QUANTUM METHODS FOR INTERACTING PARTICLE SYSTEMS II, GLAUBER DYNAMICS FOR ISING SPIN SYSTEMS

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ABSTRACT. Using the formalism and the results described in [QMIPS I] and in [QMIPS III], we discuss the approach to termodynamic equilibrium for discrete spin systems in a framework that generalizes the one originally proposed by R. Glauber. We prove a lower bound extimate for their exponetial rate of convergence to equilibrium, in the high temperature regime which is better than those previously known. We also give application to some (not necessarily ferromagnetic) Ising-spin models. Such results provide an upper bound for the critical temperature of the d-dimensional Ising model, which in dimension two coincides with the one computed with equilibrium statistical mechanics techniques.

1. INTRODUCTION

In this paper we give a first example of the application of the algebraic techniques borrowed from equilibrium quantum statistical mechanics to interacting particle systems. We refer the reader to [QMIPS I] for a general discussion and reference to other results obtained by a similar approach, and to [QMIPS III] for some application to (equilibrium) classical statistical mechanics.

Here we consider Ising spins on a $\mathbb{Z}^d$ lattice undergoing a Glauber dynamics. The aim is to derive in a straightforward way a bound for the spectral gap of the generator in the high temperature regime. The technique turns out to be useful in the case of short range interactions and provides sharper bounds than those already known. As a byproduct we get estimates for the critical temperature of the models considered.

As an introduction to the formalism, we start with the "single site dynamics" (as in [Gl]). Following the prescription given in [QMIPS I] we consider the Hilbert space of complex square summable function on the single site configuration space $\mathbb{Z}_2^{(x)}$ with respect to the symmetric Bernoulli measure. Namely $\forall x \in \mathbb{Z}^d$

$$\begin{aligned} \mathcal{H}_x \quad &: \quad = span\left\{ |\emptyset\rangle_x , |\mathrm{x}\rangle_x \right\} \cong \mathbb{C}^2 \\ |\emptyset\rangle_x \quad &\equiv \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}_x \quad |\mathrm{x}\rangle_x \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}_x \end{aligned}$$

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$\mathcal{U}_x = M(2,\mathbb{C})$ is the algebra of bounded operators on $\mathcal{H}_x$[1]. Let us define the *spin* operator

$$\boldsymbol{\sigma}_x \in \mathcal{U}_x : \boldsymbol{\sigma}_x \begin{cases} |\emptyset\rangle_x = |\mathrm{x}\rangle_x \\ |\mathrm{x}\rangle_x = |\emptyset\rangle_x \end{cases}$$

and the *spin flip* operator

$$\mathbf{f}_x \in \mathcal{U}_x : \mathbf{f}_x \begin{cases} |\emptyset\rangle_x = |\emptyset\rangle_x \\ |\mathrm{x}\rangle_x = -|\mathrm{x}\rangle_x \end{cases}$$

Consider also the automorphism $\mathbf{U}_x$ on $\mathcal{U}_x$ such that

$$\mathbf{U}_x \in \mathcal{U}_x : \mathbf{U}_x \boldsymbol{\sigma}_x \mathbf{U}_x = \mathbf{f}_x \qquad \mathbf{U}_x^2 = \mathbf{I}_x$$

Setting $\mathcal{C}_x^{(s)}, \mathcal{C}_x^{(f)} \subset \mathcal{U}_x$ to be the commutative algebras generated by the identity operator $\mathbf{I}_x$ and by, respectively, $\boldsymbol{\sigma}_x$, $\mathbf{f}_x$; one easily sees from the above definition that $\mathbf{U}_x$ maps one into another: $\mathbf{U}_x \mathcal{C}_x^{(s)} \mathbf{U}_x = \mathcal{C}_x^{(f)}$. For such reason, in the following, we will denote them both with $\mathcal{C}_x$ (the algebra of observable), that is the algebra of diagonal operator on $\mathcal{H}_x$.

This is the *kinematical structure* $(\mathcal{H}_x, \mathcal{U}_x, \mathcal{C}_x)$ for a single site system. Let Λ be any subset of $\mathbb{Z}^d$ lattice. We define its analog on Λ by tensor product $(\mathcal{H}_\Lambda, \mathcal{U}_\Lambda, \mathcal{C}_\Lambda)$ $(\mathbb{Z}_2^\Lambda = \bigotimes_{x\in\Lambda} \mathbb{Z}_2^{(x)})$. Then [QMIPS I] we have

$$\begin{aligned} |\alpha\rangle_\Lambda &= \bigotimes_{x\in\alpha} |\mathrm{x}\rangle_x \bigotimes_{x\in\Lambda\setminus\alpha} |\emptyset\rangle_x \\ \mathcal{H}_\Lambda &= span\left\{|\alpha\rangle_\Lambda : \alpha \subseteq \Lambda\right\} \end{aligned}$$

Moreover $\mathcal{U}_\Lambda = M\left(2^{|\Lambda|}, \mathbb{C}\right)$ and $\mathcal{C}_\Lambda$ is the algebra of polynomials in $\boldsymbol{\sigma}_\alpha$ $(\mathbf{f}_\alpha)$ $\forall \alpha \subset \Lambda$. Then

$$\begin{aligned} \boldsymbol{\sigma}_\alpha &= \bigotimes_{x\in\alpha} \boldsymbol{\sigma}_x \bigotimes_{x\in\Lambda\setminus\alpha} \mathbf{I}_x \\ \mathbf{f}_\alpha &= \mathbf{U}_\Lambda \boldsymbol{\sigma}_\alpha \mathbf{U}_\Lambda = \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} (-1)^{|\alpha\cap\gamma|} |\gamma\rangle\langle\gamma|_\Lambda \end{aligned}$$

where $|\gamma\rangle\langle\gamma|_\Lambda$ denotes the projector on the subspace spanned by $|\gamma\rangle_\Lambda$.The preceding construction can be extended to the configuration space of the whole particle system [QMIPS I] if we set: $\mathcal{H} = span\left\{|\alpha\rangle : \alpha \subset \mathbb{Z}^d\right\}$, $\mathcal{U} = \mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$ and $\mathcal{C}$ the algebra of observables for the system.

The following analysis involves a finite size system whose kinematical environment is $\left(\mathcal{H}_\Lambda, \mathcal{U}_\Lambda, \mathcal{C}_\Lambda, \omega_\Lambda^0\right)$, where

$$\begin{aligned} \omega_\Lambda^0(\mathbf{A}_\Lambda) &: = tr_{\mathcal{H}_\Lambda}(\mathbf{A}_\Lambda) = \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} \langle\gamma|\,\mathbf{A}_\Lambda\,|\gamma\rangle_\Lambda = 2^{|\Lambda|}\langle\emptyset|\,\mathbf{A}_\Lambda\,|\emptyset\rangle_\Lambda \\ \bar{\mathbf{A}}_\Lambda &= \mathbf{U}_\Lambda \mathbf{A}_\Lambda \mathbf{U}_\Lambda \quad \mathbf{A}_\Lambda \in \mathcal{U}_\Lambda \end{aligned}$$

[1] Here, see [QMIPS I] and [M] for more details, we think of $\mathcal{H}_x$ as spanned by two (orthonormal) vectors labelled by the "empty site" and the "full site" configurations. Consequently any operator acting on the configuration space is lifted to a linear operator acting on $\mathcal{H}_x$ and a probability measure on the configuration space becomes a convex combination of the projectors on the subspaces spanned by the basis vectors of $\mathcal{H}_x$.

