Quasilocality of Projected Gibbs Measures through Analyticity Techniques

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Abstract. We present two examples of projections of Gibbs measures. In the first example we prove high-temperature complete analyticity for the q-state Potts model. As a consequence we obtain complete analyticity also for the decimated Potts model. In the second example we prove quasilocality of the projections to the line of the pure phases of the two dimensional standard Ising system in the whole uniqueness region, and indicate why non-Gibbsianness can be expected to occur for higher dimensions.

1 Introduction

Over the last few years there has been a revival of interest in the mathematical well-definedness of real space renormalization group transformations applied to lattice spin systems. This problem originally dates back to the late '70s when Griffiths and Pearce, and Israel noticed that in certain renormalization examples one cannot take for granted the existence of an effective potential [22, 23, 31]. Later a comprehensive investigation of these so called renormalization-pathologies has been carried out by van Enter, Fernández and Sokal [14, 15, 16]. In parallel with these studies a number of other examples emerged from also other sources, such as models from nonequilibrium statistical mechanics [51, 45], percolation models [25, 46], projections of Gibbs measures to lower dimensional sublattices [48, 39, 36, 37, 38, 17], probabilistic cellular automata [35, 40], and others, showing that the measure relevant in their specific context was not a Gibbs measure. These examples are related to the inverse problem of statistical mechanics, which is the question whether for a state a reasonable potential can be found such that the state is a Gibbs measure with respect to it. The uniqueness aspect of this problem is better understood than the existence aspect: indeed, within a large class of potentials it is true that if there does exist such an interaction, then it is unique up to 'physical equivalence' [24]. In many interesting cases this inverse problem comes up for a measure which is the image of a Gibbs measure under a transformation (e.g., renormalization transformations, lower dimensional projections, restrictions to subsets of the configuration space). Given that, a way of formulating this problem is the following: What are the properties a Gibbs measure and a transformation must have in order that the transformed measure will be a Gibbs measure for some interaction? Thus far, we have only a partial answer to this question, which is apparently easier to provide when the image measure is the transform of a measure at sufficiently high temperatures than at low temperatures or near the critical point. In fact, there are typically domains of parameters

1. in the phase coexistence region where the image measure is not quasilocal, and hence non-Gibbsian. Proofs are based on the fact that by choosing a special configuration in the image system, a phase transition can be induced in the original system [22, 23, 31, 15]. A single configuration is of course negligible, i.e. it has measure zero, but it is often enough to find one and show that the system constrained by fixing this particular configuration can be described by conditional probabilities which are essentially non-quasilocal at this configuration.

2. in the uniqueness region where the potential for the image measure is an analytic function of the temperature [31, 32]. Surprisingly enough, however, there are no obvious 'safe regions', since failure of Gibbsianness occurs even deep within the uniqueness region as recent examples show (in particular, above the critical temperature or at large magnetic fields) [11, 13].

Results for regions close to the critical point are few and far between. In [33, 2] it is suggested that the critical temperature at which a phase transition can occur in the constrained system for certain majority-rule schemes and block averaging transformations, is strictly lower than for the original (unconstrained) system. These results indicate that around the critical point one may have a behaviour devoid of pathologies at least in certain cases, but they fall short of being rigorous proofs. Furthermore there are two recent more precise results available. In [26] the absence of pathologies near the critical point has been shown for decimation on a rectangular lattice and some Kadanoff transformations on a triangular lattice, both applied to the Ising model. In contrast, in [11] the presence of pathologies at the critical point is established for a block-spin transformation which is, however, not immediately linked to widely used renormalization schemes.

Analyticity is a very neat property in the sense that when it holds one can express many things by convergent expansions. However, analyticity is not the only 'good' behaviour one can have. There is actually a range of interesting regularity properties of states of which the strongest available is an analyticity property uniform in all volumes, called complete analyticity. Complete analyticity has been introduced and studied by Dobrushin and Shlosman [6, 7, 8]. A weaker version of it, that might be termed restricted complete analyticity, is a property uniform only in sufficiently regular volumes [41] (see below).

In this paper we want to investigate the behaviour of states at high temperatures or in the presence of a magnetic field in two examples in which non-Gibbsianness occurs at certain values of the parameters (see also [37]). One example is the decimated ferromagnetic q-state Potts system. Here renormalization-pathologies have been found above the critical temperature [13]. We will show, however, that at sufficiently high temperatures the decimated Potts measure is completely analytic, moreover the regimes where this behaviour and failure of Gibbsianness has been proven are separated by a narrow gap in the temperature scale. The other example we shall consider is the projection to a $d-1$ dimensional sublattice of the pure phases of the d dimensional Ising system. These projections in the absence of an external magnetic field have been shown by Schonmann to be non-Gibbsian measures for the two-dimensional case at every temperature in the phase-coexistence region [48]. Maes and Vande Velde [39], and Fernández and Pfister [17] showed that the same happens for $d-1$ dimensional projections from d-dimensional lattices. We will show below that the addition of any small magnetic field to the original two-dimensional system makes the projected measures Gibbsian, while for higher dimensions we still expect non-Gibbsianness of the projections.

2 Notation and some standard results

We will use the lattice $\mathbb{Z}^d$ throughout, with various particular choices of the dimension d. The lattice will be equipped with the l^1-metric $\mathrm{dist}(j,k) = |j-k| \equiv \sum_{n=1}^{d} |j^{(n)} - k^{(n)}|$. Accordingly, two sites j and k are called nearest neighbours whenever $|j-k| = 1$. The set of all finite volumes of the lattice will be denoted by $\mathscr{P}_f(\mathbb{Z}^d)$. For the outer r-boundary of $\Lambda \in \mathscr{P}_f(\mathbb{Z}^d)$ the symbol $\partial_r\Lambda = \{j \in \Lambda^c : \mathrm{dist}(j,\Lambda) \le r\}$ will be used, where $\mathrm{dist}(j,\Lambda) = \inf_{k\in\Lambda} \mathrm{dist}(j,k)$ and $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$. For the nearest neighbour boundary to Λ we put simply $\partial_1\Lambda = \partial\Lambda$.

To every spin we assign the same set of possible values by attaching to each site $k \in \mathbb{Z}^d$ the single spin space S. For the Ising system $S = \{-1,+1\}$, for the q-state Potts system (with $q \in \mathbb{N}$) $S = \{1,2,...,q\}$. We write $\Omega_\Lambda = S^\Lambda$ and $\Omega = S^{\mathbb{Z}^d}$ for the finite resp. infinite configuration spaces. Also, we put $\omega_{\{j\}} = \omega_j$, and use the notation $\omega_\Lambda \times \tau_{\Lambda^c}$ corresponding to a configuration which agrees with ω in Λ and with τ in Λ^c. The configuration space will be endowed with the product topology; we choose the neighbourhood basis

$$\mathscr{U}_{\omega,\Lambda} = \{\omega' : \omega'_\Lambda = \omega_\Lambda\}$$

for all $\omega \in \Omega$ and $\Lambda \in \mathscr{P}_f(\mathbb{Z}^d)$.

