THE GINZBURG-LANDAU EQUATION II. THE ENERGY OF VORTEX CONFIGURATIONS *

by

Yu. N. Ovchinnikov and I.M. Sigal

Abstract. We consider the Ginzburg-Landau equation in dimension two. We introduce a key notion of the energy of vortex configurations. It is defined by minimizing the renormalized Ginzburg-Landau (free) energy introduced in the previous paper over functions with a given set of zeros of given local indices. This notion allows us to define the vortex interaction and vortex Hamiltonian in a canonical way. We find asymptotic behaviour of this energy as the intervortex distances grow. To this end we use several novel techniques.

Keywords: Ginzburg-Landau equation, energy functional, vortex, vortex interaction, vortex Hamiltonian.

1. Introduction

The Ginzburg-Landau equation in various dimensions and for various internal symmetries plays a key rôle in condensed matter and particle physics. This equation has the form

$$-\Delta\psi + (|\psi|^2 - 1)\psi \;=\; 0\ , \tag{1.1}$$

with the boundary condition

$$|\psi(x)| \to 1 \qquad \text{as } |x| \to \infty\ . \tag{1.2}$$

It is a remarkable equation. While being one of the simplest non-linear equations, its solutions typically have a fairly complex structure.

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In this paper we study (1.1)–(1.2) in the simplest case ψ: $\mathbb{R}^2 \to \mathbb{C}$. Physically, this case is realized in thin films and high-temperature superconductors. The latter often have a layer structure with weak coupling between layers. Thus in the first approximation the layers can be considered as independent. In the case of superconductors the Ginzburg-Landau equation is coupled to a magnetic field, but in many situations the latter can be neglected which leads to Eqns (1.1)–(1.2). Moreover, many elements of the analysis of those equations are independent of whether the magnetic field is present or not.

Due to topological properties of the set of functions involved (i.e. the set of vector fields on $\mathbb{R}^2$ converging to unit vector fields at ∞) the solutions have a vortex structure. In the time dependent case it resembles the vortex structure of the Navier-Stokes and Euler equations. In the first paper of this series, [OSa], we studied single vortex solutions of (1.1)–(1.2). The present paper is devoted to static properties of multi-vortex configurations induced by this equation. The results of this paper are used in [OSb] in order to find multivortex solutions (obeying certain discrete symmetries). In [OSc] we use those results as a stepping stone in the description of the dynamics of vortices induced by solutions of the time-dependent Ginzburg-Landau equations.

Solutions of Eqns (1.1)–(1.2) are classified by the total index (winding number) of ψ, considered as a vector field on $\mathbb{R}^2$, at ∞, i.e.

$$\deg \psi \ := \ \frac{1}{2\pi} \int_{|x|=R} d(\arg \psi) \tag{1.3}$$

for R sufficiently large. We call this index (as opposed to local indices of ψ considered below) the *degree* (or *total vorticity*) of ψ.

It is shown in [H, OSa] that for any any n, Eqn (1.1) has a solution, unique modulo symmetry transformations, of the form

$$\psi^{(n)}(x) \ = \ f^{(n)}(r) e^{in\theta} \ , \tag{1.4}$$

where $1 > f^{(n)} \geq 0$ and is monotonically increasing from $f^{(n)}(0) = 0$ to 1 as r increases to ∞. Of course, $\deg \psi^{(n)} = n$. For $n = 0$, $f^{(n)}(r) = 1$. These are the most symmetric imaginable solutions to (1.1), called the *n-vortices.* n is the *degree* (or *vorticity*) of the vortex $\psi^{(n)}$. Of course, each solution $\psi^{(n)}$ generates a one-parameter for $n = 0$ and a three-parameter for $|n| > 0$ family of solutions of (1.1). The latter are obtained by applying symmetry transformations to $\psi^{(n)}$.

The Ginzburg-Landau equation is the Euler-Lagrange equation for the renormalized Ginzburg-Landau energy functional, $\mathcal{E}_{\text{ren}}(\psi)$ (introduced in [OSa] and described in Section 2). "Low" energy functions $\psi : \mathbb{R}^2 \to \mathbb{C}$ are essentially determined by their vortex structure, i.e. by their zeros and their local indices. We call a collection of these data a *vortex configuration.* More precisely, consider once differentiable functions ψ: $\mathbb{R}^2 \to \mathbb{C}$ satisfying $|\psi| \to 1$ as $|x| \to \infty$. Let $\underline{a} = (a_1, \ldots, a_K)$ and $\underline{n} = (n_1, \ldots, n_K)$, where $a_j \in \mathbb{R}^2$ and $n_j \in \mathbb{Z}$, $j = 1, \ldots, K$. We say that ψ has the vortex configuration $\underline{c} = (\underline{a}, \underline{n})$, and write $\operatorname{conf} \psi = \underline{c}$, if ψ has zeros (only) at $a_1, \ldots, a_K$ with local indices $n_1, \ldots, n_K$, respectively, i.e.

$$\int_{\gamma_j} d(\arg \psi) \; = \; 2\pi n_j$$

for any contour γ_j containing a_j, but not the other zeros of ψ and for $j = 1, \ldots, K$. Now we define

$$E(\underline{c}) \; = \; \inf \left\{ \mathcal{E}_{\text{ren}}(\psi) \mid \operatorname{conf} \psi = \underline{c} \right\} . \tag{1.5}$$

By property (c) of section 2, $E(\underline{c}) > -\infty$. We call $E(\underline{c})$ the *energy of the vortex configuration $\underline{c}$.* The force acting on a vortex configuration is $-\nabla_{\underline{a}} E(\underline{c})$. We suggest

Conjecture. *Problem (1.5) has a minimizer (and consequently, Eqns (1.1)–(1.2) have a solution with the vortex configuration $\underline{c}$) if and only if $\nabla_{\underline{a}} E(\underline{c}) = 0$*

In this paper we prove, with some mild extra assumptions, the "only if" part of this conjecture (see Section 3) and in [OSb] we present arguments in favour of the "if" part.

In Sections 4–6 we find the following asymptotics

$$E(\underline{c}) \; = \; \sum_{i=1}^{K} E_{n_i} + H(\underline{c}) + \text{Rem} \; , \tag{1.6}$$

as $r(\underline{a}) := \min_{i \neq j} |a_{ij}| \to \infty$. Here $a_{ij} = a_i - a_j$, $\underline{c} = (\underline{a}, \underline{n})$, $E_n = \mathcal{E}_{\text{ren}}(\psi^{(n)})$, the self-energy of the n-vortex,

$$H(\underline{c}) = -\pi \sum_{i \neq j} n_i n_j \ln |a_{ij}| , \tag{1.7}$$

the intervortex energy, and

$$\text{Rem} = \begin{cases} O\big(r(\underline{a})^{-2}\big) & \text{if } \nabla_{\underline{a}} H(\underline{c}) = 0 \\ O\big(r(\underline{a})^{-1}\big) & \text{otherwise.} \end{cases} \tag{1.8}$$

Eqn (1.6) can be tested as follows. Let a configuration $\underline{c} = (\underline{a}, \underline{n})$ correspond to distant vortices, i.e. $r(\underline{a}) := \min_{i \neq j} |a_{ij}| \gg 1$. Then we expect that the following function

$$\psi_{(0)}(x) = \prod_{i=1}^{K} \psi^{(n_i)}(x - a_i) ,$$

describing the K "independent" vortices, has the energy, $\mathcal{E}_{\text{ren}}(\psi_{(0)})$, close to $E(\underline{c})$. That this is indeed the case follows from our analysis in Section 4.

We demonstrate (1.6)–(1.8) by establishing upper and lower bounds. To prove the upper bound we use that $E(\underline{c}) \leq \mathcal{E}_{\text{ren}}(\psi)$ for any ψ with $\operatorname{conf} \psi = \underline{c}$ and show that for a certain class of ψ's (roughly, those which look like $\psi^{(n_j)}(x - a_j)$ for $|x - a_j| \ll r(\underline{a})$), $\mathcal{E}_{\text{ren}}(\psi) = $ r.h.s. (1.6). To the latter end we decompose the integral in $\mathcal{E}_{\text{ren}}(\psi)$ into the integrals over the discs $D_j = \{x \in \mathbb{R}^2 \mid |x - a_j| \leq r_0\}$, $j = 1, \ldots, K$, and the rest $\mathbb{R}^2 \setminus \bigcup_{j=1}^{K} D_j$ and estimate of each integral accordingly.

The lower bound, $E(\underline{c}) \geq$ r.h.s. (1.6), is more difficult. To prove it we consider a system with impurities:

$$\mathcal{E}_{\underline{\lambda}}(\psi) = \mathcal{E}_{\text{ren}}(\psi) + \sum_{j=1}^{K} \frac{\lambda_j}{2} \int \delta_{b_j} |\psi|^2 , \tag{1.9}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_K)$, $\lambda_j > 0$, are coupling constants of impurities and $\delta_{b_j} \geq 0$ are their potentials which we take to be $\delta_b(x) = \frac{1}{2\pi \bar{r}} \delta(|x - b| - \bar{r})$ with $\bar{r} = O(1)$, or a smooth version of this. We place the centers, b_j, of the impurities close to the vortex centers a_j. We argue that for $\lambda_j \geq \text{const} |\nabla_{a_j} E(\underline{c})|$ $\forall j$, the energy functional $\mathcal{E}_{\underline{\lambda}}(\psi)$ has a minimizer, $\psi_{\underline{\lambda}}$, in the class of ψ's with conf $\psi = \underline{c}$. Since we can insert in the r.h.s. of (1.5) the condition $|\psi| \leq 1$ without changing the result, we have

$$E(\underline{c}) \;\geq\; E_{\underline{\lambda}}(\underline{c}) - \sum_{j=1}^{K} \lambda_j \;, \tag{1.10}$$

where $E_\lambda(\underline{c}) = \mathcal{E}_{\underline{\lambda}}(\psi_{\underline{\lambda}})$.

