \partial_1 g).$$

Since $\mathrm{curl}\Big(A + B(-\partial_2 g, \partial_1 g)\Big) = B + B\Delta g = 0$, such ξ exists. Hence, on $\Omega_-^{2\delta}$

$$\Big(-i\partial_1 + \partial_2 - A_1 - iA_2\Big)\Pi\varphi = 0$$

implies

$$\Big(-i\partial_1 + \partial_2 - B\partial_2 g + iB\partial_1 g\Big)\Big(e^{-i\xi}\Pi\varphi\Big) = 0,$$

i.e.

$$\Big(-i\partial_1 + \partial_2\Big)\Big(e^{-Bg}e^{-i\xi}\Pi\varphi\Big) = 0,$$

which means that $e^{-Bg}e^{-i\xi}\Pi\varphi$ is analytic (identifying $\mathbf{R}^2$ with $\mathbf{C}$). Hence

$$|\Pi\varphi| = e^{Bg}|h|, \tag{4.16}$$

with some h which is analytic on $\Omega_-^{2\delta}$ (this is the core of the Aharonov-Casher argument [1]).

Let z_0 be the point on $\partial\Omega_-^\delta$ where $|h(z)|$ takes on its maximal value. By the maximum principle $|h(z)| \le |h(z_0)|$ for all $z \in \Omega_-^\delta$.

Since $z_0 \in \partial\Omega_-^\delta$, we have $B(z_0,\delta) \subset \cup_{i\in\mathcal{G}}\overline{B}(x_i,a)\bigcup S^c$ (here S^c is the complement of S) and since $\varphi \equiv 0$ on S^c, we have, by (4.14),

$$\int_{B(z_0,\delta)} |\varphi|^2 \le \frac{\lambda}{v}.$$

By (4.13) and $|(I-\Pi)\varphi|^2 \ge \frac{1}{2}|\Pi\varphi|^2 - |\varphi|^2$ pointwise, we have

$$\begin{aligned}\frac{\lambda}{B} &\ge \int_{\mathbf{R}^2} |(I-\Pi)\varphi|^2 \ge \int_{B(z_0,\delta)} |(I-\Pi)\varphi|^2 \ge \frac{1}{2}\int_{B(z_0,\delta)} |\Pi\varphi|^2 - \int_{B(z_0,\delta)} |\varphi|^2 \\ &\ge \frac{1}{2}\int_{B(z_0,\delta)} |h|^2 e^{2Bg} - \lambda v^{-1} \ge \frac{\pi}{2}|h(z_0)|^2\delta^2 - \lambda v^{-1}\end{aligned}$$

by superharmoniticity of $|h|^2$ and $g \ge 0$. Hence, by $v \le B$

$$|h(z)|^2 \le |h(z_0)|^2 \le 2\lambda\delta^{-2}v^{-1}$$

for all $z \in \Omega_-^\delta$. Hence, using (4.16), $|\Omega| \le s^2$ and $\Omega \setminus \Omega_+^{2b} \subset \Omega_-^\delta$

$$\int_{\Omega\setminus\Omega_+^{2b}} |\Pi\varphi|^2 \le 2\lambda v^{-1}s^2 e^{2B\eta}\delta^{-2}$$

and therefore

$$\int_{\Omega\setminus\Omega_+^{2b}} |\varphi|^2 \le 4\lambda v^{-1}s^2 e^{2B\eta}\delta^{-2} + \frac{2\lambda}{B} + \frac{\lambda}{v} \le 8\lambda v^{-1}s^2e^{2B\eta}\delta^{-2}$$

using (4.15), $v \le B$, $\eta \ge 0$ and $\delta \le s$. Hence from (4.12)

$$\lambda_b \le 16\lambda \cdot \frac{1 + 8b^{-2}v^{-1}s^2e^{2B\eta}\delta^{-2}}{1 - 8\lambda v^{-1}s^2e^{2B\eta}\delta^{-2}}$$

assuming that the denominator is positive. □

Lemma 4.3 *Let $\Theta := \Omega_-^{2\delta}$ for simplicity and assume that $\delta \le \min\{\frac{1}{200}, \frac{a}{6}\}$, $a \le 1 \le b$, $40b \le \ell \le s$, $r \le 1/4$, $G_\Theta \ge \ell$.*

(i.) There exist two functions, $\ell_0(\varepsilon, b) > 0$ and $1/4 \ge k(\varepsilon, b) > 0$ for all b and $0 < \varepsilon \le \varepsilon_0$, where ε_0 is a universal constant, such that

$$g_\Theta(x) \le \Big(\frac{\ell}{b}\Big)^{-k(\varepsilon,b)} G_\Theta \tag{4.17}$$

for all $\ell \ge \ell_0(\varepsilon, b)$ and for all $x \in \Theta \cap \Sigma$, where

$$\Sigma := \Big(S_-^{2\delta} \setminus (-s+2b, s-2b)^2\Big) \cup \bigcup_{i \in \mathcal{G}} \overline{B}(x_i, 2b). \tag{4.18}$$

We can assume that $\ell_0(\varepsilon, b)$ is increasing and $k(\varepsilon, b)$ is decreasing in b.

(ii.) There exists a positive number c_0 such that if $x \in \Theta$ and $x \notin A^1 \cap S$, then

$$g_\Theta(x) \le \Big[(1-c_0)^{1/r} G_\Theta + c_0^{-1} r^2 \ell^2\Big] + \sup_{y \in \Sigma \cap \Theta} g_\Theta(y). \tag{4.19}$$

Proof of part (i.) Fix $x \in \Sigma \cap \Theta$. Let T_j be the exit time from $B(x, 2 \cdot 10^j b)$ for $j \ge 1$, and $T_0 := 0$. For simplicity, let $T := T_\Theta$ in this proof. Let M be the smallest integer such that $2 \cdot 10^{M+1} b > \ell/2$, i.e. $M := [\frac{\log(\ell/4b)}{\log 10}] \ge 1$. Then

$$\mathbf{E}_x T = \sum_{j=1}^M \mathbf{E}_x \Big\{T \cdot \mathbf{1}(T_{j-1} \le T < T_j)\Big\} + \mathbf{E}_x \Big\{T \cdot \mathbf{1}(T_M \le T)\Big\} \tag{4.20}$$

$$= \sum_{j=1}^M \mathbf{E}_x \Big\{\mathbf{1}(T_{j-1} \le T) \cdot \Big[T_{j-1} + \mathbf{E}_{W_{T_{j-1}}}(T \cdot \mathbf{1}(T < T_j))\Big]\Big\} + \mathbf{E}_x \Big\{\mathbf{1}(T_M \le T) \cdot \Big[T_M + \mathbf{E}_{W_{T_M}} T\Big]\Big\}$$