We further equip both S and Ω with their associated Borel σ-fields $\mathscr{S}$ and $\mathscr{F}$, respectively. Because of separability of S and Ω, both are countably generated, moreover $\mathscr{F}$ coincides with the product σ-field $\mathscr{S}^{\mathbb{Z}^d}$. The symbol $\mathscr{F}_\Lambda$ will be used for denoting the Borel σ-field for S^Λ.

Any real measurable function on the configurations will be considered as an observable.

Observables satisfying

$$\lim_{\Lambda\to\mathbb{Z}^d} \sup_{\substack{\omega,\tau\in\Omega \\ \omega_\Lambda=\tau_\Lambda}} |f(\omega)-f(\tau)| = 0 \tag{2.1}$$

are called quasilocal observables. The set of quasilocal observables will be denoted by $B_{ql}(\Omega)$ and provided with the sup-norm topology. In fact, since S is finite, in this topology the space of quasilocal functions coincides with the space of continuous functions. With the sup-norm and under the natural operations $B_{ql}(\Omega)$ becomes a Banach space.

We will require invariance with respect to the translation group on the lattice, θ_k, $k \in \mathbb{Z}^d$. On the configuration space translation invariance reads as $(\theta_k(\omega))_j = \omega_{j+k}$, and it induces similar actions on the spaces of observables.

An interaction (also called potential) is a family of real valued functions $\Phi_\Lambda : \Omega_\Lambda \to \mathbb{R}$, with $\Lambda \in \mathscr{P}_f(\mathbb{Z}^d)$, such that Φ_Λ are $\mathscr{F}_\Lambda$-measurable and $\Phi_\emptyset = 0$. The interactions will be assumed to be translation invariant, i.e. $\Phi_{\Lambda+k} = \Phi_\Lambda \circ \theta_k$, for all k. Also, we will require a certain control on the interaction at infinity. We will talk about absolutely summable (invariant) interactions whenever

$$\sum_{\substack{\Lambda\ni 0 \\ \Lambda\in\mathscr{P}_f(\mathbb{Z}^d)}} ||\Phi_\Lambda||_\infty < \infty \tag{2.2}$$

and denote the set of such interactions by $\mathscr{B}$. Note that the summability condition (2.2) defines a norm on the set of these interactions, and thus $\mathscr{B}$ becomes a Banach space under natural operations.

The finite volume Hamiltonians corresponding to the interaction Φ are

$$\mathscr{H}^\Phi_\Lambda(\omega_\Lambda \times \omega_{\Lambda^c}) = \sum_{X\cap\Lambda\neq\emptyset} \Phi_X(\omega_X) \tag{2.3}$$

This definition also takes account of the fact that there is a boundary condition, i.e. a particular configuration fixed outside the volume Λ. Note that if $\Phi \in \mathscr{B}$, then the Hamiltonians above exist and are quasilocal, thus absolute summability is a natural regularity condition for the potential. By using the Möbius-inversion formula (a version of the inclusion-exclusion principle), one can obtain the interaction from the expression of the Hamiltonian:

$$\Phi_X(\omega) = \sum_{\Lambda\cap X\neq\emptyset} (-1)^{|X\setminus\Lambda|} \mathscr{H}^\Phi_\Lambda(\omega) \tag{2.4}$$

The Ising model (including an external magnetic field) is defined by the potential

$$\Phi_\Lambda(\omega) = \begin{cases} -h\omega_j & \text{if } \Lambda = \{j\} \\ -J_{ij}\omega_i\omega_j & \text{if } \Lambda = \{i,j\} \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

We will consider the translation invariant ferromagnetic nearest neighbour Ising model, that is, (2.5) under the constraint that the coupling factors J_{ij} are translation invariant, positive, and non-vanishing only if $|i-j|=1$.

Next, we want to introduce probability measures in order to describe physical, in particular equilibrium, states. For the non-interacting system of spins we choose the product of one-site normalized counting measures, and we regard it as a reference measure throughout in the sequel. This measure will be denoted by μ for the whole lattice, and by μ_Λ for finite volumes. Since in the thermodynamic limit the Hamiltonian is ill-defined, one cannot use Hamiltonians directly for defining correctly Maxwell-Boltzmann-Gibbs probability weights. A way of circumventing this difficulty is to consider a family of conditional probabilities which turn out to be well behaving objects in the infinite volume limit. We will use specifications, that is, a family of everywhere defined conditional probability kernels $\Pi = \{\pi_\Lambda\}_{\Lambda\in\mathscr{P}_f(\mathbb{Z}^d)}$ mapping the Borel space $(\Omega,\mathscr{F})$ to itself, and satisfying the following requirements for all $\omega\in\Omega$ and $E\in\mathscr{F}$:

1. $\pi_\Lambda(\cdot,E)$ is an $\mathscr{F}_{\Lambda^c}$-measurable function
2. $\pi_\Lambda(\omega,\cdot)$ is a probability measure on $(\Omega,\mathscr{F})$
3. $\pi_\Lambda(\omega,\cdot)$ is $\mathscr{F}_{\Lambda^c}$-proper, i.e. it reproduces the boundary conditions: for any $F\in\mathscr{F}_{\Lambda^c}$, $\pi_\Lambda(\omega,F)=\mathbb{1}_F(\omega)$
4. for any $\Lambda'\subset\Lambda$, the probability kernels are compatible, that is,

$$(\pi_\Lambda\pi_{\Lambda'})(\omega,E)\equiv\int_{\Omega_{\Lambda\setminus\Lambda'}}\pi_\Lambda(\omega,d\tau)\pi_{\Lambda'}(\tau,E)=\pi_\Lambda(\omega,E)$$

A probability measure ϱ on $(\Omega,\mathscr{F})$ is said to be consistent with the specification Π if a version of its conditional probabilities for the subfields $\mathscr{F}_{\Lambda^c}$ coincides with π_Λ.

Now we can define Gibbs measures. The specification $\Pi^\Phi=\{\pi_\Lambda^\Phi\}_{\Lambda\in\mathscr{P}_f(\mathbb{Z}^d)}$ given by the densities

$$\frac{d\pi_\Lambda^\Phi}{d\mu_\Lambda}(\omega,E)=\frac{1}{Z_\Lambda^\Phi(\omega_{\Lambda^c})}\mathbb{1}_E(\omega)e^{-\mathscr{H}_\Lambda^\Phi(\omega)} \tag{2.6}$$

defined for all $E\in\mathscr{F}$, all $\omega\in\Omega$ and every $\Lambda\in\mathscr{P}_f(\mathbb{Z}^d)$, is called a Gibbs specification for the interaction Φ and reference measure μ. Here $Z_\Lambda^\Phi(\omega_{\Lambda^c})=\int_{\Omega_\Lambda}\exp(-\mathscr{H}_\Lambda^\Phi(\omega))d\mu_\Lambda(\omega)$ is the partition function, and we used the traditional notation of mathematical physics by not making explicit the temperature in the densities. Any measure consistent with the above specification Π^Φ is called a Gibbs measure. For further details of the theory of Gibbs measures we refer to [20, 15].