On the second step, using the Euler-Lagrange equation,

$$-\Delta\psi + (|\psi|^2 - 1)\psi \;=\; -\Sigma\lambda_j \delta_{b_j}\psi \;, \tag{1.11}$$

for $\psi_{\underline{\lambda}}$, we show that $\psi_{\underline{\lambda}}$ belongs to the class of functions used in the proof of the upper bound. This completes the proof of the lower bound and therefore of (1.6).

Equation (1.11) is rather subtle. First of all there is no single reference, or unperturbed, equation for it, but rather several ones, each for a different region of $\mathbb{R}^2$. For instance, in the disc D_j the reference equation is obtained by linearizing (1.11) around the j-th vortex, $\psi^{(n_j)}(x - a_j)$. Secondly, such linearizations, besides the continuous spectrum going all the way to 0 (i.e. filling in $[0, \infty)$), have zero modes due to the spontaneous breaking of translational symmetry. Of course, perturbations due to the remaining vortices break the translational symmetry of the equation and as a result the zero mode is lifted. However, for large intervortex distances, $r(\underline{a}) \gg 1$, the translational symmetry is broken only weakly which manifests itself in the presence of zero quasimodes, i.e. in the Green's functions for the linearized operators in question being very large.

To overcome these two difficulties we, on the one hand, solve Eqn (1.11) (by perturbation theory) separately in each region, D_j, $j = 0, \ldots, K$, where $D_0 = \{x \in \mathbb{R}^2 \mid |x - a_j| \geq r_1\}$ and, recall, $D_j = \{x \in \mathbb{R}^2 \mid |x - a_j| \leq r_0\}$ for $j = 1, \ldots, K$, $r_0 \ll r(\underline{a})$ and $r_1 \gg 1$, and then patch the solutions from different regions together. On the other hand, we adjust the strength, λ_j, and location, b_j, of impurities in such a way that the r.h. sides of the resulting local linearized equations are orthogonal to the corresponding zero translational modes. This way we remove small denominators or secular terms so that the perturbation theory makes sense.

2. Renormalized Ginzburg-Landau energy

It is a straightforward observation that Eqn (1.1) is the equation for critical points of the following functional

$$\mathcal{E}(\psi) = \frac{1}{2}\int\left(|\nabla\psi|^2 + \frac{1}{2}(|\psi|^2-1)^2\right) . \tag{2.1}$$

Indeed, if we define the variational derivative, $\partial_\psi\mathcal{E}(\psi)$, of $\mathcal{E}$ by

$$\mathrm{Re}\int \xi\partial_\psi\mathcal{E}(\psi) = \frac{\partial}{\partial\lambda}\mathcal{E}(\psi_\lambda)\Big|_{\lambda=0} \tag{2.2}$$

for any path ψ_λ s.t. $\psi_0 = \psi$ and $\frac{\partial}{\partial\lambda}\psi_\lambda\big|_{\lambda=0} = \xi$, then the l.h.s. of Eqn (1.1) is equal to $\overline{\partial_\psi\mathcal{E}(\psi)} = \partial_{\bar\psi}\mathcal{E}(\psi)$ for $\mathcal{E}(\psi)$ given in (2.1).

(2.1) is the celebrated Ginzburg-Landau (free) energy. However, there is a problem with it in our context. It is shown in [OSa] that if ψ is an arbitrary C^1 vector field on $\mathbb{R}^2$ s.t. $|\psi| \to 1$ as $|x| \to \infty$ uniformly in $\hat{x} = \frac{x}{|x|}$ and $\deg\psi \neq 0$, then $\mathcal{E}(\psi) = \infty$.

We renormalize the Ginzburg-Landau energy functional as follows (see [OSa]). Let $\chi(x)$ be a smooth real function on $\mathbb{R}^2$ s.t.

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \geq 2, \\ 0 & \text{for } |x| \leq 1 \end{cases} \tag{2.3}$$

Define

$$\mathcal{E}_{\mathrm{ren}}(\psi) = \frac{1}{2}\int\left(|\nabla\psi|^2 - \frac{(\deg\psi)^2}{r^2}\chi + F(|\psi|^2)\right)d^2x \tag{2.4}$$

where

$$F(u) = \frac{1}{2}(u-1)^2 . \tag{2.5}$$

We list here the most important properties of $\mathcal{E}_{\mathrm{ren}}(\psi)$ (see [OSa] for the proofs):

(a) $\partial_{\bar\psi}\mathcal{E}_{\mathrm{ren}}(\psi) = -\Delta\psi + F'(|\psi|^2)\psi$.

(b) Given n let $M_n = \Big\{\psi = fe^{i\varphi} \mid \int\limits_{|x|\geq 2} \frac{1}{r^2}|1-f^2| < \infty$, f is continuous and $f(0) = 0$, $\int |\nabla(\varphi - n\theta)|r^{-1} < \infty$ and $\int|\nabla(\varphi-n\theta)|^2 < \infty\Big\}$. Then $\mathcal{E}_{\mathrm{ren}}(\psi) < \infty \; \forall\psi \in M_n$.

(c) We have the following bound from below

$$\mathcal{E}_{\rm ren}(\psi) \;\geq\; \mathcal{E}_{B(0,2)}(\psi) + \frac{1}{2}\int_{|x|\geq 2}\Big(\big|\nabla|\psi|\big|^2 - \frac{1}{2}|\nabla\varphi|^4\Big)\,d^2x\ , \tag{2.6}$$

where $\varphi = \arg\psi$, and for $\Omega \subset \mathbb{R}^2$,

$$\mathcal{E}_\Omega(\psi) \;=\; \frac{1}{2}\int_\Omega \Big(|\nabla\psi|^2 - \frac{(\deg\psi)^2}{r^2}\chi + F(|\psi|^2)\Big)\,d^2x\ . \tag{2.7}$$

3. The energy of vortex configurations

In this section we discuss the connection between $-\nabla E(\underline{c})$, the force acting on the vortex centers, and the existence of a minimizer for variational problem (1.5). It is clear intuitively that such a minimizer exists if and only if $\nabla E(\underline{a}) = 0$. However, to establish this fact is not so easy. In what follows $\underline{n}$ is fixed and we use the notation $E(\underline{a}) = E(\underline{c})$ and $H(\underline{a}) = H(\underline{c})$ for $\underline{c} = (\underline{a}, \underline{n})$. We begin our analysis with

Proposition 3.1. *If there is a minimizer for variational problem (1.5), then this minimizer satisfies Ginzburg-Landau equation (1.1).*

Proof. Let ψ be a minimizer for (1.5). Since for any differentiable function $\xi\colon\ \mathbb{R}^2 \to \mathbb{C}$ vanishing together with its gradient sufficiently fast at ∞ and vanishing at the points $a_1, \ldots, a_m$ we have

$$\begin{aligned} 0 \;&=\; \frac{\partial}{\partial\lambda}\mathcal{E}_{\rm ren}(\psi + \lambda\xi)\Big|_{\lambda=0} \\ &=\; \mathrm{Re}\int \bar{\xi}\big(-\Delta\psi + (|\psi|^2 - 1)\psi\big)\ , \end{aligned}$$

we conclude that ψ satisfies (1.1) for $x \neq a_1, \ldots, a_m$. On the other hand, since $\psi \in H_1^{\rm loc}(\mathbb{R}^2)$, we have that $-\Delta\psi + (|\psi|^2 - 1)\psi \in H_{-1}^{\rm loc}(\mathbb{R}^2)$. Hence $-\Delta\psi + (|\psi|^2 - 1)\psi = 0$ on $\mathbb{R}^2$. □

We assume that the function $E(\underline{a})$ is differentiable and that there are approximate minimizers $\psi_{\underline{a}}^{(\varepsilon)}$ s.t. $\nabla\mathcal{E}_{\rm ren}(\psi_{\underline{a}}^{(\varepsilon)}) \to \nabla E(\underline{a})$ as $\varepsilon \to 0$, pointwise in $\underline{a}$. Then we have

Theorem 3.2. *Let $\nabla E(\underline{a}) \neq 0$. Then variational problem (1.5) has no minimizer.*

Proof. Assume, on the contrary, that problem (1.5) has a minimizer, $\psi_{\underline{a}}$. By Proposition 3.1 it solves (1.1) and therefore is a critical point of the functional $\mathcal{E}_{\mathrm{ren}}(\psi)$. Assume first that there is a direction $\underline{e}$ in $\mathbb{R}^{2n}$ and a path $\underline{a}(t)$, $0 \leq t \leq \varepsilon$, for some $\varepsilon > 0$ starting at $\underline{a}$ in the direction $\underline{e}$ s.t. $\underline{e} \cdot \nabla E(\underline{a}) \neq 0$ and problem (1.5) has minimizers, $\psi_{\underline{a}(t)}$, for the points $\underline{a}(t)$. Then

$$\frac{d}{dt}\mathcal{E}_{\mathrm{ren}}(\psi_{\underline{a}(t)})\Big|_{t=0} \;=\; \underline{e} \cdot \nabla E(\underline{a}) \;\neq\; 0 \;,$$

which contradicts to the statement that $\psi_{\underline{a}}$ is a critical point of $\mathcal{E}_{\mathrm{ren}}(\psi)$. If (1.5) has no minimizers for any curve $\underline{a}(t)$, $0 < t \leq \varepsilon$, s.t. $a(0) = \underline{a}$ and $\dot{\underline{a}}(0) \cdot \nabla E(\underline{a}) \neq 0$, then we pick approximate minimizers in accordance with the condition above and proceed as in the argument above. $\square$

We conjecture that the assumption formulated above are always satisfied. (Approximate minimizers which we expect satisfy it are constructed in section 5 by a method of impurities.) In any case, the proof above shows that minimizers of (1.5) can be located only on a discrete set of level sets of the function $E(\underline{a})$.

