Navier-Stokes equations for a stochastic particle systems on the lattice

by

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Abstract: We introduce a class of stochastic models of particles on the cubic lattice $\mathbb{Z}^d$ with velocities and study the hydrodynamical limit on the diffusive space-time scale. Assuming special initial conditions corresponding to the incompressible regime, we prove that in dimension $d \geq 3$ there is a law of large numbers for the empirical density and the rescaled empirical velocity field. Moreover the limit fields satisfy the corresponding incompressible Navier-Stokes equations, with viscosity matrices characterized by a variational formula, formally equivalent to the Green-Kubo formula.

1. Introduction.

One of the main open problems in nonequilibrium statistical physics is the derivation of the hydrodynamical equations of fluid from the microscopic Hamiltonian dynamics. The main fluid equations are the Navier-Stokes equations and the Euler equations. The Euler equations represent the conservation of macroscopic mass, energy and momentum and have an obvious hyperbolic scaling invariance

$$x \to \alpha x, \quad t \to \alpha t. \tag{1.1}$$

The Navier-Stokes equations are more complicated and have no obvious scaling. They are given by correcting the Euler equations with viscous terms described by second order

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derivatives of conserved quantities such as energy, momentum and mass. In the incompressible regime, the Navier-Stokes equations become

$$\begin{aligned} &\text{div } u = 0, \\ &\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u \end{aligned} \tag{1.2}$$

with u the velocity field, p the pressure and $\nu > 0$ the kinematic viscosity.

Since Euler and Navier-Stokes equations involve macroscopic quantities, derivations of these equations from microscopic Hamiltonian dynamics are understood in the sense of law of large number, as the number of particles tends to infinity. We fix the space and time scales by choosing the typical interparticle distance as the unit length scale and the mean free time of particles as unit time scale. Suppose the range of molecular interactions are of the order of the typical interparticle distance. Then each particle typically interacts with at most finite number of nearby particles. Let ε denote the ratio between the microscopic and macroscopic length units. In order that particles travel a unit macroscopic length in a unit macroscopic time, the macroscopic time scale is chosen to be $\varepsilon^{-1}t$. The derivation of Euler equation can be stated as proving that the Euler equations are exact in the scaling limit $x \to \varepsilon^{-1}x, \quad t \to \varepsilon^{-1}t$. This scaling is usually called the Euler scale or hyperbolic scaling.

We now sketch Morrey's idea [1] for the derivation of the Euler equations. Since the Hamiltonian dynamics is conservative, the time derivatives of conserved quantities are given by microscopic conservation laws characterized by the corresponding microscopic currents. The currents involve microscopic interactions of particles and are not functions of local conserved quantities. To obtain closed equations, one has to represent these currents as functions of the local conserved quantities through some law of large numbers. If we assume that the system is locally in equilibrium and described by Gibbs states characterized by local conserved quantities, from the law of large numbers for Gibbs states these currents can be replaced by their expectations with respect to Gibbs states depending on local conserved quantities. Hence we obtain closed equations.

The basic ingredient in the previous heuristic derivation is a strong ergodic theorem for Hamiltonian systems. Though this ergodic theorem is a basic problem in the theory of Hamiltonian systems and is very easy to state, without further assumption, it is nevertheless still an open question. If certain random perturbations are added to the Hamiltonian dynamics, or if certain assumptions are made, it can be rigorously proved [2]. Alternatively, if instead of Hamiltonian dynamics one starts from the Boltzmann equation, very detailed derivations of Euler equation are available ([3,4,5]). Of course the Boltzmann equation is not a microscopic equation and, because the Boltzmann equation describes the low density regime, the *state equation* obtained in this case is just the one of the perfect gases. It should also be noted that all these derivations are valid only up to the times that the Euler equations have smooth solutions.

The similar program for the Navier-Stokes equations is much harder to carry out. First of all, the Navier-Stokes equations have no obvious scaling and thus they cannot be a scaling limit. Furthermore, although the basic equations in the classical physics are hyperbolic and completely reversible, the Navier-Stokes equations are irreversible due to the viscosity.

