ON THE POINT LIMIT OF THE PAULI–FIERZ MODEL

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Abstract. *We continue in this paper the analysis, begun in [NP], of the classical dynamics of the point limit of the Maxwell–Lorentz system in dipole approximation (the Pauli–Fierz model). Here, as a first step towards considering the full nonlinear system, we study the case in which a nonlinear external field of force is present. We study the flow of the regularized (namely with an extended particle) system, and show that it converges in the appropriate norm, as the radius of the particle tends to zero, to the flow of a closed coupled system of equations, containing the renormalized mass only, which so provides the very definition of the dynamics of the system in the point limit. The Abraham–Lorentz–Dirac equation for the particle position is deduced and turns out to be, in this description, a boundary condition on the vector potential, giving the evolution of its singularity. Moreover, the Hamiltonian structure of the limit system is displayed, and it is shown that the standard Hamiltonian of the Pauli–Fierz model converges to the Hamiltonian of the limit system here given.*

1. Introduction.

In a recent contribution ([NP]) the authors succeeded in defining the dynamics of the system constituted by a *free* charged *point* particle interacting with the electromagnetic field in the dipole (or linearized) approximation. As it is well known, this is a highly nontrivial problem due to the fact that the Maxwell–Lorentz system, which of course should be used to describe the dynamics, is, from a rigorous point of view, meaningless when a point particle is considered. The analogous problem shows up in quantum electrodynamics too, where it is at the origin of the divergences which led to the introduction of perturbative renormalization theory. The usual way out consists in trying to define a dynamics by taking a suitable limit ("point limit") on a regularized version of the system itself, typically obtained by attributing a formfactor ρ_r to the particle. The regularized system so obtained is often called the "Pauli–Fierz model" ([PF], [Bl], [A]). In [NP] it is shown how the flow of the regularized system converges to the flow of a well defined dynamical system if and only if mass renormalization is performed, and a suitable constraint (conserved by the evolution) between the initial data of vector potential and particle velocity is imposed. More precisely, the limit system for the vector potential A (which in the Coulomb gauge is the only relevant variable for the field) and the particle position q has the form (with c denoting the speed of light)

$$\begin{cases} \dfrac{1}{c^2}\ddot{A} = -H_m A \\ \quad \dot{q} = Q_A \\ A(0) = A_0, \quad Q_{A_0} = \dot{q}_0, \quad \dot{A}(0) = \dot{A}_0, \quad q(0) = q_0 \ . \end{cases} \tag{1.1}$$

Here H_m is a selfadjoint operator, bounded from below and containing the renormalized mass only; such an operator turns out to be of the class of the so called point interactions ([AGHH]) and its properties for the case at hand are recalled below (see theorems 2.1 and 2.2). Q_A is a quantity characterizing the singularity of the generic element A of the form domain of the operator H_m, according to the formula

$$Q_A = \frac{3c}{2e} \lim_{r \downarrow 0} r \ \frac{1}{4\pi r^2} \int_{S_r} A(x) d\mu_r(x) \ , \tag{1.2}$$

where S_r denotes the sphere of radius r about the origin, μ_r is the corresponding surface measure, and e is the particle electric charge.

So the dynamics of the system is completely specified by solving the abstract wave equation for the field given by the first of (1.1), corresponding to initial data in a suitable phase space, and then recovering the time evolution of the particle position from the second equation of the system (1.1) with the aid of formula (1.2).

The main goal would be, of course, the study of the point limit for the complete and relativistic Maxwell–Lorentz system. But this is a problem which, up to now, we are not able to tackle. As a first step, in the present paper, still remaining in the framework of the dipole approximation (i.e. linearization of the interaction), we consider the case in which a nonlinearity is introduced through an external nonlinear force field $F(q)$ acting on the particle. This still constitutes a non trivial generalization because it is not possible to prove convergence by using the same techniques exploited in [NP] which were essentially based on linear methods. In particular a fixed point method, combined with some uniform (w.r.t. the particle radius r) estimates, is needed.

By the way, a further generalization is introduced in as much as it occurs that system (1.1) turns out to describe only situations in which the total linear momentum vanishes; this is due to the fact that the constraint between initial data for particle velocity and vector potential, which in some form is necessary in order to obtain a limit at all, was chosen in [NP] in a form stronger than needed. Such a limitation will be removed here.

The procedure of the present novel approach can be briefly described as follows. Taking inspiration from the Hamiltonian structure of the regularized Pauli–Fierz equations, we write (see (3.2)) a first order system equivalent to the original one, introducing as supplementary variables the total linear momentum and the electric field

$$p_r = m_r \dot{q}_r + \frac{e}{c}\langle \rho_r, A_r \rangle \ , \qquad E_r = \frac{1}{4\pi c^2}\dot{A}_r \ .$$

It turns out that, just as in the free case, a limit system exists if and only if mass is renormalized according to the traditional prescription

$$m_r := m - \frac{8\pi e^2}{3c^2} E(\rho_r) \ , \tag{1.3}$$

where $E(\rho_r)$ is the electrostatic energy of energy of the distribution ρ_r; this energy as it is well known, diverges to $+\infty$ as $r \downarrow 0$, so that, as a consequence, the bare mass m_r diverges to $-\infty$ in the same limit. With this mass renormalization, the limit system is

$$\begin{cases} \dot{A} = 4\pi c^2 E \\ \dot{E} = -\dfrac{1}{4\pi} H_{m,p} A \\ \dot{q} = Q_A \\ \dot{p} = F(q) \\ A(0) = A_0, \quad E(0) = E_0, \quad q(0) = q_0, \quad p(0) = p_0 \end{cases} \tag{1.4}$$

(see theorem 3.4). In this set of equations $H_{m,p}$ is a family of affine operators parametrized by total linear momentum p, and defined (see (3.3) and lemma 4.1) in terms of the standard Laplacian with one point interaction, namely the operator H_m already studied in the quoted paper. The system reduces to (1.1) when $F(q) = 0$, as it is expected, but *only with the initial condition* $p_0 = 0$. Indeed, as already mentioned above, the case treated in [NP] is a particular one corresponding to solutions of the coupled system with vanishing total linear momentum (see remark 3.6).

Of course, due to the its nonlinearity, it is not possible to write down explicitly the flow of system (1.4) as was done for the free case in [NP]. Nevertheless, an alternative and expressive picture of the dynamics is possible for strong solutions of the Cauchy problem (1.4). In fact, exploiting an equivalent definition of the domain of the operator $H_{m,p}$, in which the dynamics of the particle appears as a boundary condition on the vector potential (see lemma 4.1), it turns out that the these solutions coincide with the solutions, for the same initial data, of the Cauchy problem for the system

$$\begin{cases} \frac{1}{c^2}\ddot{A} = \Delta A + \frac{4\pi e}{c} M\dot{q}\delta_0 \\ \ddot{q}(t) = c\sqrt{\lambda_0}\dot{q}(t) + \frac{3c^2}{2e}A_f(t,0) - \frac{3c^3}{2e^2}p(t) \\ \quad \dot{p} = F(q) \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0, \quad q(0) = q_0, \quad \dot{q}_0 = Q_{A_0}, \quad p(0) = p_0 \end{cases} \tag{1.5}$$

(see theorem 4.2). Here $A_f(t,0)$ is the solution of the free wave equation with the the same initial data for the field as the system (1.4). This is a more familiar problem in which a standard wave equation is coupled with an ordinary differential one. This ordinary differential equation is a low order version of the well known Abraham–Lorentz–Dirac equation ([D], [L]),

$$\begin{cases} -m\tau_0\dddot{q}(t) + m\ddot{q}(t) = -\frac{e}{c}\dot{A}_f(t,0) + F(q(t)) \\ q(0) = q_0, \quad \dot{q}(0) = Q_{A_0}, \quad \ddot{q}(0) = Q_{\dot{A}_0} \ ; \end{cases}$$

so on one hand this much questioned equation is confirmed in the present more general case, and on the other hand it is settled in a convincing mathematical context.

Finally, we show that the limit system (1.4) is in fact a Hamiltonian system, and that the corresponding Hamiltonian function is the limit in a well defined sense of the classical Hamiltonian of the Pauli–Fierz model (see theorem 4.5). This result, which answers a time honored problem (we quote only the papers [K] and [VK]), could be useful to study the many open questions related to electrodynamics of point particles (e.g. runaway solutions, quantization...), and could perhaps give valid hints to study the fully non linear system without dipole approximation.

2. Definitions and some preliminary results.

We begin by recalling some definitions and results from [NP]. We denote by $L^2_*(\mathbb{R}^3)$ the Hilbert space of square integrable, divergence–free, vector fields on $\mathbb{R}^3$. M will be the projection from $L^2_3(\mathbb{R}^3)$, the Hilbert space of square integrable vector fields on $\mathbb{R}^3$, onto $L^2_*(\mathbb{R}^3)$ and we will use the same symbol $\langle\cdot,\cdot\rangle$ ($\|\cdot\|_2$ is the corresponding Hilbert norm) to denote the scalar products in $L^2(\mathbb{R}^3)$, $L^2_3(\mathbb{R}^3)$, $L^2_*(\mathbb{R}^3)$ and also to denote the obvious pairing between an element of $L^2_3(\mathbb{R}^3)$ and one of $L^2(\mathbb{R}^3)$ (the result being a vector in $\mathbb{R}^3$). By the same abuse of notation, given two functions f and g in $L^2(\mathbb{R}^3)$, by $f\otimes g$ we will denote the operator in $L^2_3(\mathbb{R}^3)$ defined by $f\otimes g(A) := f\langle g, A\rangle$. Moreover $H^s(\mathbb{R}^3)$, $s\in\mathbb{R}$, indicates the usual scale of Sobolev–Hilbert spaces, and the meaning of $H^s_3(\mathbb{R}^3)$ and $H^s_*(\mathbb{R}^3)$ should now be clear. If γ is a continuous path in $\mathbb{R}^3$, defined on the compact time interval $I(T) := [-T,T]$, $\|\gamma\|_\infty$ denotes the usual supremum norm. With $\mathrm{Lip}(\mathbb{R}^3;\mathbb{R}^3)$ we denote the space of Lipschitz vector fields. Finally, given a measurable non negative function ρ we define its energy $E(\rho)$ as

$$E(\rho) := \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x)\rho(y)}{|x-y|}dx\ dy\ .$$