2. Free Diffusion

The operator ℓ_x (discrete derivative) acting on $\mathcal{H}_x$ is

$$\ell_x = \frac{\mathbf{I}_x - \mathbf{f}_x}{2}$$

$$\ell_x \begin{cases} |\emptyset\rangle_x = 0 \\ |\mathrm{x}\rangle_x = |\mathrm{x}\rangle_x \end{cases}$$

ℓ_x is selfadjoint and $\ell_x \ell_x = \ell_x$ so ℓ_x acts as the projection operator along the $|\mathrm{x}\rangle_x$ direction in $\mathcal{H}_x$. In the same fashion we define ℓ_x^Λ acting on $\mathcal{H}_\Lambda$, namely

$$\ell_x^\Lambda = \frac{\mathbf{I}_\Lambda - \mathbf{f}_x}{2}$$

such that

$$\ell_x^\Lambda \left|\alpha\right\rangle_\Lambda = \delta\left(x \in \alpha\right)\left|\alpha\right\rangle_\Lambda \quad \alpha \subseteq \Lambda$$

thus

$$\ell_x^\Lambda = \sum_{\emptyset \subseteq \gamma \subseteq \Lambda} \delta\left(x \in \gamma\right)\left|\gamma\right\rangle\left\langle\gamma\right|_\Lambda$$

Definition 2.1. $\forall \Lambda \subset \mathbb{Z}^d$ *the generator of the single spin flip dynamics, i.e. the dynamics of a non interacting spin system* $(\beta = 0)$ *is the operator:*

$$\mathcal{L}_\Lambda \in \mathcal{C}_\Lambda : \mathcal{L}_\Lambda = \sum_{x \in \Lambda} \ell_x^\Lambda$$

Obviously $\mathcal{L}_\Lambda$ is diagonal on $\mathcal{H}_\Lambda$

$$\begin{aligned} \mathcal{L}_\Lambda &= \sum_{x \in \Lambda} \sum_{\emptyset \subseteq \gamma \subseteq \Lambda} \delta\left(x \in \gamma\right)\left|\gamma\right\rangle\left\langle\gamma\right| = \\ &= \sum_{\emptyset \subseteq \gamma \subseteq \Lambda} \left|\gamma\right|\left|\gamma\right\rangle\left\langle\gamma\right|_\Lambda \end{aligned}$$

Since each eigenvalue has multiplicity $\binom{|\Lambda|}{|\gamma|}$,if we set P_k to be the projector along the $|\gamma\rangle$ direction in $\mathcal{H}_\Lambda$ such that $|\gamma| = k \in \mathbb{N}$ we obtain

$$\mathcal{L}_\Lambda = \sum_{n=0}^{|\Lambda|} \sum_{k=0}^{n} \binom{n}{k} k P_k$$

Similar arguments are valid for the semigroup generated by $\mathcal{L}_\Lambda$

$$\begin{aligned} \mathbf{S}_\Lambda\left(t\right) &= \sum_{\emptyset \subseteq \gamma \subseteq \Lambda} e^{-t|\gamma|}\left|\gamma\right\rangle\left\langle\gamma\right|_\Lambda = \\ &= \sum_{n=0}^{|\Lambda|} \sum_{k=0}^{n} \binom{n}{k} e^{-kt} P_k \end{aligned}$$

whose associated unique Perron-Frobenius eigenvector is $\left|\emptyset\right\rangle_\Lambda$.

Since the ℓ_x^Λ operator are defined on different lattice sites, they commute and the dynamics of the whole system is just the product dynamics, that is

$$\mathbf{S}(t) = \bigotimes_{x\in\mathbb{Z}^d} \mathbf{S}_x(t)$$

$\mathbf{S}_x(t)$ being generated by ℓ_x^Λ. Making use of the spectral representation of the semigroup one gets immediately

$$\langle\alpha|\left[\mathbf{S}(t) - I\right]|\gamma\rangle \le e^{-t} \qquad \forall\alpha,\gamma\subset\mathbb{Z}^d$$

3. Interacting Diffusion

We will write the generator of the Glauber dynamics of an interacting system in our notation. This will probably seem unfamiliar since it looks very different from the usual one (see e.g. [L]). The reader may convince herself that they are one and the same by computing a Dirichlet form for a cylinder function with both.

Definition 3.1. [QMIPS III] *The interaction for a spin system is realized through the Hamiltonian operator*

$$\mathbf{H}_\Lambda = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} J_\alpha \boldsymbol{\sigma}_\alpha$$

where $\forall\alpha\subseteq\Lambda$ J_α *is a real function on the set of subsets of* Λ. *Moreover*

$$\overline{\mathbf{H}}_\Lambda = \mathbf{U}_\Lambda\mathbf{H}_\Lambda\mathbf{U}_\Lambda = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} H_\alpha\,|\alpha\rangle\,\langle\alpha|_\Lambda$$

$$H_\alpha = \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} J_\gamma\,(-1)^{|\alpha\cap\gamma|}$$

Definition 3.2. $\forall\Lambda\subset\mathbb{Z}^d$, *the generator of the stochastic dynamics of an interacting system is*

$$\mathcal{L}_\Lambda(\beta)\in\mathcal{U}_\Lambda : \mathcal{L}_\Lambda(\beta) = e^{\beta\mathbf{H}_\Lambda}\sum_{x\in\Lambda}\ell_x^\Lambda e^{-\beta\mathbf{H}_\Lambda}\ell_x^\Lambda$$

$$\mathcal{L}_\Lambda(\beta=0) = \mathcal{L}_\Lambda$$

The right and the left eigenvectors corresponding to the 0 eigenvalue are respectively

$$|\emptyset\rangle_\Lambda$$

and

$$\left\langle \frac{e^{-\beta\mathbf{H}_\Lambda}}{Z_\Lambda^{(d)}(\beta)}\right| = \langle\emptyset|\,\frac{e^{-\beta\mathbf{H}_\Lambda}}{Z_\Lambda^{(d)}(\beta)} = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\Omega_\alpha^\Lambda(\beta)_\Lambda\,\langle\alpha| \tag{1}$$

where

$$Z_\Lambda^{(d)}(\beta) = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-\beta H_\alpha}$$

is the partition function [QMIPS III] and

$$\Omega_{\alpha}^{\Lambda}(\beta)=\sum_{\emptyset\subseteq\gamma\subseteq\Lambda}\left(\frac{e^{-\beta H_{\gamma}}(-1)^{|\alpha\cap\gamma|}}{Z_{\Lambda}^{d}(\beta)\,2^{|\Lambda|}}\right)\quad\forall\alpha\subseteq\Lambda$$