$$\le \sum_{j=1}^{M+1} \mathbf{E}_x \Big\{\mathbf{1}(T_{j-1} \le T) \cdot T_{j-1}\Big\} + \sum_{j=1}^M \mathbf{P}_x(T_{j-1} \le T) \times \sup_{y \in \Theta \cap S(x, 2 \cdot 10^{j-1} b)} \mathbf{E}_y(T \wedge T_j)$$
$$+ \mathbf{P}_x(T_M \le T) \times \sup_{y \in \Theta \cap S(x, 2 \cdot 10^M b)} \mathbf{E}_y T$$

by estimating $T \cdot \mathbf{1}(T \le T_j) \le T \wedge T_j$ and using that $T_k < T$ implies $W_{T_k} \in \Theta \cap S(x, 2 \cdot 10^k b)$ for $k = 1, 2, \ldots M$ [here $S(z, r)$ denotes the circle of radius r with center z].

Next, we claim that

$$\mathbf{P}_x(T_j \le T) \le (1 - m(\varepsilon, b))^j \tag{4.21}$$

for any $0 \le j \le M$.

To prove (4.21), let H_F denote the first hitting time of F, for any closed set F, and let

$$c(b) := \inf_{y \in S(0,2b)} \mathbf{P}_y(H_{\overline{B}(0,a-2\delta)} < T_{B(0,6b)}) > 0 \tag{4.22}$$

which is decreasing in b. Since a and δ are considered fixed, they are omitted from the notation. Let

$$m(\varepsilon, b) := c(b) \cdot \inf\Big\{ \inf_{|y|=2} \mathbf{P}_y(H_F < T_{B(0,10)}) \; : \; F \subset B(0,10), \; |F| \ge \frac{100\pi\varepsilon}{144} \Big\}$$

where the first infimum runs through all closed sets $F \subset B(0,10)$ with relative volume bigger than $\varepsilon/144$. Obviously $m(\varepsilon, b) > 0$ for $\varepsilon > 0$, $m(\varepsilon, b) \to 0$ as $\varepsilon \to 0$ uniformly in b and $m(\varepsilon, b)$ is decreasing in b. For small enough $\varepsilon \le \varepsilon_0$, we can assume that $m(\varepsilon, b) \le 1/3$.

Let $x \in \bigcup_{i \in \mathcal{G}} \overline{B}(x_i, 2b)$, i.e. $|x - x_\mu| \le 2b$ for some $\mu \in \mathcal{G}$ which we fix from now on. Let

$$\Gamma(j) := \bigcup_{\substack{i \in \mathcal{G} \\ |x_i - x_\mu| \le 10^j b}} \overline{B}(x_i, 2b) \subset B(x, 10^j b + 4b) \subset B(x, 2 \cdot 10^j b)$$

for $j \ge 1$. Then, by the definition of good points,

$$|\Gamma(j)| \ge \frac{\varepsilon}{36} |B(x_\mu, 10^j b)| = \frac{\varepsilon}{144} |B(x, 2 \cdot 10^j b)|$$

if $10^j b \le \ell/2$. Let W be a Brownian motion, started from x. If $j \ge 1$ and $W_{T_{j-1}} \in \Theta$ (i.e. $T_{j-1} < T$), then

$$\mathbf{P}_{W_{T_{j-1}}} \Big[\mathbf{1}(T_j > H_{\Gamma(j)}) \cdot \mathbf{P}_{\widetilde{W}_{H_{\Gamma(j)}}}(T_j > T) \Big] \ge m(\varepsilon, b), \tag{4.23}$$

using $W_{T_{j-1}} \in S(x, 2 \cdot 10^{j-1} b)$ and that $T < H_{\overline{B}(x_\sigma, a-2\delta)}$ and $T_{B(x_\sigma, 6b)} \le T_j$ if $\sigma \in \mathcal{G}$ is such that $|\widetilde{W}_{H_{\Gamma(j)}} - x_\sigma| = 2b$. The first relation is obvious, the second follows from $B(x_\sigma, 6b) \subset$

$B(x_\mu, 10^j b+10b) \subset B(x, 2\cdot 10^j b)$. In (4.23) $\widetilde{W}$ denotes another Brownian motion, independent of W. Hence, for $j \geq 1$

$$\mathbf{P}_x(T_j > T\,|T > T_{j-1}) \geq \mathbf{P}_x\Big\{\mathbf{P}_{W_{T_{j-1}}}\Big[\mathbf{1}(T_j > H_{\Gamma(j)})\cdot \mathbf{P}_{\widetilde{W}_{H_{\Gamma(j)}}}(T_j > T)\Big]\Big|W_{T_{j-1}} \in \Theta\Big\} \geq m(\varepsilon, b).$$

By the strong Markov property we obtain

$$\mathbf{P}_x(T_j \leq T) \leq (1-m(\varepsilon,b))^j$$

for any $0 \leq j \leq M$ (recall that for $j \leq M$ we have $10^j b \leq \ell/2$). A similar but easier argument shows that (4.21) is true if $x \in S_-^{2\delta} \setminus (-s+2b, s-2b)^2$ as well.

Let $1 \leq N \leq M$ be chosen later. Introduce

$$A_j := \mathbf{E}_x\Big\{\mathbf{1}(T_j \leq T)\cdot T_j\Big\},$$

then we can continue the estimate (4.20) as

$$\begin{aligned}\mathbf{E}_x T &\leq \sum_{j=1}^{M+1} A_{j-1} + \sum_{j=1}^{N} 2\cdot 10^{2j} b^2 (1-m(\varepsilon,b))^{j-1} + 2(M-N)G_\Theta(1-m(\varepsilon,b))^N + 2(1-m(\varepsilon,b))^M G_\Theta \\ &\leq \sum_{j=1}^{M+1} A_{j-1} + 8\cdot 10^{2N} b^2 (1-m(\varepsilon,b))^N + 2MG_\Theta(1-m(\varepsilon,b))^N.\end{aligned} \tag{4.24}$$

