Generalized Functions in Infinite Dimensional Analysis

Yuri G. KONDRATIEV
Ludwig STREIT
Werner WESTERKAMP
Jia-an YAN

November 23, 1995

Abstract

We give a general approach to infinite dimensional non-Gaussian Analysis which generalizes the work [ADKS94] to measures which possess more singular logarithmic derivative. This framework also includes the possibility to handle measures of Poisson type.

Mathematics Subject Classification (1991): 46F25

Contents

1 Background and Introduction

White Noise Analysis and – more generally – Gaussian analysis have now become of age, both date back approximately twenty years, for reviews we refer to [BeKo88, HKPS93]. Essential to both of them is an orthogonal decomposition of the underlying L^2 space — the "chaos" or "Hermite" or "normal" or "multiple Wiener integral" decomposition.

One extension of this setup has been introduced by Y.M. Berezansky: Starting from certain field operators he constructs polynomial or orthogonal decompositions with respect to the spectrum measures which need not necessary be Gaussian, see e.g., [BeLy93].

A different approach was recently proposed by [AKS93]. For smooth probability measures on infinite dimensional linear spaces a biorthogonal decomposition is a natural extension of the orthogonal one that is well known in Gaussian analysis. This biorthogonal "Appell" system has been constructed for smooth measures by Yu.L. Daletskii [Da91]. For a detailed description of its use in infinite dimensional analysis we refer to [ADKS94].

Aim of the present work. We consider the case of non–degenerate measures on co-nuclear spaces with analytic characteristic functionals. It is worth emphasizing that no further condition such as quasi–invariance of the measure or smoothness of logarithmic derivatives are required. The point here is that the important example of Poisson noise is now accessible.
For any such measure μ we construct an Appell system $\mathbb{A}^\mu$ as a pair $(\mathbb{P}^\mu, \mathbb{Q}^\mu)$ of Appell polynomials $\mathbb{P}^\mu$ and a canonical system of generalized functions $\mathbb{Q}^\mu$, properly associated to the measure μ.

Central results. Within the above framework

- we obtain an explicit description of the test function space introduced in [ADKS94]
- in particular this space is in fact identical for all the measures that we consider
- characterization theorems for generalized as well as test functions are obtained analogously as in Gaussian analysis [KLS94] for more references see [KLPSW94]
- the well known Wick product and the corresponding Wick calculus [KLS94] extends rather directly
- similarly, a full description of positive distributions (as measures) will be given.

Finally we should like to underline here the important conceptual role of holomorphy here as well as in earlier studies of Gaussian analysis (see e.g., [PS91, Ou91, KLPSW94, KLS94] as well as the references cited therein).

2 Preliminaries

2.1 Some facts on nuclear triples

We start with a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. For a given separable nuclear space $\mathcal{N}$ (in the sense of Grothendieck) densely topologically embedded in $\mathcal{H}$ we can construct the nuclear triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'.$$

The dual pairing $\langle\cdot,\cdot\rangle$ of $\mathcal{N}'$ and $\mathcal{N}$ then is realized as an extension of the inner product in $\mathcal{H}$

$$\langle f, \xi\rangle = (f,\xi) \quad f \in \mathcal{H},\ \xi \in \mathcal{N}$$

Instead of reproducing the abstract definition of nuclear spaces (see e.g., [Sch71]) we give a complete (and convenient) characterization in terms of projective limits of Hilbert spaces.

THEOREM 1 *The nuclear Fréchet space* $\mathcal{N}$ *can be represented as*

$$\mathcal{N} = \bigcap_{p \in \mathbb{N}} \mathcal{H}_p,$$

where $\{\mathcal{H}_p,\ p \in \mathbb{N}\}$ *is a family of Hilbert spaces such that for all* $p_1, p_2 \in \mathbb{N}$ *there exists* $p \in \mathbb{N}$ *such that the embeddings* $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_1}$ *and* $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_2}$ *are of Hilbert-Schmidt class. The topology of* $\mathcal{N}$ *is given by the projective limit topology, i.e., the coarsest topology on* $\mathcal{N}$ *such that the canonical embeddings* $\mathcal{N} \hookrightarrow \mathcal{H}_p$ *are continuous for all* $p \in \mathbb{N}$*.*

The Hilbertian norms on $\mathcal{H}_p$ are denoted by $|\cdot|_p$. Without loss of generality we always suppose that $\forall p \in \mathbb{N}, \forall \xi \in \mathcal{N} : |\xi| \leq |\xi|_p$ and that the system of norms is ordered, i.e., $|\cdot|_p \leq |\cdot|_q$ if $p < q$. By general duality theory the dual space $\mathcal{N}'$ can be written as

$$\mathcal{N}' = \bigcup_{p \in \mathbb{N}} \mathcal{H}_{-p}.$$

with inductive limit topology τ_{ind} by using the dual family of spaces $\{\mathcal{H}_{-p} := \mathcal{H}_p',\ p \in \mathbb{N}\}$. The inductive limit topology (w.r.t. this family) is the finest topology on $\mathcal{N}'$ such that the embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous for all $p \in \mathbb{N}$. It is convenient to denote the norm on $\mathcal{H}_{-p}$ by $|\cdot|_{-p}$. Let us mention that in our setting the topology τ_{ind} coincides with the Mackey topology $\tau(\mathcal{N}', \mathcal{N})$ and the strong topology $\beta(\mathcal{N}', \mathcal{N})$. Further note that the dual pair $\langle \mathcal{N}', \mathcal{N} \rangle$ is reflexive if $\mathcal{N}'$ is equipped with $\beta(\mathcal{N}', \mathcal{N})$. In addition we have that convergence of sequences is equivalent in $\beta(\mathcal{N}', \mathcal{N})$ and the weak topology $\sigma(\mathcal{N}', \mathcal{N})$, see e.g., [HKPS93, Appendix 5].

Further we want to introduce the notion of tensor power of a nuclear space. The simplest way to do this is to start from usual tensor powers $\mathcal{H}_p^{\otimes n}$, $n \in \mathbb{N}$ of Hilbert spaces. Since there is no danger of confusion we will preserve the notation $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on $\mathcal{H}_p^{\otimes n}$ and $\mathcal{H}_{-p}^{\otimes n}$ respectively. Using the definition

$$\mathcal{N}^{\otimes n} := \operatorname{pr}\lim_{p \in \mathbb{N}} \mathcal{H}_p^{\otimes n}$$

one can prove [Sch71] that $\mathcal{N}^{\otimes n}$ is a nuclear space which is called the n^{th} tensor power of $\mathcal{N}$. The dual space of $\mathcal{N}^{\otimes n}$ can be written

$$(\mathcal{N}^{\otimes n})' = \operatorname{ind}\lim_{p \in \mathbb{N}} \mathcal{H}_{-p}^{\otimes n}$$

Most important for the applications we have in mind is the following 'kernel theorem', see e.g., [BeKo88].

THEOREM 2 *Let* $\xi_1, ..., \xi_n \mapsto F_n(\xi_1, ..., \xi_n)$ *be an n-linear form on* $\mathcal{N}^{\otimes n}$ *which is* $\mathcal{H}_p$*-continuous , i.e.,*

$$|F_n(\xi_1, ..., \xi_n)| \leq C \prod_{k=1}^{n} |\xi_k|_p$$

for some $p \in \mathbb{N}$ *and* $C > 0$*.*
Then for all $p' > p$ *such that the embedding* $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ *is Hilbert-Schmidt there exist a unique* $\Phi^{(n)} \in \mathcal{H}_{-p'}^{\otimes n}$ *such that*

$$F_n(\xi_1, \ldots, \xi_n) = \langle \Phi^{(n)}, \xi_1 \otimes \cdots \otimes \xi_n \rangle\ , \qquad \xi_1, ..., \xi_n \in \mathcal{N}$$

and the following norm estimate holds

$$\left|\Phi^{(n)}\right|_{-p'} \leq C \;\|i_{p',p}\|^n_{HS}$$

using the Hilbert-Schmidt norm of $i_{p',p}$*.*

COROLLARY 3 *Let* $\xi_1, ..., \xi_n \mapsto F\,(\xi_1, ..., \xi_n)$ *be an* n*-linear form on* $\mathcal{N}^{\otimes n}$ *which is* $\mathcal{H}_{-p}$*-continuous, i.e.,*

$$|F_n\,(\xi_1, \ldots, \xi_n)\,| \leq C \prod_{k=1}^{n} |\xi_k|_{-p}$$

for some $p \in I\!\!N$ *and* $C > 0$*.*
Then for all $p' < p$ *such that the embedding* $i_{p,p'} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p'}$ *is Hilbert-Schmidt there exist a unique* $\Phi^{(n)} \in \mathcal{H}_{p'}^{\otimes n}$ *such that*

$$F_n\,(\xi_1, ..., \xi_n) = \langle \Phi^{(n)}, \xi_1 \otimes \cdots \otimes \xi_n \rangle, \qquad \xi_1, ..., \xi_n \in \mathcal{N}$$

and the following norm estimate holds

$$\left|\Phi^{(n)}\right|_{p'} \leq C \;\|i_{p,p'}\|^n_{HS} \;\;.$$

If in Theorem 2 (and in Corollary 3 respectively) we start from a symmetric n-linear form F_n on $\mathcal{N}^{\otimes n}$ i.e., $F_n(\xi_{\pi_1}, \ldots, \xi_{\pi_n}) = F_n\,(\xi_1, \ldots, \xi_n)$ for any permutation π, then the corresponding kernel $\Phi^{(n)}$ can be localized in $\mathcal{H}_{p'}^{\hat{\otimes} n} \subset \mathcal{H}_{p'}^{\otimes n}$ (the nth symmetric tensor power of the Hilbert space $\mathcal{H}_{p'}$). For $f_1, \ldots, f_n \in \mathcal{H}$ let $\hat{\otimes}$ also denote the symmetrization of the tensor product

$$f_1 \hat{\otimes} \cdots \hat{\otimes} f_n := \frac{1}{n!} \sum_{\pi} f_{\pi_1} \otimes \cdots \otimes f_{\pi_n} \;,$$

where the sum extends over all permutations of n letters. All the above quoted theorems also hold for complex spaces, in particular the complexified space $\mathcal{N}_{\mathbb{C}}$. By definition an element $\theta \in \mathcal{N}_{\mathbb{C}}$ decomposes into $\theta = \xi + i\eta$, $\xi, \eta \in \mathcal{N}$. If we also introduce the corresponding complexified Hilbert spaces $\mathcal{H}_{p,\mathbb{C}}$ the inner product becomes

$$(\theta_1, \theta_2)_{\mathcal{H}_{p,\mathbb{C}}} = (\theta_1, \bar{\theta}_2)_{\mathcal{H}_p} = (\xi_1, \xi_2)_{\mathcal{H}_p} + (\eta_1, \eta_2)_{\mathcal{H}_p} + i(\eta_1, \xi_2)_{\mathcal{H}_p} - i(\xi_1, \eta_2)_{\mathcal{H}_p}$$

for $\theta_1, \theta_2 \in \mathcal{H}_{p,\mathbb{C}}$, $\theta_1 = \xi_1 + i\eta_1$, $\theta_2 = \xi_2 + i\eta_2$, $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{H}_p$. Thus we have introduced a nuclear triple

$$\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \subset \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n} \subset \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$$

We also want to introduce the (Boson or symmetric) Fock space $\Gamma(\mathcal{H})$ of $\mathcal{H}$ by

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$$

with the convention $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} 0} := \mathbb{C}$ and the Hilbertian norm

$$\|\vec{\varphi}\|^2_{\Gamma(\mathcal{H})} = \sum_{n=0}^{\infty} n! \,\left|\varphi^{(n)}\right|^2 \;, \quad \vec{\varphi} = \left\{\varphi^{(n)} \;\middle|\; n \in I\!\!N_0\right\} \in \Gamma(\mathcal{H}) \;.$$

2.2 Holomorphy on locally convex spaces

We shall collect some facts from the theory of holomorphic functions in locally convex topological vector spaces $\mathcal{E}$ (over the complex field $\mathbb{C}$), see e.g., [Di81]. Let $\mathcal{L}(\mathcal{E}^n)$ be the space of n-linear mappings from $\mathcal{E}^n$ into $\mathbb{C}$ and $\mathcal{L}_s(\mathcal{E}^n)$ the subspace of symmetric n-linear forms. Also let $P^n(\mathcal{E})$ denote the n-homogeneous polynomials on $\mathcal{E}$. There is a linear bijection $\mathcal{L}_s(\mathcal{E}^n) \ni A \longleftrightarrow \widehat{A} \in P^n(\mathcal{E})$. Now let $\mathcal{U} \subset \mathcal{E}$ be open and consider a function $G : \mathcal{U} \to \mathbb{C}$.

G is said to be **G-holomorphic** if for all $\theta_0 \in \mathcal{U}$ and for all $\theta \in \mathcal{E}$ the mapping from $\mathbb{C}$ to $\mathbb{C} : \lambda \to G(\theta_0 + \lambda\theta)$ is holomorphic in some neighborhood of zero in $\mathbb{C}$. If G is G-holomorphic then there exists for every $\eta \in \mathcal{U}$ a sequence of homogeneous polynomials $\frac{1}{n!}\widehat{\mathrm{d}^n G(\eta)}$ such that

$$G(\theta + \eta) = \sum_{n=0}^{\infty} \frac{1}{n!}\widehat{\mathrm{d}^n G(\eta)}(\theta)$$

for all θ from some open set $\mathcal{V} \subset \mathcal{U}$. G is said to be **holomorphic**, if for all η in $\mathcal{U}$ there exists an open neighborhood $\mathcal{V}$ of zero such that $\sum_{n=0}^{\infty} \frac{1}{n!}\widehat{\mathrm{d}^n G(\eta)}(\theta)$ converges uniformly on $\mathcal{V}$ to a continuous function. We say that G is holomorphic at θ_0 if there is an open set $\mathcal{U}$ containing θ_0 such that G is holomorphic on $\mathcal{U}$. The following proposition can be found e.g., in [Di81].

PROPOSITION 4 *G is holomorphic if and only if it is G-holomorphic and locally bounded.*

Let us explicitly consider a function holomorphic at the point $0 \in \mathcal{E} = \mathcal{N}_{\mathbb{C}}$, then

1) there exist p and $\varepsilon > 0$ such that for all $\xi_0 \in \mathcal{N}_{\mathbb{C}}$ with $|\xi_0|_p \le \varepsilon$ and for all $\xi \in \mathcal{N}_{\mathbb{C}}$ the function of one complex variable $\lambda \to G(\xi_0 + \lambda\xi)$ is analytic at $0 \in \mathbb{C}$, and

2) there exists $c > 0$ such that for all $\xi \in \mathcal{N}_{\mathbb{C}}$ with $|\xi|_p \le \varepsilon : |G(\xi)| \le c$.

As we do not want to discern between different restrictions of one function, we consider germs of holomorphic functions, i.e., we identify F and G if there exists an open neighborhood $\mathcal{U} : 0 \in \mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ such that $F(\xi) = G(\xi)$ for all $\xi \in \mathcal{U}$. Thus we define $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ as the algebra of germs of functions holomorphic at zero equipped with the inductive topology given by the following family of norms

$$\mathrm{n}_{p,l,\infty}(G) = \sup_{|\theta|_p \le 2^{-l}} |G(\theta)|\,, \quad p, l \in \mathbb{N}.$$

Let use now introduce spaces of entire functions which will be useful later. Let $\mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$ denote the set of all entire functions on $\mathcal{H}_{-p,\mathbb{C}}$ of growth $k \in [1, 2]$ and type 2^{-l}, $p, l \in \mathbb{Z}$. This is a linear space with norm

$$\mathrm{n}_{p,l,k}(\varphi) = \sup_{z \in \mathcal{H}_{-p,\mathbb{C}}} |\varphi(z)| \exp\left(-2^{-l}|z|^k_{-p}\right), \qquad \varphi \in \mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$$

The space of entire functions on $\mathcal{N}'_{\mathbb{C}}$ of growth k and minimal type is naturally introduced by

$$\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}}) := \operatorname*{pr\,lim}_{p,l \in \mathbb{N}} \mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})\ ,$$

see e.g., [Ou91]. We will also need the space of entire functions on $\mathcal{N}_{\mathbb{C}}$ of growth k and finite type:

$$\mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}}) := \operatorname*{ind\,lim}_{p,l\in \mathbb{N}} \mathcal{E}^k_{2^l}(\mathcal{H}_{p,\mathbb{C}}) \ .$$

In the following we will give an equivalent description of $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ and $\mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}})$. Cauchy's inequality and Corollary 3 allow to write the Taylor coefficients in a convenient form. Let $\varphi \in \mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ and $z \in \mathcal{N}'_{\mathbb{C}}$, then there exist kernels $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ such that

$$\langle z^{\otimes n}, \varphi^{(n)}\rangle = \frac{1}{n!}\mathrm{d}^n\widehat{\varphi(0)}(z)$$

i.e.,

$$\varphi(z) = \sum_{n=0}^{\infty}\langle z^{\otimes n}, \varphi^{(n)}\ \rangle. \tag{1}$$

This representation allows to introduce a nuclear topology on $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$, see [Ou91] for details. Let $\mathrm{E}^{\beta}_{p,q}$ denote the space of all functions of the form (1) such that the following Hilbertian norm

$$|||\varphi|||^2_{p,q,\beta} := \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} \left|\varphi^{(n)}\right|_p^2 \ , \quad p,q \in \mathbb{N} \tag{2}$$

is finite for $\beta \in [0,1]$. (By $\left|\varphi^{(0)}\right|_p$ we simply mean the complex modulus for all p.) The space $\mathrm{E}^{-\beta}_{-p-,q}$ with the norm $|||\varphi|||_{-p,-q,-\beta}$ is defined analogously.

Theorem 5 *The following topological identity holds:*

$$\operatorname*{pr\,lim}_{p,q\in\mathbb{N}}\ \mathrm{E}^{\beta}_{p,q} = \mathcal{E}^{\frac{2}{1+\beta}}_{\min}(\mathcal{N}'_{\mathbb{C}}) \quad .$$

The proof is an immediate consequence of the following two lemmata which show that the two systems of norms are in fact equivalent.