The first difficulty is more of technical nature and there are several possible solutions. One of them is to consider the following incompressible limit. The diffusive effects, like the viscosity and the heat conduction are hard to detect, because they are small corrections to the Euler equations. They become relevant on the diffusive scale, i.e.

$$x \to \varepsilon^{-1} x, \quad t \to \varepsilon^{-2} t. \tag{1.3}$$

The Navier-Stokes equations are not invariant under such scaling, so the prescription of the diffusive scaling (1.3) cannot be used in full generality. In order to restore the scaling invariance one needs to scale also the velocity field u as

$$u \to \varepsilon u, \tag{1.4}$$

and, consequently, the pressure as $p \to \bar{p} + \varepsilon^2 p$ with $\bar{p}$ an arbitrary constant. In other words, we consider a regime where macroscopic velocities are very small compared with the sound speed. To summarize, the incompressible Navier-Stokes equations (1.2) can be understood as the scaling limit under the the rules given by (1.3) and (1.4). This limit is called the incompressible limit. It is used in [6,7] to derive the incompressible Navier-Stokes equations (1.2) from the Boltzmann equation and in [8] to derive the viscous Burgers equation as diffusive limit of the simple exclusion process.

The second difficulty is a deep conceptual question and relates to the origin of diffusivity in classical physics. Up to now, there is no good mathematical understanding for a derivation of diffusive behavior from reversible systems. A heuristic explanation, however, can be given. We follow [9] where a formal derivation of the equations (1.2) from the Hamiltonian systems are given. From Morrey's heuristic argument or from the recent work of [2], the many body Liouville equation can be approximately solved using ansatz of local Gibbs states, i.e., Gibbs states with slowly varying chemical potentials. Let us strengthen the local equilibrium assumption by demanding that the Liouville equation can be solved asymptotically in orders of ε. Then locally the solution of the Liouville equation is given by a Gibbs state with an ε order correction. Therefore, the currents should be replaced by their expectations with respect to this state. The leading order contribution to the currents will be the same as in the Euler equation case; the ε order correction will give rise to the viscosity. The difficulties in carrying out this approach rigorously are obvious. First of all, there is no rigorous proof that local Gibbs states give even the leading order solution of Liouville equation unless some noise are added to the Liouville equation. Furthermore, it is in general hard to quantify the meaning of the asymptotic expansion involving an infinite number of variables. One way to make sense of this asymptotic expansion is through the use of relative entropy [2,10]. But lack of analytic tools for general Hamiltonian systems force us to consider lattice gases models.

The models we consider are defined as follows. Let $\mathcal{V}$ be a finite set of vectors in $\mathbb{R}^d$ of fixed length $\kappa > 0$, invariant under reflections and exchanges of the coordinate axes. It will represent the set of the possible velocities of the particles. For each $v \in \mathcal{V}$ we consider a species of particles moving on the lattice $\mathbb{Z}^d$ according to a simple exclusion process (SEP), with jump intensities such that the resulting drift is v. This means that a particle of the species v in a site $x \in \mathbb{Z}^d$ jumps with exponential law to one of the neighboring sites $x + e \in \mathbb{Z}^d$, $|e| = 1$, with intensity $p_e(v)$, if $x + e$ is not already occupied by a particle with velocity v. We denote by $\eta(x, v) = 0, 1$ the occupation number of particles of the species v in x. The intensities $p_e(v)$ are chosen so that the drift of the process for the v-particles is v. The jumps of each particle are independent of the particles with different velocities. They are also independent of the particles with the same velocity, as far as the exclusion condition is satisfied. The changes of velocity are due to collisions. Particles collide when are in the same site and have suitable velocities. The essential condition on the collisions is that they conserve the number of particles involved in the collision and their total momentum, and nothing else. In the next section we define two models, called Model I and Model II, by giving the set $\mathcal{V}$ and the collision rules. Our method can be extended to a wider class of models and also to more general lattices.

We take the set $\mathcal{V}$ for Model I to be the smallest possible by including only the $2d$ unit vectors parallel to the coordinate directions; for the Model II the set $\mathcal{V}$ includes several vectors with non-coordinate directions to make the model as isotropic as possible. In both case the jump intensity is chosen of the form

$$p_e(v) = \chi + \frac{1}{2} v \cdot e. \tag{1.5}$$

with χ big enough to make positive all the jump intensities . Note that we allow a particle with velocity v to jump also in "wrong" directions, as the one orthogonal to v. This is convenient to prove ergodicity, but the assumption that such a jump happens with intensity bigger than $\kappa/2$ is unnecessary: any positive intensity would work.