Our purpose is to give a lower bound estimate for the spectral gap of the generator. We will rewrite the process in a reversible form. This can be done considering the isometry

$$e^{-\frac{\beta}{2}\mathbf{H}_{\Lambda}}:\mathcal{H}_{\Lambda}\rightarrow\mathcal{H}_{\Lambda}^{G}$$

where $\mathcal{H}_{\Lambda}^{G}$ is the Hilbert space of square summable function on $\mathbb{Z}_{2}^{\Lambda}$ with respect to the Gibbs measure, associated with the invariant vector state (1) [QMIPS I]. This map is obtained applying $e^{-\frac{\beta}{2}\mathbf{H}_{\Lambda}}$ to any vector in $\mathcal{H}_{\Lambda}$. Such isometry can be lift to an homomorphism on $\mathcal{U}_{\Lambda}$ preserving the Gibbs state, that is

$$\begin{aligned}\forall\mathbf{A}_{\Lambda}&\in\mathcal{U}_{\Lambda}\qquad\mathbf{A}_{\Lambda}\rightarrow e^{-\frac{\beta}{2}\mathbf{H}_{\Lambda}}\mathbf{A}_{\Lambda}e^{\frac{\beta}{2}\mathbf{H}_{\Lambda}}:=\mathbf{A}_{\Lambda}(\beta)\in\mathcal{U}_{\Lambda}\\ \omega_{\Lambda}^{0}\left(\boldsymbol{\rho}_{\Lambda}^{(d)}(\beta)\,\mathbf{A}_{\Lambda}\right)&=\omega_{\Lambda}^{\beta}(\mathbf{A}_{\Lambda})=\omega_{\Lambda}^{0}(\mathbf{A}_{\Lambda}(\beta))\end{aligned}\tag{2}$$

where

$$\boldsymbol{\rho}_{\Lambda}^{(d)}(\beta)=\frac{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}e^{-\beta H_{\alpha}}\left|\alpha\right\rangle\left\langle\alpha\right|_{\Lambda}}{Z_{\Lambda}^{(d)}(\beta)}$$

is the Gibbs density operator [QMIPS III]. From this follows that the action of $\mathcal{L}_{\Lambda}(\beta):\mathcal{H}_{\Lambda}\rightarrow\mathcal{H}_{\Lambda}$ is equivalent to that of $\mathcal{L}_{\Lambda}^{s}(\beta):\mathcal{H}_{\Lambda}^{G}\rightarrow\mathcal{H}_{\Lambda}^{G}$ where

$$\mathcal{L}_{\Lambda}^{s}(\beta):=e^{-\frac{\beta}{2}\mathbf{H}_{\Lambda}}\mathcal{L}_{\Lambda}(\beta)\,e^{\frac{\beta}{2}\mathbf{H}_{\Lambda}}=e^{\frac{\beta}{2}\mathbf{H}_{\Lambda}}\sum_{x\in\Lambda}\ell_{x}e^{-\beta\mathbf{H}_{\Lambda}}\ell_{x}e^{\frac{\beta}{2}\mathbf{H}_{\Lambda}}$$

Such operator is selfadjoint and has a unique eigenvector associated to the 0 eigenvalue that is

$$\begin{aligned}\left|\Omega_{\Lambda}^{\beta}\right\rangle&=\frac{e^{-\frac{\beta}{2}\mathbf{H}_{\Lambda}}}{\left(Z_{\Lambda}^{(d)}(\beta)\right)^{\frac{1}{2}}}\left|\emptyset\right\rangle_{\Lambda}=\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\omega_{\alpha}^{\Lambda}(\beta)\left|\alpha\right\rangle_{\Lambda}\\ \omega_{\alpha}^{\Lambda}(\beta)&=\left(Z_{\Lambda}^{(d)}(\beta)\right)^{\frac{1}{2}}\Omega_{\alpha}^{\Lambda}\left(\frac{\beta}{2}\right)\end{aligned}$$

Let $\mathbf{S}_{\Lambda}^{\beta}(t)$ be the semigroup generated by $\mathcal{L}_{\Lambda}^{s}(\beta)$. Consider its action on the element of $\mathcal{C}_{\Lambda}$

$$\forall\mathbf{A}_{\Lambda}\in\mathcal{C}_{\Lambda}\qquad\mathbf{A}_{\Lambda}\rightarrow\mathbf{S}_{\Lambda}^{\beta}(t)\,\mathbf{A}_{\Lambda}\mathbf{S}_{\Lambda}^{\beta}(-t):=\mathbf{A}_{\Lambda}(t)\in\mathcal{U}_{\Lambda}\tag{3}$$

such homomorphism obviously preserves the Gibbs state:

$$\left\langle\Omega_{\Lambda}^{\beta}\right|\mathbf{A}_{\Lambda}\left|\Omega_{\Lambda}^{\beta}\right\rangle=\left\langle\Omega_{\Lambda}^{\beta}\right|\mathbf{A}_{\Lambda}(t)\left|\Omega_{\Lambda}^{\beta}\right\rangle$$

From (2) (3) the generic time correlation function now reads

$$\begin{aligned}
&\omega_\Lambda^0\left(\boldsymbol{\rho}_\Lambda^{(d)}(\beta)\,\boldsymbol{\sigma}_\alpha\left[e^{-t\mathcal{L}_\Lambda(\beta)}-\mathbf{I}_\Lambda\right]\boldsymbol{\sigma}_\gamma\right)= \\
&=2^{|\Lambda|}\langle\emptyset|\,\frac{e^{-\beta\mathbf{H}_\Lambda}}{Z_\Lambda^{(d)}(\beta)}\boldsymbol{\sigma}_\alpha\left[e^{-t\mathcal{L}_\Lambda(\beta)}-\mathbf{I}_\Lambda\right]\boldsymbol{\sigma}_\gamma\,|\emptyset\rangle_\Lambda= \\
&=2^{|\Lambda|}\left\langle\Omega_\Lambda^\beta\right|\boldsymbol{\sigma}_\alpha\left[\mathbf{S}_\Lambda^\beta(t)-\mathbf{I}_\Lambda\right]\boldsymbol{\sigma}_\gamma\left|\Omega_\Lambda^\beta\right\rangle= \\
&=\left\langle\overline{\Omega}_\Lambda^\beta\right|\mathbf{f}_\alpha\left[e^{-t\overline{\mathcal{L}}_\Lambda^s(\beta)}-\mathbf{I}_\Lambda\right]\mathbf{f}_\gamma\left|\overline{\Omega}_\Lambda^\beta\right\rangle= \\
&=\langle 0|\,\frac{e^{-\frac{\beta}{2}\overline{\mathbf{H}}_\Lambda}}{\left(Z_\Lambda^{(d)}(\beta)\right)^{\frac12}}\mathbf{f}_\alpha\left[e^{-t\overline{\mathcal{L}}_\Lambda^s(\beta)}-\mathbf{I}_\Lambda\right]\mathbf{f}_\gamma\frac{e^{-\frac{\beta}{2}\overline{\mathbf{H}}_\Lambda}}{\left(Z_\Lambda^{(d)}(\beta)\right)^{\frac12}}\,|0\rangle_\Lambda= \\
&=\frac{\sum_{\emptyset\subseteq\alpha_0,\gamma_0\subseteq\Lambda}e^{-\frac{\beta}{2}\left(H_{\alpha_0}+H_{\gamma_0}\right)}(-1)^{|\alpha_0\cap\alpha|+|\gamma_0\cap\gamma|}}{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}e^{-\beta H_\alpha}}\langle\alpha_0|\left[e^{-t\overline{\mathcal{L}}_\Lambda^s(\beta)}-\mathbf{I}_\Lambda\right]|\gamma_0\rangle_\Lambda
\end{aligned}$$

where $\left|\overline{\Omega}_\Lambda^\beta\right\rangle=\mathbf{U}_\Lambda\left|\Omega_\Lambda^\beta\right\rangle=\frac{1}{2^{|\Lambda|}}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}e^{-\beta H_\alpha}\,|\alpha\rangle_\Lambda$.