Theorem 2.1. *([NP, thm. 2.1]) As $r \downarrow 0$, i.e as $\rho_r(x) := r^{-3}\rho(r^{-1}x)$, ρ a spherically symmetric probability density with bounded support, weakly converges to δ_0, the self–adjoint operator*

$$H_r := -\Delta + \frac{4\pi e^2}{m_r c^2} M \cdot \rho_r \otimes \rho_r \ ,$$

$$m_r := m - \frac{8\pi e^2}{3c^2} E(\rho_r) = m - \frac{1}{r}\frac{8\pi e^2}{3c^2} E(\rho) \ ,$$

converges in norm resolvent sense in $L^2_*(\mathbb{R}^3)$ to a self–adjoint operator H_m, where H_m has the resolvent

$$(H_m + \lambda)^{-1} = (-\Delta + \lambda)^{-1} + \Gamma_m(\lambda)^{-1} M \cdot G_\lambda \otimes G_\lambda \ ,$$

and where

$$-\lambda \in \rho(H_m), \quad \lambda > 0 \ , \qquad \Gamma_m(\lambda) = -\frac{mc^2}{4\pi e^2} + \frac{\sqrt{\lambda}}{6\pi} \ , \qquad G_\lambda(x) := \frac{1}{4\pi}\frac{e^{-\sqrt{\lambda}|x|}}{|x|} \ .$$

If m_r=const. then H_r converges in norm resolvent sense in $L^2_*(\mathbb{R}^3)$ to $-\Delta$. No other definition of the renormalized mass m_r leads to a limit for H_r.

Theorem 2.2. *([NP, thm. 2.3]) The vectors A in the operator domain $D(H_m)$ of H_m are of the type*

$$A = A_\lambda + \Gamma_m(\lambda)^{-1} M A_\lambda(0) G_\lambda, \quad A_\lambda \in H^2_*(\mathbb{R}^3), \quad -\lambda \in \rho(H_m), \quad \lambda > 0 \ .$$

The above decomposition in a regular part A_λ, and a corresponding singular one, is unique, and with $A \in D(H_m)$ of this form one has

$$(H_m + \lambda)A = (-\Delta + \lambda)A_\lambda \ .$$

Let F_m be the quadratic form corresponding to H_m. Then the vectors A in the form domain $D(F_m)$ are of the type

$$A = A_\lambda + \frac{4\pi e}{c} M Q_A G_\lambda, \qquad A_\lambda \in H^1_*(\mathbb{R}^3), \quad Q_A \in \mathbb{R}^3, \quad \lambda > 0 \ .$$

Given $A \in D(F_m)$, Q_A can be explicitly computed by the formula

$$Q_A = \frac{3c}{2e} \lim_{r\downarrow 0} r \ \frac{1}{4\pi r^2} \int_{S_r} A(x) d\mu_r(x) \ ,$$

where S_r denotes the sphere of radius r and μ_r is the corresponding surface measure. The above decomposition is unique, and with $A \in D(F_m)$ of this form one has

$$\begin{aligned} F^\lambda_m(A, A) &:= F_m(A, A) + \lambda \|A\|_2^2 \\ &= \|(-\Delta + \lambda)^{\frac{1}{2}} A_\lambda\|_2^2 + \left(\frac{4\pi e}{c}\right)^2 \Gamma_m(\lambda) |Q_A|^2 \ . \end{aligned}$$

Moreover

$$\sigma_{ess}(H_m) = \sigma_{ac}(H_m) = [0, +\infty), \quad \sigma_{sc}(H_m) = \emptyset \ ,$$

and

$$\sigma_p(H_m) = \left\{ -\left(\frac{3mc^2}{2e^2}\right)^2 \right\} \equiv \{-\lambda_0\} \ ,$$

where $-\lambda_0$ *has a threefold degeneration and*

$$X_j^0 = 2\sqrt{2\pi m}\ \frac{c}{e} M e_j G_{\lambda_0} \ ,$$

are the corresponding normalized eigenvectors, where $\{e_j\}_1^3$ *is an orthonormal basis.*

Remark 2.3. This remark would correspond to remark 2.5 in [NP]. However here we proceed in a different way avoiding the use of the fields $\tilde{X}$. By functional calculus, and by

$$\frac{1}{\sqrt{y}} = \frac{1}{\pi} \int_0^\infty \frac{1}{y+x} \frac{dx}{\sqrt{x}} \qquad y > 0 \ ,$$

we have, for any $X \in L^2_*(\mathbb{R}^3)$, $\lambda > \lambda_0$

$$\begin{aligned} (H_m + \lambda)^{-\frac{1}{2}} X &= \frac{1}{\pi} \int_0^\infty (H_m + x + \lambda)^{-1} X \frac{dx}{\sqrt{x}} \\ &= (-\Delta + \lambda)^{-\frac{1}{2}} X + \frac{1}{\pi} \int_0^\infty \Gamma_m(x+\lambda)^{-1} M \langle G_{x+\lambda}, X\rangle G_{x+\lambda} \frac{dx}{\sqrt{x}} \ . \end{aligned} \tag{2.1}$$

Analogously, if $\lambda > \lambda_0$, and r sufficiently small, we have

$$\begin{aligned} (H_r + \lambda)^{-\frac{1}{2}} X &= \frac{1}{\pi} \int_0^\infty (H_r + x + \lambda)^{-1} X \frac{dx}{\sqrt{x}} = (-\Delta + \lambda)^{-\frac{1}{2}} X \\ &+ \frac{1}{\pi} \int_0^\infty \Gamma_r(x+\lambda)^{-1} M \langle (-\Delta + x + \lambda)^{-1} \rho_r, X\rangle (-\Delta + x + \lambda)^{-1} \rho_r \frac{dx}{\sqrt{x}} \ , \end{aligned} \tag{2.2}$$

where

$$\Gamma_r(\lambda) = -\frac{m_r c^2}{4\pi e^2} - \frac{2}{3} \langle (-\Delta + \lambda)^{-1} \rho, \rho\rangle \ .$$

Moreover, by the definition of Q_A, we have

$$L_m(X) := Q_{(H_m+\lambda)^{-1/2} X} = \left(\frac{4\pi e}{c}\right)^{-1} \frac{1}{\pi} \int_0^\infty \Gamma_m(x+\lambda)^{-1} \langle G_{x+\lambda}, X\rangle \frac{dx}{\sqrt{x}} \ . \tag{2.3}$$

Lemma 2.4. *If*

$$\lim_{r \downarrow 0} \sup_{|t| \le T} \|X_r(t) - X(t)\|_2 = 0 \ ,$$

then

$$\lim_{r \downarrow 0} \sup_{|t| \le T} \left| -\frac{e}{m_r c} \langle \rho_r, (H_r + \lambda)^{-1/2} X_r(t)\rangle - L_m(X(t)) \right| = 0 \ .$$

Proof. By (2.2) we have

$$
\begin{aligned}
&-\frac{e}{m_r c}\langle\rho_r,(H_r+\lambda)^{-1/2}X_r(t)\rangle=-\frac{e}{c}\frac{\langle\rho_r,(-\Delta+\lambda)^{-\frac12}X_r(t)\rangle}{m_r}\\
&-\frac{2e}{3\pi c}\int_0^\infty\Gamma_r(x+\lambda)^{-1}\langle(-\Delta+x+\lambda)^{-1}\rho_r,X_r(t)\rangle\frac{\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}\,\frac{dx}{\sqrt{x}}\ .
\end{aligned}
$$

The thesis follows by (2.3),

$$\lim_{r\downarrow 0}\ \Gamma_r(\lambda)=\Gamma_m(\lambda)\ ,$$

$$\lim_{r\downarrow 0}\ \sup_{|t|\le T}\left|\frac{\langle\rho_r,(-\Delta+\lambda)^{-\frac12}X_r(t)\rangle}{m_r}\right|=0\ ,$$

and

$$\lim_{r\downarrow 0}\ \frac{\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}=-\frac{3c^2}{8\pi e^2}\ .$$

□

Now we recall some results on abstract second order equations (see [F] for the proof of such results). Let H be a bounded from below (this is a necessary condition) self–adjoint operator on $L^2_*(\mathbb{R}^3)$, and let F the corresponding quadratic form. Then H generates a cosine operator function $C\ :\ \mathbb{R}\to\mathcal{L}(L^2_*(\mathbb{R}^3);L^2_*(\mathbb{R}^3))$, i.e. C is a strongly continuous function such that

$$C(0)=1,\qquad C(s+t)C(s-t)=2C(s)C(t)\ ,$$

$$D(H)=\{\ A\in L^2_*(\mathbb{R}^3)\ :\ \lim_{t\to 0}\frac{2}{t^2}(C(t)A-A)\text{ exists }\}\ ,$$

$$\forall A\in D(H)\qquad HA=\lim_{t\to 0}\frac{2}{t^2}(C(t)A-A)\ ,$$

$$C(t)D(H)\subseteq D(H)\ .$$

Moreover $\forall z\in\mathbb{C}$ such that $-z^2\in\rho(H)$ and Re $z>|\inf\sigma(H)\wedge 0|^{\frac12}$

$$z(H+z^2)^{-1}A=\int_0^\infty e^{-zt}C(t)A\,dt\ ,$$

and so, by $(H+z^2)^{-1}A=z^{-2}A-z^{-2}(H+z^2)HA$, and by inverse Laplace transform, $\forall t\ge 0$, $\forall A\in D(H)$,

$$C(t)A=A-\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}e^{zt}\frac{1}{z}(H+z^2)^{-1}HA\,dz\ ,\qquad x>|\inf\sigma(H)\wedge 0|^{\frac12}\ . \tag{2.4}$$

One defines then the sine operator function $S\ :\ \mathbb{R}\to\mathcal{L}(L^2_*(\mathbb{R}^3);L^2_*(\mathbb{R}^3))$ by

$$S(t)A:=\int_0^t C(s)A\,ds=\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}e^{zt}(H+z^2)^{-1}A\,dz\ ,\qquad x>|\inf\sigma(H)\wedge 0|^{\frac12}\ . \tag{2.5}$$

Obviously, if H is strictly positive, then, by functional calculus,

$$C(t)=\cos t\,H^{\frac{1}{2}},\qquad S(t)=H^{-\frac{1}{2}}\sin t\,H^{\frac{1}{2}}\ .$$