Sometimes it is more convenient for us to work with the intervortex energy in the disc D_R defined as

$$E_R(\underline{a}) \;=\; \inf\{\mathcal{E}_R(\psi) \mid \psi \text{ has } \underline{n} \text{ vortices at } \underline{a} \text{ and } |\psi|\big|_{\partial D_R} = 1\} \tag{3.1}$$

instead of $E(\underline{a})$. Here ψ are differentiable functions from D_R to $\mathbb{C}$.

4. Asymptotics of energy of vortex configurations. Upper bound

In this section we study asymptotics of the energy, $E_R(\underline{a})$, of vortex configurations $(\underline{a}, \underline{n})$ as $r(\underline{a}) \to \infty$. Recall that $r(\underline{a}) = \min\limits_{i \neq j} |a_{ij}|$. Our main result is the following relation

$$E_R(\underline{a}) \;=\; E_R^{(0)} + \mathrm{Rem} + O\Big(\frac{1}{R^2}\Big) \;, \tag{4.1}$$

where

$$E_R^{(0)} \;=\; \sum_i E_{n_i,R} + H\Big(\frac{\underline{a}}{R}\Big) \tag{4.2}$$

and the remainder, Rem, satisfies the estimate

$$\text{Rem} \;=\; \begin{cases} O\big(r(\underline{a})^{-2}\big) & \text{if } \nabla H(\underline{a}) = 0 \\ O\big(r(\underline{a})^{-1}\big) & \text{otherwise} \end{cases} \tag{4.3}$$

We demonstrate (4.1) by verifying that its r.h.s. is an upper and lower bound for the l.h.s. The upper bound is obtained in this section, while the lower one, in the next one.

Upper bound. We prove the upper bound by using the variational inequality

$$E_R(\underline{a}) \;\le\; \mathcal{E}_R(\psi)\ , \tag{4.4}$$

for any function ψ having the given vortex configuration, and by showing that for an appropriate ψ, $\mathcal{E}_R(\psi)$ has the asymptotics given by the r.h.s. of (4.1).

Consider a class of functions ψ satisfying

$$\begin{gathered}\psi = \psi_i e^{i\varphi_{(i)}} + O\Big(\frac{1}{r\cdot r(\underline{a})^n}\Big) \text{ and } \int_0^{2\pi} \mathrm{Re}(e^{-i\varphi_0}\psi - f_i)d\theta = O\Big(\frac{1}{r(\underline{a})^{n+1}}\Big)\ , \\ \text{if } |x-a_i| \ll r(\underline{a}),\end{gathered} \tag{4.5}$$

where θ is the polar angle of $x - a_i$, $\forall\, i$, and

$$\psi \;=\; e^{i\varphi_0} + O\Big(\frac{1}{|x-\underline{a}|^2}\Big) \qquad \text{if } d(x,\underline{a}) \gg 1\ , \tag{4.6}$$

with the corresponding estimates of their first derivatives, where $n = 2$ if $\nabla H(\underline{a}) = 0$ and $n = 1$ otherwise and where $\psi_i(x) = \psi_{n_i}(x - a_i)$, $f_i = |\psi_i|$, $\varphi_i(x) = n_i\theta(x - a_i)$,

$$\varphi_{(i)} \;=\; \sum_{j,j\neq i} \varphi_j\ , \qquad \varphi_0 \;=\; \sum_j \varphi_j\ , \tag{4.7}$$

and

$$d(x,\underline{a}) \;=\; \min_j |x - a_j|\ .$$

An example of such a function is $\psi_0 = f_0 e^{i\varphi_0}$, where φ_0 is as above and f_0 is the minimizer for the functional $\mathcal{E}_{\text{ren}}(f) := \mathcal{E}_{\text{ren}}(fe^{i\varphi_0})$ (one can show that such a minimizer exists).

Proposition 4.1. *Assume ψ satisfies (4.5)–(4.6). Then*

$$\mathcal{E}_R(\psi) \;=\; E_R^{(0)} + \mathrm{Rem} + O\Big(\frac{1}{R^2}\Big)\;, \tag{4.8}$$

where, recall, $E_R^{(0)}$ is given by (4.2) and

$$\mathrm{Rem} \;=\; \begin{cases} O\big(r(\underline{a})^{-2}\big) & \text{if } \nabla H(\underline{a}) = 0 \\ O\big(r(\underline{a})^{-2}\ln\, r(\underline{a})\big) & \text{otherwise} \end{cases} \tag{4.9}$$

Proof. Let $D_j = D(a_j, r_0)$, the disc with the center at a_j and of the radius r_0. We specify r_0 later. We decompose the energy functional as

$$\mathcal{E}_R(\psi) \;=\; \sum_j \int_{D_j} e(\psi) + \int_{D_R\setminus\cup D_j} e(\psi)\;, \tag{4.10}$$

where $e(\psi)$ is the energy density,

$$e(\psi) \;=\; \frac{1}{2}|\nabla\psi|^2 + \frac{1}{4}(|\psi|^2-1)^2\;.$$

Let $e_1(\varphi) = \frac{1}{2}|\nabla\varphi|^2$ and $\langle f(\psi)\rangle = f(\psi) - \sum\limits_k f(\psi_k)$. Eqn (4.6) implies

$$\int_{D_R\setminus\cup D_k} e(\psi) \;=\; \int_{D_R\setminus\cup D_k} e_1(\varphi_0) + \int_{D_R\setminus\cup D_k} O(|x-\underline{a}|^{-4})\;. \tag{4.11}$$

Next, the estimates

$$|\psi_i| \;=\; 1 + O(|x-a_i|^{-2}) \tag{4.12}$$

and

$$\nabla|\psi_i| \;=\; O(|x-a_j|^{-3}) \tag{4.13}$$

give

$$\int_{D_R\setminus\cup D_k} e_1(\varphi_i) \;=\; \int_{D_R\setminus\cup D_k} e(\psi_i) + O(r_0^{-2})\;. \tag{4.14}$$

This together with Eqn (4.11) yields

$$\int_{D_R\setminus\cup D_k} \langle e(\psi)\rangle \;=\; \frac{1}{2}\sum_{i\neq j}\int_{D_R\setminus\cup D_k} \nabla\varphi_i\nabla\varphi_j + O(r_0^{-2})\;. \tag{4.15}$$

Next, we write ψ in the region D_i as $\psi = e^{i\varphi_0}(f_i + \xi)$, where $f_i \equiv |\psi_i|$. As a single valued harmonic function in D_i, $\varphi_{(i)}$ has the following expansion

$$\varphi_{(i)} \;=\; \sum_{m=0}^{\infty} c_m r^m \cos m(\theta - \theta_i^{(m)}) \ ,$$

where θ is the polar angle of $x - a_j$ and c_m and $\theta^{(m)}$ are some constants. This implies that

$$\int_{D_i} \nabla\varphi_i \cdot \nabla\varphi_{(i)} \;=\; 0 \ .$$

Using this relation and that

$$\int_{D_j} f_j \nabla\varphi_j \cdot \nabla\,\mathrm{Im}\,\xi \;=\; n_j \int_{D_j} f_j \frac{\partial}{\partial\theta}\mathrm{Im}\,\xi \;=\; 0 \ ,$$

we obtain

$$\int_{D_i} e(\psi) \;=\; \int_{D_i} e(\psi_i) + \int_{D_i} e_1(\varphi_{(i)}) + R_1 + R_2 \ ,$$

where

$$R_1 \;=\; \int_{D_i} (f_i^2 - 1)\Big(\nabla\varphi_i \cdot \nabla\varphi_{(i)} + \frac{1}{2}|\nabla\varphi_{(i)}|^2\Big) \ ,$$

and

$$\begin{aligned} R_2 \;=\; \int_{D_i} & \{(|\nabla\varphi_0|^2 + f_i^2 - 1)f_i\mathrm{Re}\,\xi + f_i^2(\mathrm{Re}\,\xi)^2 \\ & + \frac{1}{2}|\nabla\varphi_0|^2|\xi|^2 + \frac{1}{2}|\nabla\xi|^2 + 2\nabla f_i \cdot \nabla\mathrm{Re}\,\xi + f_i\nabla\varphi_{(i)} \cdot \nabla\mathrm{Im}\,\xi \\ & + \mathrm{Im}(\xi\nabla\varphi_0 \cdot \nabla\xi) + \frac{1}{2}(f_i^2 - 1 + 2f_i\mathrm{Re}\,\xi)|\xi|^2 + \frac{1}{4}\,|\xi|^4\Big\} \ . \end{aligned}$$

Expanding

$$\nabla\varphi_i \;=\; \nabla\varphi_i(a_i) + O\Big(\frac{|x - a_i|}{r(\underline{a})^2}\Big)$$

and using that $\int_{D_i}(1 - f_i^2)\nabla\varphi_i = 0$, we obtain that

$$R_1 \;=\; O\Big(\frac{\ln\, r_0}{r(\underline{a})^2}\Big) \ .$$

Using that, due to (4.5)–(4.6), $\xi = O\Big(\frac{1}{r \cdot r(\underline{a})}\Big)$ and $\int_0^{2\pi} \mathrm{Re}\,\xi\, d\theta = O\Big(\frac{1}{r(\underline{a})^2}\Big)$ and using that $|\nabla\varphi_i|^2 + f_i^2 - 1 = O(|x - a_i|^{-4})$, we find

$$R_2 \;=\; O\Big(\frac{\ln\, r_0}{r(\underline{a})^2}\Big) \ .$$

Finally, we observe that due to (4.14)

$$\begin{aligned}\frac{1}{2}\int_{D_k}|\nabla\varphi_{(k)}|^2 &= \sum_{j\neq k}\int_{D_k} e_1(\psi_j)+\frac{1}{2}\sum_{i\neq j}\int_{D_k}\nabla\varphi_i\cdot\nabla\varphi_j\\ &= \sum_{j\neq k}\int e(\psi_j)+\frac{1}{2}\sum_{i\neq j}\int_{D_k}\nabla\varphi_i\nabla\varphi_j+O(r_0^{-2})\ .\end{aligned}$$