Lemma 6 *Let* $\varphi \in \mathrm{E}^{\beta}_{p,q}$ *then* $\varphi \in \mathcal{E}^{\frac{2}{1+\beta}}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$ *for* $l = \frac{q}{1+\beta}$. *Moreover*

$$\mathrm{n}_{p,l,k}(\varphi) \le |||\varphi|||_{p,q,\beta} \ , \quad k = \tfrac{2}{1+\beta} \ . \tag{3}$$

Proof. We look at the convergence of the series $\varphi(z) = \sum_{n=0}^{\infty}\langle z^{\otimes n}, \varphi^{(n)}\ \rangle$, $z \in \mathcal{H}_{-p,\mathbb{C}}$, $\varphi^{(n)} \in \mathcal{H}_{p,\mathbb{C}}$ if $\sum_{n=0}^{\infty}(n!)^{1+\beta}2^{nq}|\varphi^{(n)}|_p^2 = |||\varphi|||^2_{p,q,\beta}$ is finite. The following estimate holds:

$$\begin{aligned}
\sum_{n=0}^{\infty} |\langle z^{\otimes n}, \varphi^{(n)}\ \rangle| &\le \left(\sum_{n=0}^{\infty}(n!)^{1+\beta}2^{nq}|\varphi^{(n)}|_p^2\right)^{1/2}\left(\sum_{n=0}^{\infty}\frac{1}{(n!)^{1+\beta}}2^{-nq}|z|_{-p}^{2n}\right)^{1/2} \\
&\le |||\varphi|||_{p,q,\beta}\cdot\left(\sum_{n=0}^{\infty}\left\{\frac{1}{n!}2^{-\frac{nq}{1+\beta}}|z|_{-p}^{\frac{2n}{1+\beta}}\right\}^{1+\beta}\right)^{1/2} \\
&\le |||\varphi|||_{p,q,\beta}\left(\sum_{n=0}^{\infty}\frac{1}{n!}2^{-\frac{nq}{1+\beta}}|z|_{-p}^{\frac{2n}{1+\beta}}\right)^{(1+\beta)/2} \\
&\le |||\varphi|||_{p,q,\beta}\exp\left(2^{-\frac{q}{1+\beta}}|z|_{-p}^{\frac{2}{1+\beta}}\right).
\end{aligned}$$

□

Lemma 7 *For any $p', q \in \mathbb{N}$ there exist $p, l \in \mathbb{N}$ such that*

$$\mathcal{E}^{\frac{2}{1+\beta}}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}) \subset \mathrm{E}^{\beta}_{p',q}$$

i.e., there exists a constant $C > 0$ such that

$$|\!|\!|\varphi|\!|\!|_{p',q,\beta} \le C\ \mathrm{n}_{p,l,k}(\varphi), \quad \varphi \in \mathcal{E}^{k}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}), \quad k = \tfrac{2}{1+\beta}.$$

Remark. More precisely we will prove the following: If $\varphi \in \mathcal{E}^{k}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$ then $\varphi \in \mathrm{E}^{\beta}_{p',q}$ for $k = \frac{2}{1+\beta}$ and $\rho := 2^{q-2l/k}k^{2/k}e^{2}\left\|i_{p',p}\right\|^{2}_{HS} < 1$ (in particular this requires $p' > p$ to be such that the embedding $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_{p}$ is Hilbert-Schmidt).
Moreover the following bound holds

$$|\!|\!|\varphi|\!|\!|_{p',q,\beta} \le \mathrm{n}_{p,l,k}(\varphi) \cdot (1-\rho)^{-1/2}\ . \tag{4}$$

Proof. The assumption $\varphi \in \mathcal{E}^{k}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$ implies a bound of the growth of φ :

$$|\varphi(z)| \le \mathrm{n}_{p,l,k}(\varphi)\exp(2^{-l}|z|^{k}_{-p})\ .$$

For each $\rho > 0$, $z \in \mathcal{H}_{-p,\mathbb{C}}$ the Cauchy inequality from complex analysis [Di81] gives

$$\left|\frac{1}{n!}\widehat{\mathrm{d}^{n}\varphi(0)}(z)\right| \le \mathrm{n}_{p,l,k}(\varphi)\rho^{-n}\exp(\rho^{k}2^{-l})\ |z|^{n}_{-p}\ .$$

By polarization [Di81] it follows for $z_1, \ldots, z_n \in \mathcal{H}_{-p,\mathbb{C}}$

$$\left|\frac{1}{n!}\mathrm{d}^{n}\varphi(0)(z_1, \ldots, z_n)\right| \le \mathrm{n}_{p,l,k}(\varphi)\frac{1}{n!}\left(\frac{n}{\rho}\right)^{n}\exp(\rho^{k}2^{-l})\prod_{k=1}^{n}|z_k|_{-p}\ .$$

For $p' > p$ such that $\left\|i_{p',p}\right\|_{HS}$ is finite, an application of the kernel theorem guarantees the existence of kernels $\varphi^{(n)} \in \mathcal{H}^{\hat{\otimes} n}_{p',\mathbb{C}}$ such that

$$\varphi(z) = \sum_{n=0}^{\infty}\langle z^{\hat{\otimes} n}, \varphi^{(n)}\ \rangle$$

with the bound

$$\left|\varphi^{(n)}\right|_{p'} \le \mathrm{n}_{p,l,k}(\varphi)\frac{1}{n!}\left(\frac{n}{\rho}\left\|i_{p',p}\right\|_{HS}\right)^{n}\exp(\rho^{k}\cdot 2^{-l})\ .$$

We can optimize the bound with the choice of an n-dependent ρ. Setting $\rho^{k} = 2^{l}n/k$ we obtain

$$\begin{aligned}\left|\varphi^{(n)}\right|_{p'} &\le \mathrm{n}_{p,l,k}(\varphi)\frac{1}{n!}n^{n(1-1/k)}\left(\tfrac{1}{k}2^{l}\right)^{-n/k}\left\|i_{p',p}\right\|^{n}_{HS}e^{n/k}\\ &\le \mathrm{n}_{p,l,k}(\varphi)\ (n!)^{-1/k}\left\{(k2^{-l})^{1/k}e\left\|i_{p',p}\right\|_{HS}\right\}^{n}\ ,\end{aligned}$$

where we used $n^{n} \le n!\,e^{n}$ in the last estimate. Now choose $\beta \in [0,1]$ such that $k = \frac{2}{1+\beta}$ to estimate the following norm:

$$\begin{aligned}|\!|\!|\varphi|\!|\!|^{2}_{p',q,\beta} &\le \mathrm{n}^{2}_{p,l,k}(\varphi)\sum_{n=0}^{\infty}(n!)^{1+\beta-\frac{2}{k}}2^{qn}\left\{(k2^{-l})^{1/k}e\left\|i_{p',p}\right\|_{HS}\right\}^{2n}\\ &\le \mathrm{n}^{2}_{p,l,k}(\varphi)\left(1-2^{q}\left\{(k2^{-l})^{1/k}e\left\|i_{p',p}\right\|_{HS}\right\}^{2}\right)^{-1}\end{aligned}$$

for sufficiently large l. This completes the proof. □

Analogous estimates for these systems of norms also hold if β, p, q, l become negative. This implies the following theorem. For related results see e.g., [Ou91, Prop.8.6].

THEOREM 8
If $\beta \in [0,1)$ *then the following topological identity holds:*

$$\operatorname*{ind\,lim}_{p,q\in\mathbb{N}} \ \mathrm{E}^{-\beta}_{-p,-q} = \mathcal{E}^{2/(1-\beta)}_{\max}(\mathcal{N}_{\mathbb{C}}).$$

If $\beta = 1$ *we have*

$$\operatorname*{ind\,lim}_{p,q\in\mathbb{N}} \ \mathrm{E}^{-1}_{-p,-q} = \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}) \ .$$

This theorem and its proof will appear in the context of section 8. The characterization of distributions in infinite dimensional analysis is strongly related to this theorem. From this point of view it is natural to postpone its proof to section 8.

3 Measures on linear topological spaces

To introduce probability measures on the vector space $\mathcal{N}'$, we consider $\mathcal{C}_\sigma(\mathcal{N}')$ the σ-algebra generated by cylinder sets on $\mathcal{N}'$, which coincides with the Borel σ-algebras $\mathcal{B}_\sigma(\mathcal{N}')$ and $\mathcal{B}_\beta(\mathcal{N}')$ generated by the weak and strong topology on $\mathcal{N}'$ respectively. Thus we will consider this σ-algebra as the *natural* σ-algebra on $\mathcal{N}'$. Detailed definitions of the above notions and proofs of the mentioned relations can be found in e.g., [BeKo88].

We will restrict our investigations to a special class of measures μ on $\mathcal{C}_\sigma(\mathcal{N}')$, which satisfy two additional assumptions. The first one concerns some analyticity of the Laplace transformation

$$l_\mu(\theta) = \int_{\mathcal{N}'} \exp\langle x,\theta\rangle \ \mathrm{d}\mu(x) =: \mathbb{E}_\mu(\exp\langle x,\theta\rangle) \ , \ \theta \in \mathcal{N}_{\mathbb{C}}.$$

Here we also have introduced the convenient notion of expectation $\mathbb{E}_\mu$ of a μ-integrable function.
ASSUMPTION 1 The measure μ has an analytic Laplace transform in a neighborhood of zero. That means there exists an open neighborhood $\mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ of zero, such that l_μ is holomorphic on $\mathcal{U}$, i.e., $l_\mu \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. This class of *analytic measures* is denoted by $\mathcal{M}_a(\mathcal{N}')$.

An equivalent description of analytic measures is given by the following lemma.

LEMMA 9 *The following statements are equivalent*

1) $\mu \in \mathcal{M}_a(\mathcal{N}')$

2) $\exists p_\mu \in \mathbb{N}, \quad \exists C > 0 : \qquad \left| \int_{\mathcal{N}'} \langle x,\theta\rangle^n \ \mathrm{d}\mu(x) \right| \le n!\, C^n \, |\theta|^n_{p_\mu} \ , \quad \theta \in \mathcal{H}_{p_\mu,\mathbb{C}}$

3) $\exists p'_\mu \in \mathbb{N}, \quad \exists \varepsilon_\mu > 0 : \qquad \int_{\mathcal{N}'} \exp(\varepsilon_\mu \, |x|_{-p'_\mu}) \ \mathrm{d}\mu(x) < \infty$

PROOF. The proof can be found in [KoSW95]. We give its outline in the following. The only non-trivial step is the proof of 2)⇒3).
By polarization [Di81] 2) implies

$$\left| \int_{\mathcal{N}'} \langle x^{\otimes n}, \bigotimes_{j=1}^{n} \xi_j \rangle \, \mathrm{d}\mu(x) \right| \leq n! \, C^n \prod_{j=1}^{n} |\xi_j|_{p_\mu} \ , \quad \xi_j \in \mathcal{H}_{p'} \tag{5}$$

for a (new) constant $C > 0$. Choose $p' > p_\mu$ such that the embedding $i_{p',p_\mu} : \mathcal{H}_{p'} \to \mathcal{H}_{p_\mu}$ is of Hilbert-Schmidt type. Let $\{e_k, \ k \in I\!N\} \subset \mathcal{N}$ be an orthonormal basis in $\mathcal{H}_{p'}$. Then $|x|^2_{-p'} = \sum_{k=1}^{\infty} \langle x, e_k \rangle^2$, $x \in \mathcal{H}_{-p'}$. We will first estimate the moments of even order

$$\int_{\mathcal{N}'} |x|^{2n}_{-p'} \ \mathrm{d}\mu(x) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \int_{\mathcal{N}'} \langle x, e_{k_1} \rangle^2 \ \cdots \langle x, e_{k_n} \rangle^2 \ \mathrm{d}\mu(x) \ ,$$

where we changed the order of summation and integration by a monotone convergence argument. Using the bound (5) we have

$$\begin{aligned} \int_{\mathcal{N}'} |x|^{2n}_{-p'} \ \mathrm{d}\mu(x) \ &\leq \ C^{2n} \ (2n)! \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} |e_{k_1}|^2_{p_\mu} \cdots |e_{k_n}|^2_{p_\mu} \\ &= \ C^{2n} \ (2n)! \left(\sum_{k=1}^{\infty} |e_k|^2_{p_\mu} \right)^n \\ &= \ \left(C \cdot \left\| i_{p',p_\mu} \right\|_{HS} \right)^{2n} (2n)! \end{aligned}$$

because

$$\sum_{k=1}^{\infty} |e_k|^2_{p_\mu} = \left\| i_{p',p_\mu} \right\|^2_{HS} \ .$$

The moments of arbitrary order can now be estimated by the Schwarz inequality

$$\begin{aligned} \int |x|^n_{-p'} \ \mathrm{d}\mu(x) \ &\leq \ \sqrt{\mu(\mathcal{N}')} \left(\int |x|^{2n}_{-p} \ \mathrm{d}\mu(x) \right)^{\frac{1}{2}} \\ &\leq \ \sqrt{\mu(\mathcal{N}')} \left(C \left\| i_{p',p_\mu} \right\|_{HS} \right)^n \sqrt{(2n)!} \\ &\leq \ \sqrt{\mu(\mathcal{N}')} \left(2C \left\| i_{p',p_\mu} \right\|_{HS} \right)^n n! \end{aligned}$$

since $(2n)! \leq 4^n (n!)^2$.
Choose $\varepsilon < \left(2C \left\| i_{p',p_\mu} \right\|_{HS} \right)^{-1}$ then

$$\begin{aligned} \int e^{\varepsilon |x|_{-p'}} \mathrm{d}\mu(x) \ &= \ \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int |x|^n_{-p'} \ \mathrm{d}\mu(x) \\ &\leq \ \sqrt{\mu(\mathcal{N}')} \ \sum_{n=0}^{\infty} \left(\varepsilon \ 2C \left\| i_{p',p_\mu} \right\|_{HS} \right)^n < \infty \end{aligned} \tag{6}$$

Hence the lemma is proven. □

For $\mu \in \mathcal{M}_a(\mathcal{N}')$ the estimate in statement 2 of the above lemma allows to define the moment kernels $\mathrm{M}_n^\mu \in (\mathcal{N}^{\hat{\otimes} n})'$. This is done by extending the above estimate by a simple polarization argument and applying the kernel theorem. The kernels are determined by

$$l_\mu(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \mathrm{M}_n^\mu, \theta^{\otimes n} \rangle$$

or equivalently

$$\langle \mathrm{M}_n^\mu, \theta_1 \hat{\otimes} \cdots \hat{\otimes} \theta_n \rangle = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} l_\mu(t_1\theta_1 + \cdots + t_n\theta_n) \Big|_{t_1 = \cdots = t_n = 0} .$$

Moreover, if $p > p_\mu$ is such that embedding $i_{p,p_\mu} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p_\mu}$ is Hilbert-Schmidt then

$$|\mathrm{M}_n^\mu|_{-p} \leq \left(nC \left\| i_{p,p_\mu} \right\|_{HS} \right)^n \leq n! \, \left(eC \left\| i_{p,p_\mu} \right\|_{HS} \right)^n . \tag{7}$$

Definition 10 *A function* $\varphi : \mathcal{N}' \to \mathbb{C}$ *of the form* $\varphi(x) = \sum_{n=0}^{N} \langle x^{\otimes n}, \varphi^{(n)} \rangle$, $x \in \mathcal{N}'$, $N \in \mathbb{N}$, *is called a continuous polynomial (short* $\varphi \in \mathcal{P}(\mathcal{N}')$ *) iff* $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, $\forall n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Now we are ready to formulate the second assumption:

Assumption 2 For all $\varphi \in \mathcal{P}(\mathcal{N}')$ with $\varphi = 0$ μ-almost everywhere we have $\varphi \equiv 0$. In the following a measure with this property will be called *non-degenerate*.

Note. Assumption 2 is equivalent to:
Let $\varphi \in \mathcal{P}(\mathcal{N}')$ with $\int_A \varphi \, \mathrm{d}\mu = 0$ for all $A \in \mathcal{C}_\sigma(\mathcal{N}')$ then $\varphi \equiv 0$.
A sufficient condition can be obtained by regarding admissible shifts of the measure μ. If $\mu(\cdot + \xi)$ is absolutely continuous with respect to μ for all $\xi \in \mathcal{N}$, i.e., there exists the Radon-Nikodym derivative

$$\rho_\mu(\xi, x) = \frac{\mathrm{d}\mu(x+\xi)}{\mathrm{d}\mu(x)} , \quad x \in \mathcal{N}' ,$$

Then we say that μ is $\mathcal{N}$*-quasi-invariant* see e.g., [GV68, Sk74]. This is sufficient to ensure Assumption 2, see e.g., [KoTs91, BeKo88].

Example 1 In Gaussian Analysis (especially White Noise Analysis) the Gaussian measure $\gamma_{\mathcal{H}}$ corresponding to the Hilbert space $\mathcal{H}$ is considered. Its Laplace transform is given by

$$l_{\gamma_{\mathcal{H}}}(\theta) = e^{\frac{1}{2}\langle \theta, \theta \rangle} , \qquad \theta \in \mathcal{N}_{\mathbb{C}} ,$$

hence $\gamma_{\mathcal{H}} \in \mathcal{M}_a(\mathcal{N}')$. It is well known that $\gamma_{\mathcal{H}}$ is $\mathcal{N}$–quasi-invariant (moreover $\mathcal{H}$–quasi-invariant) see e.g., [Sk74, BeKo88]. Due to the previous note $\gamma_{\mathcal{H}}$ satisfies also Assumption 2.

Example 2 *(Poisson measures)*
Let use consider the classical (real) Schwartz triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}) .$$

The Poisson white noise measure μ_p is defined as a probability measure on $\mathcal{C}_\sigma(\mathcal{S}'(I\!\!R))$ with the Laplace transform

$$l_{\mu_p}(\theta) = \exp\left\{\int_{I\!\!R} (e^{\theta(t)} - 1)\,\mathrm{d}t\right\} = \exp\left\{\langle e^\theta - 1, 1\rangle\right\}, \qquad \theta \in \mathcal{S}_{\mathbb{C}}(I\!\!R)\,,$$

see e.g., [GV68]. It is not hard to see that l_{μ_p} is a holomorphic function on $\mathcal{S}_{\mathbb{C}}(I\!\!R)$, so Assumption 1 is satisfied. But to check Assumption 2, we need additional considerations.

First of all we remark that for any $\xi \in \mathcal{S}(I\!\!R)\,,\ \xi \neq 0$ the measures μ_p and $\mu_p(\cdot + \xi)$ are orthogonal (see [VGG75] for a detailed analysis). It means that μ_p is not $\mathcal{S}(I\!\!R)$-quasi-invariant and the note after Assumption 2 is not applicable now.