Given $A_0,\dot{A}_0\in L^2_*(\mathbb{R}^3)$, $X\in C(\mathbb{R};L^2_*(\mathbb{R}^3))$, let

$$A(t):=C(t)A_0+S(t)\dot{A}_0+\int_0^t S(t-s)X(s)\,ds\ .$$

Then $A\in C(\mathbb{R};L^2_*(\mathbb{R}^3))$. If $A_0\in D(F)$, $\dot{A}_0\in L^2_*(\mathbb{R}^3)$ then

$$A\in C(\mathbb{R};D(F))\cap C^1(\mathbb{R};L^2_*(\mathbb{R}^3))\ .$$

If $A_0\in D(H)$, $\dot{A}_0\in D(F)$, $X\in C^1(\mathbb{R};L^2_*(\mathbb{R}^3))$, then

$$A\in C(\mathbb{R};D(H))\cap C^1(\mathbb{R};D(F))\cap C^2(\mathbb{R};L^2_*(\mathbb{R}^3))\ ,$$

and $A(t)$ solves the inhomogeneous Cauchy problem

$$\begin{cases}\ddot{A}(t)=-HA(t)+X(t)\\ A(0)=A_0,\quad \dot{A}(0)=\dot{A}_0\ .\end{cases}$$

We will denote by $C_m(t)$, $S_m(t)$, and by $C_r(t)$, $S_r(t)$, the cosine and sine operator functions corresponding to $c^2\,H_m$ and $c^2\,H_r$ respectively.

We conclude this section with the following lemma which is one of the main technical points of the paper.

Lemma 2.5. *For any $\gamma\in C(I(T);\mathbb{R}^3)$ one has*

$$\lim_{r\downarrow 0}\ \sup_{|t|\le T}\left\|\frac{4\pi e}{m_r c}(H_r+\lambda)^{-\frac{1}{2}}M\gamma(t)\rho_r+\frac{c}{e}\Gamma_m(\lambda)^{-1}(H_m+\lambda)^{\frac{1}{2}}M\gamma(t)G_\lambda\right\|_2=0\ .$$

Proof. By the inequality

$$\|X+Y\|_2^2\le\left|\ \|X\|_2^2-\|Y\|_2^2\ \right|+2\left|\ \langle X,Z\rangle+\langle Y,Z\rangle\ \right|+2\left(\|X\|_2^2+\|Y\|_2^2\right)\ \|Y-Z\|_2$$

it follows

$$\begin{aligned}&\left\|\frac{4\pi e}{m_r c}(H_r+\lambda)^{-\frac{1}{2}}M\gamma(t)\rho_r+\frac{c}{e}\Gamma_m(\lambda)^{-1}(H_m+\lambda)^{\frac{1}{2}}M\gamma(t)G_\lambda\right\|_2^2\\
\le&\left|\ \left\|\frac{4\pi e}{m_r c}(H_r+\lambda)^{-\frac{1}{2}}M\gamma(t)\rho_r\right\|_2^2-\frac{c^2}{e^2}\Gamma_m(\lambda)^{-2}F_m(M\gamma(t)G_\lambda,M\gamma(t)G_\lambda)\right|\\
+2&\left|\frac{4\pi e}{m_r c}\langle(H_r+\lambda)^{-\frac{1}{2}}M\gamma(t)\rho_r,\gamma(t)X_j\rangle+\frac{c}{e}\Gamma_m(\lambda)^{-1}\langle(H_m+\lambda)^{\frac{1}{2}}M\gamma(t)G_\lambda,\gamma(t)X_j\rangle\right|\\
+2&\left(\left\|\frac{4\pi e}{m_r c}(H_r+\lambda)^{-\frac{1}{2}}M\gamma(t)\rho_r\right\|_2+\left\|\frac{c}{e}\Gamma_m(\lambda)^{-1}(H_m+\lambda)^{\frac{1}{2}}M\gamma(t)G_\lambda\right\|_2\right)\times\\
\times&\left\|\frac{c}{e}\Gamma_m(\lambda)^{-1}(H_m+\lambda)^{\frac{1}{2}}M\gamma(t)G_\lambda-M\gamma(t)X_j\right\|_2\\
\equiv&K_r(t)+2\left(B_{r,j}(t)+J_r(t)D_j(t)\right)\ ,\end{aligned}$$

where $\{X_j\}_1^\infty \subset L^2(\mathbb{R}^3)$ will be defined below. Since

$$F_m(M\gamma(t)G_\lambda, M\gamma(t)G_\lambda) = \Gamma_m(\lambda)|\gamma(t)|^2 \ ,$$

$$\langle (H_r+\lambda)^{-1}\rho_r,\rho_r\rangle = \langle (-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle + \frac{2}{3}\Gamma_r(\lambda)^{-1}|\langle (-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle|^2 \ , \tag{2.6}$$

and

$$\lim_{r\downarrow 0}\frac{\langle (-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r} = -\frac{3c^2}{8\pi e^2} \ , \tag{2.7}$$

one has

$$\begin{aligned}
&\lim_{r\downarrow 0}\left\| \frac{4\pi e}{m_r c}(H_r+\lambda)^{-\frac{1}{2}}M\gamma(t)\rho_r\right\|_2 \\
&= \lim_{r\downarrow 0}\frac{2}{3}\left(\frac{4\pi e}{m_r c}\right)^2 |\gamma(t)|^2\langle (H_r+\lambda)^{-1}\rho_r,\rho_r\rangle \\
&=\frac{4}{9}\left(\frac{4\pi e}{c}\right)^2 |\gamma(t)|^2\Gamma_m(\lambda)^{-1}\left(\frac{3c^2}{8\pi e^2}\right)^2 \\
&=\frac{c^2}{e^2}\Gamma_m(\lambda)^{-1}|\gamma(t)|^2 \\
&=\frac{c^2}{e^2}\Gamma_m(\lambda)^{-2}F_m(M\gamma(t)G_\lambda, M\gamma(t)G_\lambda) \ ,
\end{aligned}$$

and so

$$\lim_{r\downarrow 0}\sup_{|t|\le T} K_r(t) = 0 \ .$$

Now let $\{f_j\}_1^\infty \subset L^2(\mathbb{R}^3)$ be a sequence such that

$$\lim_{r\downarrow 0}\|(H_m+\lambda)^{-\frac{1}{2}}f_j - \frac{c}{e}\Gamma_m(\lambda)^{-1}(H_m+\lambda)^{\frac{1}{2}}G_\lambda\|_2 = 0 \ .$$

Then $M\gamma(t)(H_m+\lambda)^{-\frac{1}{2}}f_j = (H_m+\lambda)^{-\frac{1}{2}}M\gamma(t)f_j$, and

$$\lim_{r\downarrow 0}\sup_{|t|\le T}\|M\gamma(t)(H_m+\lambda)^{-\frac{1}{2}}f_j - \frac{c}{e}\Gamma_m(\lambda)^{-1}(H_m+\lambda)^{\frac{1}{2}}M\gamma(t)G_\lambda\|_2 = 0 \ .$$

Therefore, if $X_j := (H_m+\lambda)^{-\frac{1}{2}}f_j$, one has $\lim\limits_{j\uparrow\infty}\sup\limits_{|t|\le T} D_j(t) = 0$, and

$$\begin{aligned}
B_{r,j}(t) \le & \left|\frac{4\pi e}{m_r c}\langle (H_r+\lambda)^{-1}M\gamma(t)\rho_r,\gamma(t)f_j\rangle + \frac{c}{e}\langle \Gamma_m^{-1}M\gamma(t)G_\lambda,\gamma(t)f_j\rangle\right| \\
&+\left\|\frac{4\pi e}{m_r c}(H_r+\lambda)^{-\frac{1}{2}}M\gamma(t)\rho_r\right\| \left\|(H_r+\lambda)^{-\frac{1}{2}} - (H_m+\lambda)^{-\frac{1}{2}}\right\|_{2,2}|\gamma(t)| \ \|f_j\|_2 \ ,
\end{aligned}$$

so that

$$\forall j\in\mathbb{N}\qquad \lim_{r\downarrow 0}\sup_{|t|\le T} B_{r,j}(t) = 0 \ .$$

Since

$$\sup_{r>0}\sup_{|t|\le T} J_r(t) < +\infty\ ,$$

in conclusions there follows that, for any $\epsilon > 0$, there exists j_* such that

$$\lim_{r\downarrow 0}\sup_{|t|\le T} K_r(t) + 2\left(B_{r,j_*}(t) + J_r(t)D_{j_*}(t)\right) \le \epsilon\ .$$

$\square$

3. The point limit of the Maxwell–Lorentz equations with an external force.

Let us consider the regularized Maxwell–Lorentz system in the dipole approximation with an external force F, i.e. the system of equations on $L^2_*(\mathbb{R}^3)\times\mathbb{R}^3$ given by

$$\begin{cases} \dfrac{1}{c^2}\ddot A_r = \Delta A_r + \dfrac{4\pi e}{c} M\dot q_r\rho_r \\ m_r\ddot q_r = -\dfrac{e}{c}\langle\rho_r, \dot A_r\rangle + \displaystyle\int_{\mathbb{R}^3}\rho_r(x)F(x+q)dx \\ A_r(0) = A_0\in H^1_*(\mathbb{R}^3),\quad \dot A_r(0) = \dot A_0\in L^2_*(\mathbb{R}^3),\quad q_r(0) = q_0,\quad \dot q_r(0) = \dot q_0\ . \end{cases} \tag{3.1}$$