Collecting the estimates above, we arrive at

$$\int_{D_k}\langle e(\psi)\rangle \;=\; \frac{1}{2}\sum_{j\neq i}\int_{D_k}\nabla\varphi_i\cdot\nabla\varphi_j+O\Big(\frac{\ln r_0}{r(\underline{a})^2}\Big)+O\Big(\frac{1}{r_0^2}\Big)\ , \tag{4.16}$$

which together with (4.10) and (4.15) yields

$$\mathcal{E}_R(\psi) \;=\; \sum_i\int e(\psi_i)+\sum_{i\neq j}I_{ij,R}+O\Big(\frac{\ln r_0}{r(\underline{a})^2}\Big)+O\Big(\frac{1}{r_0^2}\Big)\ , \tag{4.17}$$

where we have introduced the notation

$$I_{ij,R} \;=\; \frac{1}{2}\int_{D_R}\nabla\varphi_i\nabla\varphi_j\ .$$

Now, by the definition ($a_i \ll R$)

$$\int_{D_R} e(\psi_i) \;=\; \int_{D_{R+a_i}} e(\psi^{(n_i)}) \;=\; E_{n_i,R}+O\Big(\frac{1}{R^2}\Big)\ . \tag{4.18}$$

Next, we show that

$$I_{ij,R} \;\equiv\; \frac{1}{2}\int_{D_R}\nabla\varphi_i\nabla\varphi_j \;=\; -\pi n_i n_j \ln\Big(\frac{|a_{ij}|}{R}\Big)\ . \tag{4.19}$$

We compute

$$\int_{D_R}\nabla\varphi_i\nabla\varphi_j \;=\; n_i n_j\int_0^{2\pi}\int_0^R\frac{r-a\cos\theta}{r^2+a^2-2ar\cos\theta}dr\,d\theta\ ,$$

where $a=|a_{ij}|$. Furthermore, changing the variable of integration as $\theta\to z=e^{i\theta}$ and computing the residue, we find

$$\begin{aligned}\int_0^{2\pi}\frac{r-a\cos\theta}{r^2+a^2-2ar\cos\theta}d\theta &= \frac{\pi}{r}-\frac{r^2-a^2}{2iar^2}\oint_{|z|=1}\frac{dz}{\left(z-\frac{r}{a}\right)\left(z-\frac{a}{r}\right)}\\ &= \frac{\pi}{r}+\frac{\pi}{r}\frac{r^2-a^2}{|r^2-a^2|} \;=\; \frac{2\pi}{r}\begin{cases}1 & \text{if } r>a\\ 0 & \text{if } r<a\end{cases}\ .\end{aligned}$$

The last two equations yield (4.19). Observe also that up to a multiplicative constant expression (4.19) can be found from the symmetry considerations: the invariance of the integral on the l.h.s. under translations ($a_i \to a_i + h$ and $a_j \to a_j + h$ $\forall h \in \mathbb{R}^2$) and rotations ($a_i \to ga_i$ and $a_j \to ga_j$ $\forall g \in O(2)$) imply that it depends only on $|a_{ij}|$. Its scaling properties under the dilations ($a_i \to \lambda a_i$ and $a_j \to \lambda a_j$ $\forall \lambda \in \mathbb{R}$) imply that it is a multiple of $\ln\left(\frac{|a_{ij}|}{R}\right)$.

Eqns (4.17)–(4.19) imply (4.8) with Rem $= O\left(\frac{\ln\, r(\underline{a})}{r(\underline{a})^2}\right)$. Similarly one obtains (4.8)–(4.9) in the forceless case. □

Remark 4.2. The estimate (4.9) can be considerably improved in the forceless case, if we use instead of $e_1(\varphi) = \frac{1}{2}|\nabla\varphi|^2$ the density

$$e_2(\varphi) \;=\; \frac{1}{2}|\nabla\varphi|^2 - \frac{1}{2}|\nabla\varphi|^4 \;,$$

which is a better approximation to the density $e(\psi)$, and instead of (4.12) and (4.13) we use

$$|\psi_i| \;=\; 1 - \frac{1}{2}|\nabla\varphi_i|^2 + O(|x - a_i|^{-4}) \tag{4.20}$$

and

$$\nabla|\psi_i| \;=\; -\frac{1}{2}\nabla|\nabla\varphi_i|^2 + O(|x - a_i|^{-5}) \;, \tag{4.21}$$

respectively. Indeed, proceeding as above, we find in the forceless case that

$$\mathcal{E}(\psi) \;=\; E_R^{(0)} + K + O\left(\frac{\ln\, r(\underline{a})}{r(\underline{a})^4}\right) + O\left(\frac{1}{R^2}\right) \;, \tag{4.22}$$

where

$$K \;=\; -\frac{1}{2}\int_{D_R}\left(|\nabla\varphi_0|^4 - \sum_j |\nabla\varphi_j|^4\right) \;. \tag{4.23}$$

This result is used in [OSb].

5. Lower bound on energy of vortex configurations. Pinning effect

Lower bounds are notoriously difficult. An additional problem which faces us is that unless the condition $\nabla E_R(\underline{a}) = 0$ is satisfied minimization problem (1.5) has no minimizer. To circumvent the latter difficulty we introduce defects into the system and use the fact that sufficiently strong defects bind the vortices (the effect of pinning). More precisely, we introduce the new energy functional

$$\mathcal{E}_{\underline{\lambda}}(\psi) \;=\; \mathcal{E}_R(\psi) + \Sigma \frac{1}{2}\lambda_j \int \delta_{b_j}|\psi|^2 \;, \tag{5.1}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_K)$, $\lambda_j > 0$, are coupling constants of the defects and $\delta_{b_j} \geq 0$ are their potentials, centered at points $b_j \in \mathbb{R}^2$ depending on $\underline{a}$ and very close to the a_j's. The λ_j's and b_j's will be determined later. We take δ_b to be either

$$\delta_b = \frac{1}{2\pi\bar{r}}\,\delta(|x-b| - \bar{r}) \;, \tag{5.2}$$

where $\bar{r} = O(1)$, or a smooth version of this, i.e. δ_b is a smooth function supported in the annulus

$$\{x \in \mathbb{R}^2 \mid \bar{r} \leq |x-b| \leq \bar{r} + \delta\} \tag{5.3}$$

for some sufficiently small δ and satisfying

$$\int \delta_b \;=\; 1 \;. \tag{5.4}$$

Remark 5.1. Sometimes it is convenient to modify the definition of δ_b in such a way that $\forall j$, $\delta_{b_j} f_j$ does not contain harmonics in θ with $|m| \geq 2$, where (r, θ) are the polar coordinates of $y = x - a_j$ (see the harmonic analysis of Eqn (6.8) in the next section). To this end we replace Eqn (5.2) by

$$\delta_b \;=\; \frac{1}{2\pi\bar{r}}\frac{\partial\gamma}{\partial r}(x-b)\delta\big(\gamma(x-b)\big) \;,$$

where $\gamma(x)$ is a slight deformation (modulation) of the function $|x| - \bar{r}$, or by a smooth version of the latter function, so that

$$\int_0^{2\pi} \delta_{b_j} f_j e^{im\theta} d\theta \;=\; 0 \qquad \text{for } |m| \geq 2 \;.$$

With the potential δ_b defined as above it is argued below that $\mathcal{E}_{\underline{\lambda}}(\psi)$ has a minimizer among functions with the given vortex configuration $(\underline{a}, \underline{n})$, provided

$$\lambda_j \;\geq\; C|\nabla_{a_j} E(\underline{a})| \tag{5.5}$$

for an appropriate constant C.

We argue as follows. Consider the j-th vortex. The relevant contribution of the second term on the r.h.s. of (5.1) is $\lambda_j \int \delta_{b_j} |\psi|^2$. If the center of the vortex is at the center of the ring $\operatorname{supp} \delta_{b_j}$, i.e. $a_j = b_j$, then the contribution of this term is approximately $\lambda_j \alpha_{n_j}^2 \varepsilon^{2n_j}$, where $\varepsilon = \frac{1}{2}\operatorname{diam}(\operatorname{supp} \delta_{b_j})$ provided $\varepsilon \ll 1$ and α_{n_j} is defined by the expansion

$$|\psi_j(x)| \;=\; \alpha_{n_j}|x - a_j|^{n_j} + O(|x - a_j|^{n_j+2}) \tag{5.6}$$

for $|x - a_j| \ll 1$. On the other hand if the center of the vortex is in the middle of the ring $\operatorname{supp} \delta_{b_j}$ (i.e. $a_j \in \operatorname{supp} \delta_{b_j}$ and is equidistant from the boundaries of $\operatorname{supp} \delta_{b_j}$), then the corresponding contribution is approximately

$$\begin{aligned} \frac{1}{2}\lambda_j \int \delta_{b_j} f_j^2 \;&=\; \frac{1}{2}\lambda_j \alpha_{n_j}^2 \frac{\varepsilon^{2n_j}}{2\pi} \int_0^{2\pi} \left(2\cos\frac{\theta}{2}\right)^{2n_j} d\theta \\ &=\; \frac{1}{2}\lambda_j \alpha_{n_j}^2 \varepsilon^{2n_j} \binom{2n_j}{n_j} \;. \end{aligned}$$

Since $\binom{2n_j}{n_j} \geq 1$, this shows that it is more energetically advantageous for the vortex to be inside the ring than in its middle. Moreover, the force needed to remove the vortex from the inside of the ring is approximately

$$\frac{1}{2}\lambda_j \int \delta_{b_j} \frac{\partial}{\partial r} f^{(n_j)2} \;=\; -\frac{2}{\pi}\lambda_j n_j \alpha_{n_j}^2 \varepsilon^{2n_j - 1} \sum_{m=0}^{2n_j-1} \frac{(-1)^{n_j - m}\binom{2n_j - 1}{m}}{2(n_j - m) - 1} \;, \tag{5.7}$$

where we have used that

$$\begin{aligned} \int_0^{\pi} \left(2\cos\frac{\theta}{2}\right)^{2n_j - 1} d\theta \;&=\; 2\int_0^{\pi/2} (e^{i\theta} + e^{-i\theta})^{2n_j - 1} d\theta \\ &=\; -2 \sum_{m=0}^{2n_j-1} \frac{(-1)^{n_j - m}\binom{2n_j - 1}{m}}{2(n_j - m) - 1} \;. \end{aligned}$$

On the other hand the force with which the remaining vortices act on the j-th vortex is $-\nabla_{a_j} E(\underline{a})$. This shows that for a fixed ε, in order to keep the j-th vortex inside the ring $\operatorname{supp}\delta_{a_j}$ we need $\lambda_j = O\big(|\nabla_{a_j} E(\underline{a})|\big)$, hence condition (5.5) for the existence of minimizer for energy functional (5.1).