Here we estimated

$$\sup_{y\in\Theta\cap S(x,2\cdot 10^{j-1}b)} \mathbf{E}_y(T\wedge T_j) \leq \sup_{y \in B(x,2\cdot 10^j b)} \mathbf{E}_y T_j = 2\cdot 10^{2j} b^2$$

for $j \leq N$ and

$$\sup_{y\in\Theta\cap S(x,2\cdot 10^{j-1}b)} \mathbf{E}_y(T\wedge T_j) \leq \sup_{y\in\Theta} \mathbf{E}_y T = 2G_\Theta$$

for $j > N$ using (4.4), (4.5) and

$$\sup_{y\in B(0,1)} \mathbf{E}_y T_{B(0,1)} = \frac{1}{2}. \tag{4.25}$$

Now we estimate A_j

$$A_j = \mathbf{E}_x\Big\{\mathbf{1}(T_j \le T)\cdot T_j\Big\} \le \mathbf{E}_x\Big\{\mathbf{1}(T_{j-1}\le T)\Big[T_{j-1}\cdot \mathbf{P}_{W_{T_{j-1}}}(T_j \le T) + \mathbf{E}_{W_{T_{j-1}}}T_j\Big]\Big\}$$

$$\le (1-m(\varepsilon,b))A_{j-1} + 2\cdot 10^{2j}b^2\mathbf{P}_x(T_{j-1}\le T)$$

$$\le (1-m(\varepsilon,b))A_{j-1} + 2\cdot 10^{2j}b^2(1-m(\varepsilon,b))^{j-1}$$

for $1 \le j \le N$ using (4.23) and (4.25) in the first estimate and (4.21) in the second. Iterating this inequality, and using that $A_1 \le \mathbf{E}_x T_1 \le 200b^2$ we obtain

$$A_j \le 2\cdot 10^{2j}b^2 j(1-m(\varepsilon,b))^{j-1} \tag{4.26}$$

for $1 \le j \le N$.

For $N < j \le M$,

$$A_j = \mathbf{E}_x\Big\{\mathbf{1}(T_j \le T)\cdot T_j\Big\} \le \mathbf{E}_x\Big\{\mathbf{1}(T_{j-1}\le T)\Big[T_{j-1} + \mathbf{E}_{W_{T_{j-1}}}(T_j\cdot \mathbf{1}(T_j\le T))\Big]\Big\}$$

$$\le A_{j-1} + \mathbf{E}_x\Big\{\mathbf{1}(T_{j-1}\le T)\cdot \mathbf{E}_{W_{T_{j-1}}}T\Big\}$$

$$\le A_{j-1} + 2G_\Theta \mathbf{P}_x(T_{j-1}\le T) \le A_{j-1} + 2G_\Theta(1-m(\varepsilon,b))^N$$

by (4.21) and $T_N \le T_{j-1}$. By iteration, and using the estimate (4.26) for A_N, we have

$$A_j \le 4\Big(10^{2N}Nb^2 + MG_\Theta\Big)(1-m(\varepsilon,b))^N \tag{4.27}$$

for $j > N$. Continuing (4.24) and combining the same terms

$$\mathbf{E}_x T \le \sum_{j=1}^{N} 2\cdot 10^{2j}b^2 j(1-m(\varepsilon,b))^{j-1} + (4M+8)\Big(10^{2N}Nb^2 + MG_\Theta\Big)(1-m(\varepsilon,b))^N$$

$$\le 12M^2b^2(1-m(\varepsilon,b))^N\Big[20^{2N} + G_\Theta\Big].$$

Now choose

$$N := \min\Big\{M, \Big[\frac{\log G_\Theta}{2\log 20}\Big]\Big\}.$$

Short calculation shows that

$$\mathbf{E}_x T \le 48(\log(\ell/b))^2 b^2 G_\Theta \cdot \max\Big\{G_\Theta^{-2k(\varepsilon,b)}, \Big(\frac{\ell}{4b}\Big)^{-2k(\varepsilon,b)}\Big\}$$

with $k(\varepsilon,b) := \frac{1}{4\log 20}\log\frac{1}{1-m(\varepsilon,b)} \le \frac{1}{4}$. Since $G_\Theta \ge \ell$, $b \ge 1$, choosing a large enough function $\ell_0(\varepsilon,b)$ we finish the proof of (4.17) by (4.5).

Proof of part (ii.) Let $x \notin A^1 \cap S$ and $x \in \Theta$. We can assume that $x \notin \Sigma$, in particular $x \in S$. Consider again a Brownian motion W starting from x. Since $S_-^{2\delta} \setminus \Sigma \subset \Theta$, we have $H_\Sigma \le T_\Theta = T$. Therefore

$$\mathbf{E}_x T = \mathbf{E}_x\Big\{H_\Sigma + \mathbf{E}_{W_{H_\Sigma}} T\Big\}.$$

Obviously $\mathbf{E}_{W_{H_\Sigma}} T \le 2\sup_{y\in\Sigma\cap\Theta} g_\Theta(y)$, and it can be estimated by part (i).

To estimate $\mathbf{E}_x H_\Sigma$, let

$$F(n) := \sup_{y\in S:|x-y|\le nr\ell} \mathbf{E}_y H_\Sigma$$

for $n = 1, 2, \dots [1/r] - 1$. Fix $y \in S$ such that $|x-y| \le nr\ell$, then

$$\begin{aligned}\mathbf{E}_y H_\Sigma &= \mathbf{E}_y\Big\{H_\Sigma \cdot \mathbf{1}(H_\Sigma \le T_{B(y,r\ell)})\Big\} + \mathbf{E}_y\Big\{\mathbf{1}(H_\Sigma > T_{B(y,r\ell)}) \cdot \Big[T_{B(y,r\ell)} + \mathbf{E}_{W_{T_{B(y,r\ell)}}} H_\Sigma\Big]\Big\} \\ &\le 2\mathbf{E}_y T_{B(y,r\ell)} + F(n+1) \cdot \mathbf{P}_y(H_\Sigma > T_{B(y,r\ell)}). \qquad (4.28)\end{aligned}$$

To estimate $\mathbf{P}_y(H_\Sigma > T_{B(y,r\ell)})$, notice that the total volume of subboxes containing no good points in $B(x,\ell)$ is at most $3^{-2}\pi r^2\ell^2$. The reason is that $B(x,\ell)$ does not intersect with any clearing box, hence each of the boxes intersecting $B(x,\ell)$ (there are at most nine of them) has at most $9^{-2}\pi r^2\ell^2$ volume of subboxes which contain no good point. Hence, if $B(y,r\ell) \subset B(x,\ell)$, then

$$\Big|\Sigma \cap B(y,r\ell)\Big| \ge \frac{1}{4}\pi r^2\ell^2 - 3^{-2}\pi r^2\ell^2 = (2^{-2} - 3^{-2})|B(y,r\ell)|$$

using that if $y \in S \subset S_{-}^{2\delta}$ then at least one fourth of $B(y, r\ell)$ is in $S_{-}^{2\delta}$. So let

$$c_0 := \min\Big\{\frac{1}{2}, \inf_{\substack{F\,:\,F\subset B(0,1)\\ |F|\geq(2^{-2}-3^{-2})|B(0,1)|}} \mathbf{P}_0(H_F \leq T_{B(0,1)})\Big\} > 0$$

be a positive universal constant, then

$$\mathbf{P}_y(H_\Sigma > T_{B(y,r\ell)}) \leq (1 - c_0),$$

and by continuing (4.28) we have

$$\mathbf{E}_y H_\Sigma \leq r^2\ell^2 + (1 - c_0)F(n + 1),$$

by (4.25) if $B(y, r\ell) \subset B(x, \ell)$ which is satisfied as $n \leq [1/r] - 1$ and $|x - y| \leq nr\ell$.