Now rewrite $\overline{\mathcal{L}}_\Lambda^s(\beta)$ in the following form

$$\overline{\mathcal{L}}_\Lambda^s(\beta)=\sum_{x\in\Lambda}\bar{\ell}_x^\Lambda(-\beta)\,\bar{\ell}_x^\Lambda(\beta)$$

where from (2)

$$\begin{aligned}
\ell_x^\Lambda(\beta) &= e^{-\frac{\beta}{2}\mathbf{H}_\Lambda}\ell_x e^{\frac{\beta}{2}\mathbf{H}_\Lambda}= \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\beta}{2}\right)^n\left[..\left[\ell_x^\Lambda,\mathbf{H}_\Lambda\right]_1\,]\,]\right]_n
\end{aligned}$$

then

$$\overline{\mathcal{L}}_\Lambda^s(\beta)=\sum_{n=0}^{\infty}\frac{2}{(2n)!!}\left(\frac{\beta}{2}\right)^{2n}\left[..\left[\bar{\ell}_x^\Lambda,\bar{\mathbf{H}}_\Lambda\right]_1\,]\,]\right]_{2n}$$

which means that the operator valued function $\beta\to\overline{\mathcal{L}}_\Lambda^s(\beta)$ on $\mathcal{H}_\Lambda$ is even in β.

Given any $x\in\Lambda$

$$\begin{aligned}
\bar{\ell}_x^\Lambda(-\beta)\,\bar{\ell}_x^\Lambda(\beta) &= \frac14\left\{\left[\mathbf{I}_\Lambda-e^{-\frac{\beta}{2}\bar{\mathbf{H}}_\Lambda}\boldsymbol{\sigma}_x e^{\frac{\beta}{2}\bar{\mathbf{H}}_\Lambda}-e^{\frac{\beta}{2}\bar{\mathbf{H}}_\Lambda}\boldsymbol{\sigma}_x e^{-\frac{\beta}{2}\bar{\mathbf{H}}_\Lambda}+\right.\right. \\
&\left.\left.+e^{\frac{\beta}{2}\bar{\mathbf{H}}_\Lambda}\boldsymbol{\sigma}_x e^{-\beta\bar{\mathbf{H}}_\Lambda}\boldsymbol{\sigma}_x e^{\frac{\beta}{2}\bar{\mathbf{H}}_\Lambda}\right]\right\}
\end{aligned}$$

since

$$\boldsymbol{\sigma}_x\,|\alpha\rangle_\Lambda=\delta(x\in\alpha)\,\boldsymbol{\sigma}_x\,|\alpha\rangle_\Lambda+\delta(x\notin\alpha)\,\boldsymbol{\sigma}_x\,|\alpha\rangle_\Lambda$$

we have, considering any basis vector $|\alpha\rangle_\Lambda \in \mathcal{H}_\Lambda$

$$\begin{aligned}\bar{\ell}_x^\Lambda(-\beta)\,\bar{\ell}_x^\Lambda(\beta)\,|\alpha\rangle_\Lambda &= \frac{1}{4}\Big\{\delta(x\in\alpha)\Big[\mathbf{I}_\Lambda - \boldsymbol{\sigma}_x\Big(e^{\frac{\beta}{2}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right)} + e^{-\frac{\beta}{2}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right)}\Big) + \\ &+ e^{\beta\left(H_\alpha - H_{\alpha\setminus\{x\}}\right)}\Big] + \\ &+\delta(x\notin\alpha)\Big[\mathbf{I}_\Lambda - \boldsymbol{\sigma}_x\Big(e^{\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)} + e^{-\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)}\Big) + \\ &e^{\beta\left(H_\alpha - H_{\alpha\cup\{x\}}\right)}\Big]\Big\}\,|\alpha\rangle_\Lambda\end{aligned}$$

that is

$$\begin{aligned}\bar{\ell}_x^\Lambda(-\beta)\,\bar{\ell}_x^\Lambda(\beta)\,|\alpha\rangle_\Lambda &= \frac{1}{4}\Big\{\delta(x\in\alpha)\Big[\mathbf{I}_\Lambda - 2\boldsymbol{\sigma}_x\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right) + \\ &+ e^{\beta\left(H_\alpha - H_{\alpha\setminus\{x\}}\right)}\Big] + \\ &+\delta(x\notin\alpha)\Big[\mathbf{I}_\Lambda - 2\boldsymbol{\sigma}_x\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right) + \\ &+ e^{\beta\left(H_\alpha - H_{\alpha\cup\{x\}}\right)}\Big]\Big\}\,|\alpha\rangle_\Lambda\end{aligned}$$

from which follows

$$\begin{aligned}\overline{\mathcal{L}}_\Lambda^s(\beta)\,|\alpha\rangle_\Lambda &= \sum_{x\in\Lambda}\frac{1}{4}\big\{\delta(x\in\alpha)\big[\mathbf{I}_\Lambda + \sinh\beta\left(H_\alpha - H_{\alpha\setminus\{x\}}\right) + \cosh\beta\left(H_\alpha - H_{\alpha\setminus\{x\}}\right) + \\ &-\boldsymbol{\sigma}_x 2\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right)\Big] + \delta(x\notin\alpha)\big[\mathbf{I}_\Lambda + \sinh\beta\left(H_\alpha - H_{\alpha\cup\{x\}}\right) + \\ &+\cosh\beta\left(H_\alpha - H_{\alpha\cup\{x\}}\right) - \boldsymbol{\sigma}_x 2\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)\Big]\Big\}\,|\alpha\rangle_\Lambda\end{aligned}$$