Π is said to be a uniformly nonnull specification with respect to the reference measure μ if there is an $\varepsilon>0$ such that for every $E\in\mathscr{F}_\Lambda$, $\mu(E)>0$ implies $\pi_\Lambda(\omega,E)\geq\varepsilon$, for every $\omega\in\Omega,\Lambda\in\mathscr{P}_f(\mathbb{Z}^d)$. Measures consistent with a uniformly nonnull specification have full support.

Next we recall an important notion of locality describing the way how spins outside a finite volume have some effect on the spins inside the volume. The simplest case is when a function of ω_{Λ^c} depends only on spins in a finite neighbourhood of Λ. Whenever Π_Λ is $\mathscr{F}_{\partial_r \Lambda}$-measurable, it is called an r-Markovian specification, and in this case $\pi_\Lambda(\omega_{\Lambda^c}, \cdot)$ equals $\pi_\Lambda(\omega_{\partial_r \Lambda}, \cdot)$. If the dependence extends to the whole Λ^c, but 'smoothly' enough as the spins are spatially more and more separated, satisfying

$$\lim_{\Lambda' \to \mathbb{Z}^d} \sup_{\substack{\omega, \tau \in \Omega \\ \omega_{\Lambda'} = \tau_{\Lambda'}}} |\pi_\Lambda(\omega, f) - \pi_\Lambda(\tau, f)| = 0 \tag{2.7}$$

for all $f \in B_{ql}(\Omega)$ and $\Lambda \subset \Lambda'$, then the specification is called almost Markovian or quasilocal. (Here we used the notation $\pi_\Lambda(\omega, f) = \int_{\Omega_\Lambda} f(\tau) \pi_\Lambda(\omega, d\tau)$.) Equivalently, any quasilocal observable has quasilocal conditional expectations with respect to an almost Markovian specification. It is useful to consider besides the *uniform* notion of quasilocality discussed above also its *pointwise* counterpart [36, 17, 25]. Accordingly, π_Λ is called almost Markovian or quasilocal at the point ω if for all quasilocal observables f

$$\lim_{\Lambda' \to \mathbb{Z}^d} \sup_{\substack{\tau \in \Omega \\ \omega_{\Lambda'} = \tau_{\Lambda'}}} |\pi_\Lambda(\omega, f) - \pi_\Lambda(\tau, f)| = 0 \tag{2.8}$$

for all finite volumes $\Lambda \subset \Lambda'$. Clearly, by compactness of Ω, if (2.8) holds for all ω, we obtain the uniform version of almost Markovianness (2.7).

A particularly useful characterization of Gibbs measures based on the intuition that in equilibrium perturbations should have small direct effects on remote locations is provided by the following

Theorem 2.1 *Let Π be a specification, and suppose a reference measure is given. The following two statements imply each other:*

1. *There exists an absolutely summable interaction Φ such that Π is a Gibbs specification with respect to it.*
2. *Π is quasilocal, and uniformly nonnull with respect to the reference measure.*

Two versions of the theorem go back to Sullivan [52] and Kozlov [34]; for a discussion see [15].

We will consider below projections of Gibbs measures obtained by summing over spins living on a selected infinite sublattice $L \subset \mathbb{Z}^d$. (We also require L^c to be infinite.) The projection of a measure ϱ on $(\Omega, \mathscr{F})$ to the Borel sub σ-field $\mathscr{F}_L$ is given by

$$\varrho'(d\omega_\Lambda) = \int_{S^{L^c}} \varrho(d\tau_{L^c} \times d\omega_\Lambda) \tag{2.9}$$

for all $\omega_\Lambda \in \Omega_\Lambda$, with $\Lambda \in \mathscr{P}_f(L)$. The projection is called in particular a decimation if $\dim L = d$; we will consider decimations on the sublattice with spacing b, $b\mathbb{Z}^d = \{k \in \mathbb{Z}^d : k^{(n)} \bmod b = 0, \ \forall n = 1, ..., d\}$. In relation to such projections we will address the question whether for ϱ' an absolutely summable interaction exists such that it is a Gibbs measure for that interaction.

3 High-temperature complete analyticity of the q-state Potts model

In this section we prove complete analyticity at sufficiently high temperatures for the q-state Potts model ($q \geq 2$). As a consequence we obtain complete analyticity also for its decimations on the sublattice $b\mathbb{Z}^d$. We use the technique developed in [30, 19]. The key observation is as follows: By using Gelfand's theory, a (sufficiently large) subset of the space of continuous observables can be related to a Banach algebra of functions on the configuration space $\Omega(q) = \{1, 2, ..., q\}^{\mathbb{Z}^d}$ having absolutely convergent harmonic series. On this algebra a certain conditional expectation kernel can be defined, which is the action of the decimation transformation, and which satisfies a Gruber-Merlini type equation. When an operator related to this conditional expectation kernel is invertible within this algebra, analyticity follows, and since the result will turn out to be uniform in the sublattice we pick, it will actually imply complete analyticity.

Singularities of the thermodynamic functions such as the pressure, etc., often indicate non-uniqueness of the Gibbs measure, while the analyticity of the pressure with respect to small perturbations of the interaction implies uniqueness of the Gibbs measure. The Lee-Yang circle theorem implies a picture of phase transitions in which such singularities play a major role. In that description phase transitions occur at limit points of the zeroes of the finite volume partition functions in the complex plane when the thermodynamic limit is taken (see, e.g., [47]). The study of analyticity involves thus the introduction of complex interactions and complex measures. A stronger version of analyticity, called complete analyticity, has been introduced by Dobrushin and Shlosman as a concept of uniqueness associated with the 'best possible' regularity properties of the Gibbs state [6, 7, 8]. We will use the following definition from the set of many, mutually equivalent, conditions they introduced: A measure ϱ consistent with a specification $\{\pi_\Lambda\}$ is called completely analytic if for every finite volume Λ, and every pair of configurations ω and τ different at just one site $j \in \Lambda^c$, there exist some positive constants C and κ such that

$$||\pi_\Lambda(\omega, \cdot) - \pi_\Lambda(\tau, \cdot)||_{\text{var}} \leq C \exp\left(-\kappa \operatorname{dist}(j, \Lambda)\right) \tag{3.1}$$

Here the variation-norm $||\mu - \nu||_{\text{var}} \equiv \sup_{E \in \mathscr{F}}(\mu(E) - \nu(E))$ is used. The corresponding interaction is called a completely analytic interaction, and it is shown that all finite volume partition functions and other functions of the interaction are holomorphic for each volume.