Let some $\varphi \in \mathcal{P}(\mathcal{S}'(I\!\!R))\,,\ \varphi = 0\ \ \mu_p$-a.s. be given. We need to show that then $\varphi \equiv 0$. To this end we will introduce a system of orthogonal polynomials in the space $L^2(\mu_p)$ which can be constructed in the following way. The mapping

$$\theta(\cdot) \mapsto \alpha(\theta)(\cdot) = \log(1 + \theta(\cdot)) \in \mathcal{S}_{\mathbb{C}}(I\!\!R)\,, \quad \theta \in \mathcal{S}_{\mathbb{C}}(I\!\!R)$$

is holomorphic on a neighborhood $\mathcal{U} \subset \mathcal{S}_{\mathbb{C}}(I\!\!R)\,,\ 0 \in \mathcal{U}$. Then

$$e^\alpha_{\mu_p}(\theta; x) = \frac{e^{\langle\alpha(\theta), x\rangle}}{l_{\mu_p}(\alpha(\theta))} = \exp\{\langle\alpha(\theta), x\rangle - \langle\theta, 1\rangle\}\,, \quad \theta \in \mathcal{U}\,,\ x \in \mathcal{S}'(I\!\!R)$$

is a holomorphic function on $\mathcal{U}$ for any $x \in \mathcal{S}'(I\!\!R)$. The Taylor decomposition and the kernel theorem (just as in subsection 4.1 below) give

$$e^\alpha_{\mu_p}(\theta; x) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle\theta^{\otimes n}, C_n(x)\rangle\,,$$

where $C_n : \mathcal{S}'(I\!\!R) \to \mathcal{S}'(I\!\!R)^{\hat{\otimes} n}$ are polynomial mappings. For $\varphi^{(n)} \in \mathcal{S}_{\mathbb{C}}(I\!\!R)^{\hat{\otimes} n}\,,\ n \in I\!\!N_0$, we define Charlier polynomials

$$x \mapsto C_n(\varphi^{(n)}; x) = \langle\varphi^{(n)}, C_n(x)\rangle \in \mathbb{C}\,,\ \ x \in \mathcal{S}'(I\!\!R)\,.$$

Due to [Ito88, IK88] we have the following orthogonality property:

$$\forall\varphi^{(n)} \in \mathcal{S}_{\mathbb{C}}(I\!\!R)^{\hat{\otimes} n}\,,\ \forall\psi^{(m)} \in \mathcal{S}_{\mathbb{C}}(I\!\!R)^{\hat{\otimes} n}$$

$$\int C_n(\varphi^{(n)}) C_m(\psi^{(m)})\,\mathrm{d}\mu_p = \delta_{nm}\, n!\langle\varphi^{(n)}, \psi^{(n)}\rangle\,.$$

Now the rest is simple. Any continuous polynomial φ has a uniquely defined decomposition

$$\varphi(x) = \sum_{n=0}^{N} \langle\varphi^{(n)}, C_n(x)\rangle\,, \quad x \in \mathcal{S}'(I\!\!R)\,,$$

where $\varphi^{(n)} \in \mathcal{S}_{\mathbb{C}}(I\!\!R)^{\hat{\otimes} n}$. If $\varphi = 0$ μ_p-a.e. then

$$\|\varphi\|^2_{L^2(\mu_p)} = \sum_{n=0}^{N} n!\,\langle\varphi^{(n)}, \overline{\varphi^{(n)}}\rangle = 0.$$

Hence $\varphi^{(n)} = 0\,,\ n = 0\,, \ldots,\ N$, i.e., $\varphi \equiv 0$. So Assumption 2 is satisfied.

4 Concept of distributions in infinite dimensional analysis

In this section we will introduce a preliminary distribution theory in infinite dimensional non-Gaussian analysis. We want to point out in advance that the distribution space constructed here is in some sense too big for practical purposes. In this sense section 4 may be viewed as a stepping stone to introduce the more useful structures in §5 and §6.

We will choose $\mathcal{P}(\mathcal{N}')$ as our (minimal) test function space. (The idea to use spaces of this type as appropriate spaces of test functions is rather old see [KMP65]. They also discussed in which sense this space is "minimal".) First we have to ensure that $\mathcal{P}(\mathcal{N}')$ is densely embedded in $L^2(\mu)$. This is fulfilled because of our assumption 1 [Sk74, Sec.10 Th.1]. The space $\mathcal{P}(\mathcal{N}')$ may be equipped with various different topologies, but there exists a natural one such that $\mathcal{P}(\mathcal{N}')$ becomes a nuclear space [BeKo88]. The topology on $\mathcal{P}(\mathcal{N}')$ is chosen such that is becomes isomorphic to the topological direct sum of tensor powers $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ see e.g., [Sch71, Ch II 6.1, Ch III 7.4]

$$\mathcal{P}(\mathcal{N}') \simeq \bigoplus_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \ .$$

via

$$\varphi(x) = \sum_{n=0}^{\infty} \left\langle x^{\otimes n}, \varphi^{(n)} \right\rangle \longleftrightarrow \vec{\varphi} = \left\{ \varphi^{(n)} \;\middle|\; n \in \mathbb{N}_0 \right\} .$$

Note that only a finite number of $\varphi^{(n)}$ is non-zero. We will not reproduce the full construction here, but we will describe the notion of convergence of sequences this topology on $\mathcal{P}(\mathcal{N}')$. For $\varphi \in \mathcal{P}(\mathcal{N}')$, $\varphi(x) = \sum_{n=0}^{N(\varphi)} \left\langle x^{\otimes n}, \varphi^{(n)} \right\rangle$ let $p_n : \mathcal{P}(\mathcal{N}') \to \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ denote the mapping p_n is defined by $p_n \varphi := \varphi^{(n)}$. A sequence $\{\varphi_j,\ j \in \mathbb{N}\}$ of smooth polynomials converges to $\varphi \in \mathcal{P}(\mathcal{N}')$ iff the $N(\varphi_j)$ are bounded and $p_n \varphi_j \underset{n \to \infty}{\longrightarrow} p_n \varphi$ in $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ for all $n \in \mathbb{N}$.

Now we can introduce the dual space $\mathcal{P}'_\mu(\mathcal{N}')$ of $\mathcal{P}(\mathcal{N}')$ with respect to $L^2(\mu)$. As a result we have constructed the triple

$$\mathcal{P}(\mathcal{N}') \subset L^2(\mu) \subset \mathcal{P}'_\mu(\mathcal{N}')$$

The (bilinear) dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\mu$ between $\mathcal{P}'_\mu(\mathcal{N}')$ and $\mathcal{P}(\mathcal{N}')$ is connected to the (sesquilinear) inner product on $L^2(\mu)$ by

$$\langle\!\langle \varphi,\ \psi \rangle\!\rangle_\mu = (\varphi,\ \overline{\psi})_{L^2(\mu)} \ , \quad \varphi \in L^2(\mu),\ \psi \in \mathcal{P}(\mathcal{N}') \ .$$

Since the constant function 1 is in $\mathcal{P}(\mathcal{N}')$ we may extend the concept of expectation from random variables to distributions; for $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$

$$\mathbb{E}_\mu(\Phi) := \langle\!\langle \Phi, 1 \rangle\!\rangle_\mu \ .$$

The main goal of this section is to provide a description of $\mathcal{P}'_\mu(\mathcal{N}')$, see Theorem 18 below. The simplest approach to this problem seems to be the use of so called μ-Appell polynomials.

4.1 Appell polynomials associated to the measure μ

Because of the holomorphy of l_μ and $l_\mu(0)=1$ there exists a neighborhood of zero

$$\mathcal{U}_0 = \left\{\theta \in \mathcal{N}_{\mathbb{C}} \;\middle|\; 2^{q_0}\,|\theta|_{p_0} < 1\right\}$$

$p_0, q_0 \in I\!N$, $p_0 \geq p'_\mu$, $2^{-q_0} \leq \varepsilon_\mu$ (p'_μ, ε_μ from Lemma 9) such that $l_\mu(\theta) \neq 0$ for $\theta \in \mathcal{U}_0$ and the normalized exponential

$$e_\mu(\theta; z) = \frac{e^{\langle z,\theta\rangle}}{l_\mu(\theta)} \quad \text{for } \theta \in \mathcal{U}_0, \quad z \in \mathcal{N}'_{\mathbb{C}}\ , \tag{8}$$

is well defined. We use the holomorphy of $\theta \mapsto e_\mu(\theta; z)$ to expand it in a power series in θ similar to the case corresponding to the construction of one dimensional Appell polynomials [Bo76]. We have in analogy to [AKS93, ADKS94]

$$e_\mu(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{d}^n \widehat{e_\mu(0, z)}(\theta)$$

where $\mathrm{d}^n\widehat{e_\mu(0; z)}$ is an n-homogeneous continuous polynomial. But since $e_\mu(\theta; z)$ is not only G-holomorphic but holomorphic we know that $\theta \to e_\mu(\theta; z)$ is also locally bounded. Thus Cauchy's inequality for Taylor series [Di81] may be applied, $\rho \leq 2^{-q_0}$, $p \geq p_0$

$$\left|\frac{1}{n!}\mathrm{d}^n\widehat{e_\mu(0; z)}(\theta)\right| \leq \frac{1}{\rho^n} \sup_{|\theta|_p=\rho} |e_\mu(\theta; z)|\, |\theta|_p^n \leq \frac{1}{\rho^n} \sup_{|\theta|_p=\rho} \frac{1}{l_\mu(\theta)} e^{\rho|z|_{-p}}\, |\theta|_p^n \tag{9}$$

if $z \in \mathcal{H}_{-p,\mathbb{C}}$. This inequality extends by polarization [Di81] to an estimate sufficient for the kernel theorem. Thus we have a representation $\mathrm{d}^n\widehat{e_\mu(0; z)}(\theta) = \langle P_n^\mu(z), \theta^{\otimes n}\rangle$ where $P_n^\mu(z) \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$. The kernel theorem really gives a little more: $P_n^\mu(z) \in \mathcal{H}_{-p'}^{\hat{\otimes} n}$ for any $p'(> p \geq p_0)$ such that the embedding operator $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is Hilbert-Schmidt. Thus we have

$$e_\mu(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^\mu(z), \theta^{\otimes n}\rangle \quad \text{for } \theta \in \mathcal{U}_0,\ z \in \mathcal{N}'_{\mathbb{C}}\ . \tag{10}$$

We will also use the notation

$$P_n^\mu(\varphi^{(n)})(z) := \left\langle P_n^\mu(z), \varphi^{(n)}\right\rangle, \qquad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \quad n \in I\!N.$$

Thus for any measure satisfying Assumption 1 we have defined the $I\!P^\mu$-system

$$I\!P^\mu = \left\{\left\langle P_n^\mu(\cdot), \varphi^{(n)}\right\rangle \;\middle|\; \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n},\ n \in I\!N\right\}.$$

Let us collect some properties of the polynomials $P_n^\mu(z)$.

PROPOSITION 11 *For $x, y \in \mathcal{N}'$, $n \in I\!N$ the following holds*

(P1)

$$P_n^\mu(x) = \sum_{k=0}^{n} \binom{n}{k} x^{\otimes k} \hat{\otimes} P_{n-k}^\mu(0), \tag{11}$$

(P2)

$$x^{\otimes n} = \sum_{k=0}^{n} \binom{n}{k} P_k^{\mu}(x) \hat{\otimes} \mathrm{M}_{n-k}^{\mu} \tag{12}$$

(P3)

$$\begin{aligned} P_n^{\mu}(x+y) &= \sum_{k+l+m=n} \frac{n!}{k!\, l!\, m!} P_k^{\mu}(x) \hat{\otimes} P_l^{\mu}(y) \hat{\otimes} \mathrm{M}_m^{\mu} \\ &= \sum_{k=0}^{n} \binom{n}{k} P_k^{\mu}(x) \hat{\otimes} y^{\otimes(n-k)} \end{aligned} \tag{13}$$

(P4) Further we observe

$$\mathbb{E}_{\mu}(\langle P_m^{\mu}(\cdot), \varphi^{(m)} \rangle) = 0 \qquad \textit{for } m \neq 0\ , \varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}\ . \tag{14}$$

(P5) For all $p > p_0$ such that the embedding $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_0}$ is Hilbert–Schmidt and for all $\varepsilon > 0$ small enough $\left(\varepsilon \leq \frac{2^{-q_0}}{e \| i_{p,p_0} \|_{HS}} \right)$ there exists a constant $C_{p,\varepsilon} > 0$ with

$$|P_n^{\mu}(z)|_{-p} \leq C_{p,\varepsilon}\, n!\, \varepsilon^{-n}\, e^{\varepsilon |z|_{-p}}, \quad z \in \mathcal{H}_{-p,\mathbb{C}} \tag{15}$$

PROOF. We restrict ourselves to a sketch of proof, details can be found in [ADKS94].
(P1) This formula can be obtained simply by substituting

$$\frac{1}{l_{\mu}(\theta)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_n^{\mu}(0), \theta^{\otimes n} \right\rangle, \quad \theta \in \mathcal{N}_{\mathbb{C}}, |\theta|_q < \delta \tag{16}$$

and

$$e^{\langle x,\theta \rangle} = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle x^{\otimes n}, \theta^{\otimes n} \right\rangle, \quad \theta \in \mathcal{N}_{\mathbb{C}}, x \in \mathcal{N}'$$

in the equality $e_{\mu}(\theta; x) = e^{\langle x,\theta\rangle} l_{\mu}^{-1}(\theta)$. A comparison with (10) proves (P1). The proof of (P2) is completely analogous to the proof of (P1).
(P3) We start from the following obvious equation of the generating functions

$$e_{\mu}(\theta; x+y) = e_{\mu}(\theta; x)\, e_{\mu}(\theta; y)\, l_{\mu}(\theta)$$

This implies

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\mu}(x+y), \theta^{\otimes n} \rangle = \sum_{k,l,m=0}^{\infty} \frac{1}{k!\, l!\, m!} \langle P_k(x) \hat{\otimes} P_l(y) \hat{\otimes} \mathrm{M}_m,\ \theta^{\otimes(k+l+m)} \rangle$$

from this (P3) follows immediately.
(P4) To see this we use, $\theta \in \mathcal{N}_{\mathbb{C}}$,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_{\mu}(\langle P_m^{\mu}(\cdot), \theta^{\otimes n} \rangle) = \mathbb{E}_{\mu}(e_{\mu}(\theta; x)) = \frac{\mathbb{E}_{\mu}(e^{\langle x,\theta \rangle})}{l_{\mu}(\theta)} = 1\ .$$

Then a comparison of coefficients and the polarization identity gives the above result.

(P5) We can use

$$|P_n^\mu(z)|_{-p'} \le n!\left(\sup_{|\theta|_p=\rho}\frac{1}{l_\mu(\theta)}\right)e^{\rho|z|_{-p}}\left(\frac{e}{\rho}\,\|i_{p',p}\|_{HS}\right)^n \ , \qquad z\in\mathcal{H}_{-p,\mathbb{C}} \tag{17}$$

$p>p_0, p', \rho$ defined above. (17) is a simple consequence of the kernel theorem by (9). In particular we have

$$|P_n^\mu(0)|_{-p} \le n!\left(\sup_{|\theta|_{p_0}=\rho}\frac{1}{l_\mu(\theta)}\right)\left(\frac{e}{\rho}\,\|i_{p,p_0}\|_{HS}\right)^n$$

If $p>p_0$ such that $\|i_{p,p_0}\|_{HS}$ is finite. For $0<\varepsilon\le 2^{-q_0}/e\,\|i_{p,p_0}\|_{HS}$ we can fix $\rho=\varepsilon\, e\,\|i_{p,p_0}\|_{HS}\le 2^{-q_0}$. With

$$C_{p,\varepsilon} := \sup_{|\theta|_{p_0}=\rho}\frac{1}{l_\mu(\theta)}$$

we have

$$|P_n^\mu(0)|_{-p}\le C_{p,\varepsilon}\, n!\,\varepsilon^{-n}.$$

Using (11) the following estimates hold

$$\begin{aligned}
|P_n^\mu(z)|_{-p} &\le \sum_{k=0}^{n}\binom{n}{k}|P_k^\mu(0)|_{-p}\,|z|_{-p}^{n-k}\ , \qquad z\in\mathcal{H}_{-p,\mathbb{C}}\\
&\le C_{p,\varepsilon}\sum_{k=0}^{n}\tbinom{n}{k}k!\,\varepsilon^{-k}\,|z|_{-p}^{n-k}\\
&= C_{p,\varepsilon}\,n!\,\varepsilon^{-n}\sum_{k=0}^{n}\tfrac{1}{(n-k)!}(\varepsilon\,|z|_{-p})^{n-k}\\
&\le C_{p,\varepsilon}\,n!\,\varepsilon^{-n}\,e^{\varepsilon|z|_{-p}}\ .
\end{aligned}$$

This completes the proof. □

NOTE. The formulae (11) and (16) can also be used as an alternative definition of the polynomials $P_n^\mu(x)$.

EXAMPLE 3 Let us compare to the case of Gaussian Analysis. Here one has

$$l_{\gamma_{\mathcal{H}}}(\theta)=e^{\frac{1}{2}\langle\theta,\theta\rangle}\ , \qquad \theta\in\mathcal{N}_{\mathbb{C}}$$

Then it follows

$$\mathrm{M}_{2n}^\mu=(-1)^n P_{2n}^\mu(0)=\frac{(2n)!}{n!\,2^n}\mathrm{Tr}^{\hat{\otimes}n}\ , \qquad n\in\mathbb{N}$$

and $\mathrm{M}_n^\mu=P_n^\mu(0)=0$ if n is odd. Here $\mathrm{Tr}\in\mathcal{N}'^{\otimes 2}$ denotes the trace kernel defined by

$$\langle\mathrm{Tr},\eta\otimes\xi\rangle=(\eta,\xi)\ , \qquad \eta,\xi\in\mathcal{N} \tag{18}$$

A simple comparison shows that

$$P_n^\mu(x) =: x^{\otimes n} :$$

and

$$e_\mu(\theta; x) =: e^{\langle x,\theta\rangle} :$$

where the r.h.s. denotes usual Wick ordering see e.g., [BeKo88, HKPS93]. This procedure is uniquely defined by

$$\langle : x^{\otimes n} :, \xi^{\otimes n}\rangle = 2^{-\frac{n}{2}}|\xi|^n \, H_n\left(\tfrac{1}{\sqrt{2}|\xi|}\langle x, \xi\rangle\right) \ , \qquad \xi \in \mathcal{N}$$

where H_n denotes the Hermite polynomial of order n (see e.g., [HKPS93] for the normalization we use).

Now we are ready to give the announced description of $\mathcal{P}(\mathcal{N}')$.

LEMMA 12 *For any $\varphi \in \mathcal{P}(\mathcal{N}')$ there exists a unique representation*

$$\varphi(x) = \sum_{n=0}^{N} \left\langle P_n^\mu(x), \varphi^{(n)}\right\rangle \ , \qquad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \tag{19}$$

and vice versa, any functional of the form (19) is a smooth polynomial.