Since $\overline{\mathcal{L}}_\Lambda^s(\beta)$ is even in β the last expression reduces to

$$\begin{aligned}\overline{\mathcal{L}}_\Lambda^s(\beta)\,|\alpha\rangle_\Lambda &= \sum_{x\in\Lambda}\frac{1}{4}\big\{\delta(x\in\alpha)\big[\left(1+\cosh\beta\left(H_\alpha - H_{\alpha\setminus\{x\}}\right)\right)\mathbf{I}_\Lambda \\ &-\boldsymbol{\sigma}_x 2\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right)\Big] + \delta(x\notin\alpha)\big[\left(1+\cosh\beta\left(H_\alpha - H_{\alpha\cup\{x\}}\right)\right)\mathbf{I}_\Lambda + \\ &-\boldsymbol{\sigma}_x 2\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)\Big]\Big\}\,|\alpha\rangle_\Lambda\end{aligned}$$

and also, since

$$\begin{aligned}1+\cosh\beta\left(H_\alpha - H_{\alpha\setminus\{x\}}\right) &= 2\cosh^2\frac{\beta}{2}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right) \\ 1+\cosh\beta\left(H_\alpha - H_{\alpha\cup\{x\}}\right) &= 2\cosh^2\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)\end{aligned}$$

we have

$$\begin{aligned}\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)\,|\alpha\rangle_{\Lambda} &= \sum_{x\in\Lambda}\frac{1}{2}\Big\{\delta(x\in\alpha)\Big[\mathbf{I}_{\Lambda}\cosh^{2}\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)-\boldsymbol{\sigma}_{x}\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\Big]+\\ &\quad+\delta(x\notin\alpha)\Big[\mathbf{I}_{\Lambda}\cosh^{2}\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)-\boldsymbol{\sigma}_{x}\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\Big]\Big\}\,|\alpha\rangle_{\Lambda}\end{aligned}$$

that is

$$\begin{aligned}\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)\,|\alpha\rangle_{\Lambda} &= \sum_{x\in\Lambda}\frac{1}{2}\Big\{\delta(x\in\alpha)\Big[\mathbf{I}_{\Lambda}\Big(\cosh^{2}\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)+\\ &\quad-\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\Big)+2\bar{\ell}^{\Lambda}_{x}\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\Big]+\\ &\quad+\delta(x\notin\alpha)\Big[\mathbf{I}_{\Lambda}\Big(\cosh^{2}\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)-\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\Big)+\\ &\quad+2\bar{\ell}^{\Lambda}_{x}\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\Big]\Big\}\,|\alpha\rangle_{\Lambda}\end{aligned}$$

$\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)$ is positive and selfadjoint, moreover all the terms in the right hand side sum are positive. This implies, $\forall\,\langle\gamma|\in\mathcal{H}_{\Lambda}$

$$\begin{aligned}\langle\gamma|\,\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)\,|\alpha\rangle_{\Lambda} &\geq \langle\gamma|\sum_{x\in\Lambda}\frac{1}{2}\Big\{2\bar{\ell}_{x}+\delta(x\in\alpha)\Big[\cosh^{2}\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)+\\ &\quad-\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\Big]\,\mathbf{I}_{\Lambda}\Big\}\,|\alpha\rangle_{\Lambda}\end{aligned}$$

and also $\forall\alpha\subseteq\Lambda$

$$\begin{aligned}&\sum_{x\in\Lambda}\frac{1}{2}\delta(x\in\alpha)\Big[\cosh^{2}\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)-\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\Big]\\ \geq\ &\frac{1}{2}\sup_{x\in\alpha}\Big[\cosh^{2}\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)-\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\Big]\\ \geq\ &-\sup_{x\in\alpha}\sinh^{2}\frac{\beta}{4}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\end{aligned}$$

Then

$$\langle\gamma|\,\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)\,|\alpha\rangle_{\Lambda}\geq\langle\gamma|\left\{\overline{\mathcal{L}}_{\Lambda}-\mathbf{I}_{\Lambda}\sinh^{2}\frac{\beta}{4}\left[\sup_{\alpha\subseteq\Lambda}\sup_{x\in\alpha}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\right]\right\}|\alpha\rangle_{\Lambda}$$

Following the above procedure, we also obtain

$$\langle\gamma|\,\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)\,|\alpha\rangle_{\Lambda}\geq\langle\gamma|\left\{\overline{\mathcal{L}}_{\Lambda}-\mathbf{I}_{\Lambda}\sinh^{2}\frac{\beta}{4}\left[\sup_{\alpha\subseteq\Lambda}\sup_{x\notin\alpha}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\right]\right\}|\alpha\rangle_{\Lambda}$$

Thus $\forall \left|\alpha\right\rangle_{\Lambda}, \left|\gamma\right\rangle_{\Lambda} \in \mathcal{H}_{\Lambda}$

$$\begin{aligned}\left\langle\gamma\right| \overline{\mathcal{L}}^{s}_{\Lambda}(\beta)\left|\alpha\right\rangle_{\Lambda} &\geq \left\langle\gamma\right| \left\{ \overline{\mathcal{L}}_{\Lambda} - \mathbf{I}_{\Lambda}\frac{1}{2}\left[\sinh^{2}\frac{\beta}{4}\left[\sup_{\alpha\subseteq\Lambda}\sup_{x\in\alpha}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\right]+\right.\right.\\ &\left.\left.+\sinh^{2}\frac{\beta}{4}\left[\sup_{\alpha\subseteq\Lambda}\sup_{x\notin\alpha}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\right]\right]\right\}\left|\alpha\right\rangle_{\Lambda}\end{aligned}$$

hence we have the following:

Theorem 3.1. $\forall\Lambda\subset\mathbb{Z}^{d}$, $\forall\beta\in\left[0,\bar{\beta}^{(d)}_{\Lambda}\right]$ $\inf\left[spec\left(\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)\right)\setminus\{0\}\right]$ *is bounded from below by*

$$\begin{aligned}\overline{g}^{(d)}_{\Lambda}(\beta) := 1-\frac{1}{2}\left\{\sinh^{2}\frac{\beta}{4}\left[\sup_{\alpha\subseteq\Lambda}\sup_{x\in\alpha}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\right]+\right.\\ \left.+\sinh^{2}\frac{\beta}{4}\left[\sup_{\alpha\subseteq\Lambda}\sup_{x\notin\alpha}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\right]\right\}\end{aligned}\tag{4}$$

3.0.1. *Proof.* From the last inequality follows

$$\left\langle\alpha\right| I+\epsilon\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)\left|\gamma\right\rangle_{\Lambda} \geq \left\langle\alpha \mid I+\epsilon\bar{\mathbf{D}}_{\Lambda}(\beta)\mid\gamma\right\rangle_{\Lambda}\tag{5}$$

where by definition

$$\bar{\mathbf{D}}_{\Lambda}(\beta)=\mathbf{U}_{\Lambda}\mathbf{D}_{\Lambda}(\beta)\mathbf{U}_{\Lambda}$$

and

$$\begin{aligned}\mathbf{D}_{\Lambda}(\beta) \quad : &= \sum_{\emptyset\neq\alpha\subseteq\Lambda}\left\{\left|\alpha\right|-\frac{1}{2}\left[\sinh^{2}\frac{\beta}{4}\left[\sup_{\alpha\subseteq\Lambda}\sup_{x\in\alpha}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\right]+\right.\right.\\ &\left.\left.+\sinh^{2}\frac{\beta}{4}\left[\sup_{\alpha\subseteq\Lambda}\sup_{x\notin\alpha}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\right]\right]\right\}\left|\alpha\right\rangle\left\langle\alpha\right|_{\Lambda}\end{aligned}$$