For simplicity we will consider the case $d = 2$ and then extend the results to arbitrary dimensions in a straightforward manner. Let $S = \{1, \ldots, q\}$ and consider the group $G^0 = (S \bmod q) + 1$, and the group of its characters consisting of the roots of the unity

$$\mathscr{X}(G^0) = \{\exp\left(\frac{2\pi n i}{q}\right) : n = 1, ..., q\} \tag{3.2}$$

To each site $k \in \mathbb{Z}^2$ we assign a copy of these groups, and we consider the product groups for a finite number of sites $G_\Lambda = \otimes_{k \in \Lambda} G^0$, respectively $\mathscr{X}(G_\Lambda) = \otimes_{k \in \Lambda} \mathscr{X}(G^0)$, $\Lambda \in \mathscr{P}_f(\mathbb{Z}^2)$. We write G and $\mathscr{X}(G)$ for the product group and the character group on the full lattice, respectively. Both of these groups are compact. The character group $\mathscr{X}(G)$ consists of elements of the type $\mathrm{id}_{\mathscr{X}(G^0)} \otimes \mathrm{id}_{\mathscr{X}(G^0)} \otimes \ldots$, and $\chi_1 \otimes \chi_2 \otimes \ldots \otimes \chi_n$ with $\chi_k \neq \mathrm{id}_{\mathscr{X}(G^0)}$, where $k \in \{1, 2, \ldots, n\} \subset \mathbb{N}$. We will describe below $\Omega(q)$ in terms of the group G.

It is useful to distinguish the characters supported on the sublattice $b\mathbb{Z}^2$ from the characters supported on subsets off this sublattice. Instead of using the symbol χ we will denote them further on respectively by $\hat{\sigma}_{\Lambda \setminus b\mathbb{Z}^2}$ and $\hat{\tau}_{\Lambda \cap b\mathbb{Z}^2}$, for all $\Lambda \in \mathscr{P}_f(\mathbb{Z}^2)$. Characters supported on the empty set will be understood to be identity characters. The character set is itself too an Abelian group. Consider G^0 for the site k. Then the characters for this group can be written as

$$\hat{\sigma}_k(\omega) = \exp\left(\frac{2\pi i}{q}\sigma_k \omega_k\right) \quad \sigma_k \in G^0 \tag{3.3}$$

Suppose $f : \Omega(q) \to \mathbb{C}$ is some function. We consider harmonic expansions of such functions with respect to the character set $\mathscr{X}(G_{\Lambda \setminus b\mathbb{Z}^2} \otimes G_{\Lambda \cap b\mathbb{Z}^2})$:

$$f(\omega) = \sum_{\Lambda \in \mathscr{P}_f(\mathbb{Z}^2)} \sum_{\substack{\hat{\sigma} \in \mathscr{X}(G_{\Lambda \setminus b\mathbb{Z}^2}) \\ \hat{\tau} \in \mathscr{X}(G_{\Lambda \cap b\mathbb{Z}^2})}} a_\Lambda^f(\hat{\sigma}, \hat{\tau}) \hat{\sigma}_{\Lambda \setminus b\mathbb{Z}^2}(\omega) \hat{\tau}_{\Lambda \cap b\mathbb{Z}^2}(\omega) \tag{3.4}$$

where $a_\Lambda^f(\hat{\sigma}, \hat{\tau})$ are generalized Fourier coefficients, for all $\Lambda \subset \mathscr{P}_f(\mathbb{Z}^2)$. The characters read as follows:

$$\hat{\sigma}_{\Lambda \setminus b\mathbb{Z}^2}(\omega) = \exp\left(\frac{2\pi i}{q} \sum_{k \in \Lambda \setminus b\mathbb{Z}^2} \sigma_k \omega_k\right) \quad \sigma_k \in G^0 \quad \forall k \tag{3.5}$$

$$\hat{\tau}_{\Lambda \cap b\mathbb{Z}^2}(\omega) = \exp\left(\frac{2\pi i}{q} \sum_{k \in \Lambda \cap b\mathbb{Z}^2} \tau_k \omega_k\right) \quad \tau_k \in G^0 \quad \forall k \tag{3.6}$$

We consider furthermore the Banach space $\mathscr{A}_q$ of complex functions on $\Omega(q)$ having absolutely convergent harmonic series, equipped with the l^1-norm

$$||f||_q \equiv \sum_{\Lambda \in \mathscr{P}_f(\mathbb{Z}^2)} \sum_{\substack{\hat{\sigma} \in \mathscr{X}(G_{\Lambda \setminus b\mathbb{Z}^2}) \\ \hat{\tau} \in \mathscr{X}(G_{\Lambda \cap b\mathbb{Z}^2})}} |a_\Lambda^f(\hat{\sigma}, \hat{\tau})| < \infty \tag{3.7}$$

First we need a standard result on the structure of $\mathscr{A}_q$:

Lemma 3.1 *$\mathscr{A}_q$ is a Banach algebra (where the involution operation is complex conjugation).*

By using Gelfand's theory, the Borel space $(\Omega(q), \mathscr{F}(q))$ can be represented in terms of elements of the Banach algebra $\mathscr{A}_q$ and its spectrum. The spectrum consists of all the characters of $\mathscr{A}_q$. Since $\mathscr{A}_q$ is the algebra of functions having finite l^1-norm, the range of the Gelfand transformation acting on it is a proper subset of the Banach algebra of continuous functions on the spectrum. The range is, however, dense in the sup-norm topology, and the maximal ideal space for $\mathscr{A}_q$ is Ω. For details we refer to [27].

We are interested to see whether at high enough temperatures an interaction with certain regularity properties exists for the subsystem consisting of the τ spin variables, after the σ spin variables are integrated out. To this end we shall consider a probability kernel $E(\cdot|\tau)$ mapping $\mathscr{A}_q$ onto the subspace spanned by the $\hat{\tau}$ variables. This corresponds to a decimation transformation applied to the Gibbs measure of the original system which will be denoted by T_q.