Remark 5.2. In fact, the force exerted on the vortex by the defect is also present when the vortex is outside of the defect ring and it takes its greatest value at the distance r_0 defined by

$$\frac{\partial^2 f_j^2}{\partial r^2}\bigg|_{r=r_0} \;=\; 0 \;. \tag{5.8}$$

This greatest value is

$$F_{\max} \;=\; \lambda_j \,\frac{\partial f_j^2}{\partial r}\bigg|_{r=r_0} \;. \tag{5.9}$$

This implies, in particular, that the extent of the potential created by the defect is $O(1)$.

The minimizer, ψ_λ, of $\mathcal{E}_\lambda(\psi)$ satisfies the Euler-Lagrange equation

$$-\Delta\psi + (|\psi|^2 - 1)\psi \;=\; -\Sigma\lambda_j\delta_{b_j}\psi \;. \tag{5.10}$$

An analysis of this equation conducted in the next section shows that this minimizer satisfies estimates (4.5) and (4.6). Then Proposition 4.1 implies that the energy

$$E_{\underline{\lambda}}(\underline{a}) \;:=\; \inf\{\mathcal{E}_{\underline{\lambda}}(\psi) \mid \operatorname{conf}\psi = \underline{c}\} \tag{5.11}$$

satisfies

$$E_{\underline{\lambda}}(\underline{a}) \;=\; E_R^{(0)} + \sum \frac{1}{2}\lambda_j \int \delta_{b_j}|\psi_\lambda|^2 + \mathrm{Rem} + O\Big(\frac{1}{R^2}\Big) \;, \tag{5.12}$$

where Rem is given in (5.9). On the other hand, since the infimum can be taken over ψ's with $|\psi| \le 1$, we have that

$$E_R(\underline{a}) \;\ge\; E_{\underline{\lambda}}(\underline{a}) - \Sigma\lambda_j \;. \tag{5.13}$$

Due to (5.5), $\Sigma\lambda_j$ can be taken at least of the same order as Rem. Hence we conclude that

$$E_R(\underline{a}) \;\ge\; E_R^{(0)} + \mathrm{Rem} + O\Big(\frac{1}{R^2}\Big) \tag{5.14}$$

with Rem given in (4.3). □

6. Equation (5.10). Method of geometric solvability

In this section we show that Eqn (5.10) has a solution satisfying (4.5)–(4.6), provided condition (5.5) (or (6.11)) holds. This solution is the minimizer of variational problem (5.11). This result was used in Section 5 in order to obtain estimate (5.12).

Two key ideas underpin our method, which we call the method of geometric solvability. First of all we break the space $\mathbb{R}^2$ into the discs $D_j = \{x \in \mathbb{R}^2 \mid |x-a_j| \le r_0\}$, $j = 1, \ldots, K$, and $D_0 = \{x \in \mathbb{R}^2 \mid \mathrm{dist}(x, \underline{a}) \ge r_1\}$, where $r_0 \ll r(\underline{a})$ and $r_1 \gg 1$. We solve Eqn (5.10) (by perturbation theory) separately in each of these regions and then patch the results from different regions together.

The second idea addresses the problem of small denominators or secular terms. Indeed, the Ginzburg-Landau equation linearized around a vortex ψ_j has a (non-square integrable but decaying at ∞) zero mode due to the spontaneous breaking of translational symmetry. Of course, when several vortices are present the symmetry due to independent translations of individual vortices is broken because of the vortex interaction. However, for large intervortex distances, $r(\underline{a})$, it is broken only weakly which manifests itself in the Green's function for the linearized operator having a small denominator. In order to remove such small denominators we adjust the strength, λ_j and location, b_j, of impurities so that the r.h. sides of the resulting *local* linearized equations are orthogonal to the zero translational modes mentioned.

<u>The disc D_j.</u> In this region we linearize Eqn (5.10) around the function $\psi_j e^{i\varphi_{(j)}} = f_j e^{i\varphi_0}$, where, recall, $\varphi_{(j)}$ and φ_0 are given in (4.7) and $f_j = |\psi_j|$. To this end we substitute

$$\psi \;=\; \psi_j e^{i\varphi_{(j)}} + e^{i\varphi_0}\xi_j \tag{6.1}$$

to obtain an equation for ξ_j which we write in the form

$$L_j(\xi) \;=\; F_j + R_j(\xi)\ , \tag{6.2}$$

where L_j is the linearized operator (appropriately transformed)

$$L_j(\xi) \;=\; (-\Delta + |\nabla\varphi_j|^2 + 2f_j^2 - 1)\xi - 2i\nabla\varphi_j \cdot \nabla\xi + f_j^2\bar{\xi}\;, \tag{6.3}$$

F is a free term

$$F_j \;=\; -\lambda_j\delta_{b_j}f_j + 2ie^{-i\varphi_j}\nabla\psi_j \cdot \nabla\varphi_{(j)} \tag{6.4}$$

and $R_j(\xi) = F' + R'(\xi) + R''(\xi)$ with

$$F' \;=\; -|\nabla\varphi_{(j)}|^2 f_j\;, \tag{6.5}$$

$$R'(\xi) \;=\; -\lambda_j\delta_{b_j}\xi - (|\nabla\varphi_0|^2 - |\nabla\varphi_j|^2)\xi + 2i\nabla\varphi_{(j)} \cdot \nabla\xi\;, \tag{6.6}$$

and

$$R''(\xi) \;=\; -f_j|\xi|^2 - 2f_j(\mathrm{Re}\,\xi)\xi - |\xi|^2\xi\;. \tag{6.7}$$

Observe that we have dropped the term $\sum\limits_{i\neq j}\lambda_i\delta_{b_i}\psi$, since it is zero in the region $|x-a_j| \ll r(\underline{a})$.

We will solve Eqn (6.2) to the leading order, i.e. to $O\Big(\frac{1}{r(\underline{a})}\Big)$. To this end we drop the term $R_j(\xi)$ to obtain

$$L_j(\xi) = F_j\;. \tag{6.8}$$

The operator L_j is related to the operator L_{ψ_j}, obtained by linearizing Eqn (1.1) around the solution ψ_j (see [OSa]), as follows

$$L_j(\xi) = e^{-i\varphi_j}L_{\psi_j}(e^{i\varphi_j}\xi)\;. \tag{6.9}$$

Hence the only zero modes of the operator L, which decay at ∞, are those related to the translation symmetry of the equation, namely

$$\eta_k = e^{-i\varphi_j}\partial_{x_k}\psi_j\;, \qquad k = 1,2\;. \tag{6.10}$$

Eqn (6.8) is solvable if and only if

$$\mathrm{Re}\int\bar{\eta}_k F_j = 0 \quad \text{for } k = 1,2\;. \tag{6.11}$$

Below we will find conditions on λ_j and b_j for (6.11) to hold. For the moment we assume (6.11) and push on with our analysis.

Expand ξ in Eqn (6.8) in the Fourier series

$$\xi(x) \;=\; \sum_{m=-\infty}^{\infty} \xi_m(r) e^{im\theta} \;, \tag{6.12}$$

and define

$$\hat{\xi} \;=\; \bigoplus_{m\geq 0} \begin{pmatrix} \xi_m \\ \bar{\xi}_{-m} \end{pmatrix} . \tag{6.13}$$

Then, obviously ξ and $\hat{\xi}$ are in one-to-one correspondence, which we denote by $\xi \leftrightarrow \hat{\xi}$. Observe now that if $\xi \leftrightarrow \hat{\xi}$, then

$$L_j\xi \leftrightarrow \hat{L}\hat{\xi} := \bigoplus_{m\geq 0} L^{(m)} \begin{pmatrix} \xi_m \\ \bar{\xi}_{-m} \end{pmatrix} , \tag{6.14}$$

where

$$L^{(m)} = \begin{pmatrix} -\Delta_r + \frac{(n_j+m)^2}{r^2} + 2f_j^2 - 1 & f_j^2 \\ f_j^2 & -\Delta_r + \frac{(n_j-m)^2}{r^2} + 2f_j^2 - 1 \end{pmatrix} . \tag{6.15}$$

Here Δ_r stands for the radial Laplacian, $\Delta_r f \;=\; r^{-1}\partial_r(r\partial_r f)$ and we have used that $\varphi_j(x) = n_j\theta(x-a_j)$. Eqn (6.14) implies that Eqn (6.8) can be rewritten as

$$\hat{L}\hat{\xi} \;=\; \hat{F} \;, \tag{6.16}$$

where $\hat{F} = \bigoplus_{m\geq 0} \left(\begin{smallmatrix} F_m \\ \bar{F}_{-m} \end{smallmatrix}\right)$ with

$$F_m(r) \;=\; (2\pi)^{-1} \int_0^{2\pi} F e^{-im\theta} d\theta \;.$$

Finally, observe that the translational zero modes (6.10) in the new representation become

$$\hat{\eta}_1 \;=\; i\hat{\eta}_2 \;=\; \bigoplus_{m\geq 0} \frac{1}{2} \begin{pmatrix} f_j' - \frac{n_j}{r} f_j \\ f_j' + \frac{n_j}{r} f_j \end{pmatrix} \delta_{m,1} \;. \tag{6.17}$$