Now, as in [NP], we want to study the point limit (i.e. $r\downarrow 0$, or $\rho_r \overset{w}{\to} \delta_0$) of such a system. At first we rewrite (3.1) as

$$\begin{cases} \dot A_r = 4\pi c^2 E_r \\ \dot E_r = \dfrac{1}{4\pi}\Delta A_r - \dfrac{e^2}{m_r c^2} M\langle\rho_r, A_r\rangle\rho_r + \dfrac{e}{m_r c} M p_r\rho_r \\ \dot q_r = \dfrac{1}{m_r}p_r - \dfrac{e}{m_r c}\langle\rho_r, A_r\rangle \\ \dot p_r = \langle\rho_r, F_{q_r}\rangle \\ A_r(0) = A_0\in H^1_*(\mathbb{R}^3),\quad E_r(0) = E_0\in L^2_*(\mathbb{R}^3),\quad q_r(0) = q_0,\quad p_r(0) = p_0\ , \end{cases} \tag{3.2}$$

where $F_q(x) := F(x+q)$. Given any path p in $\mathbb{R}^3$ we define the time–dependent operator, with domain $H^2_*(\mathbb{R}^3)$,

$$\begin{aligned} H_{r,p(t)}A :=& -\Delta A + \frac{4\pi e^2}{m_r c^2} M\cdot\langle\rho_r, A\rangle\rho_r - \frac{4\pi e}{m_r c} M p(t)\rho_r \\ \equiv& H_r A - \frac{4\pi e}{m_r c} M p(t)\rho_r\ . \end{aligned}$$

Let us now try to determine the limit, as $r\downarrow 0$, of $H_{r,p}$. Given $\lambda > \lambda_0$, suppose that r is so small that $H_r+\lambda$ is strictly positive, and consider the Cauchy problem

$$\begin{cases} \dfrac{1}{c^2}\ddot A_r = -(H_r+\lambda)A_r + \dfrac{4\pi e}{m_r c} M p\rho_r \\ A(0) = A^r_0\in H^1_*(\mathbb{R}^3),\quad \dot A(0) = \dot A_0\in L^2_*(\mathbb{R}^3)\ . \end{cases}$$

Its (mild) solution is given by

$$
\begin{aligned}
A_r(t) &= \cos(ct(H_r+\lambda)^{\frac12})A_0^r + \sin(ct(H_r+\lambda)^{\frac12})(H_r+\lambda)^{-\frac12}\dot A_0 \\
&\quad + \frac{4\pi e}{m_r}\int_0^t \sin(c(t-s)(H_r+\lambda)^{\frac12})(H_r+\lambda)^{-\frac12}Mp(s)\rho_r\, ds \ ,
\end{aligned}
$$

which, after an integration by parts, can be rewritten as (here and below we are supposing that p is sufficiently regular)

$$
\begin{aligned}
A_r(t) &= \cos(ct(H_r+\lambda)^{\frac12})A_0^r + \sin(ct(H_r+\lambda)^{\frac12})(H_r+\lambda)^{-\frac12}\dot A_0 \\
&\quad - \frac{4\pi e}{m_r c}\int_0^t \cos(c(t-s)(H_r+\lambda)^{\frac12})(H_r+\lambda)^{-1}M\dot p(s)\rho_r ds \\
&\quad + \frac{4\pi e}{m_r c}(H_r+\lambda)^{-1}Mp(t)\rho_r - \frac{4\pi e}{m_r c}\cos(ct(H_r+\lambda)^{\frac12})(H_r+\lambda)^{-1}Mp(0)\rho_r \ .
\end{aligned}
$$

By thm. 2.1 and by lemma 2.5, if

$$
\lim_{r\downarrow 0}\ \|(H_r+\lambda)^{\frac12}A_0^r - (H_m+\lambda)^{\frac12}A_0\|_2 = 0 \ ,
$$

then

$$
\lim_{r\downarrow 0}\ \sup_{|t|\le T}\|(H_r+\lambda)^{\frac12}A_r(t) - (H_m+\lambda)^{\frac12}A(t)\|_2 = 0 \ ,
$$

where

$$
\begin{aligned}
A(t) &= \cos(ct(H_m+\lambda)^{\frac12})A_0 + \sin(ct(H_m+\lambda)^{\frac12})(H_m+\lambda)^{-\frac12}\dot A_0 \\
&\quad + \frac{c}{e}\Gamma_m(\lambda)^{-1}\int_0^t \cos(c(t-s)(H_m+\lambda)^{\frac12})M\dot p(s)G_\lambda\, ds \\
&\quad - \frac{c}{e}\Gamma_m(\lambda)^{-1}Mp(t)G_\lambda - \frac{c}{e}\Gamma_m(\lambda)^{-1}\cos(ct(H_m+\lambda)^{\frac12})Mp(0)G_\lambda \ .
\end{aligned}
$$

Integrating by parts we can rewrite the above expression as

$$
\begin{aligned}
A(t) + \frac{c}{e}\Gamma_m(\lambda)^{-1}Mp(t)G_\lambda &= \cos(ct(H_m+\lambda)^{\frac12})\left(A_0 + \frac{c}{e}\Gamma_m(\lambda)^{-1}Mp(0)G_\lambda\right) \\
&+ \sin(ct(H_m+\lambda)^{\frac12})(H_m+\lambda)^{-\frac12}\left(\dot A_0 + \frac{c}{e}\Gamma_m(\lambda)^{-1}M\dot p(0)G_\lambda\right) \\
&+ \frac{1}{e}\Gamma_m(\lambda)^{-1}\int_0^t \sin(c(t-s)(H_m+\lambda)^{\frac12})(H_m+\lambda)^{-\frac12}M\ddot p(s)G_\lambda\, ds \ .
\end{aligned}
$$

This tell us that

$$
A_p(t) := A(t) + \frac{c}{e}\Gamma_m(\lambda)^{-1}Mp(t)G_\lambda
$$

solves the Cauchy problem

$$
\begin{cases}
\frac{1}{c^2}\ddot A_p = -\,(H_m+\lambda)A_p + \frac{c}{e}\Gamma_m(\lambda)^{-1}M\ddot p G_\lambda \\
A_p(0) = A_0 + \frac{c}{e}\Gamma_m(\lambda)^{-1}Mp(0)G_\lambda, \quad \dot A_p(0) = \dot A_0 + \frac{c}{e}\Gamma_m(\lambda)^{-1}M\dot p(0)G_\lambda \ ,
\end{cases}
$$

and so $A(t)$ solves the Cauchy problem

$$\begin{cases} \frac{1}{c^2}\ddot{A} = -\,(H_m+\lambda)\left(A+\frac{c}{e}\Gamma_m(\lambda)^{-1}MpG_\lambda\right) \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0\ . \end{cases}$$

This induces us to define, on the time–dependent domain

$$D(H_{m,p(t)}) := D(H_m) - \frac{c}{e}\Gamma_m(\lambda)^{-1}Mp(t)G_\lambda\ ,$$

the time–dependent operator $H_{m,p(t)}$ given by

$$(H_{m,p(t)}+\lambda)A := (H_m+\lambda)\left(A+\frac{c}{e}\Gamma_m(\lambda)^{-1}Mp(t)G_\lambda\right)$$

or, alternatively, by

$$H_{m,p(t)}A := H_m\left(A+\frac{c}{e}\Gamma_m(\lambda)^{-1}Mp(t)G_\lambda\right)+\lambda\frac{c}{e}\Gamma_m(\lambda)^{-1}Mp(t)G_\lambda\ . \tag{3.3}$$

In the next two theorems we give the (local) existence results for the two dynamical systems defined by equations (3.2) and by their conjectured point limit.

Theorem 3.1. *Let $F \in \mathrm{Lip}(\mathbb{R}^3;\mathbb{R}^3)$. Then there exists $T > 0$, independent of r, such that the Cauchy problem*

$$\begin{cases} \dot{A}_r = 4\pi c^2 E_r \\ \dot{E}_r = \frac{1}{4\pi}\Delta A_r - \frac{e^2}{m_r c^2}M\langle\rho_r, A_r\rangle\rho_r + \frac{e}{m_r c}Mp_r\rho_r \\ \dot{q}_r = \frac{1}{m_r}p_r - \frac{e}{m_r c}\langle\rho_r, A_r\rangle \\ \dot{p}_r = \langle\rho_r, F_{q_r}\rangle \\ A_r(0) = A_0^r \in H_*^1(\mathbb{R}^3), \quad E_r(0) = E_0 \in L_*^2(\mathbb{R}^3), \quad q_r(0) = q_0, \quad p_r(0) = p_0 \end{cases}$$

has an unique mild solution

$$(A_r, E_r, q_r, p_r) \in C(I(T); H_*^1(\mathbb{R}^3)) \times C(I(T); L_*^2(\mathbb{R}^3)) \times C^2(I(T);\mathbb{R}^3) \times C^1(I(T);\mathbb{R}^3)\ .$$

Proof. Given $T > 0$, let us introduce the map

$$\psi_r : C_{\dot{q}_0^r}(I(T);\mathbb{R}^3) \to C_{\dot{q}_0^r}(I(T);\mathbb{R}^3) \cap C^1(I(T);\mathbb{R}^3) \tag{3.4}$$

$$\psi_r(\gamma)(t) := \frac{1}{m_r}\tilde{\gamma}(t) - \frac{e}{m_r c}\langle A_r^\gamma(t), \rho_r\rangle\ ,$$

where

$$C_{\dot{q}_0^r}(I(T);\mathbb{R}^3) := \{\gamma \in C(I(T);\mathbb{R}^3)\ :\ \gamma(0) = \dot{q}_0^r\}, \qquad \dot{q}_0^r := \frac{1}{m_r}p_0 - \frac{e}{m_r c}\langle\rho_r, A_0^r\rangle\ ,$$

$$\tilde{\gamma}(t) := p_0 + \int_0^t \int_{\mathbb{R}^3} \rho_r(x) F\left(x + q_0 + \int_0^s \gamma(u)\, du\right)\, dx\, ds\ ,$$

and where $A_r^\gamma \in C(I(T); H_*^1(\mathbb{R}^3)) \cap C^1(I(T); L_*^2(\mathbb{R}^3))$ denotes the mild solution of the Cauchy problem

$$\begin{cases} \dfrac{1}{c^2}\ddot{A}_r^\gamma = -\, H_{r,\tilde{\gamma}} A_r^\gamma \\ A_r^\gamma(0) = A_0^r, \quad \dot{A}_r^\gamma(0) = \dot{A}_0 \equiv\ 4\pi c^2 E_0\ . \end{cases}$$