Taking the supremum for all $y \in S$ such that $|x - y| \leq nr\ell$, we have

$$F(n) \leq r^2\ell^2 + (1 - c_0)F(n + 1). \tag{4.29}$$

We also have $F(n_0) \leq 2G_\Theta$ for any n_0, since $H_\Sigma \leq T_\Theta$. Iterating the inequality (4.29), we get

$$F(0) \leq c_0^{-1} r^2\ell^2 + 2(1 - c_0)^{n_0} G_\Theta.$$

Applying this to $n_0 := [1/r] - 1$ for $r \leq 1/4$ we obtain

$$\mathbf{E}_x T \leq 2[c_0^{-1} r^2\ell^2 + (1 - c_0)^{1/r} G_\Theta]$$

for all $x \notin S \cap A^1$, $x \in \Theta$, which finishes the proof of (4.19). $\square$.

Lemma 4.4 *For* $\delta \leq \frac{a}{6}$

$$G_\Omega \leq G_{\Omega_{-}^{2\delta}} \leq 8G_\Omega + 1000\delta^2.$$

Proof: The first inequality is trivial by $\Omega \subset \Omega_-^{2\delta}$. For the second inequality, let $T_- := T_{\Omega_-^{2\delta}}$ $T_+ := T_\Omega$ for simplicity, and we write

$$\mathbf{E}_x T_- = \mathbf{E}_x \Big\{ T_+ + \mathbf{E}_{W_{T_+}} T_- \Big\} \le \mathbf{E}_x T_+ + \sup_{y \in \Omega_-^{2\delta} \setminus \Omega} \mathbf{E}_y T_-$$

for any $x \in \Omega$. If $x \in \Omega_-^{2\delta} \setminus \Omega$, then we can drop the first term. Hence we get

$$2G_{\Omega_-^{2\delta}} \le 2G_\Omega + \sup_{y \in \Omega_-^{2\delta} \setminus \Omega} \mathbf{E}_y T_-. \tag{4.30}$$

Let us fix $y \in \Omega_-^{2\delta} \setminus \Omega$, let $T := T_{B(y,4\delta)}$ and let the positive number θ to be chosen later. Then

$$\mathbf{E}_y T_- = \mathbf{E}_y \Big\{ T_- \cdot \mathbf{1}(T > \theta) \cdot \mathbf{1}(T_- > \theta) \Big\} + \mathbf{E}_y \Big\{ T_- \cdot \mathbf{1}(T \le \theta) \cdot \mathbf{1}(T_- > \theta) \Big\} + \mathbf{E}_y \Big\{ T_- \cdot \mathbf{1}(T_- \le \theta) \Big\}. \tag{4.31}$$

For the last term

$$\mathbf{E}_y \Big\{ T_- \cdot \mathbf{1}(T_- \le \theta) \Big\} \le \theta.$$

For the second term, using that $T \le \theta$ and $\theta < T_-$ implies $W_T \in \Omega_-^{2\delta}$, we obtain

$$\mathbf{E}_y \Big\{ T_- \cdot \mathbf{1}(T \le \theta) \cdot \mathbf{1}(T_- > \theta) \Big\} \le \mathbf{E}_y \Big\{ \mathbf{1}(T \le \theta) \cdot \mathbf{1}(W_T \in \Omega_-^{2\delta}) \Big[T + \mathbf{E}_{W_T} T_- \Big] \Big\}$$

$$\le \theta + \mathbf{E}_y \Big\{ \mathbf{1}(W_T \in \Omega_-^{2\delta}) \mathbf{E}_{W_T} T_- \Big\} \le \theta + 2G_{\Omega_-^{2\delta}} \cdot \mathbf{P}_y(W_T \in \Omega_-^{2\delta}) \le \theta + \frac{3}{4} \cdot 2G_{\Omega_-^{2\delta}}$$

since, by elementary geometry

$$\mathbf{P}_y(W_T \in \Omega_-^{2\delta}) = \frac{|S(y, 4\delta) \cap \Omega_-^{2\delta}|}{|S(y, 4\delta)|} \le \frac{3}{4}.$$

Here we used that the boundary of $\Omega_-^{2\delta}$ consists of straight line segments and circular arcs of curvature not bigger than $(a - 2\delta)^{-1} \le (4\delta)^{-1}$ by the assumption on a.

Finally for the first term in (4.31) we use

$$\mathbf{E}_y \Big\{ T_- \cdot \mathbf{1}(T > \theta) \cdot \mathbf{1}(T_- > \theta) \Big\} \le \mathbf{E}_y \Big\{ \mathbf{1}(T > \theta) \Big[\theta + \mathbf{E}_{W_\theta} T_- \Big] \Big\}$$

$$\le \theta + \mathbf{E}_y\Big\{\mathbf{1}(T>\theta)\mathbf{E}_{W_\theta}T_-\Big\} \le \theta + 2G_{\Omega_-^{2\delta}}\cdot \mathbf{P}_y(T>\theta) \le \theta + 2G_{\Omega_-^{2\delta}} \times \Big((2\pi\theta)^{-1}\int_{\substack{z\in\mathbf{R}^2\\ |z|\le 4\delta}} e^{-\frac{z^2}{2\theta}}dz\Big)$$

$$= \theta + 2(1-e^{-8\delta^2\theta^{-1}})G_{\Omega_-^{2\delta}}.$$

Putting these three estimates together, we obtain from (4.31)

$$\mathbf{E}_y T_- \le 3\theta + \Big[\frac{3}{2} + 2(1-e^{-8\delta^2\theta^{-1}})\Big]G_{\Omega_-^{2\delta}}.$$

Combining this with (4.30), we have

$$G_{\Omega_-^{2\delta}} \le G_\Omega + \frac{3\theta}{2} + \Big[\frac{3}{4} + (1-e^{-8\delta^2\theta^{-1}})\Big]G_{\Omega_-^{2\delta}}.$$

Choose θ such that $e^{-8\delta^2\theta^{-1}} \ge 7/8$, i.e. let, say, $\theta = 80\delta^2$. This gives

$$G_{\Omega_-^{2\delta}} \le 8G_\Omega + 1000\delta^2. \quad \Box$$

Finally we shall give an upper bound on the Green's function in terms of the magnetic eigenvalue.