This formula implies that Eqn (6.11) is equivalent to the relation

$$\int_0^{\infty} \begin{pmatrix} f_j' - \frac{n_j}{r} f_j \\ f_j' + \frac{n_j}{r} f_j \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ \bar{F}_{-1} \end{pmatrix} r dr \;=\; 0 \;. \tag{6.18}$$

We analyze the operators $L^{(m)}$, $m \geq 0$. The operator-matrix $L^{(0)}$ can easily be diagonalized. A Perron-Frobenius argument given in [OSa] shows that $L^{(0)} \geq 0$ and 0 is not an eigenvalue of $L^{(0)}$. Next, a similar (but more subtle) argument shows that $L^{(1)} \geq 0$ and 0 is a non-degenerate eigenvalue of $L^{(1)}$ (with the eigenfunction $\left(f_j' - \frac{n_j}{r} f_j, f_j' + \frac{n_j}{r} f_j\right)$ corresponding to the breaking of the translational symmetry of the Ginzburg-Landau equation by ψ_j, (see [OSa] for details). Here f_j' stands for the derivative of f_j w.r. to $|x - a_j|$. Finally

$$L^{(m)} - L^{(1)} = \frac{m-1}{r^2} \begin{pmatrix} 2n_j + m + 1 & 0 \\ 0 & -2n_j + m + 1 \end{pmatrix} \geq 0$$

and $\not\equiv 0$ for $m \geq 2n_j - 1$. Hence $L^{(m)} \geq 0$ and 0 is not an eigenvalue of $L^{(m)}$ for $m \geq 2n_j - 1$. For $2 \leq m < 2n_j - 1$, $L^{(m)}$ have negative eigenvalues but still do not have an eigenvalue at zero (note that in general such eigenvalues, unless related to symmetries, are unstable and can be easily removed by small perturbations). We leave this fact without a proof since we can choose δ_{b_j} so that $F_m = 0$ for $|m| \geq 2$ (see Remark 5.1), so that we can solve Eqn (6.22) without using properties of the operators $L^{(m)}$, $m \geq 2$.

Due to condition (6.11), Eqn (6.18) has a unique solution which we write in the form

$$\begin{aligned} \hat{\xi} &= (\hat{L})^{-1} \hat{F} \\ &= \bigoplus_{m \geq 0} G_m \begin{pmatrix} F_m \\ \bar{F}_{-m} \end{pmatrix} , \end{aligned} \tag{6.19}$$

where G_m is the (regularized, if $m = 1$) inverse of $L^{(m)}$. Observe now that $L^{(m)}$ are (matrix) ordinary differential operators, their (regularized) Green's functions can be found in terms of some special solutions to the homogeneous equations. This is done in [OSe]. (It is convenient for technical reasons to include a part of $R'(\xi)$ into $L(\xi)$, namely, to replace L in Eqn (6.8) by $L + \lambda_j \delta_{b_j}$.) Results of [OSe] imply that $\hat{\xi}$ is of the same order as $\hat{F}$, i.e., as will be shown below, $O\left(\frac{1}{r(\underline{a})^2}\right)$ in the forceless case and $O\left(\frac{1}{r(\underline{a})}\right)$, otherwise. This, due to (6.1) and (6.14), implies (4.5).

Region D_0. In this region Eqn (4.10) coincides with Ginzburg-Landau equation (1.1), i.e. the r.h.s. of (5.10) vanishes. We look for solutions to (1.1) in the form

$$\psi = e^{i\varphi_0}(1+\xi_0) . \tag{6.20}$$

Substituting this into (1.1), we obtain that ξ_0 satisfies

$$L_0(\xi) = F_0 + R_0(\xi) , \tag{6.21}$$

where L_0, F_0 and R_0 are obtained from L_j, F_j and R_j, respectively, by setting $j=0$ and, consequently, $f_0 = 1$ and $\varphi_{(0)} = 0$ and setting $\lambda_0 = 0$. Assuming that ξ_0 small and that is derivatives are respectively even smaller, we drop from the resulting equation for ξ_0, the terms non-linear in ξ and the terms containing derivatives of ξ_0. Taking the real and imaginary parts of the resulting equation, we obtain (in the leading order in $\frac{1}{|x|}$)

$$\mathrm{Re}\,\xi_0 = -\frac{1}{2}|\nabla\varphi_0|^2 \tag{6.22}$$

and

$$-\Delta \mathrm{Im}\,\xi_0 = -\nabla\varphi_0\cdot\nabla|\nabla\varphi_0|^2 . \tag{6.23}$$

The last two equations show that

$$|\xi_0| = O\Big(\frac{1}{|x-\underline{a}|^2}\Big)$$

so that property (4.6) holds for the solution ψ. Moreover these equations imply that ψ is of the form

$$\psi = e^{i(\varphi_0+\mathrm{Im}\,\xi_0)}\Big(1-\frac{1}{2}|\nabla\varphi_0|^2 + O\Big(\frac{1}{|x-\underline{a}|^4}\Big)\Big) , \tag{6.24}$$

where, remember, $\mathrm{Im}\,\xi_0$ solves (6.23).

In order to solve Eqn (6.23) we have to take into account the boundary conditions on ∂D_0. Instead of this we use the solutions of Eqns (6.8) as sources. Namely, we proceed as follows. Writing

$$\varphi = \varphi_0 + \mathrm{Im}\,\xi$$

and using that λ_j are real, we derive from (5.10)

$$-\Delta \mathrm{Im}\,\xi \;=\; \nabla(\varphi_0 + \mathrm{Im}\,\xi)\cdot \nabla \ln\, f^2 \;, \tag{6.25}$$

where, recall, $f = |\psi|$. Observe that while φ_0 is a multivalued function, $\mathrm{Im}\,\xi$ is a regular function on $\mathbb{R}^2$ vanishing at ∞. Thus Eqn (6.25) has the unique solution

$$\mathrm{Im}\,\xi(x) \;=\; \frac{1}{2\pi}\int \ln|x-y|\nabla\big(\varphi_0(y) + \mathrm{Im}\,\xi(y)\big)\cdot\nabla\,\ln\, f^2(y)dy\;. \tag{6.26}$$

Let ξ_j be the solution of equation (6.8) for $j = 1,\dots,K$, and of equations (6.21)–(6.22) for $j = 0$, let $f_j = |\psi_j|$ for $j = 1,\dots,K$, and $= 1 - \frac{1}{2}|\nabla\varphi_0|^2$ for $j = 0$. In the r.h.s. of Eqn (6.35) we take $\xi = \xi_j$ in D'_j, $j = 0,\dots,K$, where $D'_j = D_j$ for $j = 1,\dots,K$ and $D'_0 = \mathbb{R}^2\backslash\bigcup_j D_j$. Plugging this into the r.h.s. of Eqn (6.25), we obtain the following formula for the solution of (6.22):

$$\begin{aligned}\mathrm{Im}\,\xi \;=\; \frac{1}{2\pi}\sum_{j=0}^{K}\int_{D'_j} \ln|x-y|\Big\{&\nabla\varphi_0(y)\cdot\nabla\,\ln\big(f_j(y)+\mathrm{Re}\,\xi_j\big)^2\\ &+\nabla\mathrm{Im}\,\xi_j(y)\cdot\nabla\ln\, f_j(y)^2\Big\}d^2y\;,\end{aligned} \tag{6.27}$$

<u>Conditions on λ_j and b_j</u>. Now we derive the restrictions on the parameters λ_j and b_j implied by solvability conditions (6.11) or (6.18). We consider separately two cases.

a. *The case* $\nabla_{a_j}H(\underline{a}) \neq 0$. Let $\vec{\eta} = \binom{\eta_1}{\eta_2}$. It is shown in Appendix 1 that

$$\mathrm{Re}\int \overline{\vec{\eta}}F \;=\; \nabla_{a_j}H(\underline{a}) - \frac{1}{2}\lambda_j\int\delta_{b_j}\nabla f_j^2\;, \tag{6.28}$$

where $\Delta a_j = b_j - a_j$. Eqns (6.11) and (6.28) imply then that

$$\frac{1}{2}\lambda_j\int\delta_{b_j}\nabla f_j^2 \;=\; \nabla_{a_j}H(\underline{a})\;. \tag{6.29}$$

This fixes the direction in which the j-th vortex center, a_j, must be shifted relative to the center, b_j, of the circle $\mathrm{supp}\,\delta_{a_j}$. Indeed, it is shown in Appendix 1 that

$$\int\delta_{b_j}\nabla f_j^2 \;\approx\; \frac{1}{2}\Delta a_j f_j^{2\prime}(r_1)\;.$$

This equation together with (6.21) implies that

$$\lambda_j = |\nabla_{a_j} H(\underline{a})|/|\Delta a_j| f_j^{2\prime}(r_1) \qquad \text{if } \nabla_{a_j} H(\underline{a}) \neq 0 \tag{6.30}$$

and that the direction $\widehat{\Delta a_j} = \Delta a_j / |\Delta a_j|$ should satisfy

$$\widehat{\Delta a_j} = \nabla_{a_j} H(\underline{a})/|\nabla_{a_j} H(\underline{a})| \qquad \text{if } \nabla_{a_j} H(\underline{a}) \neq 0 \ . \tag{6.31}$$

Consider now the free term F (see Eqn (6.4)). Let $y = x - a_j$ and let r and θ be the polar coordinates of the vector y. The definition of $H(\underline{a})$, Eqn (3.5), implies that

$$J\nabla_{a_j} H(\underline{a}) \ = \ -2\pi n_j \nabla \varphi_{(j)}(a_j) \ , \tag{6.32}$$

where, recall, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Using this equation and the fact that $J^* = -J$, we obtain that modulo $O\Big(\frac{1}{r(\underline{a})^2}\Big)$

$$\begin{aligned} &- 2ie^{-i\varphi_j} \nabla \psi_j \cdot \nabla \varphi_{(j)} \\ &= \ -\frac{1}{\pi n_j r}\Big(if_j' J\hat{y} + \frac{1}{r} f_j \hat{y}\Big) \cdot \nabla_{a_j} H(\underline{a}) \ . \end{aligned} \tag{6.33}$$

