CONVECTION–DIFFUSION EQUATION WITH SPACE-TIME ERGODIC RANDOM FLOW.

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Abstract: We prove long time diffusive behavior (homogenization) for convection–diffusion in a incompressible random flow that is time–dependent, has nonzero mean and space-time stationary bounded stream matrix.

Keywords: Homogenization, Convection–Diffusion Equations

1. Introduction.

Consider a passive scalar $u(\mathbf{x}, t)$ (like the temperature) diffusing in an incompressible fluid. It will satisfy the convection–diffusion equation

$$\partial_t u = \Delta u + \mathbf{F} \cdot \nabla u \tag{1.1}$$

with certain initial condition given. We will assume that the fluid velocity $\mathbf{F}(\mathbf{x}, t, \omega)$ is given by a random ergodic process on $\mathbb{R}^{d+1}$ satisfying

$$\nabla_x \cdot \mathbf{F} = 0.$$

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We will assume that $\mathbf{F} = \nabla_x \cdot \mathbf{H} + \mathbf{v}$ with $\mathbf{H}(\mathbf{x}, t, \omega)$ a stationary (in R^{d+1}) and bounded skew-symmetric random matrix and $\mathbf{v}$ a constant vector[4].

We want to study the effective diffusion on a macroscopic scale, i.e. introducing a scale parameter ε we show that

$$\lim_{\varepsilon \to 0} u(\varepsilon^{-1}\mathbf{x} - \varepsilon^{-2}t\mathbf{v}, \varepsilon^{-2}t, \omega) = \bar{u}(\mathbf{x}, t) \tag{1.2}$$

where $\bar{u}$ is solution of effective heat equation

$$\partial_t \bar{u} = \sum_{i,j} a^{\mathbf{v}}_{i,j} \partial_{\mathbf{x}_i} \partial_{\mathbf{x}_j} \bar{u} \tag{1.3}$$

and $a^{\mathbf{v}}$ is the corresponding effective diffusion matrix.

Such problem arises in the diffusion of contaminants in saturated porous media (cf.[BGW]) and in diffusion of particles deposited as sediment in convective flows (cf. [MC]).

For periodic flows this problem has been extensively studied (cf. [BGW], [FP], [MM], ...), In the random case, for time independent flows, only the case with zero mean drift was proved (cf. [PV], [K]), and for time dependent flows only special cases had been studied (cf. [AM], [CX]).

Our technique is based on a perturbation argument already used in [LY].

2. Notations and Results.

Let ($\Omega, \mathcal{G}, \mu$) be a probability space and $G = \{\tau_{\mathbf{x},t}\ ; (\mathbf{x}, t) \in \Re^{d+1}\}$ is a group of measure preserving transformations acting ergodically on Ω (and with stochastically continuous action).

The group G acting on Ω induce a group of operators on L^2 ($\Omega, \mathcal{G}, \mu$) given by

$$T_{\mathbf{x},t} f(\omega) = f(\tau_{\mathbf{x},t}\omega)$$

Since $\tau_{\mathbf{x},t}$ is measure preserving, the operators $T_{\mathbf{x},t}$ form a unitary group. Therefore they have closed densely defined infinitesimal generators given by

$$\begin{aligned} D_i &= \frac{\partial}{\partial x_i} \left. T_{\mathbf{x},t} \right|_{\mathbf{x}=0, t=0} \\ D_t &= \frac{\partial}{\partial t} \left. T_{\mathbf{x},t} \right|_{\mathbf{x}=0, t=0} \end{aligned} \tag{2.1}$$

[4] In other words we are considering flows with stream matrix that are stationary in time and with stationary increments in space.