Lemma 4.5 *Let $\Omega = S\setminus\bigcup_{i\in\mathcal{G}}\overline{B}(x_i,a)$ as usual and $a\le 1\le \ell\le s$, $v\le B$. Let λ be the lowest eigenvalue of $\frac{1}{2}(-i\nabla - A)^2 - B_0 + \tilde{V}$ with Dirichlet boundary condition on S. Then there exists a function $\ell_1(\varepsilon,B)>0$ such that*

$$e^{BG_\Omega} \le s^2 e^{B\ell}\lambda^{-1} \tag{4.32}$$

for all $\ell \ge \ell_1(\varepsilon,B)$.

Proof: We can assume that $G_\Omega \ge \ell$, otherwise (4.32) is trivial, since $\tilde{V}\le v\le B$, hence $\lambda \le B + \lambda(B,1) \le \ell^2 \le s^2$ is automatic if $\ell \ge (B+\lambda(B,1))^{1/2}$.

We shall construct a trial function. For any $d>0$ let

$$\Omega^{(d)} := [-s+d, s-d]^2 \setminus \bigcup_{i\in\mathcal{G}} B(x_i,d)$$

and let $g := g_\Omega$ for simplicity. Let $\gamma \geq 0$ be a fixed smooth bounded function with support in the unit ball and $\int_{\mathbf{R}^2} \gamma = 1$, $\|\nabla\gamma\|_1 \leq 4$. Let $\theta := \gamma * \mathbf{1}(\Omega^{(2)} + B(0,1))$, where $*$ denotes the convolution. Notice that $0 \leq \theta \leq 1$, $\theta \equiv 1$ on $\Omega^{(2)}$, $\theta \equiv 0$ on Ω^c and $\|\nabla\theta\|_\infty \leq 4$.

Let $\varphi(x) := \theta(x)e^{Bg(x)}$ for $x \in S$, obviously $\varphi \in H_0^1(\Omega) \subset H_0^1(S)$, hence we can use it as a trial function for λ. Notice that φ is zero on the support of $\tilde{V}$. Choose a gauge $\tilde{A}$ corresponding to g, i.e. let $(\tilde{A}_1, \tilde{A}_2) := B(\partial_2 g, -\partial_1 g)$, which is well defined on Ω, and φ is zero outside of Ω anyway. Since λ is gauge invariant,

$$\lambda \leq \frac{\frac{1}{2}\int_\Omega |(-i\partial_1 + \partial_2 - \tilde{A}_1 - i\tilde{A}_2)\varphi|^2}{\int_\Omega |\varphi|^2}.$$

By the IMS localization

$$\int_\Omega |(-i\partial_1 + \partial_2 - \tilde{A}_1 - i\tilde{A}_2)\varphi|^2 \leq 2\int_\Omega \theta^2 |(-i\partial_1 + \partial_2 - \tilde{A}_1 - i\tilde{A}_2)e^{Bg}|^2 + 2\|\nabla\theta\|_\infty^2 \int_{\mathrm{supp}\,\nabla\theta} e^{2Bg}.$$

We know that $(-i\partial_1 + \partial_2 - \tilde{A}_1 - i\tilde{A}_2)e^{Bg(x)} \equiv 0$ for $x \in \Omega$. We can apply part (i) of Lemma 4.3 with $b = b_0 := (a+3)/2 \leq 2$, $\delta := 0$ to obtain

$$g(x) = g_\Omega(x) \leq G_\Omega \Big(\frac{\ell}{b_0}\Big)^{-k(\varepsilon,b_0)} \leq 2G_\Omega \ell^{-k(\varepsilon,b_0)} \tag{4.33}$$

for $\ell \geq \ell_0(\varepsilon, 2) \geq \ell_0(\varepsilon, b_0)$ and for all $x \in \Omega \setminus \Omega^{(3+a)}$, in particular for all $x \in \mathrm{supp}\,\nabla\theta$, since $\mathrm{supp}\,\nabla\theta \subset \Omega \setminus \Omega^{(2)} \subset \Omega \setminus \Omega^{(3+a)}$. Hence

$$\lambda \leq \frac{16s^2 \exp\Big(4BG_\Omega \ell^{-k(\varepsilon,b_0)}\Big)}{\int_\Omega |\varphi|^2} \tag{4.34}$$

(the extra s^2 comes from the volume $|\mathrm{supp}\,\nabla\theta| \leq s^2$).

Let $x_0 \in \Omega$ be the point where $g(x)$ takes on its maximum. By (4.33) we know that $x_0 \in \Omega^{(3+a)}$ (in particular $\Omega^{(3+a)}$ is not empty) if $\ell \geq \frac{\log 2}{k(\varepsilon,2)} \geq \frac{\log 2}{k(\varepsilon,b_0)}$.

Moreover, $B(x_0, \pi^{-1/2}) \subset \Omega^{(2)} \subset \Omega$, hence $\theta \equiv 1$ on $B(x_0, \pi^{-1/2})$. Therefore, by Jensen's inequality

$$\int_\Omega |\varphi|^2 \geq \int_{B(x_0,\pi^{-1/2})} e^{2Bg} \geq \exp\Big(2B\int_{B(x_0,\pi^{-1/2})} g\Big). \tag{4.35}$$

Since $\Delta g = -1$, we have

$$\Delta\Big(g + \frac{1}{4}(x - x_0)^2\Big) = 0$$

on $B(x_0, \pi^{-1/2})$, hence

$$\int_{B(x_0,\pi^{-1/2})} \Big(g(x) + \frac{1}{4}(x - x_0)^2\Big)\mathrm{d}x = g(x_0) = G_\Omega.$$