Then, taking into account Remark 5.1, we compute that

$$F_{\pm m} \ = \ F_{\pm 1} \delta_{m,\pm 1} \ . \tag{6.34}$$

Let $k_j = |k_j| e^{-i\theta_j}$ be the complex number corresponding to the vector $-\frac{1}{2\pi n_j} J\nabla_{a_j} H(\underline{a})$. We compute

$$F_{\pm 1} \ = \ i|k_j| e^{\mp i\theta_j}\Big(f_j' \mp \frac{n_j}{r} f_j\Big) - \frac{1}{2\pi}\int_0^{2\pi} \lambda_j \delta_{b_j} f_j e^{\mp i\theta} d\theta \tag{6.35}$$

b. *The case* $\nabla_{a_j} H(\underline{a}) = 0$. First of all we observe that since $\varphi_{(j)} = \sum_{k\neq j} \varphi_k$ is a single valued function in the region $|x - a_j| < r(\underline{a})$ and is harmonic in $\mathbb{R}^2$, it has in this region the following Fourier series expansion

$$\varphi_{(j)} \ = \ \sum_{m=0}^{\infty} c_m r^m \cos\, m(\theta - \theta_j^{(m)}) \tag{6.36}$$

for some amplitudes c_m and phases $\theta_j^{(m)}$ (with $c_m = O\big(r(\underline{a})^{-m}\big)$). Moreover in the forceless configuration $c_1 = 0$. Using this we find that in the forceless case, Eqns (6.34)–(6.35) for the r.h.s. of (6.16) must be replaced by the equation

$$\begin{aligned} F_{\pm m} \;&=\; \pm\, 2i\alpha r e^{\mp 2i\theta_j^{(2)}}\Big(\pm f_j' - \frac{n_j}{r} f_j\Big)\delta_{m,2} \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi} \lambda_j \delta_{b_j} f_j e^{\mp i\theta}\delta_{m,1}\ , \end{aligned} \tag{6.37}$$

where the coefficient α is of the order $O\big(r(\underline{a})^{-2}\big)$ (see Eqn (6.36)). Since the operator $L^{(2)}$ does not have a bounded zero mode, we can set $\lambda_j = 0$ in this approximation. Hence Eqn (6.16) has a unique solution of the order $O\big(r(\underline{a})^{-2}\big)$. The need for λ_j arises only in the next step of the perturbation theory in the small term $R(\xi)$, neglected previously, i.e. in the order $O\big(r(\underline{a})^{-3}\big)$. Thus in this case we can take $\lambda_j = O\big(r(\underline{a})^{-3}\big)$ and so on.

Iteration scheme. Now we derive an equation allowing us to go beyond the first order perturbation theory. To this end we use the method of geometric parametrices of [Sig]. Let $\{\chi_j\}_0^K$ be the partition of unity, i.e. $\sum_{j=0}^{K}\chi_j = 1$, s.t. $\operatorname{supp}\chi_j \subset D_j$. Let G_j be the (regularized) Green's function for $M_j = L_j - 2i\nabla\varphi_{(j)}\cdot\nabla + \lambda_j\delta_{b_j}$, $j = 0,\dots,K$. Applying the operator $\sum_{j=0}^{K}\chi_j G_j$ to Eqn (5.10) and using Eqns (6.2) and the equations $G_j M_j(\xi_j) = \xi_j$ for $j = 0,\dots,K$, we obtain

$$\sum_{j=0}^{K}\chi_j\xi_j \;=\; \sum_{j=0}^{K}\chi_j G_j F_j + \sum_{j=0}^{K}\chi_j G_j \overline{R}_j(\chi_j\xi_j)\ , \tag{6.38}$$

where $\overline{R}_j(\xi) = R_j(\xi) - 2i\nabla\varphi_{(j)}\cdot\nabla\xi + \lambda_j\delta_{b_j}\xi$ and where we used that $\overline{R}_j(\xi) = R_j(\chi_j\xi)$. If χ_j are characteristic functions (of some sets, i.e. $\chi_j^2 = \chi_j$), then the last equation implies the $K+1$ equations

$$\overline{\xi}_j \;=\; \chi_j G_j F_j + \chi_j G_j \overline{R}_j(\overline{\xi}_j)\ , \tag{6.39}$$

where $\overline{\xi}_j = \chi_j\xi_j$ and $j = 0,\dots,K$. The latter equations can be used to find the $\overline{\xi}_j$'s iteratively to an arbitrary order in $\frac{1}{r(\underline{a})}$. After that using the equations $\psi = \sum_{j=0}^{K}\chi_j\psi$ and

$\psi = e^{i\varphi_0}(f_j + \xi_j)$ one reconstructs ψ as

$$\psi \;=\; \sum_{j=0}^{K} e^{i\varphi_0}(\chi_j f_j + \overline{\xi}_j)\ . \tag{6.40}$$

This analysis shows that in the leading order in perturbation theory Eqn (5.10) has a solution satisfying (4.5)–(4.6) with $n = 1$, provided the λ_j's and b_j's are s.t. (6.11) holds.

Appendix 1

In this appendix we perform some computations required in section 6.

First we claim that

$$\mathrm{Re}\int \bar{\vec{\eta}}(F + \lambda_j \delta_{b_j} f_j) \;=\; \nabla_{a_j} H(\underline{a})\ . \tag{A1.1}$$

Indeed, Eqn (6.17) shows that η_k, $k = 1, 2$, have only the $m = \pm 1$ harmonics. Using this equation together with Eqn (6.4), or Eqn (6.34), we obtain

$$\mathrm{Re}\int \bar{\vec{\eta}}(F + \lambda_j \delta_{b_j} f_j) \;=\; -\mathrm{Re}\,\pi i k_j \int_0^\infty \eta\cdot J\eta r dr \begin{pmatrix} 1 \\ i \end{pmatrix}\ , \tag{A1.2}$$

where $\eta = \binom{f_j' - \frac{n_j}{r} f_j}{f_j' + \frac{n_j}{r} f_j}$. The r.h.s. can be computed explicitly:

$$\begin{aligned} \text{R.H.S. (A1.2)} \;&=\; -\mathrm{Re}\,\pi k_j \int_0^\infty \Big(-4\frac{n_j}{r} f_j f_j' \Big) r dr \begin{pmatrix} i \\ -1 \end{pmatrix} \\ &=\; 2\pi k_j n_j \mathrm{Re}\begin{pmatrix} i \\ -1 \end{pmatrix} \;=\; 2\pi n_j J k_j\ . \end{aligned}$$

Since $k_j = -\frac{1}{2\pi n_j} J \nabla_{a_j} H(\underline{a})$ and $J^2 = -id$, Eqn (A1.1) follows. $\square$

Now we compute the term $\mathrm{Re}\int \bar{\xi}_k \lambda_j \delta_{b_j} f_j$ under the assumption that $|\Delta a_j| \ll \bar{r}$, where, recall, $\Delta a_j = b_j - a_j$. To simplify the computations we let δ_{b_j} be the true δ-function, not a smeared one. Under the first assumption the equation $|x - b_j| = \bar{r}$ for the circle ($= \mathrm{supp}\,\delta_{b_j}$) can be written in the leading approximation in Δa_j as

$$r \approx \bar{r} + \Delta a_j \cdot \hat{y}\ , \tag{A1.3}$$

where, remember, r and θ are the polar coordinates of $y = x - a_j$. Thus $\delta_{b_j} = \delta(|x - b_j| - \bar{r})$ can be replaced by $\delta(r - \bar{r} - \Delta a_j \cdot \hat{y})$. This yields

$$\mathrm{Re}\int \bar{\vec{\eta}}\delta_{b_j} f_j \approx \int f_j f_j' \hat{y}\delta(r - \bar{r} - \Delta a_j \cdot \hat{y})$$
$$\approx f_j(\bar{r})f_j'(\bar{r})\int_0^{2\pi} \hat{y}(\bar{r} + \Delta a_j \cdot \hat{y})d\theta \ .$$

Hence

$$\mathrm{Re}\int \vec{\eta}\delta_{b_j} f_j \ \approx \ \pi f_j(\bar{r})f_j'(\bar{r})\Delta a_j \ . \tag{A1.4}$$

Appendix 2. Region $D_j \cap D_0$

Now we investigate behaviour of the solution $\hat{\xi}$ in the regions $D_j \cap D_0$. We require that asymptotics in $D_j \cap D_0$ of the solutions found in D_j and D_0 match. We consider two cases.

a. <u>The case $\nabla_{a_j} H(a) \neq 0$</u>. Now we find the leading asymptotic of $\hat{\xi}$ in the region $1 \ll r \ll r(\underline{a})$. Since only the operator $L^{(1)}$ has a zero mode, the leading term for $r \gg 1$ comes from the $m = 1$ sector. Thus we consider Eqn (6.16) in this sector:

$$L^{(1)}\begin{pmatrix} \xi_1 \\ \bar{\xi}_{-1} \end{pmatrix} \ = \ \begin{pmatrix} F_1 \\ \bar{F}_{-1} \end{pmatrix} . \tag{A2.1}$$

We write the solution of this equation in the form

$$\begin{pmatrix} \xi_1 \\ \bar{\xi}_{-1} \end{pmatrix} \ = \ \begin{pmatrix} \xi_1^0 \\ \bar{\xi}_{-1}^0 \end{pmatrix} + c\begin{pmatrix} f_j' - \frac{n_j}{r} f_j \\ f_j' + \frac{n_j}{r} f_j \end{pmatrix} , \tag{A2.2}$$

where $\begin{pmatrix} \xi_1^0 \\ \bar{\xi}_{-1}^0 \end{pmatrix}$ is a special solution, c is a constant and the vector which multiplies c is, recall, the translational zero mode. The value of c is fixed from original non-linear equation (6.2) (say, by the perturbation theory). We show in [OSe] that $\xi_{\pm 1}^0$ are bounded.