Since we are actually interested in the analyticity of the image interaction, we associate to $\mathscr{A}_q$ the complex interaction space $\mathscr{B}(\mathscr{A}_q)$. We assume translation invariance of the interactions. The (complex) Hamiltonian for the ferromagnetic nearest neighbour q-state Potts model is

$$\mathscr{H}_\Lambda(\xi) = -J \sum_{\substack{j,k\in\Lambda \\ |j-k|=1}} \delta_{\xi_j \xi_k} \tag{3.8}$$

where $J \in \mathscr{A}_q$ and $\mathrm{Re}\, J > 0$. The (complex) Gibbs specification constructed for this Hamiltonian will be denoted by $\{\gamma_\Lambda^J\}$. We will consider the local field given by the contribution to the Hamiltonian from a site k belonging to the internal system, and its neighbourhood:

$$u_k \equiv u_k(\xi) = -J \sum_{j:|j-k|=1} \delta_{\xi_j \xi_k} \tag{3.9}$$

The representation of this function in terms of characters goes as follows. By using the identity

$$\frac{1}{q} \sum_{\sigma_k \in S} \exp\left(\frac{2\pi i}{q} \sigma_k(\eta_k - \zeta_k)\right) = \delta_{\eta_k \zeta_k} \tag{3.10}$$

we find

$$u_k(\xi) = \sum_{\hat{\sigma}_k \in \mathscr{X}(G^0)} v_k(\xi) \hat{\sigma}_k(\xi) \tag{3.11}$$

thereby separating the contribution depending on the site k and ∂k. Here

$$v_k \equiv v_k(\xi) = -\frac{J}{q}\Big[\sum_{\substack{j:|j-k|=1 \\ j \in \mathbb{Z}^2 \setminus b\mathbb{Z}^2}} \sum_{\substack{\hat{\sigma}_j \in \mathscr{X}(G^0) \\ \hat{\sigma}_k^* \in \mathscr{X}(G^0)}} \hat{\sigma}_j(\xi) \delta_{\hat{\sigma}_j \hat{\sigma}_k^*} + \sum_{\substack{j:|j-k|=1 \\ j \in b\mathbb{Z}^2}} \sum_{\substack{\hat{\tau}_j \in \mathscr{X}(G^0) \\ \sigma_k^* \in \mathscr{X}(G^0)}} \hat{\tau}_j(\xi) \delta_{\hat{\tau}_j \hat{\sigma}_k^*} \Big] \tag{3.12}$$

Note that complex conjugation of the characters is a one-to-one map in the character group, and it is more convenient to work with the complex conjugated characters rather than with the inverse characters.

Next, a linear operator K on $\mathscr{A}_q$ will be defined so as to integrate out the spin at site k, by putting

$$\begin{aligned} &K \hat{\tau}_X = 0 \\ &K \hat{\sigma}_X \hat{\tau}_Y = H(v_k) \hat{\sigma}_{X \setminus \{k\}} \hat{\tau}_Y \quad \text{if } X \neq \emptyset \end{aligned} \tag{3.13}$$

where

$$H(v_k) = \int_{\xi_k \in S} \sum_{\sigma_k \in \mathscr{X}(G^0)} \hat{\sigma}_k(\xi) \gamma_k^J(\xi_{\{k\}^c}, d\xi_k) \tag{3.14}$$

is the conditional expectation of $\hat{\sigma}_k(\xi)$ with boundary conditions $\xi_{\{k\}^c}$, and we convene to choose k as the first site of the volume X according to some, e.g. lexicographic, ordering. (Once q is fixed, the operator, and the functions v_k and H are well defined, so for simplicity we do not label them by q.) The conditional expectation for the site k is

$$H(v_k) = \frac{e^{v_k} - 1}{e^{v_k} + q - 1} \tag{3.15}$$

For the conditional expectations a Gruber-Merlini-type equation can be established [30]

$$E(f|\tau) - E(Kf|\tau) = R(f|\tau) \tag{3.16}$$

by making the identifications

$$E(\hat{\tau}_Y|\tau) = \hat{\tau}_Y \tag{3.17}$$

$$E(\hat{\sigma}_X \hat{\tau}_Y|\tau) = E(\hat{\sigma}_{X \setminus \{k\}} \hat{\tau}_Y H(v_k)|\tau) \quad \text{if } X \neq \emptyset \tag{3.18}$$

$$R(\hat{\tau}_Y|\tau) = \hat{\tau}_Y \tag{3.19}$$

$$R(\hat{\sigma}_X \hat{\tau}_Y|\tau) = 0 \qquad \text{if } |X| \geq 2 \tag{3.20}$$

By iterating (3.16) and taking the limit over the iterates we obtain

$$E(f|\tau) = R((\mathrm{id}_{\mathscr{A}_q} - K)^{-1} f|\tau) \tag{3.21}$$

whenever

$$||K||_q < 1 \tag{3.22}$$

The left hand side in (3.21) is a complex function measurable with respect to the spins living on $b\mathbb{Z}^2$, and it is the unique solution of (3.16). The iteration of the operator K actually means that next we integrate out the spin at the second site of the volume X (if there is any). Conditions (3.18) and (3.20) change by changing the set X to $X \setminus \{k\}$, moreover we get $E(\hat{\sigma}_{X \setminus \{k,l\}} \hat{\tau}_Y H(v_k) H(v_l)|\tau)$ at the right hand side of (3.18), where l denotes the second site. If X has no more elements, then we get zero at both sides, and we can go on to larger volumes according to (3.4).

By the definition of the decimation transformation, whenever an effective interaction exists, the equation

$$E(\hat{\tau}_l|\tau) = H(v_l') \tag{3.23}$$

holds, for all $l \in \mathbb{Z}^2 \setminus b\mathbb{Z}^2$. Moreover, by (3.21) and (3.23)

$$H(v_l') = R(\sum_{n=1}^{\infty} K^n \hat{\tau}_l|\tau) \tag{3.24}$$

follows. Thus whenever the operator K is invertible, that is, at high enough temperatures (small-norm interactions), the image local field v'_l is well defined and is an analytic function of v_k, given by the series expansion cf. (3.24). Then by the Möbius transform (2.4) the image coupling constants $J' \in \mathscr{A}_q$ can be reconstructed.

The above considerations lead thus to the following conclusion:

Theorem 3.2 *For every $q \geq 2$ there is a neighbourhood of $J = 0$ in which the decimated interaction $T_q(J) = J'$ is an analytic function of J.*

Proof: It remains to show that $\mathrm{id}_{\mathscr{A}_q} - K$ can be made indeed invertible in some domain. We shall actually construct a neighbourhood of $J = 0$ where complete analyticity holds.

First let us notice that the $|| \cdot ||_q$ norm dominates the sup-norm, for all q, therefore condition (3.22) will automatically involve a control also with respect to the sup-norm. Indeed, we have for all $\omega \in \Omega(q)$

$$\begin{aligned} |f(\omega)| &= |\sum_{\Lambda \in \mathscr{P}_f(\mathbb{Z}^2)} \sum_{\substack{\hat{\sigma} \in \mathscr{X}(G_{\Lambda \setminus b\mathbb{Z}^2}) \\ \hat{\tau} \in \mathscr{X}(G_{\Lambda \cap b\mathbb{Z}^2})}} a^f_\Lambda(\hat{\sigma}, \hat{\tau}) \exp\,[\frac{2\pi i}{q}(\sum_{k \in \Lambda \setminus b\mathbb{Z}^2} \sigma_k \omega_k + \sum_{k \in \Lambda \cap b\mathbb{Z}^2} \tau_k \omega_k)]| \\ &\leq \sum_{\Lambda \in \mathscr{P}_f(\mathbb{Z}^2)} \sum_{\substack{\hat{\sigma} \in \mathscr{X}(G_{\Lambda \setminus b\mathbb{Z}^2}) \\ \hat{\tau} \in \mathscr{X}(G_{\Lambda \cap b\mathbb{Z}^2})}} |a^f_\Lambda(\hat{\sigma}, \hat{\tau})| = ||f||_q \end{aligned}$$

For a discussion on the relation between possible norms and uniqueness properties (uniqueness, exponential decay of correlations, analyticity) in general, see also [12, 3].