Therefore

$$\int_{B(x_0,\pi^{-1/2})} g \geq G_\Omega - \frac{1}{8\pi},$$

which finally gives

$$\int_\Omega |\varphi|^2 \geq e^{-B/(4\pi)} e^{2BG_\Omega}$$

from (4.35). Hence, from (4.34) and $G_\Omega \geq \ell \geq 1$

$$\lambda \leq 16 e^{B/(4\pi)} s^2 \exp\Big[-2BG_\Omega\Big(1 - 2\ell^{-k(\varepsilon,b_0)}\Big)\Big] \leq s^2 e^{-BG_\Omega}$$

for large enough $\ell \geq \ell_2(\varepsilon, B)$ (recall that $b_0 \leq 2$). Choosing

$$\ell_1(\varepsilon, B) := \max\Big\{(B + \lambda(B,1))^{1/2}, \ell_0(\varepsilon, 2), \frac{\log 2}{k(\varepsilon, 2)} + 1, \ell_2(\varepsilon, B)\Big\}$$

we finish the proof of Lemma 4.5. □.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. First notice that $K \leq 1$ and (4.9) imply $\lambda \leq 1$. Then, consider the estimate on η which is defined in (4.11). We can assume that $G_{\Omega_-^{2\delta}} \geq 1$, otherwise $\eta \leq 1$ and Lemma 4.2 immediately implies Theorem 4.1 for large enough ℓ. Hence

$$G_\Omega \leq G_{\Omega_-^{2\delta}} \leq 9 G_\Omega \tag{4.36}$$

by Lemma 4.4 and $\delta \leq \frac{1}{200}$. Furthermore, we can assume that $G_\Omega \geq \ell$, otherwise $\eta \leq 9\ell$ using (4.36), and Theorem 4.1 follows directly from Lemma 4.2 thanks to the factor $e^{20B\ell}$ in the definition of K. In particular $G_{\Omega_-^{2\delta}} \geq \ell$.

Therefore using Lemma 4.3, the fact that $\Omega_-^{2\delta} \setminus \Omega_+^{2b} \subset \Sigma \cup (A^1)^c$ and (4.36) we obtain that

$$\eta \leq \Big(\frac{\ell}{b}\Big)^{-k(\varepsilon,b)} G_{\Omega_-^{2\delta}} + \Big[(1-c_0)^{1/r} G_{\Omega_-^{2\delta}} + c_0^{-1} r^2 \ell^2\Big] \leq 10\Big[(1-c_0)^{1/r} G_{\Omega} + c_0^{-1} r^2 \ell^2\Big] \tag{4.37}$$

if $\ell \geq \ell_3(\varepsilon, b, r)$. Combining this estimate with (4.32), we obtain

$$e^{2B\eta} \leq \Big(s^2 e^{B\ell} \lambda^{-1}\Big)^{20(1-c_0)^{1/r}} e^{20Bc_0^{-1} r^2 \ell^2} \tag{4.38}$$

if, in addition, $\ell \geq \ell_1(\varepsilon, B)$.

Now we can estimate λ_b. By Lemma 4.2, (4.38) and the definitions of K and $w(r)$ given in (4.7) and (4.8) we immediately obtain (4.6), if, in addition to all the previous conditions, the denominator in (4.10) is bigger than $\frac{1}{2}$. This is implied by (4.9) if ℓ is large enough. The function $\tilde{\ell}$ is obtained by taking the maximum of all previous lower bounds on ℓ. $\square$.

5 Upper bound on the Lifschitz tail

We need an easy combinatorial lemma. Let $\mathbf{G}$ be the following graph: its vertex set is $\mathbf{Z}^2$, and for $p, q \in \mathbf{Z}^2$ let (p, q) be an edge if and only if $0 < \mathrm{dist}(p, q) < 2$ (i.e. $\mathbf{G}$ is the "at most second nearest neighbor graph" on $\mathbf{Z}^2$). A subset $\mathcal{S}$ of $\mathbf{Z}^2$ is called *connected* if there is a connected subgraph of $\mathbf{G}$ whose vertex set is $\mathcal{S}$.

Lemma 5.1 *Let $\mathcal{G}_N$ be the collection of connected subsets of $\mathbf{Z}^2$ with N elements which contain the origin. Then $|\mathcal{G}_N| \leq 8^{2N-2}$*

Proof: It is enough to count the number of subtrees of $\mathbf{G}$ with N points, containing the origin, since every connected graph has a spanning tree, which may not be unique. We can encode a subtree by a sequence $\{a_j\}$ of length $2N-2$ which is built from eight symbols; $\searrow \rightarrow \nearrow \uparrow \nwarrow \leftarrow \swarrow$ and $\downarrow$. The sequence is generated as follows. Starting from the origin $P_0 = 0$, let the first element of the sequence a_1 be the first arrow (in the order listed above) which points towards a vertex connected with P_0 in the subgraph. Let P_1 be the endpoint of a_1. Inductively, if a_1, a_2, $\dots a_k$ and P_k have been defined, then let P_{k+1} be that neighboring vertex of P_k for which the angle $P_{k-1}P_kP_{k+1}$ is the smallest measured clockwise. If P_k has only one neighbor (P_{k-1}), then this angle is considered to be 360^o and let $P_{k+1} := P_{k-1}$. Let a_{k+1} be $\searrow \rightarrow \nearrow \uparrow \nwarrow \leftarrow \swarrow$ and $\downarrow$ if this angle is 45^o, 90^o, 135^o, 180^o, 225^o, 270^o, 315^o or 360^o, respectively. In plain words, we travel along the edges of the tree with the rule that at each vertex we always go "to the right". Obviously, after $2N-2$ steps we arrive back to the origin and we have visited each vertex (some of them several times) and each edge exactly twice. From the sequence $a_1, \dots a_{2N-2}$ one can reconstruct the tree uniquely (though not every sequence on these eight symbols corresponds to a tree). The number of sequences is obviously 8^{2N-2}. $\square$.

5.1 Hard-core potential

We complete the upper bound for the hard core potential case. Choose $q(t) := (\log t)^2$, $\ell(t) = 10\sqrt{\frac{\log t}{B_0}}$, $0 < \varepsilon < \varepsilon_0$, $0 < r \le 1/4$, $r \le \log\left(\frac{1}{1-c_0}\right)/(\log 30)$, $b > 1$ and $\delta := \min\{\frac{a}{6}, \frac{1}{200}\}$. From (3.18) and

$$\log(a+b) \le 2\max\{\log a, \log b\} \tag{5.1}$$

we obtain

$$\limsup_{t\to\infty} \frac{\log L^{hc}(t)}{\log t} \le \limsup_{t\to\infty} (\log t)^{-1} \log \mathcal{E}_\omega e^{-t\lambda^{hc}_{Q,\omega}} \tag{5.2}$$

since

$$\limsup_{t\to\infty} (\log t)^{-1} \log\Big(C_2 e^{-C_3 q(t)} + e^{-B_0 t}\Big) = -\infty.$$