PROOF. The representations from Definition 10 and equation (19) can be transformed into one another using (11) and (12). □

4.2 The dual Appell system and the representation theorem for $\mathcal{P}'_\mu(\mathcal{N}')$

To give an internal description of the type (19) for $\mathcal{P}'_\mu(\mathcal{N}')$ we have to construct an appropriate system of generalized functions, the $\mathbb{Q}^\mu$-system. The construction we propose here is different from that of [ADKS94] where smoothness of the logarithmic derivative of μ was demanded and used for the construction of the $\mathbb{Q}^\mu$-system. To avoid this additional assumption (which excludes e.g., Poisson measures) we propose to construct the $\mathbb{Q}^\mu$-system using differential operators.

For $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$ define a differential operator of order n with constant coefficients $\varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}$, $m \in \mathbb{N}$

$$D(\Phi^{(n)})\langle x^{\otimes m}, \varphi^{(m)}\rangle = \begin{cases} \dfrac{m!}{(m-n)!}\langle x^{\otimes(m-n)}\hat{\otimes}\Phi^{(n)}, \varphi^{(m)}\rangle & \text{for } m \geq n \\ 0 & \text{for } m < n \end{cases}$$

and extend by linearity from the monomials to $\mathcal{P}(\mathcal{N}')$.

LEMMA 13 $D(\Phi^{(n)})$*is a continuous linear operator from $\mathcal{P}(\mathcal{N}')$ to $\mathcal{P}(\mathcal{N}')$.*

REMARK. For $\Phi^{(1)} \in \mathcal{N}'$ we have the usual Gâteaux derivative as e.g., in white noise analysis [HKPS93]

$$D(\Phi^{(1)})\varphi = D_{\Phi^{(1)}}\varphi := \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\cdot + t\Phi^{(1)})|_{t=0}$$

for $\varphi \in \mathcal{P}(\mathcal{N})$ and we have $D(\left(\Phi^{(1)}\right)^{\otimes n}) = (D_{\Phi^{(1)}})^n$ thus $D(\left(\Phi^{(1)}\right)^{\otimes n})$ is in fact a differential operator of order n.

PROOF. By definition $\mathcal{P}(\mathcal{N}')$ is isomorphic to the topological direct sum of tensor powers $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$

$$\mathcal{P}(\mathcal{N}') \simeq \bigoplus_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \ .$$

Via this isomorphism $D(\Phi^{(n)})$ transforms each component $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}$, $m \geq n$ by

$$\varphi^{(m)} \mapsto \frac{n!}{(m-n)!} (\Phi^{(n)}, \ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes} n}}$$

where the contraction $(\Phi^{(n)}, \ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes} n}} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}(m-n)}$ is defined by

$$\langle x^{\otimes(m-n)}, \ (\Phi^{(n)}, \ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes} n}} \rangle := \langle x^{\otimes(m-n)} \hat{\otimes} \Phi^{(n)}, \varphi^{(m)} \rangle \tag{20}$$

for all $x \in \mathcal{N}'$. It is easy to verify that

$$|(\Phi^{(n)}, \ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes} n}}|_q \leq |\Phi^{(n)}|_{-q} |\varphi^{(m)}|_q \ , \qquad q \in I\!N$$

which guarantees that $(\Phi^{(n)}, \ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes} n}} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}(m-n)}$ and shows at the same time that $D(\Phi^{(n)})$ is continuous on each component. This is sufficient to ensure the stated continuity of $D(\Phi^{(n)})$ on $\mathcal{P}(\mathcal{N}')$. □

LEMMA 14 *For* $\Phi^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\hat{\otimes} n}$, $\varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}$ *we have*

$$\text{(P6)} \qquad D(\Phi^{(n)}) \langle P_m^{\mu}(x), \varphi^{(m)} \rangle = \begin{cases} \dfrac{m!}{(m-n)!} \left\langle P_{m-n}^{\mu}(x) \hat{\otimes} \Phi^{(n)}, \ \varphi^{(m)} \right\rangle & \text{for } m \geq n \\ 0 & \text{for } m < n \end{cases} \tag{21}$$

PROOF. This follows from the general property of Appell polynomials which behave like ordinary powers under differentiation. More precisely, by using

$$\langle P_m^{\mu}, \theta^{\otimes m} \rangle = \left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^m e_{\mu}(t\theta; \cdot) \Big|_{t=0} \ , \qquad \theta \in \mathcal{N}_{\mathbb{C}}$$

we have

$$\begin{aligned} D(\Phi^{(1)}) \langle P_m(x), \theta^{\otimes m} \rangle &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \langle P_m^{\mu}(x + \lambda \Phi^{(1)}), \theta^{\otimes m} \rangle \Big|_{\lambda=0} \\ &= \left(\frac{\partial}{\partial t} \right)^m \frac{\partial}{\partial \lambda} e_{\mu}(t\theta; x + \lambda \Phi^{(1)}) \Big|_{\substack{t=0 \\ \lambda=0}} \\ &= \langle \Phi^{(1)}, \theta \rangle \left(\tfrac{\partial}{\partial t} \right)^m t \ e_{\mu}(t\theta; x) \Big|_{t=0} \\ &= \langle \Phi^{(1)}, \theta \rangle \sum_{k=0}^{m} \tbinom{m}{k} \left(\left(\tfrac{\mathrm{d}}{\mathrm{d}t} \right)^k t \right) \left(\tfrac{\mathrm{d}}{\mathrm{d}t} \right)^{m-k} e_{\mu}(t\theta; x) \Big|_{t=0} \\ &= m \, \langle \Phi^{(1)}, \theta \rangle \left(\tfrac{\mathrm{d}}{\mathrm{d}t} \right)^{m-1} e_{\mu}(t\theta; x) \Big|_{t=0} \\ &= m \, \langle \Phi^{(1)}, \theta \rangle \left\langle P_{m-1}(x), \theta^{\otimes(m-1)} \right\rangle \ . \end{aligned}$$

This proves

$$D(\Phi^{(1)})\langle P^{\mu}_{m}, \varphi^{(m)}\rangle = m\left\langle P^{\mu}_{m-1}\hat{\otimes}\Phi^{(1)},\ \varphi^{(m)}\right\rangle .$$

The property (21), then follows by induction. □

In view of Lemma 16 it is possible to define the adjoint operator $D(\Phi^{(n)})^* : \mathcal{P}'_{\mu}(\mathcal{N}') \to \mathcal{P}'_{\mu}(\mathcal{N}')$ for $\Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes}n}_{\mathbb{C}}$. Further we can introduce the constant function $1\!\!1 \in \mathcal{P}'_{\mu}(\mathcal{N}')$ such that $1\!\!1(x) \equiv 1$ for all $x \in \mathcal{N}'$, so

$$\langle\!\langle 1\!\!1,\ \varphi\rangle\!\rangle_{\mu} = \int_{\mathcal{N}'} \varphi(x)\,\mathrm{d}\mu(x) = \mathbb{E}_{\mu}(\varphi).$$

Now we are ready to define our $\mathbb{Q}$-system.

DEFINITION 15 *For any* $\Phi^{(n)} \in \left(\mathcal{N}^{\hat{\otimes}n}_{\mathbb{C}}\right)'$ *we define* $Q^{\mu}_{n}(\Phi^{(n)}) \in \mathcal{P}'_{\mu}(\mathcal{N}')$ *by*

$$Q^{\mu}_{n}(\Phi^{(n)}) = D(\Phi^{(n)})^* 1\!\!1 \ .$$

We want to introduce an additional formal notation $Q^{\mu}_{n}(x)$ which stresses the linearity of $\Phi^{(n)} \mapsto Q^{\mu}_{n}(\Phi^{(n)}) \in P'_{\mu}(\mathcal{N}')$:

$$\langle Q^{\mu}_{n}, \Phi^{(n)}\rangle := Q^{\mu}_{n}(\Phi^{(n)}) \ .$$

EXAMPLE 4 It is possible to put further assumptions on the measure μ to ensure that the expression is more than formal. Let us assume a smooth measure (i.e., the logarithmic derivative of μ is infinitely differentiable, see [ADKS94] for details) with the property

$$\exists q \in \mathbb{N}\ ,\ \exists\{C_n \geq 0,\ n \in \mathbb{N}\} : \forall \xi \in \mathcal{N}$$

$$\left|\int D^{n}_{\xi}\varphi \,\mathrm{d}\mu(x)\right| \leq C_n \,\|\varphi\|_{L^2(\mu)}\,|\xi|^{n}_{q}$$

where φ is any finitely based bounded $\mathcal{C}^{\infty}$-function on $\mathcal{N}'$. This obviously establishes a bound of the type

$$\|Q^{\mu}_{n}(\xi_1 \otimes \cdots \otimes \xi_n)\|_{L^2(\mu)} \leq C'_n \prod_{j=1}^{n} |\xi_j|_q\ , \qquad \xi_1, \ldots, \xi_n \in \mathcal{N}\ ,\ n \in \mathbb{N}$$

which is sufficient to show (by means of kernel theorem) that there exists $Q^{\mu}_{n}(x) \in \left(\mathcal{N}^{\hat{\otimes}n}_{\mathbb{C}}\right)'$ for almost all $x \in \mathcal{N}'$ such that we have the representation

$$Q^{\mu}_{n}(\varphi^{(n)})(x) = \langle Q^{\mu}_{n}(x), \varphi^{(n)}\rangle\ , \qquad \varphi^{(n)} \in \mathcal{N}^{\hat{\otimes}n}_{\mathbb{C}}$$

for almost all $x \in \mathcal{N}'$. For any smooth kernel $\varphi^{(n)} \in \mathcal{N}^{\hat{\otimes}n}_{\mathbb{C}}$ we have then that the function

$$x \mapsto \langle Q^{\mu}_{n}(x), \varphi^{(n)}\rangle \ = Q^{\mu}_{n}\left(\varphi^{(n)}\right)(x)$$

belongs to $L^2(\mu)$.

EXAMPLE 5 The simplest non trivial case can be studied using finite dimensional real analysis. We consider $I\!R$ as our basic Hilbert space and as our nuclear space $\mathcal{N}$. Thus the nuclear "triple" is simply

$$I\!R \subseteq I\!R \subseteq I\!R$$

and the dual pairing between a "test function" and a "distribution" degenerates to multiplication. On $I\!R$ we consider a measure $\mathrm{d}\mu(x) = \rho(x)\,\mathrm{d}x$ where ρ is a positive $\mathcal{C}^\infty$–function on $I\!R$ such that Assumptions 1 and 2 are fulfilled. In this setting the adjoint of the differentiation operator is given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^* f(x) = -\left(\tfrac{\mathrm{d}}{\mathrm{d}x} + \beta(x)\right) f(x)\ , \qquad f \in \mathcal{C}^1(I\!R)$$

where the logarithmic derivative β of the measure μ is given by

$$\beta = \frac{\rho'}{\rho}$$

This enables us to calculate the $\mathbb{Q}^\mu$-system. One has

$$\begin{aligned} Q_n^\mu(x) &= \left(\left(\tfrac{\mathrm{d}}{\mathrm{d}x}\right)^*\right)^n 1\!\!1 = (-1)^n \left(\tfrac{\mathrm{d}}{\mathrm{d}x} + \beta(x)\right)^n 1\!\!1 \\ &= (-1)^n \frac{\rho^{(n)}(x)}{\rho(x)}\ . \end{aligned}$$

The last equality can be seen by simple induction.
If $\rho = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is the Gaussian density Q_n^μ is related to the nth Hermite polynomial:

$$Q_n^\mu(x) = 2^{-n/2} H_n\left(\tfrac{x}{\sqrt{2}}\right)\ .$$

DEFINITION 16 *We define the $\mathbb{Q}^\mu$-system in $\mathcal{P}'_\mu(\mathcal{N}')$ by*

$$\mathbb{Q}^\mu = \left\{ Q_n^\mu(\Phi^{(n)}) \;\middle|\; \Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)',\ n \in I\!N_0 \right\}\ ,$$

and the pair $(I\!P^\mu, \mathbb{Q}^\mu)$ will be called the Appell system $I\!A^\mu$ generated by the measure μ.

Now we are going to discuss the central property of the Appell system $I\!A^\mu$.

THEOREM 17 (Biorthogonality w.r.t. μ)

$$\left\langle\!\!\left\langle \langle Q_n^\mu(\Phi^{(n)}),\ \langle P_m^\mu, \varphi^{(m)}\rangle \right\rangle\!\!\right\rangle_\mu = \delta_{m,n}\ n!\ \langle \Phi^{(n)}, \varphi^{(n)}\rangle \tag{22}$$

for $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$ and $\varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}$.

PROOF. It follows from (14) and (21) that

$$\begin{aligned}\left\langle\!\!\left\langle Q_n^\mu(\Phi^{(n)}),\ \langle P_m^\mu, \varphi^{(m)}\rangle \right\rangle\!\!\right\rangle_\mu &= \left\langle\!\!\left\langle 1\!\!1, D(\Phi^{(n)})\langle P_m^\mu, \varphi^{(m)}\rangle \right\rangle\!\!\right\rangle_\mu \\ &= \frac{m!}{(m-n)!} I\!\!E_\mu \left(\langle P^\mu_{(m-n)} \hat{\otimes} \Phi^{(n)},\ \varphi^{(m)} \rangle \right) \\ &= m!\, \delta_{m,n}\, \langle \Phi^{(m)}, \varphi^{(m)} \rangle \ .\end{aligned}$$

□

Now we are going to characterize the space $\mathcal{P}'_\mu(\mathcal{N}')$

THEOREM 18 *For all* $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ *there exists a unique sequence* $\{\Phi^{(n)} |\ n \in I\!\!N_0\}$, $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$ *such that*

$$\Phi = \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)}) \equiv \sum_{n=0}^{\infty} \langle Q_n^\mu, \Phi^{(n)} \rangle \tag{23}$$

and vice versa, every series of the form (23) generates a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$.

PROOF. For $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ we can uniquely define $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$ by

$$\langle \Phi^{(n)}, \varphi^{(n)} \rangle = \frac{1}{n!} \langle\!\langle \Phi,\ \langle P_n^\mu, \varphi^{(n)} \rangle \rangle\!\rangle_\mu \ , \qquad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$$

This definition is possible because $\langle P_n^\mu, \varphi^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$. The continuity of $\varphi^{(n)} \mapsto \langle \Phi^{(n)}, \varphi^{(n)} \rangle$ follows from the continuity of $\varphi \mapsto \langle\!\langle \Phi, \varphi \rangle\!\rangle$, $\varphi \in \mathcal{P}(\mathcal{N}')$. This implies that $\varphi \mapsto \sum_{n=0}^\infty n!\, \langle \Phi^{(n)}, \varphi^{(n)} \rangle$ is continuous on $\mathcal{P}(\mathcal{N}')$. This defines a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$, which we denote by $\sum_{n=0}^\infty Q_n^\mu(\Phi^{(n)})$. In view of Theorem 17 it is obvious that

$$\Phi = \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)}) \ .$$

To see the converse consider a series of the form (23) and $\varphi \in \mathcal{P}(\mathcal{N}')$. Then there exist $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in I\!\!N$ and $N \in I\!\!N$ such that we have the representation

$$\varphi = \sum_{n=0}^{N} P_n^\mu(\varphi^{(n)}) \ .$$

So we have

$$\left\langle\!\!\left\langle \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)}), \varphi \right\rangle\!\!\right\rangle_\mu := \sum_{n=0}^{N} n!\, \langle \Phi^{(n)}, \varphi^{(n)} \rangle$$

because of Theorem 17. The continuity of $\varphi \mapsto \langle\!\langle \sum_{n=0}^\infty Q_n^\mu(\Phi^{(n)}), \varphi \rangle\!\rangle_\mu$ follows because $\varphi^{(n)} \mapsto \langle \Phi^{(n)}, \varphi^{(n)} \rangle$ is continuous for all $n \in I\!\!N$. □

5 Test functions on a linear space with measure

In this section we will construct the test function space $(\mathcal{N})^1$ and study its properties. On the space $\mathcal{P}(\mathcal{N}')$ we can define a system of norms using the representation from Lemma 12. Let

$$\varphi = \sum_{n=0}^{N} \langle P_n^\mu, \varphi^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$$

be given, then $\varphi^{(n)} \in \mathcal{H}_{\mathbb{C},p}^{\hat{\otimes} n}$ for each $p \geq 0$ $(n \in \mathbb{N})$. Thus we may define for any $p, q \in \mathbb{N}$ a Hilbertian norm on $\mathcal{P}(\mathcal{N}')$ by

$$\|\varphi\|_{p,q,\mu}^2 = \sum_{n=0}^{N} (n!)^2 \, 2^{nq} \, |\varphi^{(n)}|_p^2$$

The completion of $\mathcal{P}(\mathcal{N}')$ w.r.t. $\|\cdot\|_{p,q,\mu}$ is called $(\mathcal{H}_p)_{q,\mu}^1$.

Definition 19 *We define*

$$(\mathcal{N})_\mu^1 := \operatorname{pr\,lim}_{p,q \in \mathbb{N}} (\mathcal{H}_p)_{q,\mu}^1 \ .$$

This space has the following properties

Theorem 20 $(\mathcal{N})_\mu^1$ *is a nuclear space. The topology* $(\mathcal{N})_\mu^1$ *is uniquely defined by the topology on* $\mathcal{N}$*: It does not depend on the choice of the family of norms* $\{|\cdot|_p\}$.

Proof. Nuclearity of $(\mathcal{N})_\mu^1$ follows essentially from that of $\mathcal{N}$. For fixed p, q consider the embedding

$$I_{p',q',p,q} : (\mathcal{H}_{p'})_{q',\mu}^1 \to (\mathcal{H}_p)_{q,\mu}^1$$

where p' is chosen such that the embedding

$$i_{p',p} : \mathcal{H}_{p'} \to \mathcal{H}_p$$

is Hilbert–Schmidt. Then $I_{p',q',p,q}$ is induced by

$$I_{p',q',p,q}\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, i_{p',p}^{\otimes n} \varphi^{(n)} \rangle \quad \text{for} \quad \varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)} \rangle \in (\mathcal{H}_{p'})_{q',\mu}^1 \, .$$

Its Hilbert–Schmidt norm is easily estimated by using an orthonormal basis of $(\mathcal{H}_{p'})_{q',\mu}^1$. The result is the bound

$$\|I_{p',q',p,q}\|_{HS}^2 \leq \sum_{n=0}^{\infty} 2^{n(q-q')} \, \|i_{p',p}\|_{HS}^{2n}$$

which is finite for suitably chosen q'.