Let $H_{i,j}(\omega)$ a skew-symmetric matrix of "smooth" bounded functions on Ω. We will consider divergence free field given by

$$\mathbf{F} = \{F_i = \sum_j D_j H_{i,j},\ i = 1, \dots, d\}$$

Consider the solution $u(\mathbf{x}, t, \omega)$ of the parabolic equation:

$$\partial_t u = \Delta u + \mathbf{v} \cdot \nabla u \ + \ \mathbf{F}(\mathbf{x}, t, \omega) \cdot \nabla u \tag{2.2}$$

where $\mathbf{v} \in \Re^d$ is a constant vector and $\mathbf{F}(\mathbf{x}, t, \omega) = \mathbf{F}(\tau_{\mathbf{x},t}\omega)$.

We will show that in μ–probability

$$\lim_{\varepsilon \to 0} u(\varepsilon^{-1}\mathbf{x} - \varepsilon^{-2}t\mathbf{v}, \varepsilon^{-2}t, \omega) = \bar{u}(\mathbf{x}, t) \tag{2.3}$$

where $\bar{u}$ is the (deterministic) solution of effective heat equation

$$\partial_t \bar{u} = \sum_{i,j} a^{\mathbf{v}}_{i,j} \partial_{\mathbf{x}_i} \partial_{\mathbf{x}_j} \bar{u} \tag{2.4}$$

and $a^{\mathbf{v}}$ is the corresponding effective diffusion matrix. We will also give a variational formula for $a^{\mathbf{v}}$.

Consider the diffusion process on $\Re^d$ solution of the SDE

$$\begin{aligned} d\mathbf{y}^\omega(t) &= \sqrt{2} d\mathbf{w}_t + \{\mathbf{v} + \mathbf{F}(\mathbf{y}^\omega(t), t, \omega)\}\, dt \\ \mathbf{y}^\omega(0) &= 0\ , \end{aligned} \tag{2.5}$$

where $\mathbf{w}_t$ is a standard Wiener process on $\Re^d$. We will prove that the rescaled diffusion

$$\mathbf{x}^\varepsilon(t) \ = \ \varepsilon \left(\ \mathbf{y}^\omega(\varepsilon^{-2}t) - \varepsilon^{-2}t\mathbf{v} \ \right)$$

converge in law to a brownian motion with diffusion matrix $a^{\mathbf{v}}$.

3. Probabilistic Convergence Theorem.

To the diffusion process defined by (2.5) one can associate a canonical Markov process on Ω by

$$\begin{cases} \eta(t) = \tau_{\mathbf{y}^\omega(t),t}\, \omega \\ \eta(0) = \omega\ . \end{cases}$$

The generator of this process is given by

$$L \;=\; \mathbf{D}^2 + \mathbf{v}\cdot\mathbf{D} + \mathbf{F}\cdot\mathbf{D} \;+ D_t.$$

Observe that μ is an ergodic invariant measure for this process (cf. [O]).

Denote by $<\cdot,\cdot>$ the inner product in $L^2(\mu)$. Define $\mathcal{H}_1$ as

$$\mathcal{H}_1 = clos\{\phi \in \mathcal{D}(\mathbf{D})\}$$

where the closure is made with respect the norm $\|\phi\|_1 =< |\mathbf{D}\phi|^2 >^{1/2}$ and

$$\mathcal{H}_{-1} = \left\{\psi : \|\psi\|_{-1}^2 = \sup_{\phi}\left(\, 2 < \psi, \phi > - \|\phi\|_1^2 \,\right)\right\}.$$

Proposition 3.1 . *For any $g \in \mathcal{H}_{-1}$ there exists a sequence $u_n \in L^2 \cap \mathcal{H}_1$ such that*

$$\|Lu_n - g\|_{-1} \longrightarrow 0 \tag{3.1}$$

Corollary 3.2 . *The sequence u_n defined in Proposition 3.1 converge strongly in $\mathcal{H}_1$.*

Proof of corollary 3.2

It follows immediately from

$$\|u_n - u_m\|_1 =\| \; S(u_n - u_m) \; \|_{-1} \leq \| \; L(u_n - u_m) \; \|_{-1} \rightarrow 0$$