Dropping in (A2.1) the derivatives of ξ which are of a higher order in r^{-1} as well as the terms $f_j^2 - 1$ and $\frac{(n_j \pm 1)^2}{r^2}$ in $L^{(1)}$ and dropping the terms coming from $\lambda_j \delta_{b_j}$ and the terms containing f_j' from $F_{0,1}$ and $\bar{F}_{0,-1}$, we arrive at the asymptotic equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \bar{\xi}_{-1} \end{pmatrix} \ = \ -ik_j \frac{n_j}{r}\begin{pmatrix} 1 \\ 1 \end{pmatrix} . \tag{A2.3}$$

A particular solution of this equation, $\frac{n_j k_j}{2ir}\binom{1}{1}$, and the asymptotics, $\frac{n_j}{r}\binom{-1}{1}$, of the translational zero mode for $r \gg 1$ lead to the asymptotics of the general solution to Eqn (6.18):

$$\begin{pmatrix} \xi_1 \\ \bar{\xi}_{-1} \end{pmatrix} = \frac{n_j k_j}{2ir}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\nu e^{-i\theta_j}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad (A2.4)$$

plus higher order terms in r^{-1}. Here ν is a function of r (incorporating $c\frac{n_j}{r}$), it cannot be found from Eqn (A2.3), an explicit expression for it will be given below. This together with Eqn (6.12) yields the asymptotic expression for the solution to Eqn (6.8):

$$\xi = \frac{n_j|k_j|}{r}\sin(\theta-\theta_j) - 2i\nu\cdot\cos(\theta-\theta_j) \qquad (A2.5)$$

plus higher order terms in r^{-1}. The second term on the r.h.s. yields, in the leading order, a correction to the phase of ψ (due to a small translation of the center of the vortex), while the first term, to $|\psi|$:

$$\varphi = \varphi_j + 2\nu\cos(\theta-\theta_j) \qquad (A2.6)$$

and

$$|\psi| = 1 - \frac{n_j^2}{2r^2} + \frac{n_j|k_j|}{r}\sin(\theta-\theta_j) \ .$$

Consider now Eqn (6.22) in the region $1 \ll |x-a_j| \ll r(\underline{a})$. We obtain

$$-\Delta \mathrm{Im}\xi_0 = \frac{4n_j^2|k_j|}{r^3}\cos(\theta-\theta_j) \ .$$

The solution to this equation decreasing at infinity is

$$\mathrm{Im}\,\xi_0 = 2|k_j|n_j^2\cos(\theta-\theta_j)\frac{1}{r}\ln\left(\frac{r}{r_*}\right) , \qquad (A2.7)$$

where r_* is a constant, which can be found from solving Eqn (A2.1) at $r = O(1)$. Comparing Eqns (A2.5) and (A2.7) and using that $\xi = \xi_0$, modulo higher order terms, we find

$$\nu = -|k_j|n_j^2\frac{1}{r}\ln\left(\frac{r}{r_*}\right) \ . \qquad (A2.8)$$

Observe now that in the region $1 \ll r \ll r(\underline{a})$ (which is a part of $|x-\underline{a}| \gg 1$)

$$\nabla\varphi_0 \;=\; \frac{n_j}{r}(-\sin\,\theta, \cos\,\theta) + k_j\;, \tag{A2.9}$$

which implies that ψ is, modulo $O\big(r(\underline{a})^{-2}\big)$, of the form (6.39) (remember that $k_j = O\big(|r(\underline{a})|^{-1}\big)$). Thus the obtained solutions in the regions D_j, $j = 1,\ldots,K$, and D_0 match (modulo higher order terms) in the common domain.

Thus we have shown that it has solutions of Eqn (5.10) in the regions D_j, $j = 1,\ldots,K$, have asymptotics in $D_j \cap D_0$ given by (A2.6), while the solution in D_0 has asymptotics in the same domains $D_j \cap D_0$ given by (6.23) and (A2.7). Thus in the overlapping regions the obtained local solutions match.

b. *The case* $\nabla_{a_j}H(\underline{a}) = 0$. Equation (6.36) with $\lambda_j = 0$ shows that in this approximation $\xi_m = 0$ for $m \neq 2$. Consider the sector $m = 2$. In the region $1 \ll |x - a_j| \ll r(\underline{a})$, Eqn (6.16) in the sector $m = 2$ leads to

$$\xi_{\pm 2} \;=\; \pm i n_j \alpha e^{\mp 2i\theta_j^{(2)}}\Big[-1 - \frac{4n_j^2-2}{r^2} \pm \frac{3n_j}{2}\Big]\;. \tag{A2.10}$$

Combining this with (6.1) and (6.13) we obtain the following expression for the correction to the phase, φ_0, in this region

$$\delta\varphi \;=\; 3\alpha n_j^2 \cos\big(2(\theta - \theta_j^{(2)})\big)\;. \tag{A2.11}$$

Such a correction leads to the correction to the energy of the order $O\Big(\frac{\ln\, r(\underline{a})}{r(\underline{a})^4}\Big)$. The rest of the analysis of the general case can be carried over into the forceless case without a change. This shows that in the case $\nabla H(\underline{a}) = 0$, in the leading order in perturbation theory, the solutions of Eqn (5.10) in the regions D_j, $j = 0,\ldots,K$, match in the overlaps $D_0 \cap D_j$ of these regions.

The analysis above can be carried out to an arbitrary order of perturbation theory with the conclusions not changed. This concludes our argument that Eqn (5.10) has a unique solution satisfying (4.5) and (4.6).

Supplement

In this supplement we show in the first order of perturbation theory, that the solution of Eqn (5.10) found in Section 6 has in fact the vortex configuration $\underline{c}$. We consider two cases.

(a) *The case* $\nabla_{a_j} H(\underline{a}) \neq 0$. Consider Eqn (A2.1) for $r \ll 1$. We have the leading order

$$\begin{pmatrix} -\Delta_r + \frac{(n_j+1)^2}{r^2} & \alpha_{n_j}^2 r^{2n_j} \\ \alpha_{n_j}^2 r^{2n_j} & -\Delta_r + \frac{(n_j-1)^2}{r^2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = -2in_j k_j \alpha_{n_j} r^{n_j-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{S.1}$$

where, recall, α_{n_j} is determined by Eqn (5.6). The general solution regular at $r \to 0$ is

$$\frac{i\alpha_{n_j} k_j}{2} r^{n_j+1} \begin{pmatrix} \frac{\alpha_{n_j}^2}{8(n_j+1)^2} r^{2(n_j+1)} \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} \frac{\alpha_{n_j}^2}{4n_j(2n_j+1)} r^{3n_j+1} \\ r^{n_j-1} \end{pmatrix} + c_2 \begin{pmatrix} r^{n_j+1} \\ \frac{\alpha_{n_j}^2}{4(n_j+2)(2n_j+5)} r^{3(n_j+1)} \end{pmatrix}, \tag{S.2}$$

where c_1 and c_2 are some constants. Note that the translational mode $\begin{pmatrix} f_j' - \frac{n_j}{r} f_j \\ f_j' + \frac{n_j}{r} f_j \end{pmatrix} = \begin{pmatrix} 0 \\ 2\alpha_{n_j} n_j r^{n_j-1} \end{pmatrix} + O(r^{n_j+1})$ as $r \to 0$. Hence (S.2) can be written as

$$\frac{1}{2} i\alpha_{n_j} k_j r^{n_j+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 2\alpha_{n_j} n_j r^{n_j-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} r^{n_j+1} \\ 0 \end{pmatrix} + O(r^{n_j+1}), \tag{S.3}$$

where $c_3 = c_1/2\alpha_{n_j} n_j$. Thus in this approximation the n_j-vortex, shifted to some distance from the initial position, does not split.

(b) *The case* $\nabla_a H(\underline{a}) = 0$. On the small distances, $r \ll 1$, we obtain from Eqn (6.16) in the sector $m = 2$ that

$$\xi_{\pm 2} = \pm i\alpha\alpha_{n_j} e^{\mp 2i\theta_j^{(2)}} [r^{n_j+2} C_{\pm 1} + r^{|n_j-2|} C_{\pm 2}], \tag{S.4}$$

where $C_{-1}=1$, $C_2=0$ and C_1 and C_{-2} are some real constants of the order $O(1)$, while the numbers α_{n_j} are defined from Eqn (5.6).

If $|n_j|=1$, then it follows from Eqn (S.4) that the j-th vortex is slightly deformed. In the case $|n_j|\geq 2$, if we know from, say, symmetry considerations that the j-th vortex does not split, then we should add to the function ψ a shift solution (the $m=1$ sector) with such a coefficient, c, that the term given by Eqn (S.4) and the new term added produce only a shift z_0 of the j-th zero of ψ, i.e.

$$\begin{aligned} & z^{n_j}+z^{n_j-2}\alpha e^{2i(\theta_j^{(2)}-\pi/4)}c_2+\frac{cz^{n_j-1}}{\alpha_{n_j}} \\ = & \ z^{n_j}-n_j z_0 z^{n_j-1}+z_0^2 n_j(n_j-1)\frac{z^{n_j-2}}{2} . \end{aligned} \qquad (S.5)$$

Using this equation, we obtain

$$\begin{aligned} z_0 &= \pm e^{i(\theta_j^{(2)}-\pi/4)}\left(\frac{2\alpha C_2}{n_j(n_j-1)}\right)^{1/2} \\ &= O\big(r(\underline{a})^{-1}\big) . \end{aligned} \qquad (S.6)$$

Such a shift of the function ψ contributes $O\big(r(\underline{a})^{-4}\big)$ to the energy (see Remark 5.2). Choosing the centers b_j of our potentials δ_{b_j} appropriately, we move the zeros of ψ to the old positions.

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