Let us assume that we are given two different systems of Hilbertian norms $|\cdot|_p$ and $|\cdot|_k'$, such that they induce the same topology on $\mathcal{N}$. For fixed k and l we have to estimate $\|\cdot\|_{k,l,\mu}'$ by $\|\cdot\|_{p,q,\mu}$ for some p, q (and vice versa which is completely analogous). Since $|\cdot|_k'$ has to be continuous with respect to the projective limit topology on $\mathcal{N}$, there exists p and a constant C such that $|f|_k' \leq C\,|f|_p$, for all $f \in \mathcal{N}$, i.e., the injection i from $\mathcal{H}_p$ into the completion $\mathcal{K}_k$ of $\mathcal{N}$ with respect to $|\cdot|'_k$ is a mapping bounded by C. We denote by i also its linear extension from $\mathcal{H}_{\mathbb{C},p}$ into $\mathcal{K}_{\mathbb{C},k}$. It follows that $\imath^{\otimes n}$ is bounded by C^n from $\mathcal{H}_{\mathbb{C},p}^{\otimes n}$ into $\mathcal{K}_{\mathbb{C},k}^{\otimes n}$. Now we choose q such that $2^{\frac{q-l}{2}} \geq C$. Then

$$\begin{aligned}
\|\cdot\|'^2_{k,l,\mu} &= \sum_{n=0}^{\infty} (n!)^2\, 2^{nl}\, |\cdot|'^2_k \\
&\leq \sum_{n=0}^{\infty} (n!)^2\, 2^{nl} C^{2n}\, |\cdot|^2_p \\
&\leq \|\cdot\|^2_{p,q,\mu} \ ,
\end{aligned}$$

which had to be proved. □

Lemma 21 *There exist $p, C, K > 0$ such that for all n*

$$\int |P_n^\mu(x)|^2_{-p}\, \mathrm{d}\mu(x) \leq (n!)^2\, C^n\, K \tag{24}$$

Proof. The estimate (17) may be used for $\rho \leq 2^{-q_0}$ and $\rho \leq 2\varepsilon_\mu$ (ε_μ from Lemma 9). This gives

$$\int |P_n^\mu(x)|^2_{-p}\, \mathrm{d}\mu(x) \leq (n!)^2 \left(\frac{e}{\rho}\, \|i_{p,p_0}\|_{HS}\right)^{2n} \int e^{2\rho|x|_{-p_0}} \mathrm{d}\mu(x)$$

which is finite because of Lemma 9. □

Theorem 22 *There exist $p', q' < 0$ such that for all $p \geq p'$, $q \geq q'$ the topological embedding $(\mathcal{H}_p)^1_q \subset L^2(\mu)$ holds.*

Proof. Elements of the space $(\mathcal{N})^1_\mu$ are defined as series convergent in the given topology. Now we need to study the convergence of these series in $L^2(\mu)$. Choose q' such that $C > 2^{q'}$ (C from estimate (24)). Let us take an arbitrary

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$$

For $p > p'$ (p' as in Lemma 21) and $q > q'$ the following estimates hold

$$\begin{aligned}
\|\varphi\|_{L^2(\mu)} &\leq \sum_{n=0}^{\infty} \left\| \langle P_n^\mu, \varphi^{(n)} \rangle \right\|_{L^2(\mu)} \\
&\leq \sum_{n=0}^{\infty} |\varphi^{(n)}|_{-p}\, \|\, |P_n^\mu|_{-p} \|_{L^2(\mu)} \\
&\leq K \sum_{n=0}^{\infty} n!\, 2^{nq/2} \left|\varphi^{(n)}\right|_{-p} (C 2^{-q})^{n/2} \\
&\leq K \left(\sum_{n=0}^{\infty} (C\, 2^{-q})^n \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} (n!)^2\, 2^{qn} \left|\varphi^{(n)}\right|^2_{-p} \right)^{\frac{1}{2}} \\
&= K \left(1 - C\, 2^{-q}\right)^{-1/2} \|\varphi\|_{p,q,\mu} \, .
\end{aligned}$$

Taking the closure the inequality extends to the whole space $(\mathcal{H}_p)^1_q$. □

Corollary 23 *$(\mathcal{N})^1_\mu$ is continuously and densely embedded in $L^2(\mu)$.*

Example 6 *(μ-exponentials as test functions)*
The μ-exponential given in (10) has the following norm

$$||e_\mu(\theta;\cdot)||^2_{p,q,\mu} = \sum_{n=0}^\infty 2^{nq}\,|\theta|_p^{2n}\;, \qquad \theta\in\mathcal{N}_{\mathbb{C}}$$

This expression is finite if and only if $2^q|\theta|_p^2 < 1$. Thus we have $e_\mu(\theta;\cdot)\notin(\mathcal{N})^1_\mu$ if $\theta\neq 0$. But we have that $e_\mu(\theta;\cdot)$ is a test function of finite order i.e., $e_\mu(\theta;\cdot)\in(\mathcal{H}_p)^1_q$ if $2^q|\theta|_p^2<1$. This is in contrast to some useful spaces of test functions in Gaussian Analysis, see e.g., [BeKo88, HKPS93].

The set of all μ–exponentials $\{e_\mu(\theta;\cdot)\mid 2^q|\theta|_p^2<1,\ \theta\in\mathcal{N}_{\mathbb{C}}\}$ is a total set in $(\mathcal{H}_p)^1_q$. This can been shown using the relation $\mathrm{d}^n e_\mu(0;\cdot)(\theta_1,...,\theta_n)=\langle P_n^\mu,\theta_1\hat{\otimes}\cdots\hat{\otimes}\theta_n\rangle$.

Proposition 24 *Any test function φ in $(\mathcal{N})^1_\mu$ has a uniquely defined extension to $\mathcal{N}'_{\mathbb{C}}$ an element of $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$*

Proof. Any element φ in $(\mathcal{N})^1_\mu$ is defined as a series of the following type

$$\varphi=\sum_{n=0}^\infty\langle P_n^\mu,\varphi^{(n)}\rangle\;,\qquad \varphi^{(n)}\in\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}$$

such that

$$||\varphi||^2_{p,q,\mu}=\sum_{n=0}^\infty(n!)^2\,2^{nq}\,|\varphi^{(n)}|_p^2$$

is finite for each $p,q\in I\!N$. In this proof we will show the convergence of the series

$$\sum_{n=0}^\infty\langle P_n^\mu(z),\varphi^{(n)}\rangle,\quad z\in\mathcal{H}_{-p,\mathbb{C}}$$

to an entire function in z.

Let $p>p_0$ such that the embedding $i_{p,p_0}:\mathcal{H}_p\hookrightarrow\mathcal{H}_{p_0}$ is Hilbert-Schmidt. Then for all $0<\varepsilon\le 2^{-q_0}/e\,\|i_{p,p_0}\|_{HS}$ we can use (15) and estimate as follows

$$\begin{aligned}\sum_{n=0}^\infty|\langle P_n^\mu(z),\varphi^{(n)}\rangle| &\le \sum_{n=0}^\infty|P_n^\mu(z)|_{-p}|\varphi^{(n)}|_p\\ &\le C_{p,\varepsilon}\,e^{\varepsilon|z|_{-p}}\sum_{n=0}^\infty n!\,|\varphi^{(n)}|_p\,\varepsilon^{-n}\\ &\le C_{p,\varepsilon}\,e^{\varepsilon|z|_{-p}}\left(\sum_{n=0}^\infty(n!)^2 2^{nq}|\varphi^{(n)}|_p^2\right)^{1/2}\left(\sum_{n=0}^\infty 2^{-nq}\varepsilon^{-2n}\right)^{1/2}\\ &= C_{p,\varepsilon}\,\left(1-2^{-q}\varepsilon^{-2}\right)^{-1/2}\,||\varphi||_{p,q,\mu}\;\;e^{\rho|z|_{-p}}\end{aligned}$$

if $2^q>\varepsilon^{-2}$. That means the series $\sum_{n=0}^\infty\langle P_n^\mu(z),\varphi^{(n)}\rangle$ converges uniformly and absolutely in any neighborhood of zero of any space $\mathcal{H}_{-p,\mathbb{C}}$. Since each term $\langle P_n^\mu(z),\varphi^{(n)}\rangle$ is entire in z the uniform convergence implies that $z \mapsto \sum_{n=0}^{\infty} \langle P_n^{\mu}(z), \varphi^{(n)} \rangle$ is entire on each $\mathcal{H}_{-p,\mathbb{C}}$ and hence on $\mathcal{N}'_{\mathbb{C}}$. This completes the proof. □

The following corollary is an immediate consequence of the above proof and gives an explicit estimate on the growth of the test functions.

COROLLARY 25 *For all $p > p_0$ such that the norm $\|i_{p,p_0}\|_{HS}$ of the embedding is finite and for all $0 < \varepsilon \leq 2^{-q_0}/e \, \|i_{p,p_0}\|_{HS}$ we can choose $q \in I\!N$ such that $2^q > \varepsilon^{-2}$ to obtain the following bound.*

$$|\varphi(z)| \leq C \; \|\varphi\|_{p,q,\mu} \; e^{\varepsilon |z|_{-p}} \;, \qquad \varphi \in (\mathcal{N})^1, \; z \in \mathcal{H}_{-p,\mathbb{C}} \;,$$

where

$$C = C_{p,\varepsilon} \left(1 - 2^{-q} \varepsilon^{-2} \right)^{-1/2} .$$

Let us look at Proposition 24 again. On one hand any function $\varphi \in (\mathcal{N})^1_{\mu}$ can be written in the form

$$\varphi(z) = \sum_{n=0}^{\infty} \langle P_n^{\mu}(x), \varphi^{(n)} \rangle \;, \qquad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \;, \tag{25}$$

on the other hand it is entire, i.e., it has the representation

$$\varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \tilde{\varphi}^{(n)} \rangle \;, \qquad \tilde{\varphi}^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \;, \tag{26}$$

To proceed further we need the explicit correspondence $\left\{ \varphi^{(n)}, n \in I\!N \right\} \longleftrightarrow \left\{ \tilde{\varphi}^{(n)}, n \in I\!N \right\}$ which is given in the next lemma.

LEMMA 26 **(Reordering)**
Equations (25) and (26) hold iff

$$\tilde{\varphi}^{(k)} = \sum_{n=0}^{\infty} \binom{n+k}{k} \left(P_n^{\mu}(0), \varphi^{(n+k)} \right)_{\mathcal{H}^{\hat{\otimes} n}}$$

or equivalently

$$\varphi^{(k)} = \sum_{n=0}^{\infty} \binom{n+k}{k} \left(\mathrm{M}_n^{\mu}, \tilde{\varphi}^{(n+k)} \right)_{\mathcal{H}^{\hat{\otimes} n}}$$

where $\left(P_n^{\mu}(0), \varphi^{(n+k)} \right)_{\mathcal{H}^{\hat{\otimes} n}}$ and $\left(\mathrm{M}_n^{\mu}, \tilde{\varphi}^{(n+k)} \right)_{\mathcal{H}^{\hat{\otimes} n}}$ denote contractions defined by (20).

This is a consequence of (11) and (12). We omit the simple proof.

Proposition 24 states

$$(\mathcal{N})^1_{\mu} \subseteq \mathcal{E}^1_{\min}(\mathcal{N}')$$

as sets, where

$$\mathcal{E}^1_{\min}(\mathcal{N}') = \left\{ \varphi|_{\mathcal{N}'} \;\middle|\; \varphi \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}}) \right\} .$$

Corollary 25 then implies that the embedding is also continuous. Now we are going to show that the converse also holds.

THEOREM 27 *For all measures* $\mu \in \mathcal{M}_a(\mathcal{N}')$ *we have the topological identity*

$$(\mathcal{N})^1_\mu = \mathcal{E}^1_{\min}(\mathcal{N}') \; .$$

To prove the missing topological inclusion it is convenient to use the nuclear topology on $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ (given by the norms $|\!|\!|\cdot|\!|\!|_{p,q,1}$) introduced in section 2. Theorem 5 ensures that this topology is equivalent to the projective topology induced by the norms $\mathrm{n}_{p,l,k}$. Then the above theorem is an immediate consequence of the following norm estimate.

PROPOSITION 28 *Let* $p > p_\mu$ *(*p_μ *as in Lemma 9) such that* $\|i_{p,p_\mu}\|_{HS}$ *is finite and* $q \in I\!N$ *such that* $2^{q/2} > K_p$ *(*$K_p := eC\,\|i_{p,p_\mu}\|_{HS}$ *as in (7)). For any* $\varphi \in \mathrm{E}^1_{p,q}$ *the restriction* $\varphi|_{\mathcal{N}'}$ *is a function from* $(\mathcal{H}_p)^1_{q',\mu}$, $q' < q$*. Moreover the following estimate holds*

$$||\varphi||_{p,q',\mu} \le |\!|\!|\varphi|\!|\!|_{p,q,1}\,(1-2^{q/2}K_p)^{-1}(1-2^{q'-q})^{-1/2} \; .$$

PROOF. Let $p, q \in I\!N$, K_p be defined as above. A function $\varphi \in \mathrm{E}^1_{p,q}$ has the representation (25). Using the Reordering lemma combined with (7) and

$$\left|\tilde{\varphi}^{(n)}\right|_p \le \frac{1}{n!}\,2^{-nq/2}\,|\!|\!|\varphi|\!|\!|_{p,q,1}$$

we obtain

$$\begin{aligned}
\left|\varphi^{(n)}\right|_p &\le \sum_{k=0}^{\infty}\binom{n+k}{k}\,|\mathrm{M}^\mu_k|_{-p}\left|\tilde{\varphi}^{(n+k)}\right|_p \\
&\le |\!|\!|\varphi|\!|\!|_{p,q,1}\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{k!}{(n+k)!}K_p^k\,2^{-(n+k)q/2} \\
&\le |\!|\!|\varphi|\!|\!|_{p,q,1}\,\frac{1}{n!}2^{-nq/2}\sum_{k=0}^{\infty}(2^{-q/2}K_p)^k \\
&\le |\!|\!|\varphi|\!|\!|_{p,q,1}\,\frac{1}{n!}2^{-nq/2}(1-2^{-q/2}K_p)^{-1}
\end{aligned}$$

For $q' < q$ this allows the following estimate

$$\begin{aligned}
||\varphi||^2_{p,q',\mu} &= \sum_{n=0}^{\infty}(n!)^2\,2^{q'n}\,|\varphi^{(n)}|_p^2 \\
&\le |\!|\!|\varphi|\!|\!|^2_{p,q,1}\,(1-2^{-q/2}K_p)^{-2}\sum_{k=0}^{\infty}2^{n(q'-q)} < \infty
\end{aligned}$$

This completes the proof. □

Since we now have proved that the space of test functions $(\mathcal{N})^1_\mu$ is isomorphic to $\mathcal{E}^1_{\min}(\mathcal{N}')$ for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$, we will now drop the subscript μ. The test function space $(\mathcal{N})^1$ is the same for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$.

COROLLARY 29 $(\mathcal{N})^1$ *is an algebra under pointwise multiplication.*

COROLLARY 30 $(\mathcal{N})^1$ *admits 'scaling' i.e., for* $\lambda \in \mathbb{C}$ *the scaling operator* $\sigma_\lambda : (\mathcal{N})^1 \to (\mathcal{N})^1$ *defined by* $\sigma_\lambda\varphi(x) := \varphi(\lambda x)$, $\varphi \in (\mathcal{N})^1$, $x \in \mathcal{N}'$ *is well-defined.*

COROLLARY 31 *For all* $z \in \mathcal{N}'_{\mathbb{C}}$ *the space* $(\mathcal{N})^1$ *is invariant under the shift operator* τ_z : $\varphi \mapsto \varphi(\cdot + z)$.

6 Distributions

In this section we will introduce and study the space $(\mathcal{N})^{-1}_{\mu}$ of distributions corresponding to the space of test functions $(\mathcal{N})^{1}$. Since $\mathcal{P}(\mathcal{N}') \subset (\mathcal{N})^{1}$ the space $(\mathcal{N})^{-1}_{\mu}$ can be viewed as a subspace of $\mathcal{P}'_{\mu}(\mathcal{N}')$

$$(\mathcal{N})^{-1}_{\mu} \subset \mathcal{P}'_{\mu}(\mathcal{N}')$$

Let us now introduce the Hilbertian subspace $(\mathcal{H}_{-p})^{-1}_{-q,\mu}$ of $\mathcal{P}'_{\mu}(\mathcal{N}')$ for which the norm

$$\|\Phi\|^{2}_{-p,-q,\mu} := \sum_{n=0}^{\infty} 2^{-qn} \left|\Phi^{(n)}\right|^{2}_{-p}$$

is finite. Here we used the canonical representation

$$\Phi = \sum_{n=0}^{\infty} Q^{\mu}_{n}(\Phi^{(n)}) \in \mathcal{P}'_{\mu}(\mathcal{N}')$$

from Theorem 18. The space $(\mathcal{H}_{-p})^{-1}_{-q,\mu}$ is the dual space of $(\mathcal{H}_{p})^{1}_{q}$ with respect to $L^{2}(\mu)$ (because of the biorthogonality of $\mathbb{P}-$and $\mathbb{Q}-$systems). By general duality theory

$$(\mathcal{N})^{-1}_{\mu} := \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q,\mu}$$

is the dual space of $(\mathcal{N})^{1}$ with respect to $L^{2}(\mu)$. As we noted in section 2 there exists a natural topology on co-nuclear spaces (which coincides with the inductive limit topology). We will consider $(\mathcal{N})^{-1}_{\mu}$ as a topological vector space with this topology. So we have the nuclear triple

$$(\mathcal{N})^{1} \subset L^{2}(\mu) \subset (\mathcal{N})^{-1}_{\mu} .$$

The action of $\Phi = \sum_{n=0}^{\infty} Q^{\mu}_{n}(\Phi^{(n)}) \in (\mathcal{N})^{-1}_{\mu}$ on a test function $\varphi = \sum_{n=0}^{\infty}\langle P^{\mu}_{n}, \varphi^{(n)}\rangle \in (\mathcal{N})^{1}$ is given by

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_{\mu} = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle .$$

For a more detailed characterization of the singularity of distributions in $(\mathcal{N})^{-1}_{\mu}$ we will introduce some subspaces in this distribution space. For $\beta \in [0,1]$ we define

$$(\mathcal{H}_{-p})^{-\beta}_{-q,\mu} = \left\{ \Phi \in \mathcal{P}'_{\mu}(\mathcal{N}') \;\middle|\; \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} \left|\Phi^{(n)}\right|^{2}_{-p} < \infty \text{ for } \Phi = \sum_{n=0}^{\infty} Q^{\mu}_{n}(\Phi^{(n)}) \right\}$$

and

$$(\mathcal{N})^{-\beta}_{\mu} = \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-\beta}_{-q,\mu} ,$$

It is clear that the singularity increases with increasing β:

$$(\mathcal{N})^{-0} \subset (\mathcal{N})^{-\beta_1} \subset (\mathcal{N})^{-\beta_2} \subset (\mathcal{N})^{-1}$$

if $\beta_1 \leq \beta_2$.We will also consider $(\mathcal{N})^{\beta}_{\mu}$ as equipped with the natural topology.