If we prove that ψ_r has an unique fixed point γ_r^*, then

$$\left(A_r^{\gamma_r^*},\ \frac{1}{4\pi c^2}\dot{A}_r^{\gamma_r^*},\ q_0 + \int_0^t \gamma_r^*(s)\, ds,\ \tilde{\gamma}_r^*\right)$$

gives the solution. Let us now show that ψ_r is a contraction on $C_{\dot{q}_0^r}(I(T); \mathbb{R}^3)$. We have

$$|\psi_r(\gamma_1)(t) - \psi_r(\gamma_2)(t)| \leq c_F T^2 \frac{1}{m_r}\|\gamma_1 - \gamma_2\|_\infty + \frac{e}{|m_r|c}|\langle A_r^{\gamma_1}(t) - A_r^{\gamma_2}(t), \rho_r\rangle|\ ,$$

where c_F denotes the Lipschitz constant of F. Since

$$A_r^\gamma(t) = C_r(t)A_0^r + S_r(t)\dot{A}_0 + \frac{4\pi ec}{m_r}\int_0^t S_r(t-s)M\tilde{\gamma}(s)\rho_r\, ds\ , \tag{3.5}$$

we have

$$\frac{e}{|m_r|c}|\langle A_r^{\gamma_1}(t) - A_r^{\gamma_2}(t), \rho_r\rangle| \leq\ c_F T^2\, \frac{8\pi e^2}{3m_r^2}\, \|\gamma_1 - \gamma_2\|_\infty \left|\int_0^t \langle S_r(t-s)\rho_r, \rho_r\rangle\, ds\right|\ .$$

Since (see (2.5))

$$\langle S_r(t)\rho_r, \rho_r\rangle = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{zt}\langle (H_r + z^2)^{-1}\rho_r, \rho_r\rangle\, dz\ ,$$

by (2.6) and (2.7) it follows that

$$\frac{8\pi e^2}{3m_r^2}\ \left|\int_0^t \langle S_r(t-s)\rho_r, \rho_r\rangle\, ds\right|$$

converges, uniformly in t over compact intervals, and so there exists $c(T)$, with $c(T)\downarrow 0$ as $T \downarrow 0$, such that

$$\sup_{|t|\leq T}\sup_{r>0}\ \frac{8\pi e^2}{3m_r^2}\ \left|\int_0^t \langle S_r(t-s)\rho_r, \rho_r\rangle\, ds\right| \leq c(T)\ . \tag{3.6}$$

Therefore

$$\|\psi_r(\gamma_1) - \psi_r(\gamma_2)\|_\infty \leq c_F T^2\left(c(T) + \frac{1}{|m_r|}\right)\|\gamma_1 - \gamma_2\|_\infty\ .$$

Since $1/|m_r| \to 0$, we can choose, independently of r, a $T > 0$ such that

$$c_F T^2 \left(c(T) + \frac{1}{m_r} \right) < 1 \ ,$$

and the proof is concluded by the contraction mapping principle. □

Theorem 3.2. *Let $F \in \mathrm{Lip}(\mathbb{R}^3; \mathbb{R}^3)$. Then there exists $T > 0$ such that the Cauchy problem*

$$\begin{cases} \dot{A} = 4\pi c^2 E \\ \dot{E} = -\dfrac{1}{4\pi} H_{m,p} A \\ \dot{q} = Q_A \\ \dot{p} = F(q) \\ A(0) = A_0 \in D(F_m), \quad E(0) = E_0 \in L^2_*(\mathbb{R}^3), \quad q(0) = q_0, \quad p(0) = p_0 \ , \end{cases}$$

has an unique mild solution

$$(A, E, q, p) \in C(I(T); D(F_m)) \times C(I(T); L^2_*(\mathbb{R}^3)) \times C^1(I(T); \mathbb{R}^3) \times C^1(I(T); \mathbb{R}^3) \ .$$

Proof. Given $T > 0$, let us introduce the map

$$\psi : C_{Q_{A_0}}(I(T); \mathbb{R}^3) \to C_{Q_{A_0}}(I(T); \mathbb{R}^3) \tag{3.7}$$

$$\psi(\gamma)(t) := Q_{A^\gamma(t)} \ ,$$

where $C_{Q_{A_0}}(I(T); \mathbb{R}^3) := \{\gamma \in C(I(T); \mathbb{R}^3) \ : \ \gamma(0) = Q_{A_0}\}$, and A^γ denotes the mild solution of the Cauchy problem

$$\begin{cases} \dfrac{1}{c^2} \ddot{A}^\gamma = - H_{m,\tilde{\gamma}} A^\gamma \\ A^\gamma(0) = A_0, \quad \dot{A}^\gamma(0) = \dot{A}_0 \equiv \ 4\pi c^2 E_0 \ , \end{cases} \tag{3.8}$$

with

$$\tilde{\gamma}(t) := p_0 + \int_0^t F\left(q_0 + \int_0^s \gamma(u) \, du \right) \, ds \ .$$

If we prove that ψ has an unique fixed point γ^* then

$$\left(A^{\gamma^*}, \ \frac{1}{4\pi c^2} \dot{A}^{\gamma^*}, \ q_0 + \int_0^t \gamma^*(s) \, ds, \ \tilde{\gamma}^* \right)$$

gives the solution. Let us now prove that ψ is a contraction. Defining

$$\tilde{A} := A^\gamma + \frac{c}{e} \Gamma_m(\lambda)^{-1} M \tilde{\gamma} G_\lambda \ ,$$

by (3.3), A^γ solves the Cauchy problem (3.8) if and only if $\tilde A$ solves

$$\begin{cases} \frac{1}{c^2}\ddot{\tilde A} = - H_m\tilde A - \lambda\frac{c}{e}\Gamma_m(\lambda)^{-1}M\tilde\gamma G_\lambda + \frac{c}{e}\Gamma_m(\lambda)^{-1}M\ddot{\tilde\gamma}G_\lambda \\ \tilde A(0) = A_0 + \frac{c}{e}\Gamma_m(\lambda)^{-1}Mp_0G_\lambda, \quad \dot{\tilde A}(0) = \dot A_0 + \frac{c}{e}\Gamma_m(\lambda)^{-1}MF(q_0)G_\lambda\ . \end{cases}$$

Therefore the solution of (3.8) is given by

$$\begin{aligned} A^\gamma(t) =& C_m(t)\left(A_0 + \frac{c}{e}\Gamma_m(\lambda)^{-1}Mp_0G_\lambda\right) + S_m(t)\left(\dot A_0 + \frac{c}{e}\Gamma_m(\lambda)^{-1}MF(q_0)G_\lambda\right) \\ &+\frac{c}{e}\Gamma_m(\lambda)^{-1}\int_0^t S_m(t-s)\left(-\lambda M\tilde\gamma G_\lambda + M\ddot{\tilde\gamma}(s)G_\lambda\right)\,ds \\ &-\frac{c}{e}\Gamma_m(\lambda)^{-1}M\tilde\gamma(t)G_\lambda\ , \end{aligned}$$

which, after an integration by parts, can be rewritten as

$$\begin{aligned} A^\gamma(t) =& C_m(t)\left(A_0 + \frac{c}{e}\Gamma_m(\lambda)^{-1}Mp_0G_\lambda\right) + S_m(t)\dot A_0 \\ &+\frac{c}{e}\Gamma_m(\lambda)^{-1}\int_0^t C_m(t-s)M\dot{\tilde\gamma}(s)G_\lambda\,ds \\ &+\lambda\frac{c}{e}\Gamma_m(\lambda)^{-1}\int_0^t S_m(t-s)M\tilde\gamma(s)G_\lambda\,ds - \frac{c}{e}\Gamma_m(\lambda)^{-1}M\tilde\gamma(t)G_\lambda\ . \end{aligned}$$

Therefore we have

$$\begin{aligned} &|\psi(\gamma_1)(t) - \psi(\gamma_2)(t)| \\ &\le \Gamma_m(\lambda)^{-1}\frac{1}{6\pi^2}\int_0^t |\dot{\tilde\gamma}_1(s) - \dot{\tilde\gamma}_2(s)|\ |L_m(C_m(t-s)(H_m+\lambda)^{\frac12}G_\lambda)|\,ds \\ &+ \lambda\Gamma_m(\lambda)^{-1}\frac{1}{6\pi^2}\int_0^t |\tilde\gamma_1(s) - \tilde\gamma_2(s)|\ |L_m(S_m(t-s)(H_m+\lambda)^{\frac12}G_\lambda)|\,ds \\ &+\frac{1}{4\pi}\Gamma_m(\lambda)^{-1}|\tilde\gamma_1(t) - \tilde\gamma_2(t)| \\ &\le \Gamma_m(\lambda)^{-1}\,\frac{c_FT^2}{4\pi}\left(\frac{2}{3\pi}\int_0^t |L_m(C_m(t-s)(H_m+\lambda)^{\frac12}G_\lambda)|\,ds \right.\\ &\left. + \lambda T\frac{2}{3\pi}\int_0^t |L_m(S_m(t-s)(H_m+\lambda)^{\frac12}G_\lambda)|\,ds + 1\right)\|\gamma_1-\gamma_2\|_\infty\ , \end{aligned}$$

and so there exists a T sufficiently small such that the map ψ is a contraction. The proof is then concluded by the contraction mapping principle. □

Before stating our convergence theorem we need the following preliminary result:

Theorem 3.3. *For any* $\gamma_r \in C^1(I(T);\mathbb{R}^3)$, $r>0$, $T>0$, *let*

$$A_r \in C(I(T);H^1_*(\mathbb{R}^3)) \cap C^1(I(T);L^2_*(\mathbb{R}^3))$$

be the mild solution of the Cauchy problem

$$\begin{cases} \dfrac{1}{c^2}\ddot{A}_r = - H_{r,\gamma_r} A_r \\ A_r(0) = A_0^r \in H^1_*(\mathbb{R}^3), \quad \dot{A}_r(0) = \dot{A}_0 \in L^2_*(\mathbb{R}^3) \ , \end{cases}$$

with $\lambda > \lambda_0$. *If*

$$\lim_{r\downarrow 0} \|\gamma_r - \gamma\|_\infty + \|\dot\gamma_r - \dot\gamma\|_\infty = 0 \ ,$$