Let us replace e^{v_k} by the complex function e^{z_k}, such that $|e^{z_k}| = e^{v_k}$. Then the condition it has to satisfy is

$$||\frac{e^{z_k} - 1}{e^{z_k} - 1 + q}||_q < 1 \tag{3.25}$$

Consider the case

$$||\frac{e^{z_k} - 1}{q}||_q < 1$$

and use the notation $w_k = (e^{z_k} - 1)/q$. Then by a Taylor series expansion we get

$$||\frac{w_k}{w_k + 1}||_q \leq \frac{||w_k||_q}{1 - ||w_k||_q} \tag{3.26}$$

Thus a sufficient condition for (3.25) to hold is

$$||w_k||_q < \frac{1}{2} \tag{3.27}$$

On the other hand

$$||e^{z_k} - 1||_q \leq e^{||v_k||_q} - 1 \tag{3.28}$$

hence for a sufficient condition for (3.25) to hold we get

$$||v_k||_q < \ln(\frac{q}{2} + 1) \tag{3.29}$$

In case that $q = 2$, better bounds can be obtained. Then the left hand side in (3.25) can be written as $\tanh z_k$, and we get (as in [30])

$$||\tanh z_k||_2 \leq \tan ||v_k||_2 < 1 \tag{3.30}$$

which implies

$$||v_k||_2 < \frac{\pi}{4} \tag{3.31}$$

Because of translation invariance, we actually have a uniform control over the volumes of the sublattice, hence the estimates (3.29) and (3.31) imply complete analyticity of the decimated Gibbs measure for the ferromagnetic Potts and Ising models at sufficiently high temperatures. □

Remark 3.3 By [13] the decimated Gibbs measure for the high-q Potts model on $2\mathbb{Z}^2$ is non-Gibbsian for $\beta J \geq \frac{3}{8} \ln q$. Note that Th. 3.2 is consistent with this result since complete analyticity occurs for $\beta J \leq \frac{1}{4} \ln q$, for large q.

Remark 3.4 The limiting case $q \to 1$ corresponds to independent bond percolation. Formally, a sufficient condition for complete analyticity that we get in this case is $||v_k||_1 < \ln 2$, which corresponds to occupation probabilities $p < 0.32 p_c$, where $p_c = 1/2$ is the critical percolation probability.

Remark 3.5 The proof of Th. 3.2 extends to arbitrary dimensions d. Then conditions (3.29) and (3.31) given in terms of the sup-norm become as follows: For $q \geq 3$

$$\beta J < \frac{1}{z(d)} \ln(\frac{q}{2} + 1) \tag{3.32}$$

respectively for $q = 2$

$$\beta J < \frac{\pi}{4z(d)} \tag{3.33}$$

where $z(d)$ denotes the coordination number of the d-dimensional lattice.

4 Lower dimensional projections of the states of the Ising system in the uniqueness region

Suppose $\beta > \beta_c$, and consider μ_J^+, the + phase of the two dimensional ferromagnetic nearest neighbour Ising system. R. Schonmann proved that there is no absolutely summable interaction for which the projection of μ_J^+ onto the line would be Gibbsian [48]. We will denote this projection by ν^+, obtained by putting $L = \mathbb{Z} \otimes \{0\}$ in (2.9). A similar behaviour of μ_J^- under projection can be established. The mechanism of the break-down of Gibbsianness at low temperatures becomes transparent by noticing the fact that the alternating configuration is a point of non-quasilocality for ν^+ and a complete wetting occurs [15], giving rise to long range order in the observables off the line.

At sufficiently high temperatures, however, there exists an absolutely summable interaction such that ν^+ is a Gibbs measure, moreover the interaction is an analytic function of the temperature. This can be proven by adapting Th. 3.2 to this particular situation (see also Rem. 3.5). Our result is

Proposition 4.1 *In the temperature regime $\beta J < \frac{\pi}{8}$ the potential for the Gibbs measure ν^+ depends analytically on β.*

High temperature analyticity has been proven by an expansion method also in [39]. The inverse temperature below which this method yields analyticity is $\beta = \operatorname{arctanh} 1/3e^2 = 0.0451$ (in units of J), a value lower by a factor 8 than ours (equal to 0.3927).

In [17] it was shown that a similar wetting phenomenon occurs for general $d-1$ dimensional projections of the pure phases of the $d \geq 3$ dimensional Ising system. (For details about wetting see also [18].) Also, high-temperature analyticity of the projections can be established for higher dimensions too, cf. Th. 3.2 and Rem. 3.5.

Now we want to investigate the projection of the Ising system in the presence of an external magnetic field. We will state our conclusions separately in the two-dimensional and the higher dimensional cases. In all dimensions, however, if a sufficiently strong magnetic field is applied, then the projection is Gibbsian since it corresponds to the system in a low density regime (compare with [23]). The question is if the same behaviour would occur in the presence of small magnetic fields. Without restricting generality we choose an external magnetic field $h > 0$.

Denote by μ^h the Gibbs measure for the d-dimensional Ising system when a magnetic field $h > 0$ is switched on. The corresponding specification will be denoted by Γ^h. The projection to the line as defined by (2.9) will be denoted by ν^h. First we want to construct the specification Π^h for this projection.

The configuration space of the Ising system on the $d-1$ dimensional sublattice will be denoted by $\Omega = \{-1, +1\}^{\mathbb{Z}^{d-1}}$, and the Borel σ-field for Ω by $\mathscr{F}$. Suppose $W \subset \mathbb{Z}^d$. Following the suggestion of Fernández and Pfister [17] to define the specification for the projection by properly restricting the global specification for the original measure, we first define a global specification $\tilde{\Gamma}^h$ to the local specification Γ^h by putting

$$\tilde{\gamma}^h_W(\omega, E) = \begin{cases} \gamma^h_W(\omega, E) & \text{if } |W| < \infty \\ \lim_{\Lambda \to \mathbb{Z}^d} \gamma^h_{\Lambda \cap W}(\omega_{\Lambda \triangle W} \times +_{W^c \cap \Lambda^c}, E) & \text{if } |W| = \infty \end{cases} \tag{4.1}$$

where $\triangle$ means symmetric difference. The 'globalization' in fact consists in extending the conditional probability kernels defined for finite volumes to infinite subsets of the lattice. Here the notation $+_\Lambda = +1$, $\forall\, j \in \Lambda$, has been used. Note that a global specification does exist since for the Ising potential the global Markov property is known to hold in the whole uniqueness regime, in arbitrary dimensions [21, 1]. The 'globalization' cf. (4.1) is made in such a way that μ^h is consistent with $\tilde{\Gamma}^h$.