EXAMPLE 7 *(Generalized Radon–Nikodym derivative)*
We want to define a generalized function $\rho_\mu(z,\cdot) \in (\mathcal{N})^{-1}_\mu$, $z \in \mathcal{N}'_{\mathbb{C}}$ with the following property

$$\langle\!\langle \rho_\mu(z,\cdot), \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x-z)\,\mathrm{d}\mu(x) \ , \qquad \varphi \in (\mathcal{N})^1 \ .$$

That means we have to establish the continuity of $\rho_\mu(z,\cdot)$. Let $z \in \mathcal{H}_{-p,\mathbb{C}}$. If $p' \geq p$ is sufficiently large and $\varepsilon > 0$ small enough, Corollary 25 applies i.e., $\exists q \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned} \left| \int_{\mathcal{N}'} \varphi(x-z)\mathrm{d}\mu(x) \right| &\leq C\,\|\varphi\|_{p',q,\mu} \int_{\mathcal{N}'} e^{\varepsilon|x-z|_{-p'}}\mathrm{d}\mu(x) \\ &\leq C\,\|\varphi\|_{p',q,\mu}\, e^{\varepsilon|z|_{-p'}} \int_{\mathcal{N}'} e^{\varepsilon|x|_{-p'}}\mathrm{d}\mu(x) \end{aligned}$$

If ε is chosen sufficiently small the last integral exists. Thus we have in fact $\rho(z,\cdot) \in (\mathcal{N})^{-1}_\mu$. It is clear that whenever the Radon–Nikodym derivative $\frac{\mathrm{d}\mu(x+\xi)}{\mathrm{d}\mu(x)}$ exists (e.g., $\xi \in \mathcal{N}$ in case μ is $\mathcal{N}$-quasi-invariant) it coincides with $\rho_\mu(\xi,\cdot)$ defined above. We will now show that in $(\mathcal{N})^{-1}_\mu$ we have the canonical expansion

$$\rho_\mu(z,\cdot) = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n Q^\mu_n(z^{\otimes n}).$$

It is easy to see that the r.h.s. defines an element in $(\mathcal{N})^{-1}_\mu$. Since both sides are in $(\mathcal{N})^{-1}_\mu$ it is sufficient to compare their action on a total set from $(\mathcal{N})^1$. For $\varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}}$ we have

$$\begin{aligned} \left\langle\!\!\left\langle \rho_\mu(z,\cdot), \langle P^\mu_n, \varphi^{(n)}\rangle \right\rangle\!\!\right\rangle_\mu &= \int_{\mathcal{N}'} \langle P^\mu_n(x-z), \varphi^{(n)}\rangle\,\mathrm{d}\mu(x) \\ &= \sum_{n=0}^{\infty} \binom{n}{k} (-1)^{n-k} \int_{\mathcal{N}'} \langle P^\mu_k(x) \hat{\otimes} z^{\otimes n-k}, \varphi^{(n)}\rangle\,\mathrm{d}\mu(x) \\ &= (-1)^n \langle z^{\otimes n}, \varphi^{(n)}\rangle \\ &= \left\langle\!\!\left\langle \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k Q^\mu_k(z^{\otimes k}), \langle P^\mu_n, \varphi^{(n)}\rangle \right\rangle\!\!\right\rangle_\mu \ , \end{aligned}$$

where we have used (13), (14) and the biorthogonality of $\mathbb{P}$- and $\mathbb{Q}$-systems. This had to be shown. In other words, we have proven that $\rho_\mu(-z,\cdot)$ is the generating function of the $\mathbb{Q}$-functions

$$\rho_\mu(-z,\cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} Q^\mu_n(z^{\otimes n}) \ . \tag{27}$$

Let use finally remark that the above expansion allows for more detailed estimates. It is easy to see that $\rho_\mu \in (\mathcal{N})^{-0}_\mu$.

EXAMPLE 8 *(Delta distribution)*
For $z \in \mathcal{N}'_{\mathbb{C}}$ we define a distribution by the following $\mathbb{Q}$-decomposition:

$$\delta_z = \sum_{n=0}^{\infty} \frac{1}{n!} Q^\mu_n(P^\mu_n(z))$$

If $p \in \mathbb{N}$ is large enough and $\varepsilon > 0$ sufficiently small there exists $C_{p,\varepsilon} > 0$ according to (15) such that

$$\begin{aligned} \|\delta_z\|^2_{-p,-q,\mu} &= \sum_{n=0}^{\infty} (n!)^{-2} 2^{-nq} \, |P_n^\mu(z)|^2_{-p} \\ &\leq C^2_{p,\varepsilon} \, e^{2\varepsilon|z|_{-p}} \sum_{n=0}^{\infty} 2^{-nq} \varepsilon^{-2n} \ , \qquad z \in \mathcal{H}_{-p,\mathbb{C}} \ , \end{aligned}$$

which is finite for sufficiently large $q \in \mathbb{N}$. Thus $\delta_z \in (\mathcal{N})^{-1}_\mu$.

For $\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)} \rangle \in (\mathcal{N})^1$ the action of δ_z is given by

$$\langle\!\langle \delta_z, \varphi \rangle\!\rangle_\mu = \sum_{n=0}^{\infty} \langle P_n^\mu(z), \varphi^{(n)} \rangle = \varphi(z)$$

because of (22). This means that δ_z (in particular for z real) plays the role of a "δ-function" (evaluation map) in the calculus we discuss.

7 Integral transformations

We will first introduce the Laplace transform of a function $\varphi \in L^2(\mu)$. The global assumption $\mu \in \mathcal{M}_a(\mathcal{N}')$ guarantees the existence of $p'_\mu \in \mathbb{N}$, $\varepsilon_\mu > 0$ such that $\int_{\mathcal{N}'} \exp(\varepsilon_\mu |x|_{-p'_\mu}) \, \mathrm{d}\mu(x) < \infty$ by Lemma 9. Thus $\exp(\langle x, \theta \rangle) \in L^2(\mu)$ if $2|\theta|_{p'_\mu} \leq \varepsilon_\mu$, $\theta \in \mathcal{H}_{p'_\mu,\mathbb{C}}$. Then by Cauchy–Schwarz inequality the Laplace transform defined by

$$L_\mu \varphi(\theta) := \int_{\mathcal{N}'} \varphi(x) \exp\langle x, \theta \rangle \, \mathrm{d}\mu(x)$$

is well defined for $\varphi \in L^2(\mu)$, $\theta \in \mathcal{H}_{p'_\mu,\mathbb{C}}$ with $2|\theta|_{p'_\mu} \leq \varepsilon_\mu$. Now we are interested to extend this integral transform from $L^2(\mu)$ to the space of distributions $(\mathcal{N})^{-1}_\mu$.

Since our construction of test function and distribution spaces is closely related to $\mathbb{P}$- and $\mathbb{Q}$-systems it is useful to introduce the so called S_μ-transform

$$S_\mu \varphi(\theta) := \frac{L_\mu \varphi(\theta)}{l_\mu(\theta)} \ .$$

Since $e_\mu(\theta; x) = e^{\langle x, \theta \rangle} / l_\mu(\theta)$ we may also write

$$S_\mu \varphi(\theta) = \int_{\mathcal{N}'} \varphi(x) \, e_\mu(\theta; x) \, \mathrm{d}\mu(x) \ .$$

The μ-exponential $e_\mu(\theta, \cdot)$ is not a test function in $(\mathcal{N})^1$, see Example 6 . So the definition of the S_μ-transform of a distribution $\Phi \in (\mathcal{N})^{-1}_\mu$ must be more careful. Every such Φ is of finite order i.e., $\exists p, q \in \mathbb{N}$ such that $\Phi \in (\mathcal{H}_{-p})^{-1}_{-q,\mu}$. As shown in Example 6 $e_\mu(\theta, \cdot)$ is in the corresponding dual space $(\mathcal{H}_p)^1_{q,\mu}$ if $\theta \in \mathcal{H}_{p,\mathbb{C}}$ is such that $2^q |\theta|_p^2 < 1$. Then we can define a consistent extension of S_μ-transform.

$$S_\mu \Phi(\theta) := \langle\!\langle \Phi, \ e_\mu(\theta, \cdot) \rangle\!\rangle_\mu$$

if θ is chosen in the above way. The biorthogonality of $I\!\!P$- and $Q\!\!\!\!Q$-system implies

$$S_\mu\Phi(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n}\rangle \ .$$

It is easy to see that the series converges uniformly and absolutely on any closed ball $\left\{\theta \in \mathcal{H}_{p,\mathbb{C}} \mid \ |\theta|_p^2 \le r,\ r < 2^{-q}\right\}$, see the proof of Theorem 34. Thus $S_\mu\Phi$ is holomorphic a neighborhood of zero, i.e., $S_\mu\Phi \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$. In the next section we will discuss this relation to the theory of holomorphic functions in more detail.

The third integral transform we are going to introduce is more appropriate for the test function space $(\mathcal{N})^1$. We introduce the convolution of a function $\varphi \in (\mathcal{N})^1$ with the measure μ by

$$C_\mu\varphi(y) := \int_{\mathcal{N}'} \varphi(x+y)\,\mathrm{d}\mu(x), \quad y \in \mathcal{N}'.$$

From Example 7 the existence of a generalized Radon–Nikodym derivative $\rho_\mu(z,\cdot)$, $z \in \mathcal{N}'_\mathbb{C}$ in $(\mathcal{N})^{-1}_\mu$ is guaranteed. So for any $\varphi \in (\mathcal{N})^1$, $z \in \mathcal{N}'_\mathbb{C}$ the convolution has the representation

$$C_\mu\varphi(z) = \langle\!\langle \rho_\mu(-z,\cdot),\ \varphi\rangle\!\rangle_\mu \ .$$

If $\varphi \in (\mathcal{N})^1$ has the canonical representation

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu,\ \varphi^{(n)}\rangle$$

we have by equation (27)

$$C_\mu\varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)}\rangle \ .$$

In Gaussian Analysis C_μ- and S_μ-transform coincide. It is a typical non-Gaussian effect that these two transformations differ from each other.

8 Characterization theorems

Gaussian Analysis has shown that for applications it is very useful to characterize test and distribution spaces by the integral transforms introduced in the previous section. In the non-Gaussian setting first results in this direction have been obtained by [AKS93, ADKS94].

We will start to characterize the space $(\mathcal{N})^1$ in terms of the convolution C_μ.

THEOREM 32 *The convolution* C_μ *is a topological isomorphism from* $(\mathcal{N})^1$ *on* $\mathcal{E}^1_{\mathrm{min}}(\mathcal{N}'_\mathbb{C})$.

REMARK. Since we have identified $(\mathcal{N})^1$ and $\mathcal{E}^1_{\mathrm{min}}(\mathcal{N}')$ by Theorem 27 the above assertion can be restated as follows. We have

$$C_\mu : \mathcal{E}^1_{\mathrm{min}}(\mathcal{N}') \to \mathcal{E}^1_{\mathrm{min}}(\mathcal{N}'_\mathbb{C})$$

as a topological isomorphism.

Proof. The proof has been well prepared by Theorem 5, because the nuclear topology on $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ is the most natural one from the point of view of the above theorem. Let $\varphi \in (\mathcal{N})^1$ with the representation

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu}, \varphi^{(n)} \rangle \ .$$

From the previous section it follows

$$C_\mu \varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)} \rangle$$

It is obvious from (2) that

$$|||C_\mu \varphi|||_{p,q,1} = \|\varphi\|_{p,q,\mu}$$

for all $p, q \in I\!N_0$, which proves the continuity of

$$C_\mu : (\mathcal{N})^1 \to \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}}) \ .$$

Conversely let $F \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$. Then Theorem 5 ensures the existence of a sequence of generalized kernels $\left\{ \varphi^{(n)} \in \mathcal{N}'_{\mathbb{C}} \mid n \in I\!N_0 \right\}$ such that

$$F(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)} \rangle \ .$$

Moreover for all $p, q \in I\!N_0$

$$|||F|||^2_{p,q,1} = \sum_{n=0}^{\infty} (n!)^2 \, 2^{nq} \left| \varphi^{(n)} \right|^2_p$$

is finite. Choosing

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu}, \varphi^{(n)} \rangle$$

we have $\|\varphi\|_{p,q,\mu} = |||F|||_{p,q,1}$. Thus $\varphi \in (\mathcal{N})^1$. Since $C_\mu \varphi = F$ we have shown the existence and continuity of the inverse of C_μ. □

To illustrate the above theorem in terms of the natural topology on $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ we will reformulate the above theorem and add some useful estimates which relate growth in $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ to norms on $(\mathcal{N})^1$.

Corollary 33

1) Let $\varphi \in (\mathcal{N})^1$ then for all $p, l \in I\!N_0$ and $z \in \mathcal{H}_{-p,\mathbb{C}}$ the following estimate holds

$$|C_\mu \varphi(z)| \leq \|\varphi\|_{p,2l,\mu} \exp(2^{-l} |z|_{-p})$$

i.e., $C_\mu \varphi \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$.

2) Let $F \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$. Then there exists $\varphi \in (\mathcal{N})^1$ with $C_\mu \varphi = F$. The estimate

$$|F(z)| \leq C \exp(2^{-l} |z|_{-p})$$

for $C > 0,\ p, q \in \mathbb{N}_0$ *implies*

$$\|\varphi\|_{p',q,\mu} \le C\left(1 - 2^{q-2l}e^2 \left\|i_{p',p}\right\|_{HS}^2\right)^{-1/2}$$

if the embedding $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ *is Hilbert-Schmidt and* $2^{l-q/2} > e\left\|i_{p',p}\right\|_{HS}$.

Proof. The first statement follows from

$$|C_\mu\varphi(z)| \le \mathrm{n}_{p,l,1}(C_\mu\varphi)\cdot \exp(2^{-l}|z|_{-p})$$

which follows from the definition of $\mathrm{n}_{p,l,1}$ and estimate (3). The second statement is an immediate consequence of Lemma 7. □

The next theorem characterizes distributions from $(\mathcal{N})_\mu^{-1}$ in terms of S_μ-transform.

Theorem 34 *The* S_μ*-transform is a topological isomorphism from* $(\mathcal{N})_\mu^{-1}$ *on* $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$.

Remark. The above theorem is closely related to the second part of Theorem 8. Since we left the proof open we will give a detailed proof here.

Proof. Let $\Phi \in (\mathcal{N})_\mu^{-1}$. Then there exists $p, q \in \mathbb{N}$ such that

$$\|\Phi\|_{-p,-q,\mu}^2 = \sum_{n=0}^{\infty} 2^{-nq}|\Phi^{(n)}|_{-p}^2$$

is finite. From the previous section we have

$$S_\mu\Phi(\theta) = \sum_{n=0}^{\infty}\langle\Phi^{(n)}, \theta^{\otimes n}\rangle\ . \tag{28}$$

For $\theta \in \mathcal{N}_{\mathbb{C}}$ such that $2^q|\theta|_p^2 < 1$ we have by definition (Formula (2))

$$|\!|\!|S_\mu\Phi|\!|\!|_{-p,-q,-1} = \|\Phi\|_{-p,-q,\mu}\ \ .$$

By Cauchy–Schwarz inequality

$$\begin{aligned}
|S_\mu\Phi(\theta)| &\le \sum_{n=0}^{\infty}|\Phi^{(n)}|_{-p}|\theta|_p^n \\
&\le \left(\sum_{n=0}^{\infty}2^{-nq}|\Phi^{(n)}|_{-p}^2\right)^{1/2}\left(\sum_{n=0}^{\infty}2^{nq}|\theta|_p^{2n}\right)^{1/2} \\
&= \|\Phi\|_{-p,-q,\mu}\left(1 - 2^q|\theta|_p^2\right)^{-1/2}\ .
\end{aligned}$$

Thus the series (28) converges uniformly on any closed ball $\left\{\theta \in \mathcal{H}_{p,\mathbb{C}} \mid\ |\theta|_p^2 \le r,\ r < 2^{-q}\right\}$. Hence $S_\mu\Phi \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ and

$$\mathrm{n}_{p,l,\infty}(S_\mu\Phi) \le \|\Phi\|_{-p,-q,\mu}\,(1 - 2^{q-2l})^{-1/2}$$

if $2l > q$. This proves that S_μ is a continuous mapping from $(\mathcal{N})_\mu^{-1}$ to $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. In the language of section 2.2 this reads

$$\operatorname*{ind\,lim}_{p,q\in \mathbb{N}} \mathrm{E}_{-p,-q}^{-1} \subset \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$$

topologically.

Conversely, let $F \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ be given, i.e., there exist $p, l \in \mathbb{N}$ such that $\mathrm{n}_{p,l,\infty}(F) < \infty$. The first step is to show that there exists $p', q \in \mathbb{N}$ such that

$$|||F|||_{-p',-q,-1} < \mathrm{n}_{p,l,\infty}(F) \cdot C \ ,$$

for sufficiently large $C > 0$. This implies immediately

$$\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}) \subset \operatorname*{ind\,lim}_{p,q\in \mathbb{N}} \mathrm{E}_{-p,-q}^{-1}$$

topologically, which is the missing part in the proof of the second statement in Theorem 8.