As for the collisions, we consider binary collisions, which exchange the velocities of any pairs of particles at the same site into another pairs of velocities having the same total momentum chosen at random among all the possible ones (if any), provided that the exclusion condition is not violated. The evolution of our model will be described in terms of a generator $\mathcal{L}$ which will be the sum of the exclusion part $\mathcal{L}^{ex}$ and the collision part $\mathcal{L}^c$.

Lattice gas models for the hydrodynamics have been widely considered before. We refer to [11] and references quoted therein for a comprehensive review. The aim of these works was to construct simple models for hydrodynamical equations which could be carefully and fastly simulated on the computer. Hence the effort is to make them as deterministic as possible. The numerical results provided a remarkably good agreement with hydrodynamical equations. The macroscopic behaviors were also analyzed theoretically, under the assumption of propagation of chaos. This produced a type of Boltzmann equation for the models, although no explicit low density condition was introduced and no proofs were given. After adding a small amount of stochasticity, the Euler regime was studied in [12] using the approach of [2]. In [13,14] the Euler and Navier-Stokes regimes were considered for models with discrete space, time and velocities, and with stochastic free motion and deterministic or stochastic collisions. However the kinetic limit was used as an intermediate step. In the present paper we deal directly with the hydrodynamic limit and no intermediate step or low density assumption is needed.

As discussed before, our analysis is confined to the incompressible regime. The incompressibility assumption corresponds to choose the chemical potentials of the local equilibrium suitably small i.e. of order ε. Following the heuristic derivation of viscosity in the previous paragraph, we have to compute the next order correction term of this state, which is of order ε^2. This is the key element of this paper. The viscosity will be determined through this correction term.

This analysis allows us to prove our main result, the validity of the law of large numbers for the empirical density and the rescaled empirical velocity field as the scaling parameter ε goes to 0. In fact, Theorem 2.1 states their weak convergence in probability to deterministic fields solving the incompressible Navier-Stokes equation with the viscosity determined according to the previous argument. A very short sketch of the idea of the proof is the following.

Recall the dynamics has two parts: the exclusion part $\mathcal{L}^{ex}$ and the collision part $\mathcal{L}^{c}$. Without the collisions our process is just the asymmetric simple exclusion process. The parallel problem for the asymmetric simple exclusion processes has been studied in [8] using the spectral gap and the logarithmic Sobolev inequality. Unfortunately, these estimates are hard to prove for the present models. We use an alternative approach. Let ρ_x^v denote the "local density" of particles with velocity $v \in \mathcal{V}$. We first use the spectral gap and the logarithmic Sobolev inequality of the exclusion part of the dynamics, $\mathcal{L}^{ex}$, to reduce the local relevant parameters to $\rho_x^v, v \in \mathcal{V}$. On the other hand, only certain linear combinations of ρ_x^v, $v \in \mathcal{V}$ representing local mass and momentum are conserved by the collisions. We then use the collision part to eliminate the irrelevant modes. Details will be presented in Section 4 and Section 6.

Finally we comment on the history of the methods used in this paper. The first mathematical rigorous proof of hydrodynamical limit for reversible nongradient models is the pioneering work of [15]. Though the Green-Kubo formula has widely been used to derive the hydrodynamic equation and certain mathematical rigorness had been achieved before (see, for example, [16]), this work represents the first completely mathematical rigorous proof of Green-Kubo formula and it gives a variational formula which is much more stable to work with. The key assumptions of this work are the reversibility of the process and the spectral gap of the dynamics. The spectral gap, though difficult, is more of technical nature. It is proved in several models by [17,18,19] and a general approach is outlined in [18]. A class of nonreversible models is treated by [20] but the final hydrodynamical equation is required to be a nonlinear heat equation without drift. The restriction on the drift is removed in [8] provided the space dimension $d > 2$ and the logarithmic Sobolev inequality can be proved. A general argument to obtain the logarithmic Sobolev inequality is given in [21]. The restriction on the space dimension $d > 2$ in [8] is optimal and one expects completely different behavior for $d \leq 2$. In this case, the scaling is not diffusive and the hydrodynamical equation probably is nonlocal. There is no mathematical results concerning this phenomenon up to now. A similar phenomenon is expected also in the real hydrodynamics for $d \leq 2$, due to the long time tails which cause the divergence of the transport coefficients [22].