$\gamma \in C^1(I(T), \mathbb{R}^3)$, *and*

$$\lim_{r\downarrow 0} \ \|(H_r + \lambda)^{\frac{1}{2}} A_0^r - (H_m + \lambda)^{\frac{1}{2}} A_0\|_2 = 0 \ ,$$

then

$$\lim_{r\downarrow 0} \ \sup_{|t|\le T} \|(H_r + \lambda)^{\frac{1}{2}} A_r(t) - (H_m + \lambda)^{\frac{1}{2}} A(t)\|_2 = 0,$$
$$\lim_{r\downarrow 0} \ \sup_{|t|\le T} \|\dot{A}_r(t) - \dot{A}(t)\|_2 = 0 \ ,$$

where $A \in C(I(T); D(F_m)) \cap C^1(\mathbb{R}; L^2_*(\mathbb{R}^3))$ *is the mild solution of the Cauchy problem*

$$\begin{cases} \dfrac{1}{c^2}\ddot{A} = -H_{m,\gamma} A \\ A(0) = A_0 \in D(F_m), \quad \dot{A}(0) = \dot{A}_0 \in L^2_*(\mathbb{R}^3) \ . \end{cases}$$

Proof. By proceeding as in the proof of [NP, thm.2.8], and by the definitions of H_{r,γ_r} and $H_{m,\gamma}$, it will suffice to prove the analogous statements for the mild solutions of the Cauchy problems ($\lambda > \lambda_0$)

$$\begin{cases} \dfrac{1}{c^2}\ddot{A}_r = - (H_r + \lambda) A_r + \dfrac{4\pi e}{m_r c} M \gamma_r \rho_r \\ A(0) = A_0^r, \quad \dot{A}(0) = \dot{A}_0 \ , \end{cases}$$

and

$$\begin{cases} \dfrac{1}{c^2}\ddot{A} = - (H_{m,\gamma} + \lambda) A \equiv -(H_m + \lambda)(A + \dfrac{c}{e}\Gamma_m(\lambda)^{-1} M\gamma G_\lambda) \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0 \ . \end{cases}$$

Proceeding as in the calculations above leading to the definition of $H_{m,p}$ (see the beginning of this section), we have that the solution of the first problem is given by

$$\begin{aligned} A_r(t) &= \cos(ct(H_r + \lambda)^{\frac{1}{2}}) A_0^r + \sin(ct(H_r + \lambda)^{\frac{1}{2}})(H_r + \lambda)^{-\frac{1}{2}} \dot{A}_0 \\ &\quad - \frac{4\pi e}{m_r c} \int_0^t \cos(c(t-s)(H_r + \lambda)^{\frac{1}{2}})(H_r + \lambda)^{-1} M \dot\gamma_r(s) \rho_r ds \\ &\quad + \frac{4\pi e}{m_r c}(H_r + \lambda)^{-1} M \gamma_r(t) \rho_r \ , \end{aligned}$$

and the solution of the second one is given by

$$\begin{aligned}A(t) =& \cos(ct(H_m+\lambda)^{\frac{1}{2}})A_0+\sin(ct(H_m+\lambda)^{\frac{1}{2}})(H_m+\lambda)^{-\frac{1}{2}}\dot{A}_0\\ &+\frac{c}{e}\Gamma_m(\lambda)^{-1}\int_0^t \cos(c(t-s)(H_m+\lambda)^{\frac{1}{2}})M\dot{\gamma}(s)G_\lambda\, ds\\ &-\frac{c}{e}\Gamma_m(\lambda)^{-1}M\gamma(t)G_\lambda\ .\end{aligned}$$

Now, by the known strong resolvent convergence of H_r to H_m (see thm. 2.1), and by the C^1 convergence of $\{\gamma_r\}_{r>0}$ to γ, the proof is concluded by lemma 2.5. □

We now come to the main result of this paper:

Theorem 3.4. *Let $F\in \mathrm{Lip}(\mathbb{R}^3;\mathbb{R}^3)$, $|F(x)|\le M_0(1+|x|)$, $\lambda>\lambda_0$, and $E_0\in L^2_*(\mathbb{R}^3)$. Let $A_0^r\in H^1_*(\mathbb{R}^3)$, $A_0\in D(F_m)$, such that*

$$\lim_{r\downarrow 0}\ \|(H_r+\lambda)^{\frac{1}{2}}A_0^r-(H_m+\lambda)^{\frac{1}{2}}A_0\|_2=0\ . \tag{3.9}$$

Then there exists $T>0$ such that, if

$$(A_r,E_r,q_r,p_r)\in C(I(T);H^1_*(\mathbb{R}^3))\times C(I(T);L^2_*(\mathbb{R}^3))\times C^2(I(T);\mathbb{R}^3)\times C^1(I(T);\mathbb{R}^3)$$

denotes the mild solution of the Cauchy problem

$$\begin{cases}\dot{A}_r =&4\pi c^2E_r\\ \dot{E}_r =&\frac{1}{4\pi}\Delta A_r-\frac{e^2}{m_rc^2}M\langle\rho_r,A_r\rangle\rho_r+\frac{e}{m_rc}Mp_r\rho_r\\ \dot{q}_r =&\frac{1}{m_r}p_r-\frac{e}{m_rc}\langle\rho_r,A_r\rangle\\ \dot{p}_r =&\langle\rho_r,F_{q_r}\rangle\\ A_r(0) =&A_0^r,\quad E_r(0)=E_0,\quad q_r(0)=q_0,\quad p_r(0)=p_0\ ,\end{cases}$$

then

$$\begin{aligned}&\lim_{r\downarrow 0}\ \sup_{|t|\le T}\|(H_r+\lambda)^{\frac{1}{2}}A_r(t)-(H_m+\lambda)^{\frac{1}{2}}A(t)\|_2=0,\\ &\lim_{r\downarrow 0}\ \sup_{|t|\le T}\|E_r(t)-E(t)\|_2=0,\\ &\lim_{r\downarrow 0}\ \sup_{|t|\le T}|\dot{q}_r(t)-\dot{q}(t)|+\sup_{|t|\le T}|q_r(t)-q(t)|=0,\\ &\lim_{r\downarrow 0}\ \sup_{|t|\le T}|\dot{p}_r(t)-\dot{p}(t)|+\sup_{|t|\le T}|p_r(t)-p(t)|=0\ ,\end{aligned}$$

where

$$(A,E,q,p)\in C(I(T);D(F_m))\times C(I(T);L^2_*(\mathbb{R}^3))\times C^1(I(T);\mathbb{R}^3)\times C^1(I(T);\mathbb{R}^3)$$

denotes the mild solution of the Cauchy problem

$$\begin{cases} \dot{A} = 4\pi c^2 E \\ \dot{E} = -\dfrac{1}{4\pi} H_{m,p} A \\ \dot{q} = Q_A \\ \dot{p} = F(q) \\ A(0) = A_0, \quad E(0) = E_0, \quad q(0) = q_0, \quad p(0) = p_0 \ . \end{cases} \tag{3.10}$$

Proof. Let us denote by γ_r^* and γ^* the fixed points of the maps ψ_r (see (3.4)) and ψ (see (3.7)) respectively. If we prove that

$$\lim_{r\downarrow 0} \|\gamma_r^* - \gamma^*\|_\infty = 0 \ , \tag{3.11}$$

then, obviously,

$$\lim_{r\downarrow 0} \|\tilde{\gamma}_r^* - \tilde{\gamma}^*\|_\infty + \|\dot{\tilde{\gamma}}_r^* - \dot{\tilde{\gamma}}^*\|_\infty = 0 \ ,$$

and so the proof is concluded by thm. 3.3. Let us consider an arbitrary family $\{\gamma_r\}_{r>0} \subset C_{\dot{q}_0^r}(I(T);\mathbb{R}^3)$, such that $\|\gamma_r - \gamma\|_\infty \to 0$, $\gamma \in C_{Q_{A_0}}(I(T);\mathbb{R}^3)$. Then, writing $A_r^{\gamma_r} = (H_r + \lambda)^{-\frac{1}{2}} X_r$ and $A^\gamma = (H_m + \lambda)^{-\frac{1}{2}} X$, by thm. 3.3 we have

$$\lim_{r\downarrow 0} \sup_{|t|\leq T} \|X_r(t) - X(t)\|_2 = 0 \ .$$

Therefore, by lemma 2.4, we have

$$\lim_{r\downarrow 0} \left\| \frac{e}{m_r c} \langle A_r^{\gamma_r}, \rho_r\rangle + Q_{A^\gamma} \right\|_\infty = 0 \ .$$

This gives

$$\lim_{r\downarrow 0} \|\psi_r(\gamma_r) - \psi(\gamma)\|_\infty = 0 \ ,$$

and so, for any $n \in \mathbb{N}$,

$$\lim_{r\downarrow 0} \|\psi_r^n(\gamma_r) - \psi^n(\gamma)\|_\infty = 0 \ .$$

Since $\gamma_r^* = \psi_r(\gamma_r^*)$, by (3.5) and (3.6), defining $A_r^0(t) := C_r(t)A_0^r + 4\pi c^2 S_m(t) E_0$, we have

$$\begin{aligned} &|\psi_r(\gamma_r^*)(t)| \leq \frac{e}{|m_r|c} |\langle A_r^{\gamma_r^*}(t), \rho_r\rangle| + \frac{1}{|m_r|} |\tilde{\gamma}_r^*(t)| \\ \leq& \frac{e}{|m_r|c} \|\langle A_r^0, \rho_r\rangle\|_\infty + \frac{8\pi e^2}{3m_r^2} \|\tilde{\gamma}_r^*\|_\infty \left| \int_0^t \langle S_r(t-s)\rho_r, \rho_r\rangle \, ds \right| + \frac{1}{|m_r|} \|\tilde{\gamma}_r^*\|_\infty \\ \leq& K_0 + \left(c(T) + \frac{1}{|m_r|} \right) \|\tilde{\gamma}_r^*\|_\infty \\ \leq& K_0 + \left(c(T) + \frac{1}{|m_r|} \right) \left(|p_0| + M_0 T \int_{\mathbb{R}^3} |x| \rho_r(x) \, dx + M_0 T(1 + |q_0| + T\|\gamma_r^*\|_\infty) \right) \ . \end{aligned}$$