Let's set $\overline{g}^{(d)}_{\Lambda}(\beta)$ to be $\inf\left[spec\left(\mathbf{D}_{\Lambda}(\beta)\right)\setminus\{0\}\right]$

$$\begin{aligned}\overline{g}^{(d)}_{\Lambda}(\beta) &= 1-\frac{1}{2}\left\{\sinh^{2}\frac{\beta}{4}\left[\sup_{\alpha\subseteq\Lambda}\sup_{x\in\alpha}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\right]+\right.\\ &\left.+\sinh^{2}\frac{\beta}{4}\left[\sup_{\alpha\subseteq\Lambda}\sup_{x\notin\alpha}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\right]\right\}\end{aligned}$$

$\mathbf{D}_{\Lambda}(\beta)$ is of course selfadjoint and will stay positive for β ranging from 0 to a certain value $\bar{\beta}^{(d)}_{\Lambda}$ such that $\overline{g}^{(d)}_{\Lambda}\left(\bar{\beta}^{(d)}_{\Lambda}\right)=0$. (5) implies

$$\left\langle\alpha\right|\left(I+\epsilon\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)\right)^{-1}\left|\gamma\right\rangle_{\Lambda} \leq \left\langle\alpha\right|\left(I+\epsilon\bar{\mathbf{D}}_{\Lambda}(\beta)\right)^{-1}\left|\gamma\right\rangle_{\Lambda}$$

iterating n times the last inequality we get

$$\left\langle\alpha\right|\left(I+\epsilon\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)\right)^{-n}\left|\gamma\right\rangle_{\Lambda} \leq \left\langle\alpha\right|\left(I+\epsilon\bar{\mathbf{D}}_{\Lambda}(\beta)\right)^{-n}\left|\gamma\right\rangle_{\Lambda}$$

Setting $\epsilon = \frac{t}{n}$, the last expression will be valid at least $\forall t \leq n^{\alpha}$, $\alpha < 1$. By the Hille-Yosida theorem, we obtain

$$\langle \alpha | \, e^{-t\overline{\mathcal{L}}^{s}_{\Lambda}(\beta)} \, |\gamma\rangle_{\Lambda} \leq \langle \alpha | \, e^{-t\bar{\mathbf{D}}_{\Lambda}(\beta)} \, |\gamma\rangle_{\Lambda} \leq 1 + e^{-t\overline{g}^{(d)}_{\Lambda}(\beta)}$$

which proves the theorem. ■

Since (4) involves the difference of two cylindric functions whose supports differ by just one lattice site, it is clear that if $\forall \alpha \subseteq \Lambda$ $H_{\alpha} \propto |\Lambda|^{p}$; $p = 1$ and also $\sup_{\alpha \subseteq \Lambda} \sup_{x \in \alpha} \left(H_{\alpha} - H_{\alpha \setminus \{x\}} \right)$ $\left(\sup_{\alpha \subseteq \Lambda} \sup_{x \notin \alpha} \left(H_{\alpha} - H_{\alpha \cup \{x\}} \right) \right)$ do not depend on α, we would obtain a volume independent expression for the spectral gap of $\mathbf{D}_{\Lambda}(\beta)$; that is $\overline{g}^{(d)}_{\Lambda}(\beta) \rightarrow \overline{g}^{(d)}(\beta)$. A sufficient condition for this argument to be valid is to assume $\mathbf{H}_{\Lambda}(\beta)$ to be traslation invariant. This proves the following

Proposition 3.2. *The spectral gap for the Glauber dynamics of an interacting spin system of the type described above is bounded from below by* $\overline{g}^{(d)}(\beta)$ $\forall \beta \in \left[0, \bar{\beta}^{(d)}\right]$, *where* $\bar{\beta}^{(d)} := \inf \left\{ \beta \geq 0 : \overline{g}^{(d)}\left(\bar{\beta}^{(d)}\right) = 0 \right\}$.

Now we will apply these results to some selected models. For how to treat these models in our framework see [QMIPS III] and reference quoted therein. We will always assume periodic b.c., but it is clear that in the thermodynamic limit, for small β's, the estimate for the gap will not be sensitive w.r.t. b.c.

Example 3.1. Nearest neighbor Ising model without external source. *The eigenvalue of the Hamiltonian operator*

$$\mathbf{H}^{I}_{\Lambda} := - \sum_{\emptyset \subseteq \alpha \subseteq \Lambda} \delta\left(d\left(\alpha\right) = 1\right) \boldsymbol{\sigma}_{\alpha}$$

are

$$H^{I}_{\alpha} = -d\,|\Lambda| + 2\,|\partial\alpha| \qquad \forall \alpha \subseteq \Lambda$$

where $|\partial\alpha|$ *is the area of the surface bounding* α *(the length of the contour surrounding* α *if* $d = 2$*). Then*

$$H^{I}_{\alpha} - H^{I}_{\alpha \setminus \{x\}} = 2\left(|\partial\alpha| - |\partial\left(\alpha \setminus \{x\}\right)|\right)$$
$$H^{I}_{\alpha} - H^{I}_{\alpha \cup \{x\}} = 2\left(|\partial\alpha| - |\partial\left(\alpha \cup \{x\}\right)|\right)$$

where the right hand sides of the above expressions ranges from $-4d$ *to* $4d$*. Thus*

$$\overline{g}^{(d)}_{I}(\beta) = 1 - \sinh^{2}\beta d$$

and

$$\bar{\beta}^{(d)}_{I} = \frac{1}{d}\ln\left(1 + \sqrt{2}\right)$$

Fig.s 1 and 2 show a sketch of $\overline{g}^{(d)}_{I}(\beta)$ as a function of β ($d =, 2, 3, 4$), in comparison to equivalent estimates for *Glauber* ($\overline{g}^{(d)}_{G}(\beta)$) and *heat bath* ($\overline{g}^{(d)}_{hb}(\beta)$) dynamics [L].

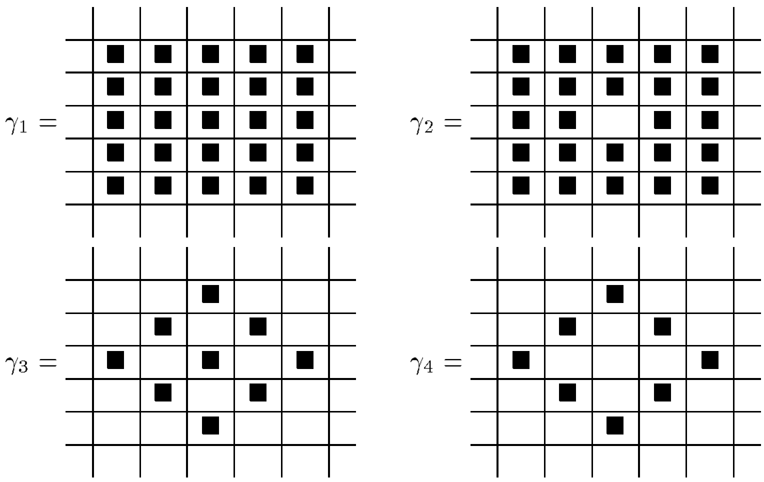
Example 3.2. Ising model with second neighbors antiferromagnetic interaction in the presence of an external source.