Now we apply the estimate from Theorem 4.1 for $S := Q$, $s(t) := q(t) = (\log t)^2$. Notice that $\ell \ge \tilde{\ell}(\varepsilon, b, B, r)$ if $t \ge t_0(\varepsilon, B, b, r)$. We obtain

$$\limsup_{t\to\infty} (\log t)^{-1} \log \mathcal{E}_\omega e^{-t\lambda^{hc}_{Q,\omega}} \le \limsup_{t\to\infty} (\log t)^{-1} \log\Big[\mathcal{E}_\omega \exp\Big(-tK\lambda_b^{1+w(r)}\Big) + \exp\Big(-tK\cdot 2^{-(1+w(r))}\Big)\Big] \tag{5.3}$$

recalling the condition (4.9) in Theorem 4.1 and distinguishing the cases $(\lambda^{hc}_{Q,\omega}/K)^{1/(1+w(r))} \leq 1/2$ or $> 1/2$. Using (5.1) and

$$\limsup_{t\to\infty}(\log t)^{-1}\log\exp\Big(-tK\cdot 2^{-(1+w(r))}\Big) = -\infty \tag{5.4}$$

we get from (5.2) and (5.3)

$$\limsup_{t\to\infty}\frac{\log L^{hc}(t)}{\log t} \leq \limsup_{t\to\infty}(\log t)^{-1}\log\mathcal{E}_\omega\exp\Big(-tK\lambda_b^{1+w(r)}\Big). \tag{5.5}$$

Following Sznitman's construction let

$$D := S\cap\bigcup_{\overline{C}_m\cap A\neq\emptyset}\overline{C}_m \supset S\cap A^1.$$

Let U (respectively $\tilde{U}$) be the complement in D of the union over $m\in\mathbf{Z}^2$ of closed subboxes intersecting $\mathrm{int}(C_m)$ and containing a point (respectively a good point) from ω. The interior of D may have several components, $D_0, D_1, \ldots$, each has its own sets $U_0, U_1, \ldots$ and $\tilde{U}_0, \tilde{U}_1 \ldots$ defined as $U_i := U\cap D_i$, $\tilde{U}_i := \tilde{U}\cap D_i$, which are certainly disjoint. Since $\Omega^b_+ \subset \tilde{U} = \cup_i\tilde{U}_i$, let us choose the labelling of the components such that $\lambda(\Omega^b_+\cap\tilde{U}_0) \leq \lambda(\Omega^b_+\cap\tilde{U}_i)$ for all $i\geq 1$. As before, $\lambda(\mathcal{U}) = \lambda^{(B_0)}(\mathcal{U})$ denotes the lowest eigenvalue of the free magnetic operator $\frac{1}{2}(-i\nabla - A)^2 - B_0$ with Dirichlet boundary conditions on $\mathcal{U}$ for any open set $\mathcal{U}$. Hence $\lambda_b = \lambda(\Omega^b_+) = \lambda(\Omega^b_+\cap\tilde{U}_0) \geq \lambda(\tilde{U}_0)$. By the definition of good points and the construction of A and D we have

$$U_0\subset\tilde{U}_0\subset D_0 \quad |\tilde{U}_0| \leq |U_0| + \varepsilon|D_0|. \tag{5.6}$$

Let $N = |D_0|/\ell^2$ which is an integer. For fixed N, the number of possibilities for D_0 is at most $(s/\ell)^2 64^N$ since D_0 is connected, and after having chosen one box within S in $(s/\ell)^2$ ways, we have to count the number of connected subsets of $\mathbf{G}$ with N vertices, which contain a fixed point (obviously $\mathbf{G}$ is isomorphic to the "connectedness" graph of the closed boxes). This number was estimated in Lemma 5.1.

For fixed D_0 the number of possibilities for U_0 and $\tilde{U}_0$ is at most $2^{2N\cdot([\sqrt{2}\ell/b]+1)^2} \leq 2^{8N(\ell/b)^2}$. Finally $|U_0| \geq |\tilde{U}_0| - \varepsilon N\ell^2$, and we know, that once $D_0, U_0, \tilde{U}_0$ is chosen, the total volume which receives no point is at least $\max\{|U_0|, Nr^2\ell^2\pi/729\} \geq \max\{|\tilde{U}_0|, Nr^2\ell^2\pi/729\} - \varepsilon N\ell^2$. Here we used the definition of good points and that $|A| \geq \frac{1}{9}|D| \geq \frac{1}{9}|D_0|$ by construction.

Hence

$$\mathcal{E}_\omega \exp\left(-tK\lambda_b^{1+w(r)}\right) \tag{5.7}$$

$$\leq \sum_{N=0}^{(s/\ell)^2} (s/\ell)^2 64^N 2^{8N(\ell/b)^2} e^{\nu\varepsilon N\ell^2} \exp\left(-tK\lambda(\tilde{U}_0)^{1+w(r)}\right) e^{-\nu\cdot\max\{|\tilde{U}_0|, Nr^2\ell^2\pi/729\}}.$$

The sum above is majorated by

$$s^2 \sum_{N=0}^{(s/\ell)^2} 64^N e^{-N\ell^2 E} \leq s^2\left(1 - 64e^{-\ell^2 E}\right)^{-1}$$

if

$$E = E(\varepsilon, b, r, \nu) := \frac{\pi r^2 \nu}{729} - \nu\varepsilon - 8(\log 2)b^{-2} > 0$$

and $t \geq t_1(\varepsilon, B, b, r, \nu)$.

Fix $\kappa > 0$. We minimize

$$tK\lambda(\tilde{U}_0)^{1+w(r)} + \nu|\tilde{U}_0|$$

by the magnetic isoperimetric inequality [8]

$$\lambda(\tilde{U}_0) \geq \lambda(B, \pi^{-1/2}|\tilde{U}_0|^{1/2})$$

and by the lower bound on the lowest eigenvalue (3.9). The result is

$$tK\lambda(\tilde{U}_0)^{1+w(r)} + \nu|\tilde{U}_0| \geq \min_X\left\{tKe^{-B_0(1+\kappa)\pi^{-1}X(1+w(r))} + \nu X\right\} \geq F\log(tK/F)$$

if $t \geq t_2(\kappa, B, \nu)$, where

$$F = F(\kappa, B, r, \nu) := \frac{2\pi\nu}{B(1+\kappa)(1+w(r))}.$$

For any $r > 0$ there exists a small enough ε and large enough b such that $E \geq 10^{-3}r^2\nu$. In this case

$$\limsup_{t\to\infty}(\log t)^{-1}\log\mathcal{E}_\omega\exp\left(-tK\lambda_b^{1+w(r)}\right) \leq \limsup_{t\to\infty}(\log t)^{-1}\log\frac{s^2\exp\Big(-F\log(tK/F)\Big)}{1-64\exp(-10^{-3}r^2\ell^2\nu)}. \tag{5.8}$$