□

We will use the CLT for martingales in order to conclude the argument. In fact by proposition 3.1, applied at the occurrence to the functions $g = \mathbf{F}\cdot\mathbf{e}$, we can write

$$\begin{aligned}\varepsilon\int_0^{t\varepsilon^{-2}} g(\eta(s))ds \;&=\; \varepsilon\int_0^{t\varepsilon^{-2}} (Lu_n^{\mathbf{e}})(\eta(s))ds \;+\; \varepsilon\int_0^{t\varepsilon^{-2}} f_n(\eta(s))ds \\ &=\; \varepsilon\big\{u_n^{\mathbf{e}}(\eta(t\varepsilon^{-2})) - u_n^{\mathbf{e}}(\eta(0))\big\} \;+\; \varepsilon M^{u_n^{\mathbf{e}}}(t\varepsilon^{-2}) + \varepsilon\int_0^{t\varepsilon^{-2}} f_n(\eta(s))ds\end{aligned} \tag{3.2}$$

where $\|f_n\|_{-1} \rightarrow 0$ and $M^{u_n^{\mathbf{e}}}(t)$ is a martingale with quadratic variation $\int_0^t |\mathbf{D}u_n^{\mathbf{e}}(\eta(s))|^2 ds$.

Since $u_n^{\mathbf{e}}$ is in L^2, the first term in (3.2) goes to 0 as $\varepsilon \to 0$ in L^2. For the last term observe that

$$\lim_{\varepsilon\to 0} E\left[\left(\varepsilon\int_0^{t\varepsilon^{-2}} f_n(\eta(s))ds\right)^2\right] \le \|f_n\|_{-1}^2 \underset{n}{\longrightarrow} 0$$

While for the martingale term $\varepsilon M^{u_n^{\mathbf{e}}}(t\varepsilon^{-2})$, by CLT for martingales and the ergodic theorem (cf. [O] for example), it converges (in law, for a.e. ω) as $\varepsilon \to 0$ to a martingale with constant quadratic variation given by $\|u_n^{\mathbf{e}}\|_1^2$ that converge as $n \to \infty$ to $\|u_0^{\mathbf{e}}\|_1^2$ by corollary 3.2.

This gives the convergence of the finite dimensional distribution. For the compactness, since $H_{i,j}$ is bounded, it follows by a Varadhan's argument using the GRR inequality (cf. [V], [O]).

The diffusion matrix along the direction $\mathbf{e}$ is then given by

$$\mathbf{e}\cdot a^{\mathbf{v}}\mathbf{e} \ = \ \|\mathbf{e}\|^2 + < \|Du_0^{\mathbf{e}}\|^2 > .$$

4. A Perturbation Argument.

First let us remark that (3.1) can be explicitly solved if $\mathbf{F} = 0$.

Proposition 4.1 . *For any $g \in \mathcal{H}_{-1}$ there exists a sequence $u_n \in L^2 \cap \mathcal{H}_1$ such that*

$$\|\left(\ \mathbf{D}^2 + \mathbf{v}\cdot\mathbf{D} + D_t\ \right)u_n - g\|_{-1} \longrightarrow 0 \tag{4.1}$$

Proof.

Since $L^2 \cap \mathcal{H}_{-1}$ is dense in $\mathcal{H}_{-1}$, we may assume $g \in L^2 \cap \mathcal{H}_{-1}$. Let $T_{\mathbf{x},t}$ be the translation operator on L^2 defined by

$$T_{\mathbf{x},t}f(\omega) = f(\tau_{\mathbf{x},t}\omega)$$

Then $\{T_{\mathbf{x},t}, (\mathbf{x},t) \in \mathbb{R}^{d+1}\}$ is a unitary group that has spectral resolution

$$T_{\mathbf{x},t} = \int_{\mathbb{R}^{d+1}} e^{i(\mathbf{k}\cdot\mathbf{x}+ht)}U(d\mathbf{k},dh)$$