$$
\begin{aligned}
\mathbf{H}_{\Lambda}(h,J) &:= -\sum_{\emptyset \subseteq \alpha \subseteq \Lambda} \left[\delta\left(d(\alpha)=1\right)+J_{\alpha}+h\delta\left(d(\alpha)=0\right)\right]\boldsymbol{\sigma}_{\alpha} \\
J_{\alpha} &:= -J\delta\left(\alpha=\alpha_1\cup\alpha_2 : d(\alpha_1)=d(\alpha_2)=0\ ;\ dist(\alpha_1,\alpha_2)=2\right) \\
H_{\gamma}(h,J) &= \left(-d|\Lambda|+2|\partial\gamma|\right)(1-J)-h|\Lambda|+2h|\gamma|-2JY_{\gamma} \\
Y_{\gamma}(J) &:= \sum_{\emptyset \subseteq \alpha \subseteq \Lambda} \delta\left(\alpha=\alpha_1\cup\alpha_2 : d(\alpha_1)=d(\alpha_2)=0\ ;\ dist(\alpha_1,\alpha_2)=2\right)\times \\
&\quad \times\left[\delta\left(\alpha_1 \subset\subset \gamma\right)\delta\left(\alpha_2\cap\gamma=\emptyset\right)+\delta\left(\alpha_2\subset\subset\gamma\right)\delta\left(\alpha_1\cap\gamma=\emptyset\right)\right]
\end{aligned}
$$

where $h, J \geq 0$ *and* $\delta(\alpha_i \subset\subset \gamma) = \delta(\alpha_i \subset \gamma)\,\delta(\alpha_i \cap \partial\gamma = \emptyset)$ $i = 1, 2$. *Then*

$$
\begin{aligned}
H_{\alpha}(h,J)-H_{\alpha\setminus\{x\}}(h,J) &= +2\left(|\partial\alpha|-|\partial(\alpha\setminus\{x\})|\right)(1-J)+ \\
&\quad +2J\left(Y_{\alpha\setminus\{x\}}-Y_{\alpha}\right)+2h \qquad (6)\\
H_{\alpha}(h,J)-H_{\alpha\cup\{x\}}(h,J) &= +2\left(|\partial\alpha|-|\partial(\alpha\cup\{x\})|\right)(1-J)+ \\
&\quad +2J\left(Y_{\alpha\cup\{x\}}-Y_{\alpha}\right)-2h \qquad (7)
\end{aligned}
$$

The maximum value of the r.h.s. of the above expressions is realized, for example, considering the sets sketched below. If $J > 1$ *we set* $\alpha = \gamma_1$, $\alpha\setminus\{x\} = \gamma_2$ *and* $\alpha = \gamma_2$, $\alpha \cup \{x\} = \gamma_1$ *respectively in (6) and in (7). If* $J < 1$ *we set* $\alpha = \gamma_3$, $\alpha\setminus\{x\} = \gamma_4$ *in (6) and* $\alpha = \gamma_4$, $\alpha \cup \{x\} = \gamma_3$ *in (7)*



So we get

$$
\begin{aligned}
\overline{g}^{(d)}_{h,J}(\beta) &= 1-\sinh^2\beta\left[d\left(|J-1|+J\right)+2h\right] \\
\bar{\beta}^{(d)}_{h,J} &= \frac{1}{d\left(|J-1|+J\right)+2h}\ln\left(1+\sqrt{2}\right)
\end{aligned}
$$

Finally if $J = 1$ we have

$$\overline{g}_{h,J}^{(d)}(\beta) = 1 - \sinh^2 \beta (dJ + 2h)$$
$$\bar{\beta}^{(d)}(h, J) = \frac{\ln(1 + \sqrt{2})}{dJ + 2h}$$

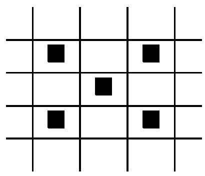
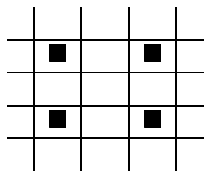
Example 3.3. Dobrushin-Gertsik model. *The lattice dimension is $d = 2$ and the Hamiltonian operator involves a nearest-neighbor interaction term with coupling J_1, a next-nearest neighbor interaction term with coupling J_2 and an external field h.*

$$\begin{aligned}
\mathbf{H}_{\Lambda}^{DG}(h, J_1, J_2) \quad &: \quad = -\sum_{\emptyset \subseteq \alpha \subseteq \Lambda} \left[J_1 \delta (d(\alpha) = 1) + J_\alpha + h\delta (d(\alpha) = 0)\right] \boldsymbol{\sigma}_\alpha \\
J_\alpha \quad &: \quad = J_2 \delta \left(\alpha = \alpha_1 \cup \alpha_2 : d(\alpha_1) = d(\alpha_2) = 0 \ ; \ dist(\alpha_1, \alpha_2) = \sqrt{2}\right) \\
H_{\gamma}^{DG}(h, J_1, J_2) \quad &= \quad (-2|\Lambda| + 2|\partial\gamma|) J_1 + (-2|\Lambda| + 2|\not\partial\gamma|) J_2 + (-|\Lambda| + 2|\gamma|) h \\
|\not\partial\gamma| \quad &: \quad = \sum_{\emptyset \subseteq \alpha \subseteq \Lambda} \delta \left(\alpha = \alpha_1 \cup \alpha_2 : d(\alpha_1) = d(\alpha_2) = 0 \ ; \ dist(\alpha_1, \alpha_2) = \sqrt{2}\right) \times \\
&\qquad \times \delta (|\alpha \cap \gamma| = 1)
\end{aligned}$$

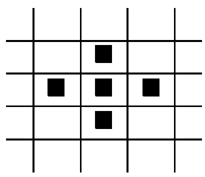
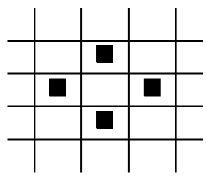
$J_i \in \mathbb{R}$ $i = 1, 2$ and

$$\begin{aligned}
(8)\, H_{\alpha}^{DG}(h, J_1, J_2) - H_{\alpha \setminus \{x\}}^{DG}(h, J_1, J_2) \quad &= \quad 2J_1 (|\partial\alpha| - |\partial (\alpha \setminus \{x\})|) + \\
&\quad +2J_2 (|\not\partial\alpha| - |\not\partial (\alpha \setminus \{x\})|) + 2h \\
(9) H_{\alpha}^{DG}(h, J_1, J_2) - H_{\alpha \cup \{x\}}^{DG}(h, J_1, J_2) \quad &= \quad 2J_1 (|\partial\alpha| - |\partial (\alpha \cup \{x\})|) + \\
&\quad +2J_2 (|\not\partial\alpha| - |\not\partial (\alpha \cup \{x\})|) - 2h
\end{aligned}$$