We conclude with a few remarks on the viscosity matrix. The viscosity matrix we obtain does not reduce to the contribution due to the stochasticity of the model, corresponding to the symmetric part of the generator. Actually, we prove that the contribution due to the "deterministic motion", i.e. to the anti-symmetric part of the generator, is nonnegative and nonvanishing. For the real incompressible Navier-Stokes fluid all the eigenvalues coincide, but we cannot prove that this is true for Model II. We do not know if this is due to the lack of our analytic tools or due to the nonisotropy of the underlying cubic lattice. Our analysis can be extended to lattices with other geometries, but we do not formulate an abstract framework for all possible lattices.

2. The models and the main theorem.

Let $\Lambda_L \in \mathbb{Z}^d$ be the cubic sublattice $\{-L, \dots, L\}^d$ with periodic boundary conditions. Let $L = \varepsilon^{-1}$. We denote by e_i, $i = 1, \dots, d$, the coordinate vectors with component

$$(e_i)_j = \delta_{i,j}$$

Moreover, we put $\mathcal{E} = \{e = \pm e_i \text{ for some } i = 1, \dots, d\}$. $R_i v$ will denote the reflection of the vector $v \in \mathbb{R}^d$ w.r.t. the plane π_i orthogonal to e_i, i.e. the vector with coordinates

$$(R_i v)_j = \begin{cases} -v_i \text{ if } i = j \\ v_j \text{ otherwise.} \end{cases}$$

Denote by σ be any element of $\mathcal{J}$, the permutation group of the set $\{1, \dots, d\}$, and let $\sigma v = (v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(d)})$.

Let $\mathcal{V}$ be a finite subset of $\mathbb{R}^d$ representing the possible velocities and N be its cardinality. Assume that $\mathcal{V}$ is symmetric w.r.t. reflections and permutations:

$$R_i \mathcal{V} = \mathcal{V} \quad i = 1, \dots, d, \quad \sigma \mathcal{V} = \mathcal{V}, \, \sigma \in \mathcal{J}$$

For our purposes it will be sufficient to consider velocities of fixed modulus, so we also assume

$$|v| = \kappa \text{ for any } v \in \mathcal{V}$$

On each site of the lattice at most one particle for each velocity is allowed, with no exclusion rule between different velocities. A configuration of particles on the lattice is denoted by $\eta = \{\eta_x, x \in \Lambda_L\}$ where $\eta_x = \{\eta(x, v), v \in \mathcal{V}\}$ and $\eta(x, v) = 0, 1, \ x \in \mathbb{Z}^d, \ v \in V$ is the number of particles in x with velocity v. The generator of the dynamics is defined by

$$\mathcal{L} f = \mathcal{L}^{ex} f + \mathcal{L}^c f \tag{2.1}$$

$\mathcal{L}^{ex}$ is the generator of the exclusion process for particles with several colors (velocities), without exclusion between different colors

$$\mathcal{L}^{ex} f(\eta) = \sum_{w \in \mathcal{V}} \sum_{\substack{y \in \mathbb{Z}^d \\ e \in \mathcal{E}}} p(y, y+e, w, \eta) \left[f(\eta^{y, y+e, w}) - f(\eta) \right] \tag{2.2}$$

where

$$\eta^{x,y,w}(z, v) = \begin{cases} \eta(z, v) \text{ if } v \neq w \text{ or } z \neq x, y \\ \eta(x, w) \text{ if } v = w \text{ and } z = y \\ \eta(y, w) \text{ if } v = w \text{ and } z = x \end{cases}$$

The jump intensity is given by

$$p(x,y,w,\eta) = \eta(x,w)p(x-y,w), \quad p(x-y,w) = [\chi + (y-x)\cdot w/2], \quad v \in \mathcal{V}, \tag{2.3}$$

for some $\chi > \kappa/2$. The constant χ has to be chosen large enough so that the jump rates are nonnegative.