with $U(d\mathbf{k},dh)$ the spectral projections. Defining $U(d\mathbf{k},dh)g(\omega) = \hat{g}(d\mathbf{k},dh,\omega)$, the condition $g \in \mathcal{H}_{-1}$ implies that its energy–spectrum measure

$$\nu_g(d\mathbf{k},dh)\delta(\mathbf{k}-\mathbf{k}')\delta(h-h') =< \hat{g}(d\mathbf{k},dh,\omega)\overline{\hat{g}(d\mathbf{k}',dh',\omega)} >$$

satisfies

$$\int \frac{1}{|\mathbf{k}|^2}\nu_g(d\mathbf{k}, dh) = \|g\|_{-1}^2 < +\infty$$

Consider the solution of

$$(\ \lambda 1 - \mathbf{D}^2 - \mathbf{v}\cdot\mathbf{D} - D_t\ )\, u_\lambda = g\ .$$

Then we have explicitly

$$\widehat{\mathbf{D}u_\lambda}(d\mathbf{k}, dh, \omega) = \frac{\mathbf{k}}{\lambda - |\mathbf{k}|^2 - i(\mathbf{v}\cdot\mathbf{k} + h)}\hat{g}(d\mathbf{k}, dh, \omega)$$

and we can conclude that $\mathbf{D}u_\lambda$ converge strongly in L^2 as $\lambda \to 0$. $\square$

In the following $J = \mathbf{F}\cdot\mathbf{D}$, $M = \mathbf{v}\cdot\mathbf{D} + D_t$, $S = \mathbf{D}^2$, $L_0 = S + M$.

Lemma 4.2 . *JL_0^{-1} is a bounded operator in $\mathcal{H}_{-1}$.*

Proof

Since J is a bounded operator

$$\|Ju\|_{-1}\ \le\ C_1\|u\|_1\ .$$

On the other hand it is immediate to show that

$$\|u\|_1\ \le\ \|L_0 u\|_{-1}$$

because M is an asymmetric operator. So we have

$$\|JL_0^{-1}v\|_{-1} \le C_1\|v\|_{-1}$$

for any $v = L_0 u$. But by proposition 4.1 $\{L_0 u, u \in \mathcal{H}_1 \cap L^2\}$ is dense in $\mathcal{H}_{-1}$. $\square$

Lemma 4.3 . *For any $\delta < C_1^{-1}$ and $\psi \in \mathcal{H}_{-1}$ there exists a sequence $u_n \in L^2 \cap \mathcal{H}_1$ such that*

$$\|(L_0 + \delta J)u_n - \psi\|_{-1} \longrightarrow 0 \tag{4.2}$$

Proof

Since the norm of JL_0^{-1} is bounded by a finite constant C_1, we may define the bounded operator $\left(1 + \delta J L_0^{-1} \right)^{-1}$ by the serie $\sum_{n \geq 0} \left(\delta J L_0^{-1} \right)^n$. Let $g = \left(1 + \delta J L_0^{-1} \right)^{-1} \psi$. By Proposition 4.1 there exists a sequence $u_n \in \mathcal{H}_1 \cap L^2$ such that $\| L_0 u_n - g \|_{-1} \to 0$. Then we have

$$\|(L_0 + \delta J) u_n - \psi\|_{-1} = \| \left(1 + \delta J L_0^{-1} \right) \left(L_0 u_n - g \right) \|_{-1} \leq C(\delta) \| L_0 u_n - g \|_{-1} \to 0.$$

□

Proof of Proposition 3.1

We have to show that Lemma 4.3 is valid with $\delta = 1$. Observe that Proposition 4.1 and everything said above is valid if $L_\delta = S + \delta M$ is substituted to L_0[5]. Formally we have

$$L = S + A = \frac{1}{\delta} \left(1 - (1 - \delta) S \left(S + \delta A \right)^{-1} \right) \left(S + \delta A \right) .$$