Therefore, if r and T are sufficiently small, then

$$\|\gamma_r^*\|_\infty \leq \left(1 - K_1\left(c(T) + \frac{1}{|m_r|}\right)\right)^{-1} K_2 \ ,$$

and so

$$\sup_{r>0} \|\gamma_r^*\|_\infty < +\infty \ .$$

These results imply, denoting by $\sigma < 1$ the common, independent of r, constant of contractivity of ψ_r and ψ,

$$\begin{aligned}\|\gamma_r^* - \gamma^*\|_\infty &\leq \|\psi_r^n(\gamma_r) - \psi^n(\gamma)\|_\infty + \sup_{r>0} \|\psi_r^n(\gamma_r) - \gamma_r^*\|_\infty + \|\psi^n(\gamma) - \gamma^*\|_\infty \\ &\leq \|\psi_r^n(\gamma_r) - \psi^n(\gamma)\|_\infty + \sigma^n \sup_{r>0} \|\gamma_r - \gamma_r^*\|_\infty + \|\psi^n(\gamma) - \gamma^*\|_\infty \ .\end{aligned}$$

Therefore (3.11) is proven, and the proof is done. □

Remark 3.6. Here we make the connection with the results obtained, for the case $F = 0$, in [NP]. The results there stated are a particular case of the ones given here: indeed in [NP, corollary 2.9] the initial fields A_0^r and A_0 are not arbitrary elements in $H_*^1(\mathbb{R}^3)$ and $D(F_m)$ respectively (however they belong to L^2–dense subsets of $H_*^1(\mathbb{R}^3)$ and $D(F_m)$ respectively). Moreover the convergence conditions (3.9) are weaker than the (2.8) used in [NP]. In the present setting, writing $A_0^r = (H_r + \lambda)^{-\frac{1}{2}} X_0^r$, these would correspond, besides (3.9), to

$$\lim_{r\downarrow 0} \langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_0^r \rangle = \frac{4\pi e}{c} \Gamma_m(\lambda) Q_{A_0} \equiv \frac{4\pi e}{c} \Gamma_m(\lambda) \dot{q}_0 \ . \tag{3.12}$$

Since the relations (2.8) in [NP] are declared to be necessary and sufficient for the convergence, a contradiction seems to appear. This is not the case. Indeed in [NP] we studied the limit of the Maxwell–Lorentz system when the limit total linear momentum p_0 is vanishing. In fact one has, by (2.2.), (2.3), and (3.9)

$$\begin{aligned}&\lim_{r\downarrow 0} p_0^r = \lim_{r\downarrow 0} \ m_r \dot{q}_0 + \frac{e}{c} \langle \rho_r, A_0^r \rangle \\ &= \lim_{r\downarrow 0} \ \left(-\frac{e}{c}\Gamma_r(\lambda) - \frac{2e}{3c}\langle(-\Delta + \lambda)^{-1}\rho_r, \rho_r\rangle\right) \frac{1}{\pi} \int_0^\infty \Gamma_m(x+\lambda)^{-1} \langle G_{x+\lambda}, X_0^r \rangle \frac{dx}{\sqrt{x}} \\ &\quad + \frac{e}{c} \langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_0^r \rangle \\ &\quad + \frac{2e}{3c\pi} \int_0^\infty \Gamma_r(x+\lambda)^{-1} \langle (-\Delta + x + \lambda)^{-1} \rho_r, X_0^r \rangle \langle (-\Delta + x + \lambda)^{-1} \rho_r, \rho_r \rangle \frac{dx}{\sqrt{x}} \\ &= \lim_{r\downarrow 0} \ -\frac{4\pi e^2}{c^2} \Gamma_m(\lambda) \dot{q}_0 + \frac{e}{c} \langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_0^r \rangle \ ,\end{aligned}$$

and so, given (3.9), (3.12) is equivalent to $p_0 = 0$. This is consistent with the fact that the limit equations (3.10) coincide, in the case $F = 0$, $p_0 = 0$, with the ones obtained in [NP]. As a last remark regarding [NP] let us point out that, since, when $F = 0$, the linear momentum is conserved, the condition (3.12) is preserved by the flow.

4. The limit dynamics.

In this section we want to give a more detailed description of the solution of the Cauchy problem (3.10). Let us begin with an alternative characterization of the domain of $H_{m,p}$. By thm. 2.2 we have that the vectors in $D(H_{m,p})$ are of the type

$$A = A_\lambda + \Gamma_m(\lambda)^{-1} M \left(A_\lambda(0) - \frac{c}{e}p\right) G_\lambda, \quad A_\lambda \in H^2_*(\mathbb{R}^3), \quad -\lambda \in \rho(H_m), \quad \lambda > 0 \; .$$

This obviously implies

$$A_\lambda(0) = \left(\frac{4\pi e}{c}\right) \Gamma_m(\lambda) Q_A + \frac{c}{e}p \; ,$$

and, by the definition of $\Gamma_m(\lambda)$, we have

$$A_\lambda(0) - \frac{2}{3}\frac{e}{c}\sqrt{\lambda} Q_A = -\frac{mc}{e} Q_A + \frac{c}{e}p \; .$$

Since

$$\lim_{r\downarrow 0} \frac{1}{4\pi r^2} \int_{S_r} \left(A - \frac{4\pi e}{c} M Q_A G_0\right) d\mu_r(x) = A_\lambda(0) - \frac{2}{3}\frac{e}{c}\sqrt{\lambda} Q_A \; ,$$

in conclusion we have the following

Lemma 4.1. *$A(t) \in D(H_{m,p(t)})$ if and only if $A(t) - \frac{4\pi e}{c} M Q_{A(t)} G_\lambda \in H^2_*(\mathbb{R}^3)$, $-\lambda \in \rho(H_m)$, $\lambda > 0$, and the following boundary condition holds:*

$$\lim_{r\downarrow 0} \frac{1}{4\pi r^2} \int_{S_r} \left(A(t) - \frac{4\pi e}{c} M Q_{A(t)} G_0\right) d\mu_r(x) = -\frac{mc}{e} Q_{A(t)} + \frac{c}{e}p(t) \; . \tag{4.1}$$

Moreover

$$H_{m,p(t)} A(t) = -\Delta\left(A(t) - \frac{4\pi e}{c} M Q_{A(t)} G_\lambda\right) - \lambda \frac{4\pi e}{c} M Q_{A(t)} G_\lambda \; .$$

The next result is analogous to thm. 3.3 in [NP]: it characterizes the solutions of the limit equation (3.10).

Theorem 4.2. *Given $F \in \mathrm{Lip}(\mathbb{R}^3;\mathbb{R}^3)$, let*

$$(A, E, q, p) \in C^1([0,T]; D(F_m)) \times C^1([0,T]; L^2_*(\mathbb{R}^3)) \times C^2([0,T];\mathbb{R}^3) \times C^1([0,T];\mathbb{R}^3)$$

be the strict solution of the Cauchy problem

$$\begin{cases} \dot A = 4\pi c^2 E \\ \dot E = -\dfrac{1}{4\pi} H_{m,p} A \\ \dot q = Q_A \\ \dot p = F(q) \\ A(0) = A_0 \in D(H_{m,p_0}), \quad E(0) = E_0 \in D(F_m), \quad q(0) = q_0, \quad p(0) = p_0 \; , \end{cases} \tag{4.2}$$

Then

$$A(t) = A_f(t) + \frac{4\pi e}{c} M A_\delta(t) \ ,$$

where $A_f(t)$ is the solution of the free wave equation with initial data A_0, E_0, and

$$A_\delta(t,x) = \frac{1}{4\pi} \frac{\theta(ct - |x|)}{|x|} Q_{A(t-|x|/c)} \ ,$$

where θ denotes the Heaviside function. Moreover Q_A satisfies the equation

$$\dot{Q}_{A(t)} = c\sqrt{\lambda_0} Q_{A(t)} + \frac{3c^2}{2e} A_f(t,0) - \frac{3c^3}{2e^2} p(t) \ . \tag{4.3}$$

Proof. Let

$$A(t) = A_f(t) + \frac{4\pi e}{c} M A_\delta(t) \ ,$$

where

$$A_\delta(t,x) = \frac{1}{4\pi} \frac{\theta(ct - |x|)}{|x|} Q\left(t - \frac{|x|}{c}\right)$$

denotes the retarded potential of the source $Q\delta_0$, and $A_f(t)$ denotes the solution of the free wave equation with initial data A_0, and E_0. Therefore $A(t)$ satisfies the distributional equation

$$\frac{1}{c^2}\ddot{A} = \Delta A + \frac{4\pi e}{c} M Q \delta_0 \ .$$

By the Kirchhoff formula one can verify that A_f gives no contribution to Q_A (see [NP]). Therefore $Q_{A(t)} = Q(t)$. Since, by an elementary integration,

$$\begin{aligned}
&\lim_{r\downarrow 0} \frac{1}{4\pi r^2} \int_{S_r} \left(A - \frac{4\pi e}{c} M Q G_0 \right) d\mu_r(x) \\
=&A_f(t,0) - \frac{2}{3}\frac{e}{c^2} \lim_{r\downarrow 0} \left(\frac{Q(t - r/c) - Q(t)}{r} \right) \\
=&A_f(t,0) + \frac{2}{3}\frac{e}{c^2} \dot{Q}(t)
\end{aligned}$$

we have that A satisfies the boundary condition (4.1) if and only if (4.3) holds true. By lemma 4.1 (applied in the case $p = 0$) and by [NP, thm.3.3] it follows that

$$A - \frac{4\pi e}{c} M Q G_\lambda \in H^2_*(\mathbb{R}^3) \ ,$$

and so, by lemma 4.1 again, we have $A \in D(H_{m,p})$. The proof is then concluded by the distributional identity

$$\Delta A + \frac{4\pi e}{c} M Q \delta_0 = \Delta \left(A - \frac{4\pi e}{c} M Q G_\lambda \right) + \lambda \frac{4\pi e}{c} M Q G_\lambda \equiv -H_{m,p} A \ .$$

$\square$

Remark 4.3. Alternatively thm. 4.2 can be rephrased by saying that the strict solution of (4.2) coincides with the solution of the Cauchy problem

$$\begin{cases} \quad \dot{A} = 4\pi c^2 E \\ \quad \dot{E} = \dfrac{1}{4\pi}\Delta A + \dfrac{e}{c}M\dot{q}\delta_0 \\ \ddot{q}(t) = c\sqrt{\lambda_0}\dot{q}(t) + \dfrac{3c^2}{2e}A_f(t,0) - \dfrac{3c^3}{2e^2}p(t) \\ \quad \dot{p} = F(q) \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0, \quad q(0) = q_0, \quad \dot{q}(0) = Q_{A_0}, \quad p(0) = p_0 \ . \end{cases} \tag{4.4}$$

Differentiating (when possible) equation (4.4), we obtain the classical Abraham–Lorentz-Dirac equation

$$\begin{cases} -m\tau_0\dddot{q}(t) + m\ddot{q}(t) = -\dfrac{e}{c}\dot{A}_f(t,0) + F(q(t)) \\ q(0) = q_0, \quad \dot{q}(0) = Q_{A_0}, \quad \ddot{q}(0) = Q_{\dot{A}_0} \ , \end{cases}$$

where $\tau_0 := \dfrac{2e^2}{3mc^3}$.