We construct the specification $\Pi^h = \{\pi^h_V\}$ for the projection given on $(\Omega, \mathscr{F})$ by using the global specification. By choosing $W = \mathbb{Z}^{d-1}$ and $V \subset \mathbb{Z}^{d-1}$ (possibly infinite) in (4.1), we write

$$\pi^h_V(\omega, E) := \lim_{\Lambda \to \mathbb{Z}^d} \int \gamma^h_{\Lambda \cap V}(\omega_{\Lambda \triangle \mathbb{Z}^{d-1}} \times +_{\Lambda^c \cap (\mathbb{Z}^d \setminus \mathbb{Z}^{d-1})}, E)\mu^h(d\omega_{\Lambda \setminus \mathbb{Z}^{d-1}}) \tag{4.2}$$

Π^h is indeed a specification for the spin system living on the $d-1$ dimensional sublattice. Clearly, $\pi^h_V(\cdot, E)$ is a function dependent only on spins in $V^c = \mathbb{Z}^{d-1} \setminus V$. It is also proper, every event measurable with respect to $\mathscr{F}_{V^c}$ reproduces the boundary condition. Compatibility follows by direct inspection. Moreover, as Γ^h was FKG, the projection Π^h becomes FKG too.

Lemma 4.2 *Fix $h > 0$, and consider the specification Π^h and the measure ν^h. Let $\{V_n\} \subset \mathbb{Z}^{d-1}$ be a sequence of volumes. The following hold true:*

1. *(a) $\pi^h_{V_m}(+, \cdot) \geq \pi^h_{V_n}(+, \cdot)$ for all $V_m \subset V_n$.*

   (b) $\pi^h_{V_m}(-, \cdot) \leq \pi^h_{V_n}(-, \cdot)$ for all $V_m \subset V_n$.

2. *Suppose $\{V_n\}$ is a van Hove sequence of finite volumes in the $d-1$ dimensional sublattice. Then there exist the limits in van Hove sense*

$$\lim_{V_n \to \mathbb{Z}^{d-1}} \pi^h_{V_n}(+, \cdot) = \inf_{V_n \to \mathbb{Z}^{d-1}} \pi^h_{V_n}(+, \cdot)$$
$$\lim_{V_n \to \mathbb{Z}^{d-1}} \pi^h_{V_n}(-, \cdot) = \sup_{V_n \to \mathbb{Z}^{d-1}} \pi^h_{V_n}(-, \cdot)$$

3. *With probability 1 we have*

$$\lim_{V_n \to \mathbb{Z}^{d-1}} \pi^h_{V_n}(+, \cdot) = \lim_{V_n \to \mathbb{Z}^{d-1}} \pi^h_{V_n}(-, \cdot) = \nu^h$$

Proof: The first statements follow by the FKG inequality and the fact that π_V^h is a specification:

$$\begin{aligned}\pi_{V_n}^h(+,E) &= \pi_{V_n}^h \pi_{V_m}^h(+,E)\\ &= \int \pi_{V_n}^h(+,d\eta)\pi_{V_m}^h(\eta,E)\\ &\le \int \pi_{V_n}^h(+,d\eta)\pi_{V_m}^h(+,E)\\ &= \pi_{V_m}^h(+,E)\end{aligned}$$

and similarly for the $-$ boundary conditions. The existence of the limits follows by monotonicity and boundedness. The fact that both specifications are consistent with the same measure ν^h comes from the uniqueness of the measure μ^h, and a combination of (2.9) and (4.2) with the consistency of μ^h with $\tilde{\Gamma}^h$. $\square$

Consider now the case $d = 2$ with the same notations as above.

Theorem 4.3 *For every $h \neq 0$, ν^h is a Gibbs measure for some absolutely summable interaction, at any temperature.*

First we need two lemmas. In [44] Gibbs measures for the two dimensional Ising model are characterized by mixing conditions (see also [41]). These properties give detailed information about the way local fluctuations are decoupled in separate volumes. In particular these conditions imply some neat properties of the Gibbs measures related to a weak dependence on the boundary conditions also far outside the Dobrushin uniqueness regime.

Denote an $(2L+1) \times (2L+1)$ square by

$$\Lambda_L = \{(x,y) \in \mathbb{Z}^2 : -L \le x, y \le L\}$$

Take $\Lambda' \subset \Lambda \in \mathscr{P}_f(\mathbb{Z}^2)$, and a site $j \in \Lambda^c$. The Gibbs measure μ^h is said to fulfil the *weak mixing condition for the volume* Λ with constants $C > 0$ and $\kappa > 0$ if for every Λ'

$$\sup_{\tau,\tau^j \in \Omega_{\Lambda^c}} ||\gamma_{\Lambda'}^h(\tau,\cdot) - \gamma_{\Lambda'}^h(\tau^j,\cdot)||_{\text{var}} \le C \sum_{\substack{k \in \Lambda' \\ j \in \partial\Lambda}} \exp\left(-\kappa \operatorname{dist}(j,k)\right) \tag{4.3}$$

The Gibbs measure μ^h is said to fulfil the *strong mixing condition for the volume* Λ with constants $C > 0$ and $\kappa > 0$ if for each Λ' and $j \in \Lambda^c$

$$\sup_{\tau,\tau^j \in \Omega_{\Lambda^c}} ||\gamma_{\Lambda'}^h(\tau,\cdot) - \gamma_{\Lambda'}^h(\tau^j,\cdot)||_{\text{var}} \le C \exp\left(-\kappa \operatorname{dist}(\Lambda',j)\right) \tag{4.4}$$

The notation

$$(\tau^j)_k = \begin{cases} \tau_k & \text{if } j \neq k \\ -\tau_k & \text{if } j = k \end{cases} \tag{4.5}$$

has been used.

Lemma 4.4 *If μ^h satisfies the weak mixing condition with constants C and κ for each $\Lambda \in \mathscr{P}_f(\mathbb{Z}^2)$, then some constants $C' > 0$ and $\kappa' > 0$ can be found, such that μ^h satisfies the strong mixing condition with respect to them, for sufficiently large squares $\Lambda_L \in \mathscr{P}_f(\mathbb{Z}^2)$, at every temperature.*

The next lemma gives sufficient conditions for the weak mixing property.