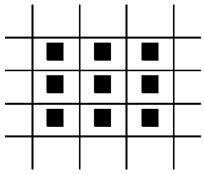
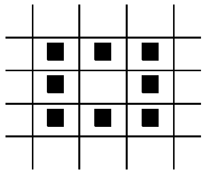
The maximum value of the previous expressions is obtained considering, for example, sets of the following type. If $J_1 > 0$; $J_2 < 0$ we set $\alpha = \gamma_1$, $\alpha \setminus \{x\} = \gamma_2$ in (8) and $\alpha = \gamma_2$, $\alpha \cup \{x\} = \gamma_1$ in (9)

$\gamma_1 =$  $\gamma_2 =$ 

If $J_1 < 0$; $J_2 > 0$ we set $\alpha = \gamma_3$, $\alpha \setminus \{x\} = \gamma_4$ in (8) and $\alpha = \gamma_4$, $\alpha \cup \{x\} = \gamma_3$ in (9)

$\gamma_3 =$  $\gamma_4 =$ 

If $J_1, J_2 < 0$ we set $\alpha = \gamma_5$, $\alpha \backslash \{x\} = \gamma_6$ in (8) and $\alpha = \gamma_6$, $\alpha \cup \{x\} = \gamma_5$ in (9)

$\gamma_5 =$  $\gamma_6 =$ 

while if $J_1, J_2 > 0$ we set α and $\alpha \cup \{x\}$ as a singleton in both (8) and (9). Finally we obtain

$$\begin{aligned} \overline{g}^{(d)}_{DG}(\beta, h, J_1, J_2) &= 1 - \sinh^2 2\beta\left(|J_1| + |J_2| + h\right) \\ \bar{\beta}^{(d)}_{DG}(h, J_1, J_2) &= \frac{\ln\left(1+\sqrt{2}\right)}{2\left(|J_1| + |J_2| + h\right)} \end{aligned}$$

4. Conclusions

Clearly our bounds is not very meaningful in dimension one, since we have ergodicity for any positive β [L]. This is due to the fact that in our approach dynamics is written in form of a Markov chain whose space state is the set of the parts of Λ, $\mathfrak{P}_\Lambda$. Therefore, the dependence of the generator of the process on the lattice dimension, is hidden in the cardinality of the state space. This is also why the preceding analysis works better in higher dimensions.

5. Acknowledgments

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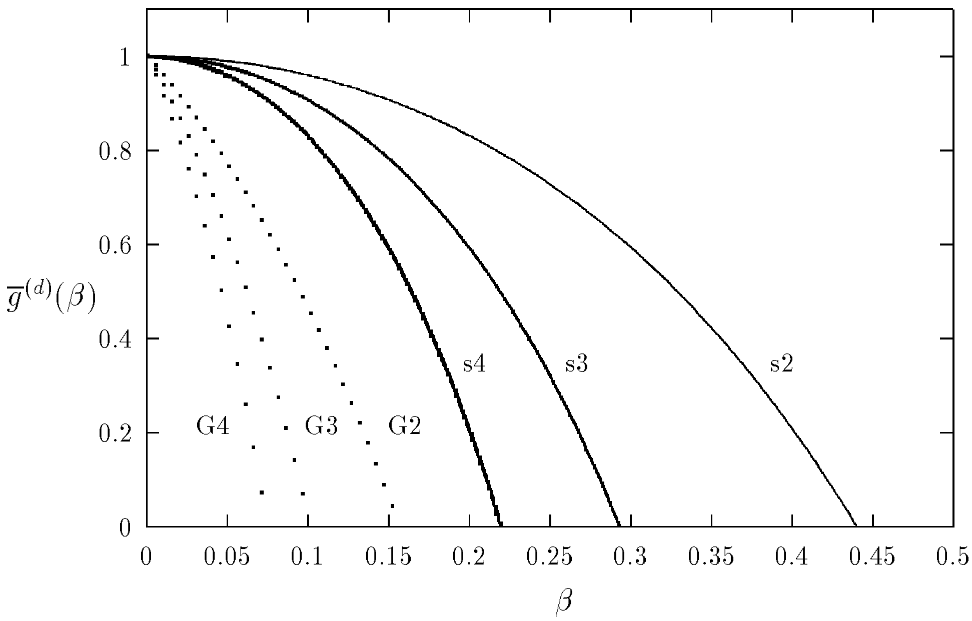

Figure 1:

$$\mathrm{G}d: \quad {\overline{g}_G}^{(d)}(\beta) = 1 - de^{2d\beta}(1 - e^{-2\beta})$$

$$\mathrm{s}d: \quad {\overline{g}_I}^{(d)}(\beta) = 1 - sinh^2(d\beta)$$

$$d = 2, 3, 4$$

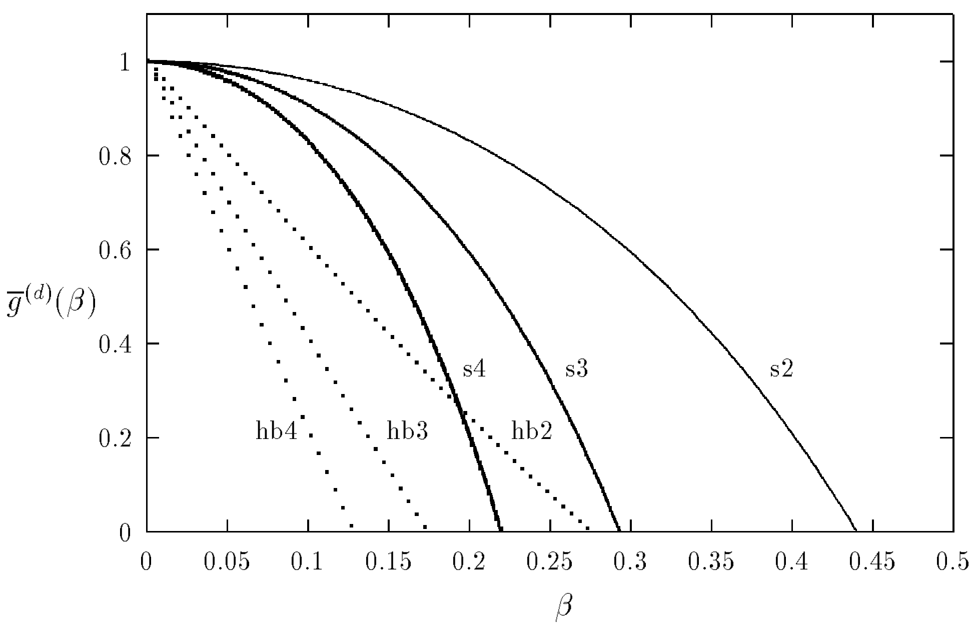

Figure 2:

$$\text{hb}d: \quad \overline{g}_{hb}{}^{(d)}(\beta) = 1 - d tanh(2\beta)$$

$$\text{s}d: \quad \overline{g}_I{}^{(d)}(\beta) = 1 - sinh^2(d\beta)$$

$$d = 2, 3, 4$$