By assumption the Taylor expansion

$$F(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{d}^n \widehat{F(0)}(\theta)$$

converges uniformly on any closed ball $\left\{ \theta \in \mathcal{H}_{p,\mathbb{C}} \middle| \ |\theta|_p^2 \leq r, \ r < 2^{-l} \right\}$ and

$$|F(\theta)| \leq \mathrm{n}_{p,l,\infty}(F) \ .$$

Proceeding analogously to Lemma 6, an application of Cauchy's inequality gives

$$\begin{aligned} \frac{1}{n!} \mathrm{d}^n \widehat{F(0)}(\theta) &\leq 2^l |\theta|_p^n \sup_{|\theta|_p \leq 2^{-l}} |F(\theta)| \\ &\leq \mathrm{n}_{p,l,\infty}(F) \ \cdot 2^{nl} \cdot |\theta|_p^n \end{aligned}$$

The polarization identity gives

$$\left| \frac{1}{n!} \mathrm{d}^n F(0)(\theta_1, \ldots, \theta_n) \right| \leq \mathrm{n}_{p,l,\infty}(F) \ \cdot e^n \cdot 2^{nl} \prod_{j=1}^{n} |\theta_j|_p$$

Then by kernel theorem (Theorem 2) there exist kernels $\Phi^{(n)} \in \mathcal{H}_{-p',\mathbb{C}}^{\hat{\otimes} n}$ for $p' > p$ with $\|i_{p',p}\|_{HS} < \infty$ such that

$$F(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n} \rangle \ .$$

Moreover we have the following norm estimate

$$\left| \Phi^{(n)} \right|_{-p'} \leq \mathrm{n}_{p,l,\infty}(F) \ \left(2^l e \, \|i_{p',p}\|_{HS} \right)^n$$

Thus

$$\begin{aligned}\|\!|F|\!\|^2_{-p',-q,-1} &= \sum_{n=0}^{\infty} 2^{-nq} \left|\Phi^{(n)}\right|^2_{-p'} \\ &\leq \mathrm{n}^2_{p,l,\infty}(F) \sum_{n=0}^{\infty} \left(2^{2l-q} e^2 \left\|i_{p',p}\right\|^2_{HS}\right)^n \\ &= \mathrm{n}^2_{p,l,\infty}(F) \left(1 - 2^{2l-q} e^2 \left\|i_{p',p}\right\|^2_{HS}\right)^{-1}\end{aligned}$$

if $q \in I\!N$ is such that $\rho := 2^{2l-q} e^2 \left\|i_{p',p}\right\|^2_{HS} < 1$. So we have in fact

$$\|\!|F|\!\|_{-p',-q,-1} \leq \mathrm{n}_{p,l,\infty}(F)(1-\rho)^{-1/2}.$$

Now the rest is simple. Define $\Phi \in (\mathcal{N})^{-1}_{\mu}$ by

$$\Phi = \sum_{n=0}^{\infty} Q^{\mu}_{n}(\Phi^{(n)})$$

then $S_\mu \Phi = F$ and

$$\|\Phi\|_{-p',-q,\mu} = \|\!|F|\!\|_{-p',-q,-1}$$

This proves the existence of a continuous inverse of the S_μ–transform. Uniqueness of Φ follows from the fact that μ-exponentials are total in any $(\mathcal{H}_p)^1_q$. □

We can extract some useful estimates from the above proof which describe the degree of singularity of a distribution.

COROLLARY 35 *Let* $F \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ *be holomorphic for all* $\theta \in \mathcal{N}_{\mathbb{C}}$ *with* $|\theta|_p \leq 2^{-l}$. *If* $p' > p$ *with* $\left\|i_{p',p}\right\|_{HS} < \infty$ *and* $q \in I\!N$ *is such that* $\rho := 2^{2l-q} e^2 \left\|i_{p',p}\right\|^2_{HS} < 1$. *Then* $\Phi \in (\mathcal{H}_{-p'})^{-1}_{-q}$ *and*

$$\|\Phi\|_{-p',q,\mu} \leq \mathrm{n}_{p,l,\infty}(F) \cdot (1-\rho)^{-1/2}.$$

For a more detailed discussion of the degree of singularity the spaces $(\mathcal{N})^{-\beta}$, $\beta \in [0,1)$ are useful. In the following theorem we will characterize these spaces by means of S_μ-transform.

THEOREM 36 *The* S_μ*-transform is a topological isomorphism from* $(\mathcal{N})^{-\beta}_{\mu}$, $\beta \in [0,1)$ *on* $\mathcal{E}^{2/(1-\beta)}_{\max}(\mathcal{N}_{\mathbb{C}})$.

REMARK. The proof will also complete the proof of Theorem 8.

PROOF. Let $\Phi \in (\mathcal{H}_{-p})^{-\beta}_{-q,\mu}$ with the canonical representation $\Phi = \sum_{n=0}^{\infty} Q^{\mu}_{n}(\Phi^{(n)})$ be given. The S_μ-transform of Φ is given by

$$S_\mu \Phi(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n} \rangle.$$

Hence

$$\|\!|S_\mu \Phi|\!\|^2_{-p,-q,-\beta} = \sum_{n=0}^{\infty} (n!)^{1-\beta}\, 2^{-nq} |\Phi^{(n)}|^2_{-p}$$

is finite. We will show that there exist $l \in I\!N$ and $C < 0$ such that

$$\mathrm{n}_{-p,-l,2/(1-\beta)}(S_\mu\Phi) \leq C \; \| S_\mu\Phi \|_{-p,-q,-\beta} \; .$$

We can estimate as follows

$$\begin{aligned}
S_\mu\Phi(\theta) &\leq \sum_{n=0}^{\infty} \left|\Phi^{(n)}\right|_{-p} |\theta|_p^n \\
&\leq \left(\sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-nq} |\Phi^{(n)}|_{-p}^2\right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{1}{(n!)^{1-\beta}} 2^{nq} \, |\theta|_p^{2n}\right)^{1/2} \\
&= \| S_\mu\Phi \|_{-p,-q,-\beta} \left(\sum_{n=0}^{\infty} \rho^{n\beta} \cdot \frac{1}{(n!)^{1-\beta}} 2^{nq} \rho^{-n\beta} \, |\theta|_p^{2n} \cdot\right)^{1/2} ,
\end{aligned}$$

where we have introduced a parameter $\rho \in (0,1)$. An application of Hölder's inequality for the conjugate indices $\frac{1}{\beta}$ and $\frac{1}{1-\beta}$ gives

$$\begin{aligned}
|S_\mu\Phi(\theta)| &\leq \| S_\mu\Phi \|_{-p,-q,-\beta} \left(\sum_{n=0}^{\infty} \rho^n\right)^{\beta/2} \cdot \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(2^q \rho^{-\beta} |\theta|_p^2\right)^{\frac{n}{1-\beta}}\right)^{\frac{1-\beta}{2}} \\
&= \| S_\mu\Phi \|_{-p,-q,-\beta} \, (1-\rho)^{-\beta/2} \exp\left(\tfrac{1-\beta}{2} \, 2^{\frac{q}{1-\beta}} \, \rho^{-\frac{\beta}{1-\beta}} \, |\theta|_p^{\frac{2}{1-\beta}}\right)
\end{aligned}$$

If $l \in I\!N$ is such that

$$2^{l-\frac{q}{1-\beta}} > \tfrac{1-\beta}{2} \rho^{-\frac{\beta}{1-\beta}}$$

we have

$$\begin{aligned}
\mathrm{n}_{-p,-l,2/(1-\beta)}(S_\mu\Phi) &= \sup_{\theta \in \mathcal{H}_{p,\mathbb{C}}} |S_\mu\Phi(\theta)| \exp\left(-2^l |\theta|_p^{2/(1-\beta)}\right) \\
&\leq (1-\rho)^{-\beta/2} \, \| S_\mu\Phi \|_{-p,-q,-\beta}
\end{aligned}$$

This shows that S_μ is continuous from $(\mathcal{N})_\mu^{-\beta}$ to $\mathcal{E}_{\max}^{2/(1-\beta)}(\mathcal{N}_{\mathbb{C}})$. Or in the language of Theorem 8

$$\operatorname*{ind\,lim}_{p,q \in I\!N} \mathrm{E}_{-p,-q}^{-\beta} \subset \mathcal{E}_{\max}^{2/(1-\beta)}(\mathcal{N}_{\mathbb{C}})$$

topologically.

The proof of the inverse direction is closely related to the proof of Lemma 7. So we will be more sketchy in the following.
Let $F \in \mathcal{E}_{\max}^k(\mathcal{N}_{\mathbb{C}})$, $k = \frac{2}{1-\beta}$. Hence there exist $p, l \in I\!N_0$ such that

$$|F(\theta)| \leq \mathrm{n}_{-p,-l,k}(F) \exp(2^l |\theta|_p^k) \; , \qquad \theta \in \mathcal{N}_{\mathbb{C}}$$

From this we have completely analogous to the proof of Lemma 7 by Cauchy inequality and kernel theorem the representation

$$F(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n} \rangle$$

and the bound

$$\left|\Phi^{(n)}\right|_{-p'} \leq \mathrm{n}_{-p,-l,k}(F)\ (n!)^{-1/k}\left\{(k2^{l})^{1/k} e \left\|i_{p',p}\right\|_{HS}\right\}^{n} ,$$

where $p' > p$ is such that $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is Hilbert–Schmidt. Using this we have

$$\begin{aligned}
\|\!|F|\!\|^2_{-p',-q,-\beta} &= \sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-qn}\left|\Phi^{(n)}\right|^2_{-p'} \\
&\leq \mathrm{n}^2_{-p,-l,k}(F)\sum_{n=0}^{\infty}(n!)^{1-\beta-2/k}2^{-qn}\left\{(k2^{l})^{1/k} e \left\|i_{p',p}\right\|_{HS}\right\}^{2n} \\
&\leq \mathrm{n}^2_{-p,-l,k}(F)\sum_{n=0}^{\infty}\rho^n
\end{aligned}$$

where we have set $\rho := 2^{-q+2l/k}k^{2/k}e^2\left\|i_{p',p}\right\|^2_{HS}$. If $q \in \mathbb{N}$ is chosen large enough such that $\rho < 1$ the sum on the right hand side is convergent and we have

$$\|\!|F|\!\|_{-p',-q,-\beta} \leq \mathrm{n}_{-p,-l,2/(1-\beta)}(F)\cdot(1-\rho)^{-1/2} . \tag{29}$$

That means

$$\mathcal{E}^{2/(1-\beta)}_{\max}(\mathcal{N}_{\mathbb{C}}) \subset \operatorname*{ind\,lim}_{p,q\in\mathbb{N}_0}\ \mathrm{E}^{-\beta}_{-p,-q}$$

topologically.

If we set

$$\Phi := \sum_{n=0}^{\infty} Q^{\mu}_{n}(\Phi^{(n)})$$

then $S_\mu\Phi = F$ and $\Phi \in (\mathcal{H}_{-p'})^{-\beta}_{-q}$ since

$$\sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-qn}|\Phi^{(n)}|^2_{-p'}$$

is finite. Hence

$$S_\mu : (\mathcal{N})^{-\beta}_{\mu} \to \mathcal{E}^{2/(1-\beta)}_{\max}(\mathcal{N}_{\mathbb{C}})$$

is one to one. The continuity of the inverse mapping follows from the norm estimate (29). □

9 The Wick product

In Gaussian Analysis it has been shown that $(\mathcal{N})^{-1}_{\gamma_\mathcal{H}}$ (and other distribution spaces) is closed under so called Wick multiplication (see [KLS94] and [BeS95, Øk95, Va95] for applications). This concept has a natural generalization to the present setting.

Definition 37 *Let $\Phi, \Psi \in (\mathcal{N})^{-1}_{\mu}$. Then we define the Wick product $\Phi \diamond \Psi$ by*

$$S_\mu(\Phi\diamond\Psi) = S_\mu\Phi\cdot S_\mu\Psi \ .$$

This is well defined because $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ is an algebra and thus by the characterization Theorem 34 there exists an element $\Phi \diamond \Psi \in (\mathcal{N})_{\mu}^{-1}$ such that $S_{\mu}(\Phi \diamond \Psi) = S_{\mu}\Phi \cdot S_{\mu}\Psi$.

By definition we have

$$Q_n^{\mu}(\Phi^{(n)}) \diamond Q_m^{\mu}(\Psi^{(m)}) = Q_{n+m}^{\mu}(\Phi^{(n)} \hat{\otimes} \Psi^{(m)}) \ ,$$

$\Phi^{(n)} \in (\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n})'$ and $\Psi^{(m)} \in (\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n})'$. So in terms of $\mathbb{Q}$–decompositions $\Phi = \sum_{n=0}^{\infty} Q_n^{\mu}(\Phi^{(n)})$ and $\Psi = \sum_{n=0}^{\infty} Q_n^{\mu}(\Psi^{(n)})$ the Wick product is given by

$$\Phi \diamond \Psi = \sum_{n=0}^{\infty} Q_n(\Xi^{(n)})$$

where

$$\Xi^{(n)} = \sum_{k=0}^{n} \Phi^{(k)} \hat{\otimes} \Psi^{(n-k)}$$

This allows for concrete norm estimates.

Proposition 38 *The Wick product is continuous on* $(\mathcal{N})_{\mu}^{-1}$. *In particular the following estimate holds for* $\Phi \in (\mathcal{H}_{-p_1})_{-q_1,\mu}^{-1}$, $\Psi \in (\mathcal{H}_{-q_2})_{-q_2}^{-1}$ *and* $p = \max(p_1, p_2)$, $q = q_1 + q_2 + 1$

$$\|\Phi \diamond \Psi\|_{-p,-q,\mu} = \|\Phi\|_{-p_1,-q_1,\mu} \, \|\Psi\|_{-p_2,-q_2,\mu} \ .$$

Proof. We can estimate as follows

$$\begin{aligned}
\|\Phi \diamond \Psi\|_{-p,-q,\mu}^2 &= \sum_{n=0}^{\infty} 2^{-nq} \left|\Xi^{(n)}\right|_{-p}^2 \\
&= \sum_{n=0}^{\infty} 2^{-nq} \left(\sum_{k=0}^{n} \left|\Phi^{(k)}\right|_{-p} \left|\Psi^{(n-k)}\right|_{-p} \right)^2 \\
&\leq \sum_{n=0}^{\infty} 2^{-nq} \, (n+1) \sum_{k=0}^{n} \left|\Phi^{(k)}\right|_{-p}^2 \left|\Psi^{(n-k)}\right|_{-p}^2 \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} 2^{-nq_1} \left|\Phi^{(n)}\right|_{-p}^2 2^{-nq_2} \left|\Psi^{(n-k)}\right|_{-p}^2 \\
&\leq \left(\sum_{n=0}^{\infty} 2^{-nq_1} \left|\Phi^{(k)}\right|_{-p_1}^2 \right) \left(\sum_{n=0}^{\infty} 2^{-nq_2} \left|\Psi^{(n)}\right|_{-p_2}^2 \right) \\
&= \|\Phi\|_{-p_1,-q_1,\mu}^2 \, \|\Psi\|_{-p_2,-q_2,\mu}^2 \ .
\end{aligned}$$

□

Similar to the Gaussian case the special properties of the space $(\mathcal{N})_{\mu}^{-1}$ allow the definition of *Wick analytic functions* under very general assumptions. This has proven to be of some relevance to solve equations e.g., of the type $\Phi \diamond X = \Psi$ for $X \in (\mathcal{N})_{\mu}^{-1}$. See [KLS94] for the Gaussian case.

Theorem 39 *Let* $F : \mathbb{C} \to \mathbb{C}$ *be analytic in a neighborhood of the point* $z_0 = \mathbb{E}_{\mu}(\Phi)$, $\Phi \in (\mathcal{N})_{\mu}^{-1}$. *Then* $F^{\diamond}(\Phi)$ *defined by* $S_{\mu}(F^{\diamond}(\Phi)) = F(S_{\mu}\Phi)$ *exists in* $(\mathcal{N})^{-1}$.

PROOF. By the characterization Theorem 34 $S_\mu \Phi \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. Then $F(S_\mu \Phi) \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ since the composition of two analytic functions is also analytic. Again by characterization Theorem we find $F^\diamond(\Phi) \in (\mathcal{N})_\mu^{-1}$. □

REMARK. If $F(z) = \sum_{n=0}^\infty a_k (z - z_0)^n$ then the *Wick series* $\sum_{n=0}^\infty a_k (\Phi - z_0)^{\diamond n}$ (where $\Psi^{\diamond n} = \Psi \diamond \ldots \diamond \Psi$ n-times converges in $(\mathcal{N})^{-1}$ and $F^\diamond(\Phi) = \sum_{n=0}^\infty a_k (\Phi - z_0)^{\diamond n}$ holds.

EXAMPLE 9 The above mentioned equation $\Phi \diamond X = \Psi$ can be solved if $\mathbb{E}_\mu(\Phi) = S_\mu \Phi(0) \neq 0$. That implies $(S_\mu \Phi)^{-1} \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. Thus $\Phi^{\diamond(-1)} = S_\mu^{-1}\left((S_\mu \Phi)^{-1}\right) \in (\mathcal{N})_\mu^{-1}$. Then $X = \Phi^{\diamond(-1)} \diamond \Psi$ is the solution in $(\mathcal{N})_\mu^{-1}$. For more instructive examples we refer the reader to [KLS94].

10 Positive distributions

In this section we will characterize the positive distributions in $(\mathcal{N})_\mu^{-1}$. We will prove that the positive distributions can be represented by measures in $\mathcal{M}_a(\mathcal{N}')$. In the case of the Gaussian Hida distribution space $(\mathcal{S})'$ similar statements can be found in works of Kondratiev [Ko80a, b] and Yokoi [Yok90, Yok93], see also [Po87] and [Lee91]. In the Gaussian setting also the positive distributions in $(\mathcal{N})^{-1}$ have been discussed, see [KoSW95].

Since $(\mathcal{N})^1 = \mathcal{E}^1_{\min}(\mathcal{N}')$ we say that $\varphi \in (\mathcal{N})^1$ is positive ($\varphi \geq 0$) if and only if $\varphi(x) \geq 0$ for all $x \in \mathcal{N}'$.

DEFINITION 40 *An element* $\Phi \in (\mathcal{N})_\mu^{-1}$ *is positive if for any positive* $\varphi \in (\mathcal{N})^1$ *we have* $\langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu \geq 0$. *The cone of positive elements in* $(\mathcal{N})_\mu^{-1}$ *is denoted by* $(\mathcal{N})_{\mu,+}^{-1}$.

THEOREM 41 *Let* $\Phi \in (\mathcal{N})_{\mu,+}^{-1}$. *Then there exists a unique measure* $\nu \in \mathcal{M}_a(\mathcal{N}')$ *such that* $\forall \varphi \in (\mathcal{N})^1$

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x) \, \mathrm{d}\nu(x) \ . \tag{30}$$

Vice versa, any (positive) measure $\nu \in \mathcal{M}_a(\mathcal{N}')$ *defines a positive distribution* $\Phi \in (\mathcal{N})_{\mu,+}^{-1}$ *by (30).*

REMARKS.

1. For a given measure ν the distribution Φ may be viewed as the generalized Radon-Nikodym derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ of ν with respect to μ. In fact if ν is absolutely continuous with respect to μ then the usual Radon-Nikodym derivative coincides with Φ.

2. Note that the cone of positive distributions generates the same set of measures $\mathcal{M}_a(\mathcal{N}')$ for all initial measures $\mu \in \mathcal{M}_a(\mathcal{N}')$.