Lemma 4.5 *Consider the measures μ^h, and μ for $h = 0$ and $\beta < \beta_c$.*

1. *There exist some constants C and κ such that μ^h satisfies the weak mixing condition for every finite volume Λ, for each $h > 0$, at every temperature.*
2. *There exist some constants C and κ such that μ satisfies the weak mixing condition for every finite volume Λ, at each $\beta < \beta_c$.*

For sufficiently low temperatures in a non-zero field, and all temperatures above the critical point at any field the proof of these lemmas was given in [44]. The gap in the proof for the rest of the uniqueness region was closed in [49], relying on results in [10, 9, 28, 29]. Notice that the strong mixing behaviour above actually coincides with complete analyticity restricted to large squares.

Proof of the theorem: By the results of [44, 49], the strong mixing condition for large squares entails that neither a bulk, nor a surface transition in the neighbourhood of the line can occur. Note that, in fact, instead of the squares Λ_L the same would hold for also other subsets provided that they were sufficiently regular, e.g. unions of squares (see also the remark after Th. 1.1 in [44]). Strong mixing thus corresponds to complete analyticity restricted to a class of sufficiently regular volumes. We will think of the line as the boundary of a union of large squares that will enable us to use this result. We have this strong mixing, or restricted complete analyticity, in the uniqueness region by Lemmas 4.4 and 4.5. This then implies that all points for the projection ν^h are continuity points, that is, ν^h is a Gibbs measure for all $h > 0$, at every temperature. The same conclusion holds for $h < 0$.

We show how strong mixing in two dimensions implies quasilocality in one dimension in the sense of (2.7). We have to investigate how the flip of remote spins affects the probability in finite volumes $V \subset \mathbb{Z}$. Take $W \supset V$, a piece of the line. Since for any ξ

$$||\pi_V^h(\tau, \cdot) - \pi_V^h(\tau_{W \setminus V} \times \xi_{W^c}, \cdot)||_{\text{var}} \leq \sum_{x \in W^c} ||\pi_V^h(\tau, \cdot) - \pi_V^h(\tau^x, \cdot)||_{\text{var}} \tag{4.6}$$

it is actually enough to show that for any possible choice of W

$$\lim_{W \to \mathbb{Z}} \sum_{x \in W^c} \sup_{\tau \in \Omega_{V^c}} ||\pi_V^h(\tau, \cdot) - \pi_V^h(\tau^x, \cdot)||_{\text{var}} = 0 \tag{4.7}$$

for every τ (where $W^c = \mathbb{Z} \setminus W$). We will put for a shorthand

$$\begin{aligned} &\delta_{\Lambda_L \cap V}(\omega, \tau, \tau^x) = \\ &\quad \gamma^h_{\Lambda_L \cap V}(\omega_{\Lambda_L \setminus \mathbb{Z}} \times \tau_{\Lambda_L \cap V^c} \times +_{\Lambda_L^c \cap \mathbb{Z}^c}, \cdot) - \gamma^h_{\Lambda_L \cap V}(\omega_{\Lambda_L \setminus \mathbb{Z}} \times \tau^x_{\Lambda_L \cap V^c} \times +_{\Lambda_L^c \cap \mathbb{Z}^c}, \cdot) \end{aligned} \tag{4.8}$$

Pick a sequence of squares $\{\Lambda_L\}_{L\geq 1} \subset \mathbb{Z}^2$. Then we have

$$\begin{aligned}
&\lim_{W\to\mathbb{Z}} \sum_{x\in W^c} \sup_{\tau\in\Omega_{V^c}} ||\pi_V^h(\tau,\cdot) - \pi_V^h(\tau^x,\cdot)||_{\text{var}} \\
&= \lim_{W\to\mathbb{Z}} \sum_{x\in W^c} \sup_{\tau\in\Omega_{V^c}} || \lim_{L\to\infty} \int \delta_{\Lambda_L\cap V}(\omega,\tau,\tau^x)\mu^h(d\omega)||_{\text{var}} \\
&\leq \lim_{W\to\mathbb{Z}} \lim_{L\to\infty} \sum_{x\in W^c} \int \sup_{\tau\in\Omega_{V^c}} ||\delta_{\Lambda_L\cap V}(\omega,\tau,\tau^x)||_{\text{var}}\mu^h(d\omega) \\
&\leq \lim_{W\to\mathbb{Z}} \lim_{L\to\infty} \sum_{x\in W^c} Ce^{-\kappa\,\text{dist}(x,\Lambda_L\cap V)} \\
&= C' \lim_{W\to\mathbb{Z}} e^{-\kappa\,\text{dist}(W^c,V)} = 0
\end{aligned}$$

where C' is a positive constant. □

Clearly, the proof above is substantially relying on the fact that for the special case of the two dimensional Ising model weak mixing implies strong mixing. Actually the crucial point is that the boundaries here are one dimensional. In contrast, for three dimensions layering transitions can not be ruled out. Indeed, in the three dimensional case it is thought that the so called Basuev phenomenon occurs which is a layering transition. This consists in the following. Consider the semi-infinite Ising model, say, above the plane $(x, y, 0)$. Suppose there is a small external field $h > 0$ switched on. When $-$ boundary conditions are applied, an interface appears as a consequence of the competition between the effect of the field and the entropic repulsion of the boundary condition. In these circumstances a coexistence of two semi-infinite Gibbs measures will appear, describing two layers near the plane $(x, y, 0)$, of average height r and $r+1$. Note, however, that at the same time the infinite-volume Gibbs measure is unique. Layering phase transitions are long range order phenomena localized in 'shells' of sites of negligible volume compared to the size of the whole system [5, 50, 4, 44, 42, 43]. Since layering transitions have after all no effect on the bulk phase diagram, this phenomenon suggests a weaker notion of uniqueness in the bulk. If the Basuev phenomenon would occur, then it presumably would give rise to discontinuities of certain conditional expectations of some observables for the two dimensional projection. One can expect that those configurations which belong to the layers are mostly sheets of $-$ spins with small patches of $+$ spins. Having a positive magnetic field switched on, these events are of course exceptional. Indeed, by general results in [17] (Prop. 4.1) it follows that those configurations for which the Basuev transition may occur must belong to a set of measure zero.

The Gibbs measures one has on the line in the presence of external fields can be used for approximating the non-Gibbsian measures in zero field in the coexistence region:

Proposition 4.6 *Consider a sequence of positive magnetic fields $\{h_n\}$ convergent to zero, and fix the temperature below the critical point of the two dimensional system. Then ν^{h_n} is weakly convergent to ν^+. Similarly, ν^- can be weakly approximated by the sequence ν^{-h_n}.*

Proof: For the measure μ^h, describing the d-dimensional Ising system in a field h, it is known that $\mu^{h_n} \to \mu$ in the weak topology, for every sequence $h_n \to 0$. Then weak convergence of ν^{h_n} to ν^+ follows by taking the marginal on the line of μ^{h_n}. $\square$

Finally, for the projection of the supercritical state in two dimensions we have

Corollary 4.7 *Part 2 of Lemma 4.5 and Lemma 4.4 imply similarly that the projection of the measure for the two dimensional zero-field Ising model at every temperature above the critical point is a Gibbs measure.*

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