The collision operator is defined as follows: let

$$\mathcal{Q} = \{q = (v,w,v',w') \in \mathcal{V}^4 : v + w = v' + w'\}.$$

We call the quadruples $q \in \mathcal{Q}$ "collisions". $\mathcal{Q}$ is the set of all the possible collisions. The first two arguments of q, v and w are the incoming velocities, while v', w' are the outgoing velocities of the collision q. We define

$$\mathcal{L}^c f(\eta) = \sum_{y\in\mathbb{Z}^d} \sum_{q\in\mathcal{Q}} \left[f(\eta^{y,q}) - f(\eta)\right]. \tag{2.4}$$

Here, for $q = (v,w,v',w')$ we put

$$\tilde{p}(y,q,\eta) = \eta(y,v)\eta(y,w)(1-\eta(y,v'))(1-\eta(y,w')) \tag{2.5}$$

and

$$\eta^{y,q}(z,u) = \begin{cases} \eta(z,u) \text{ if } z \neq y \text{ or } u \notin q \text{ or } \tilde{p}(y,q,\eta) = 0 \\ 0 \text{ if } \tilde{p}(y,q,\eta) = 1 \text{ and } z = y \text{ and } (u = v \text{ or } u = w) \\ 1 \text{ if } \tilde{p}(y,q,\eta) = 1 \text{ and } z = y \text{ and } (u = v' \text{ or } u = w') \end{cases}$$

Note that if η does not contain in y two particles with the incoming velocities v and w, or there are particles with outgoing velocities v' and w', $\eta^{y,q} = \eta$. Putting $(Q_y^q f)(\eta) \equiv \left[f(\eta^{y,q}) - f(\eta)\right]$, we rewrite (2.4) as

$$\mathcal{L}^c f(\eta) = \sum_{y} \sum_{q\in\mathcal{Q}} (Q_y^q f)(\eta) \tag{2.6}$$

Specific choices of the the set $\mathcal{V}$ will be given later on in this section. For the moment, we simply note that the collision operator conserves the total mass and the total momentum. More precisely, the quantities

$$I_0(\eta_x) := \sum_{v\in\mathcal{V}} \eta(x,v), \tag{2.7}$$

and, for $\alpha = 1,\ldots,d$:

$$I_\alpha(\eta_x) := \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)\eta(x,v) \tag{2.8}$$

are conserved in a collision, i.e., $\mathcal{L}^c\big[I_\alpha(\eta_x)\big] = 0$ for all $x \in \Lambda_L$. The choices of $\mathcal{V}$ will ensure that these are the only conserved quantities during a collision.

Since jumps conserve the total number of particles for each color, we have

$$\mathcal{L}\big[\sum_x I_\alpha(\eta_x)\big] = 0, \quad \alpha = 0, \dots, d.$$

As a consequence, the following measure is invariant for $\mathcal{L}$

$$d\mu_{L,r,n} = Z_{L,r,n}^{-1} \prod_{x\in\Lambda_L} \exp\{rI_0(\eta_x) + \sum_{\alpha=1}^d n_\alpha I_\alpha(\eta_x)\} \tag{2.9}$$

with n the d-dimensional vector $\{n_\alpha,\ \alpha = 1, \dots d\}$. All the measures we will consider in the sequel are absolutely continuous w.r.t. the global equilibrium measure μ_r corresponding to $n = 0$, namely the one defined as

$$\mu_r(\eta) = Z_r^{-1} \prod_{x\in\Lambda_L} \exp\{rI_0(\eta_x)\}, \qquad r \in \mathbb{R}. \tag{2.10}$$

When possible we also omit the subscript r and denote by $\big\langle \cdot \big\rangle$ the expectation w.r.t. μ_r.

The generator $\mathcal{L}$ is not symmetric w.r.t. this measure, because the generator of the exclusion part is not symmetric. However, one can check immediately that

$$E^{\mu_r}[f\mathcal{L}^c g] = E^{\mu_r}[g\mathcal{L}^c f],$$

so $\mathcal{L}^c$ is symmetric w.r.t μ_r.