Remark 4.4. By the definition (3.3), and by $D(H_{m,p}) \subset D(F_m)$, we have

$$\begin{aligned} \langle H_{m,p}A_1, A_2\rangle =& F_m\left(A_1 + \frac{c}{e}\Gamma_m(\lambda)^{-1}pG_\lambda, A_2\right) + \lambda\frac{c}{e}\Gamma_m(\lambda)^{-1}p\cdot\langle G_\lambda, A_2\rangle \\ =& F_m(A_1, A_2) + 4\pi p\cdot Q_{A_2} \ . \end{aligned}$$

Therefore, in the case $F = -\nabla V$, equations (4.2) are nothing but the Hamilton equations corresponding to the (degenerate) Hamiltonian

$$\mathcal{H}_m(A, E, q, p) := 2\pi c^2\|E\|_2^2 + \frac{1}{8\pi}F_m(A, A) + p\cdot Q_A + V(q) \ ;$$

this is defined on the symplectic vector space $(D(F_m) \times L^2_*(\mathbb{R}^3) \times \mathbb{R}^6, \Omega_0)$, where Ω_0 denotes the standard symplectic form

$$\Omega_0((A_1, E_1, q_1, p_1), (A_2, E_2, q_2, p_2)) = \langle A_1, E_2\rangle - \langle A_2, E_1\rangle + q_1\cdot p_2 - q_2\cdot p_1 \ .$$

Moreover one has the following convergence result:
Theorem 4.5. *Let*

$$\mathcal{H}_r(A, E, q, p) = 2\pi c^2\|E\|_2^2 + \frac{1}{8\pi}\|\nabla A\|_2^2 + \frac{1}{2m_r}\left|p - \frac{e}{c}\langle\rho_r, A\rangle\right|^2 + V(q)$$

be the Hamiltonian giving the equations (2.2), defined on the symplectic vector space $(H^1_*(\mathbb{R}^3) \times L^2_*(\mathbb{R}^3) \times \mathbb{R}^6, \Omega_0)$. *Let* $E \in L^2_*(\mathbb{R}^3)$, $(q,p) \in \mathbb{R}^6$, *and let* $A_r \in H^1_*(\mathbb{R}^3)$, $A \in D(F_m)$ *satisfy the condition (3.9). Then*

$$\lim_{r\downarrow 0} \mathcal{H}_r(A_r, E, q, p) = \mathcal{H}_m(A, E, q, p) \ .$$

Proof. Let us write $A_r=(H_r+\lambda)^{-\frac{1}{2}}X_r$, $A=(H_m+\lambda)^{-\frac{1}{2}}X$, $\lambda>\lambda_0$. Since, by (3.9) and lemma 2.4, one has

$$\lim_{r\downarrow 0}\frac{e}{m_r c}p\cdot\langle\rho_r,A_r\rangle=p\cdot Q_A\ ,$$

obviously it will suffice to prove that

$$\begin{aligned}&\lim_{r\downarrow 0}\|(-\Delta+\lambda)^{\frac{1}{2}}(H_r+\lambda)^{-\frac{1}{2}}X_r\|_2^2+\frac{4\pi e^2}{m_r c^2}|\langle\rho_r,(H_r+\lambda)^{-\frac{1}{2}}X_r\rangle|^2\\ &=\|X\|_2^2+\left(\frac{4\pi e}{c}\right)^2\Gamma_m(\lambda)|L_m(X)|^2\ .\end{aligned}$$

By remark 2.3 one obtains

$$\begin{aligned}&\lim_{r\downarrow 0}\ \|(-\Delta+\lambda)^{\frac{1}{2}}A_r\|_2^2+\frac{4\pi e^2}{m_r c^2}|\langle\rho_r,A_r\rangle|^2=\lim_{r\downarrow 0}\ \|X_r\|_2^2\\ &+\frac{2}{\pi}\int_{\mathbb{R}_+}\Gamma_r(x+\lambda)^{-1}\langle(-\Delta+x+\lambda)^{-1}\rho_r,X_r\rangle\cdot\langle(-\Delta+\lambda)^{\frac{1}{2}}(-\Delta+x+\lambda)^{-1}\rho_r,X_r\rangle\frac{dx}{\sqrt{x}}\\ &+\frac{2}{3\pi^2}\int_{\mathbb{R}^2_+}\Gamma_r(x+\lambda)^{-1}\Gamma_r(y+\lambda)^{-1}\langle(-\Delta+x+\lambda)^{-1}\rho_r,X_r\rangle\cdot\langle(-\Delta+y+\lambda)^{-1}\rho_r,X_r\rangle\times\\ &\qquad\times\langle(-\Delta+\lambda)(-\Delta+x+\lambda)^{-1}\rho_r,(-\Delta+y+\lambda)^{-1}\rho_r\rangle\,\frac{dx\,dy}{\sqrt{xy}}\\ &+\frac{4\pi e^2}{m_r c^2}\Bigg(|\langle\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}X_r\rangle|^2\\ &+\frac{4}{3\pi}\int_{\mathbb{R}_+}\Gamma_r(x+\lambda)^{-1}\langle(-\Delta+x+\lambda)^{-1}\rho_r,X_r\rangle\cdot\langle\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}X_r\rangle\times\\ &\qquad\times\langle(-\Delta+x+\lambda)^{-1}\rho_r,\rho_r\rangle\frac{dx}{\sqrt{x}}+\\ &\frac{4}{9\pi^2}\int_{\mathbb{R}^2_+}\Gamma_r(x+\lambda)^{-1}\Gamma_r(y+\lambda)^{-1}\langle(-\Delta+x+\lambda)^{-1}\rho_r,X_r\rangle\cdot(-\Delta+y+\lambda)^{-1}\rho_r,X_r\rangle\times\\ &\qquad\times\langle(-\Delta+x+\lambda)^{-1}\rho_r,\rho_r\rangle\langle(-\Delta+y+\lambda)^{-1}\rho_r,\rho_r\rangle\,\frac{dx\,dy}{\sqrt{xy}}\Bigg)\ .\end{aligned}$$

By (2.3), (2.7), (3.9), and by

$$\begin{aligned}&\frac{2}{3}\lim_{r\downarrow 0}\frac{8\pi e^2}{3c^2}\frac{\langle(-\Delta+x+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}\langle(-\Delta+y+\lambda)^{-1}\rho_r,\rho_r\rangle\\ &\quad+\langle(-\Delta+\lambda)(-\Delta+x+\lambda)^{-1}\rho_r,(-\Delta+y+\lambda){-}1\rho_r\rangle\\ &=-\Gamma_m(\lambda)\ ,\end{aligned}$$

$$\begin{aligned}&\lim_{r\downarrow 0}\frac{8\pi e^2}{3c^2}\frac{\langle(-\Delta+x+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}\langle(\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}X_r\rangle\\ &\quad+\langle(-\Delta+\lambda)^{\frac{1}{2}}(-\Delta+x+\lambda)^{-1}\rho_r,X_r\rangle=0\ ,\end{aligned}$$

the proof is done if we prove that

$$\lim_{r\downarrow 0}\frac{|\langle\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}X_r\rangle|^2}{m_r}=0\ .$$

Obviously this is equivalent to prove that $\{\sqrt{r}(-\Delta+\lambda)^{-\frac{1}{2}}\rho_r\}_{r>0}$ weakly converges to zero in $L^2(\mathbb{R}^3)$. Evidently such a family weakly converges to zero on a dense set, and so we only need to prove that it is bounded in $L^2(\mathbb{R}^3)$. This is true since, by standard Sobolev estimates, one has

$$\begin{aligned}&|\langle\sqrt{r}(-\Delta+\lambda)^{-\frac{1}{2}}\rho_r,f\rangle|\le\sqrt{r}\|(-\Delta+\lambda)^{-\frac{1}{2}}\rho_r\|_2\|f\|_2\\ \le&C\sqrt{r}\|\rho_r\|_{6/5}\|f\|_2=C\|\rho\|_{6/5}\|f\|_2\ .\end{aligned}$$

□

References

[AGHH] Albeverio S., Gesztesy F., Høegh–Krohn R., Holden H.: *Solvable Models in Quantum Mechanics.* New York: Springer–Verlag 1988

[A] Arai A.: Rigorous Theory of Spectra and Radiation for a Model in Quantum Electrodynamics. *J. Math. Phys.* **24** (1983), 1896–1910

[Bl] Blanchard P.: Discussion Mathematique du Modèle de Pauli–Fierz relatif au Catastrophe Infrarouge. *Commun. Math. Phys.* **15** (1969), 156–172

[D] Dirac P.A.M.: Classical Theory of Radiating Electrons. *Proc. Roy. Soc.* **A167** (1938), 148–168

[F] Fattorini H.O.: *Second Order Linear Differential Equations in Banach Spaces.* Amsterdam: North–Holland 1985

[Kr] Kramers H.A.: Nonrelativistic Quantum Electrodynamics and the Correspondence Principle, in: R. Stoops (ed.), *Rapports et Discussions du huitieme Conseil de Physique Solvay.* Brussels 1950

[L] Lorentz H.A.: *The Theory of Electrons (2nd edition).* New York: Dover 1952

[NP] Noja D., Posilicano A.: The Wave Equation with One Point Interaction and the (Linearized) Classical Electrodynamics of a Point Particle. To appear in *Ann. Inst. Henri Poincaré*

[PF] Pauli W., Fierz M.: Zur Theorie der Emission Langwelliger Lichtquanten. *Nuovo Cimento* **15** (1938), 168–188

[VK] VanKampen N.G.: Contributions to the Quantum Theory of Light Scattering. *Kgl. Danske. Videnskab. Selskab, Mat. Fys. Medd* **26** (1951), No.15