PROOF. To prove the first part we define moments of a distribution Φ and give bounds on their growth. Using this we construct a measure ν which is uniquely defined by given moments*. The next step is to show that any test functional $\varphi \in (\mathcal{N})^1$ is integrable with respect to ν.

---

*Since the algebra of exponential functions is not contained in $(\mathcal{N})_\mu^1$ we cannot use Minlos' theorem to construct the measure. This was the method used in Yokoi's work [Yok90].

Since $\mathcal{P}(\mathcal{N}') \subset (\mathcal{N})^1$ we may define moments of a positive distribution $\Phi \in (\mathcal{N})^{-1}_{\mu}$ by

$$\mathrm{M}_n(\xi_1, ..., \xi_n) = \left\langle\!\!\left\langle \Phi,\ \prod_{j=1}^{n} \langle \cdot, \xi_j \rangle \right\rangle\!\!\right\rangle_{\mu} \quad , \qquad n \in \mathbb{N}, \quad \xi_j \in \mathcal{N},\ 1 \le j \le n$$

$$\mathrm{M}_0 = \langle\!\langle \Phi,\ 1\!\!1 \rangle\!\rangle \ .$$

We want to get estimates on the moments. Since $\Phi \in (\mathcal{H}_{-p})^{-1}_{-q,\mu}$ for some $p, q > 0$ we may estimate as follows

$$\left| \left\langle\!\!\left\langle \Phi, \left\langle x^{\otimes n}, \bigotimes_{j=1}^{n} \xi_j \right\rangle \right\rangle\!\!\right\rangle_{\mu} \right| \le \|\Phi\|_{-p,-q,\mu} \left\| \left\langle x^{\otimes n}, \bigotimes_{j=1}^{n} \xi_j \right\rangle \right\|_{p,q,\mu} \ .$$

To proceed we use the property (12) and the estimate (7) to obtain

$$\begin{aligned}
\left\| \left\langle x^{\otimes n}, \bigotimes_{j=1}^{n} \xi_j \right\rangle \right\|^2_{p,q,\mu} &= \sum_{k=0}^{n} \binom{n}{k}^2 \left\| \left\langle P^{\mu}_k \hat{\otimes} \mathrm{M}^{\mu}_{n-k}, \bigotimes_{j=1}^{n} \xi_j \right\rangle \right\|^2_{p,q,\mu} \\
&\le \sum_{k=0}^{n} \binom{n}{k}^2 (k!)^2\, 2^{kq}\, |\mathrm{M}^{\mu}_{n-k}|^2_{-p} \prod_{j=1}^{n} |\xi_j|^2_p \\
&= \prod_{j=1}^{n} |\xi_j|^2_p \sum_{k=0}^{n} \binom{n}{k}^2 (k!)^2\, ((n-k)!)^2\, K^{2(n-k)} 2^{kq} \\
&\le \prod_{j=1}^{n} |\xi_j|^2_p\, (n!)^2\, 2^{nq} \sum_{k=0}^{n} 2^{-(n-k)q} K^{2(n-k)} \\
&\le \prod_{j=1}^{n} |\xi_j|^2_p\, (n!)^2\, 2^{nq} \sum_{k=0}^{\infty} 2^{-kq} K^{2k}
\end{aligned}$$

which is finite for p, q large enough. Here K is determined by equation (7). Then we arrive at

$$\Big| \mathrm{M}_n(\xi_1, ...\xi_n) \Big| \le K\ C^n\ n! \prod_{j=1}^{n} |\xi_j|_p \tag{31}$$

for some $K, C > 0$.

Due to the kernel theorem 2 we then have the representation

$$\mathrm{M}_n(\xi_1, ...\xi_n) = \left\langle \mathrm{M}^{(n)}, \xi_1 \otimes ... \otimes \xi_n \right\rangle \ ,$$

where $\mathrm{M}^{(n)} \in (\mathcal{N}^{\hat{\otimes} n})'$. The sequence $\left\{ \mathrm{M}^{(n)},\ n \in \mathbb{N}_0 \right\}$ has the following property of positivity: for any finite sequence of smooth kernels $\left\{ g^{(n)}, n \in \mathbb{N} \right\}$ (i.e., $g^{(n)} \in \mathcal{N}^{\hat{\otimes} n}$ and $g^{(n)} = 0\ \forall\ n \ge n_0$ for some $n_0 \in \mathbb{N}$) the following inequality is valid

$$\sum_{k,j}^{n_0} \left\langle \mathrm{M}^{(k+j)}\ , g^{(k)} \otimes \overline{g^{(j)}} \right\rangle \ge 0 \ . \tag{32}$$

This follows from the fact that the left hand side can be written as $\langle\!\langle \Phi, |\varphi|^2 \rangle\!\rangle$ with

$$\varphi(x) = \sum_{n=0}^{n_0} \left\langle x^{\otimes n}, g^{(n)} \right\rangle, \quad x \in \mathcal{N}' \ ,$$

which is a smooth polynomial. Following [BS71, BeKo88] inequalities (31) and (32) are sufficient to ensure the existence of a uniquely defined measure ν on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$, such that for any $\varphi \in \mathcal{P}(\mathcal{N}')$ we have

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x) \, \mathrm{d}\nu(x) \ .$$

From estimate (31) we know that $\nu \in \mathcal{M}_a(\mathcal{N}')$. Then Lemma 9 shows that there exists $\varepsilon > 0$, $p \in \mathbb{N}$ such that $\exp(\varepsilon |x|_{-p})$ is ν−integrable. Corollary 25 then implies that each $\varphi \in (\mathcal{N})^1$ is ν-integrable.

Conversely let $\nu \in \mathcal{M}_a(\mathcal{N}')$ be given. Then the same argument shows that each $\varphi \in (\mathcal{N})^1$ is ν-integrable and from Corollary 25 we know that

$$\left| \int_{\mathcal{N}'} \varphi(x) \mathrm{d}\nu(x) \right| \leq C \, \|\varphi\|_{p,q,\mu} \int_{\mathcal{N}'} \exp(\varepsilon |x|_{-p}) \, \mathrm{d}\nu(x)$$

for some $p, q \in \mathbb{N}$, $C > 0$. Thus the continuity of $\varphi \mapsto \int_{\mathcal{N}'} \varphi \, \mathrm{d}\nu$ is established, showing that Φ defined by equation (30) is in $(\mathcal{N})^{-1}_{\mu,+}$. □

11 Change of measure

Suppose we are given two measures $\mu, \hat{\mu} \in \mathcal{M}_a(\mathcal{N}')$ both satisfying Assumption 2. Let a distribution $\hat{\Phi} \in (\mathcal{N})^{-1}_{\hat{\mu}}$ be given. Since the test function space $(\mathcal{N})^1$ is invariant under changes of measures in view of Theorem 27, the continuous mapping

$$\varphi \mapsto \langle\!\langle \hat{\Phi}, \varphi \rangle\!\rangle_{\hat{\mu}} \ , \qquad \varphi \in (\mathcal{N})^1$$

can also be represented as a distribution $\Phi \in (\mathcal{N})^{-1}_\mu$. So we have the implicit relation $\Phi \in (\mathcal{N})^{-1}_\mu \leftrightarrow \hat{\Phi} \in (\mathcal{N})^{-1}_{\hat{\mu}}$ defined by

$$\langle\!\langle \hat{\Phi}, \varphi \rangle\!\rangle_{\hat{\mu}} = \langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu \ .$$

This section will provide formulae which make this relation more explicit in terms of re-decomposition of the $\mathbb{Q}$-series. First we need an explicit relation of the corresponding $\mathbb{P}$-systems.

LEMMA 42 *Let* $\mu, \hat{\mu} \in \mathcal{M}_a(\mathcal{N}')$ *then*

$$P_n^\mu(x) = \sum_{k+l+m=n} \frac{n!}{k! \, l! \, m!} P_k^{\hat{\mu}}(x) \hat{\otimes} P_l^\mu(0) \hat{\otimes} \mathrm{M}_m^\mu \ .$$

PROOF. Expanding each factor in the formula

$$e_\mu(\theta, x) = e_{\hat{\mu}}(\theta, x) l_\mu^{-1}(\theta) l_{\hat{\mu}}(\theta) \ ,$$

we obtain

$$\sum_{n=0}^\infty \frac{1}{n!} \langle P_n^\mu(x), \theta^{\otimes n} \rangle = \sum_{k,l,m=0}^\infty \frac{1}{k! \, l! \, m!} \langle P_k^{\hat{\mu}}(x) \otimes P_l^{\hat{\mu}}(0) \otimes \mathrm{M}_m^\mu, \theta^{\otimes(k+l+m)} \rangle \ .$$

A comparison of coefficients gives the above result. □

An immediate consequence is the next reordering lemma.

Lemma 43 *Let $\varphi \in (\mathcal{N})^1$ be given. Then φ has representations in $I\!\!P^{\mu}$-series as well as $I\!\!P^{\hat{\mu}}$-series:*

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu}, \varphi^{(n)} \rangle = \sum_{n=0}^{\infty} \langle P_n^{\hat{\mu}}, \hat{\varphi}^{(n)} \rangle$$

where $\varphi^{(n)}, \hat{\varphi}^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ for all $n \in I\!\!N_0$, and the following formula holds:

$$\hat{\varphi}^{(n)} = \sum_{l,m=0}^{\infty} \frac{(l+m+n)!}{l!\, m!\, n!} \left(P_l^{\mu}(0) \hat{\otimes} \mathrm{M}_m^{\hat{\mu}}, \varphi^{(l+m+n)} \right)_{\mathcal{H}^{\otimes(l+m)}} . \tag{33}$$

Now we may prove the announced theorem.

Theorem 44 *Let $\hat{\Phi} = \sum_{n=0}^{\infty} \langle Q_n^{\hat{\mu}}, \hat{\Phi}^{(n)} \rangle \in (\mathcal{N})_{\hat{\mu}}^{-1}$. Then $\Phi = \sum_{n=0}^{\infty} \langle Q_n^{\mu}, \Phi^{(n)} \rangle$ defined by*

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_{\mu} = \langle\!\langle \hat{\Phi}, \varphi \rangle\!\rangle_{\hat{\mu}} \ , \qquad \varphi \in (\mathcal{N})^1$$

is in $(\mathcal{N})_{\mu}^{-1}$ and the following relation holds

$$\Phi^{(n)} = \sum_{k+l+m=n} \frac{1}{l!\, m!} \hat{\Phi}^{(k)} \hat{\otimes} P_l^{\mu}(0) \hat{\otimes} \mathrm{M}_m^{\hat{\mu}}$$

Proof. We can insert formula (33) in the formula

$$\sum_{n=0}^{\infty} n!\, \langle \Phi^{(n)}, \varphi^{(n)} \rangle = \sum_{n=0}^{\infty} n!\, \langle \hat{\Phi}^{(n)}, \hat{\varphi}^{(n)} \rangle$$

and compare coefficients again. □

Acknowledgements. The following results were presented at the workshop "Mathematical Approach to Fluctuations" organized at IIAS by Professor T. Hida. L.S. would like to express his gratitude to IIAS not only for the opportunity to do so, but also for its academic hospitality in the weeks after the workshop which provided a unique opportunity for continued work and collaboration. The first author was supported by the European program 'Human capital and mobility' and by the 'Deutsche Forschungsgemeinschaft'. W.W. gratefully acknowledges financial support of a scholarship from 'Graduiertenförderung des Landes Nordrhein–Westfalen'. The last author (J.A.Y) was partially supported by STRIDE and the National Natural Science Foundation of China.

References

[AKS93] Albeverio, S., Kondratiev, Yu.G. and Streit, L. (1993), *How to generalize White Noise Analysis to Non-Gaussian Spaces.* In: 'Dynamics of Complex and Irregular Systems'. Eds.: Ph. Blanchard et al., World Scientific.

[ADKS94] Albeverio, S., Daletzky, Y., Kondratiev, Yu. G. and Streit, L. (1994), *Non-Gaussian infinite dimensional analysis,* preprint, to appear in J. Func. Anal..

[BeS95] Benth, F. and Streit, L. (1995), *The Burgers Equation with a Non-Gaussian Random Force.* UMa preprint.

[BeKo88] Berezansky, Yu. M. and Kondratiev, Yu. G. (1988), *Spectral Methods in Infinite-Dimensional Analysis*, (in Russian), Naukova Dumka, Kiev. English translation, 1995, Kluwer Academic Publishers, Dordrecht.

[BeLy93] Berezansky, Yu.M. and Lytvynov, E.V. (1993), *Generalized White Noise Analysis connected with pertubed field operators*, Dopovidy AN Ukrainy, No **10**.

[BS71] Berezansky, Yu. M. and Shifrin, S.N. (1971), *The generalized degree symmetric Moment Problem,* Ukrainian Math. J. 23 N3, 247-258.

[Bo76] Bourbaki, N. (1976), *Elements of mathematics. Functions of a real variable.* Addison-Wesley.

[Da91] Daletsky, Yu.L. (1991), *A biorthogonal analogy of the Hermite polynomials and the inversion of the Fourier transform with respect to a non Gaussian measure*, Funct. Anal. Appl. **25**, 68-70.

[Di81] Dineen, S. (1981), *Complex Analysis in Locally Convex Spaces,* Mathematical Studies 57, North Holland, Amsterdam.

[GV68] Gel'fand, I. M. and Vilenkin, N.Ya. (1968), *Generalized Functions*, Vol. IV, Academic Press, New York and London.

[Hi75] Hida, T. (1975), *Analysis of Brownian Functionals,* Carleton Math. Lecture Notes No. 13, Carleton.

[Hi80] Hida, T. (1980), *Brownian Motion.* Springer, New York.

[HKPS93] Hida, T., Kuo, H.H., Potthoff, J. and Streit, L. (1993), *White Noise. An infinite dimensional calculus.* Kluwer, Dordrecht.

[Ito88] Ito, Y. (1988), *Generalized Poisson Functionals.* Prob. Th. Rel. Fields **77** 1-28.

[IK88] Ito, Y. and Kubo, I. (1988), *Calculus on Gaussian and Poisson White Noises.* Nagoya Math. J. **111** 41-84.

[Ko78] Kondratiev, Yu.G. (1978), *Generalized functions in problems of infinite dimensional analysis.* Ph.D. thesis, Kiev University.

[Ko80a] Kondratiev, Yu.G. (1980), *Spaces of entire functions of an infinite number of variables, connected with the rigging of a Fock space.* In: 'Spectral Analysis of Differential Operators.' Math. Inst. Acad. Sci. Ukrainian SSR, p. 18-37. English translation: Selecta Math. Sovietica **10** (1991), 165-180.

[Ko80b] Kondratiev, Yu.G. (1980), *Nuclear spaces of entire functions in problems of infinite dimensional analysis.* Soviet Math. Dokl. **22**, 588-592.

[KLPSW94] Kondratiev, Yu.G., Leukert, P., Potthoff, J., Streit, L., Westerkamp, W. (1994), *Generalized Functionals in Gaussian Spaces – the Characterization Theorem Revisited.* Manuskripte 175/94, Uni Mannheim.

[KLS94] Kondratiev, Yu.G., Leukert, P., Streit, L. (1994), *Wick Calculus in Gaussian Analysis,* BiBoS preprint 637, to appear in Acta Applicandae Mathematicae.

[KoSa78] Kondratiev, Yu.G. and Samoilenko, Yu.S. (1978), *Spaces of trial and generalized functions of an infinite number of variables*, Rep. Math. Phys. **14**, No.3, 325-350.

[KoS93] Kondratiev, Yu.G. and Streit, L. (1993), *Spaces of White Noise distributions: Constructions, Descriptions, Applications.* I. Rep. Math. Phys. **33**, 341-366.

[KoSW95] Kondratiev, Yu.G. , Streit, L. and Westerkamp, W.(1995), *A Note on Positive Distributions in Gaussian Analysis*, Ukrainian Math. J. **47** No. 5.

[KoTs91] Kondratiev, Yu.G. and Tsykalenko T.V. (1991), *Dirichlet Operators and Associated Differential Equations.* Selecta Math. Sovietica **10**, 345-397.

[KMP65] Kristensen, P., Mejlbo, L., and Poulsen, E.T. (1965), *Tempered Distributions in Infinitely Many Dimensions. I. Canonical Field Operators.* Commun. math. Phys. **1**, 175–214.

[Kuo92] Kuo, H.-H. (1992), *Lectures on white noise analysis.* Soochow J. Math. **18**, 229-300.

[Lee91] Lee, Y.-J. (1991), *Analytic Version of Test Functionals, Fourier Transform and a Characterization of Measures in White Noise Calculus.* J. Funct. Anal. **100**, 359-380.

[Lu70] Lukacs, E. (1970), *Characteristic Functions*, 2nd edition, Griffin, London.

[Øk95] Øksendal, B. (1995), *Stochastic Partial Differential Equations and Applications to Hydrodynamics.* In: 'Stochastic Analysis and Applications to Physics' Eds.: A.I. Cardoso et al; Kluwer, Dordrecht.

[Ou91] Ouerdiane, H. (1991), *Application des méthodes d'holomorphie et de distributions en dimension quelconque á l'analyse sur les espaces Gaussiens.* BiBoS preprint 491.

[Po87] Potthoff, J. (1987), *On positive generalized functionals.* J. Funct. Anal. **74**, 81-95.

[PS91] Potthoff, J. and Streit, L. (1991), *A characterization of Hida distributions.* J. Funct. Anal. **101**, 212-229.

[Sch71] Schaefer, H.H. (1971), *Topological Vector Spaces,* Springer, New York.

[Sk74] Skorohod, A.V. (1974), *Integration in Hilbert Space,* Springer, Berlin.

[Va95] Våge, G. (1995), *Stochastic Differential Equations and Kondratiev Spaces.* Ph.D. thesis, Trondheim University.

[VGG75] Vershik, A.M., Gelfand, I.M. and Graev, M.I. (1975), *Representations of diffeomorphisms groups.* Russian Math. Surveys **30**, No 6, 3-50.

[Yok90] Yokoi, Y.(1990), *Positive generalized white noise functionals.* Hiroshima Math. J. **20**, 137-157.

[Yok93] Yokoi, Y. (1993), *Simple setting for white noise calculus using Bargmann space and Gauss transform.* Preprint.

Yuri G. Kondratiev[1,2]
Ludwig Streit[1,3]
Werner Westerkamp[1]
Jia-an Yan[4]

[1] *BiBoS, Universität Bielefeld,*
D 33615 Bielefeld, Germany

[2] *Institute of Mathematics,*
Kiev, Ukraine

[3] *CCM, Universidade da Madeira,*
P9000 Funchal, Portugal

[4] *Institute of Applied Mathematics,*
Academica Sinica,
100080 Beijing, China