Since A is an asymmetric operator, $\|u\|_1 \leq \|(S + \delta A) u\|_{-1}$ for all u in $\mathcal{H}_1$. Since, by lemma 4.3, $\{ \left(S + \delta A \right) u, u \in \mathcal{H}_1 \cap L^2 \}$ is dense in $\mathcal{H}_{-1}$, $\| \left(S + \delta A \right)^{-1} \phi \|_1 \leq \|\phi\|_{-1}$ for all ϕ in $\mathcal{H}_{-1}$. In particular,

$$\|S \left(S + \delta A \right)^{-1} \phi\|_{-1} = \| \left(S + \delta A \right)^{-1} \phi \|_1 \leq \|\phi\|_{-1}$$

i.e. that $S \left(S + \delta A \right)^{-1}$ is a bounded operator in $\mathcal{H}_{-1}$ with norm bounded by 1. It follows that also $\left(1 - (1 - \delta) S \left(S + \delta A \right)^{-1} \right)^{-1}$ (defined as a power series as in lemma 4.3) is a bounded operator in $\mathcal{H}_{-1}$.

Define $\psi = \delta \left(1 - (1 - \delta) S \left(S + \delta A \right)^{-1} \right)^{-1} g$, then by lemma 4.3 there exists a sequence $u_n \in \mathcal{H}_1 \cap L^2$ such that

$$\|(S + \delta A) u_n - \psi\|_{-1} \longrightarrow 0$$

so considering this sequence we have

$$\begin{aligned} \|L u_n - g\|_{-1} &= \|\frac{1}{\delta} \left(1 - (1 - \delta) S \left(S + \delta A \right)^{-1} \right) \left[\left(S + \delta A \right) u_n - \psi \right] \|_{-1} \\ &\leq C(\delta) \|(S + \delta A) u_n - \psi\|_{-1} \end{aligned}$$

and this conclude the proof of Proposition 3.1. □

[5] Observe that $\|J L_\delta^{-1}\| \leq C_1$ for any δ

5. Bounds on the effective diffusion.

It will be interesting to study the effective diffusion matrix as function of the parameters of the original microscopic equation (2.2). In dimension 2 and for time independent fields one can obtain an elementary bound on the effective diffusivity on the orthogonal direction of the mean drift $\mathbf{v}$.

For $d = 2$, chose a direction $\mathbf{e}$ such that $\mathbf{v}\cdot\mathbf{e} = 0$. In this case $\mathbf{F}$ can be written as $\mathbf{F} = \mathbf{D}^{\perp}H$ for a bounded smooth stream function H. Then, as result of what proven in the previous sections, there exists $u^{\mathbf{e}}$ such that

$$\mathbf{D}^2 u^{\mathbf{e}} + \mathbf{e}_{\perp}\cdot(|\mathbf{v}|\mathbf{D}u^{\mathbf{e}} - \mathbf{D}H) + \mathbf{D}^{\perp}H\cdot\mathbf{D}u^{\mathbf{e}} = 0$$

in $\mathcal{H}_{-1}$. Then testing this equation against $u^{\mathbf{e}}$ and H one obtains the relations

$$< |\mathbf{D}u^{\mathbf{e}}|^2 > = < He_{\perp}\cdot\mathbf{D}u^{\mathbf{e}} > = \frac{1}{|\mathbf{v}|} < \mathbf{D}H\cdot(\mathbf{I}+\mathbf{H})Du^{\mathbf{e}} >$$

so using Schwarz inequality one obtains

$$\mathbf{e}\cdot a(v\mathbf{e}_{\perp})\mathbf{e} = < |\mathbf{D}u^{\mathbf{e}}|^2 > \leq \frac{1}{v^2}(1+\|H\|_{\infty}) < |\mathbf{D}H|^2 >$$

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