Using that

$$\limsup_{t\to\infty}\frac{\log s^2}{\log t} = 0, \quad \limsup_{t\to\infty}\frac{-F\log(K/F)}{\log t} = 4000c_0^{-1}r^2(1+w(r))F$$

if ε is small e nough and b is large enough, depending on r, we obtain from (5.5) and (5.8)

$$\limsup_{\substack{b\to\infty\\ \varepsilon\to 0}}\limsup_{t\to\infty}\frac{\log L^{hc}(t)}{\log t} \leq -\frac{2\pi\nu}{B(1+\kappa)(1+w(r))} + \frac{8000\pi c_0^{-1}r^2\nu}{B(1+\kappa)}. \tag{5.9}$$

Hence, recalling that $w(r)\to 0$ as $r\to 0$,

$$\limsup_{\substack{\kappa\to 0\\ r\to 0}}\limsup_{\substack{b\to\infty\\ \varepsilon\to 0}}\limsup_{t\to\infty}\frac{\log L^{hc}(t)}{\log t} \leq -\frac{2\pi\nu}{B}, \tag{5.10}$$

which completes the proof of the upper bound in (2.9).

5.2 Soft potential

The upper bound for the soft potential is similar to the previous section. Fix $0 < \beta < B_0$. Let $2(24B_0)^{1/2}t \leq q(t) < 2(24B_0)^{1/2}t + \beta^{-1/2}$ such that $\beta^{1/2}q(t)$ is an integer, and let $s(t) = (\log t)^2$.

From (3.16) and Proposition 3.4

$$\limsup_{t\to\infty}\frac{L(t)}{\log t} \leq \limsup_{t\to\infty}(\log t)^{-1}\log\mathcal{E}_\omega e^{-t\lambda_{Q,\omega}}$$

$$\leq \limsup_{t\to\infty}(\log t)^{-1}\log\mathcal{E}_\omega\exp\Big(-\frac{t}{100}\min_{j\in\beta^{-1/2}\mathbf{L}}\lambda_{S_j,\omega}^{(B_0+\beta)}\Big). \tag{5.11}$$

Then by Theorem 4.1 and the analogue of (5.3) and (5.4), we have

$$\limsup_{t\to\infty}\frac{L(t)}{\log t} \leq \limsup_{t\to\infty}(\log t)^{-1}\log\mathcal{E}_\omega\exp\Big(-tK\min_{j\in\beta^{-1/2}\mathbf{L}}\lambda_{j,b}^{1+w(r)}\Big), \tag{5.12}$$

where $\lambda_{j,b} := \lambda^{(B_0+\beta)}(\Omega_+^{j,b})$ denotes the lowest eigenvalue of $\Omega_+^{j,b} \subset S_j$ obtained within each $S = S_j$ by the procedure of Section 4 and with magnetic field $B + 2\beta = 2(B_0 + \beta)$ instead of B.

Let

$$D := Q \cap \bigcup_{\overline{C}_m \cap A \neq \emptyset} \overline{C}_m \supset Q \cap A^1$$

(notice that here we have Q instead of S) and let U, $\tilde{U}$ defined exactly as in the hard core case. Consider the sets $D \cap S_j$ for all $j \in \beta^{-1/2}\mathbf{L}$. Each of these sets may have several components, let us call them $D_0^j, D_1^j, \ldots$. The sets $U_0^j, U_1^j, \ldots$, $\tilde{U}_0^j, \tilde{U}_1^j, \ldots$ are defined analogously within each D_i^j. We know that each $\Omega_+^{j,b}$ is within $\cup_i \tilde{U}_i^j$. Let us choose $j_0 \in \beta^{-1/2}\mathbf{L}$ and the labelling of the sets $U_i^{j_0}$ such that $\lambda^{(B_0+\beta)}(\Omega_+^{j_0,b} \cap \tilde{U}_0^{j_0}) \leq \lambda^{(B_0+\beta)}(\Omega_+^{j,b} \cap \tilde{U}_i^j)$ for all $j \in \beta^{-1/2}\mathbf{L}$ and for all i. Then $\min_{j \in \beta^{-1/2}\mathbf{L}} \lambda_{j,b}^{(B_0+\beta)} \geq \lambda^{(B_0+\beta)}(\tilde{U}_0^{j_0})$. Finally, let $N = |D_0^{j_0}|/\ell^2$ and we also have $|U_0^{j_0}| \geq |\tilde{U}_0^{j_0}| - \varepsilon N \ell^2$. From now on we omit the superscripts j_0 and $(B_0 + \beta)$.

The only difference between the cases of the soft and the hard-core potential is in the combinatorics. The number of possibilities for D_0 with fixed N is $(q/\ell)^2 64^N$ instead of $(s/\ell)^2 64^N$ as in the hard core case. Otherwise, once D_0 is fixed, the combinatorics for U_0 and $\tilde{U}_0$ remains unchanged. Hence

$$\mathcal{E}_\omega \exp\Big(-tK \min_{j \in \beta^{-1/2}\mathbf{L}} \lambda_{j,b}^{1+w(r)}\Big)$$

$$\leq \sum_{N=0}^{(q/\ell)^2} (q/\ell)^2 64^N 2^{8N(\ell/b)^2} e^{\nu \varepsilon N \ell^2} \exp\Big(-t\lambda(\tilde{U}_0^{j_0})^{1+w(r)}\Big) \exp\Big(-\nu \cdot \max\{|\tilde{U}_0^{j_0}|, N r^2 \ell^2 \pi/729\}\Big).$$

The rest of the calculation is identical to the hard core case, except that in the analogue of (5.8) we have

$$\limsup_{t \to \infty} \frac{\log q^2}{\log t} = 2$$

instead of $\limsup_{t \to \infty} \frac{\log s^2}{\log t} = 0$.

For the analogue of (5.9) we obtain

$$\limsup_{\substack{b\to\infty\\ \varepsilon\to 0}}\limsup_{t\to\infty}\frac{\log L(t)}{\log t}\leq 2-\frac{2\pi\nu}{(B+2\beta)(1+\kappa)(1+w(r))}+\frac{8000\pi c_0^{-1}r^2\nu}{(B+2\beta)(1+\kappa)}. \tag{5.13}$$

Hence

$$\limsup_{\beta\to 0}\limsup_{\substack{\kappa\to 0\\ r\to 0}}\limsup_{\substack{b\to\infty\\ \varepsilon\to 0}}\limsup_{t\to\infty}\frac{\log L(t)}{\log t}\leq -\frac{2\pi\nu}{B}+2, \tag{5.14}$$

which completes the proof of (2.1). $\square$.

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