The formal adjoint w.r.t μ is given by

$$\mathcal{L}^* = (\mathcal{L}^{ex})^* + \mathcal{L}^c$$

with

$$(\mathcal{L}^{ex})^* f(\eta) = \sum_{w\in\mathcal{V}} \sum_{\substack{y\in\mathbb{Z}^d \\ e\in\mathcal{E}}} p^*(y, y+e, w, \eta)\big[f(\eta^{y,y+e,w}) - f(\eta)\big] \tag{2.11}$$

and

$$p^*(x, y, w, \eta) = \eta(y, w)[\chi + (y - x)\cdot w/2] = p(y, x, -w, \eta). \tag{2.12}$$

There are $d + 1$ locally conserved quantities I_α (2.8) for this dynamics and the corresponding currents $w^\alpha_{x,\beta}$, $\alpha = 0, \dots, d$ are defined by

$$\mathcal{L}\big[I_\alpha(\eta_x)\big] = \sum_{\beta=1}^d \nabla^-_\beta w^\alpha_{x,\beta} \tag{2.13}$$

with

$$\nabla_\beta^- g(x) \equiv g(x) - g(x - e_\beta), \qquad \nabla_\beta g(x) \equiv g(x + e_\beta) - g(x),$$

Note that the collision operator plays no role in (2.13) because $\mathcal{L}^c\big[I_\alpha(\eta_x)\big] = 0$. Similarly we define $w^{*,\alpha}_{x,\beta}$ as

$$\mathcal{L}^*\big[I_\alpha(\eta_x)\big] = \sum_{\beta=1}^d \nabla_\beta^- w^{*,\alpha}_{x,\beta} \tag{2.14}$$

Since the current depends only on the exclusion part of the process and the exclusion dynamics is defined for each color separately, we can compute the currents using formulae for the asymmetric simple exclusion. More precisely, we can compute the current of the asymmetric simple exclusion process by

$$\mathcal{L}^{ex}\eta(x,v) = \sum_{\beta=1}^d \nabla_e^- w_{x,\beta}(v), \quad \mathcal{L}^*\eta(x,v) = \sum_{\beta=1}^d \nabla_\beta^- w^*_{x,\beta}(v).$$

The currents $w_{x,\beta}(v)$ and $w^*_{x,\beta}(v)$ are given explicitly by

$$\begin{aligned}
w_{x,\beta}(v) &= \frac{1}{2}(p(e_\beta,v) + p(-e_\beta,v))\nabla_\beta\eta(x,v) + w^{(a)}_{x,\beta}(v),\\
w^*_{x,\beta}(v) &= \frac{1}{2}(p(e_\beta,v) + p(-e_\beta,v))\nabla_\beta\eta(x,v) - w^{(a)}_{x,\beta}(v),\\
w^{(a)}_{x,\beta}(v) &= (p(e_\beta,v) - p(-e_\beta,v))b_{x,\beta}(v),
\end{aligned} \tag{2.15}$$

with

$$b_{x,\beta}(v) = \eta(x+e_\beta,v)\eta(x,v) - \frac{1}{2}\Big[\eta(x+e_\beta,v) + \eta(x,v)\Big] \qquad (2.16).$$

Here $p(e_\beta, v)$ is the jump rate of the color v in the direction e_β. With the convention (2.3), $p(e_\beta,v) - p(-e_\beta,v) = e_\beta\cdot v$ and $\frac{1}{2}(p(e_\beta,v) + p(-e_\beta,v)) = \chi$

Using these equations, we obtain the mass current $w^0_{x,\beta}$ and the momentum current $w^\alpha_{x,\beta}$ as

$$\begin{aligned}
w^0_{x,\beta} &= \sum_{v\in\mathcal{V}} w_{x,\beta}(v) = \chi\nabla_\beta I_0(\eta_x) + \sum_{v\in\mathcal{V}}(e_\beta\cdot v)b_{x,\beta}(v)\\
w^\alpha_{x,\beta} &= \sum_{v\in\mathcal{V}}(e_\alpha\cdot v)w_{x,\beta}(v) = \chi\nabla_\beta I_\alpha + \sum_{v\in\mathcal{V}}(e_\alpha\cdot v)(e_\beta\cdot v)b_{x,\beta}(v), \quad \alpha,\beta = 1,\dots,d.
\end{aligned} \tag{2.17}$$

Similarly,

$$\begin{aligned}
w^{*,0}_{x,\beta} &= \chi\nabla_\beta I_0(\eta_x) - \sum_{v\in\mathcal{V}}(e_\beta\cdot v)b_{x,\beta}(v)\\
w^{*,\alpha}_{x,\beta} &= \chi\nabla_\beta I_\alpha - \sum_{v\in\mathcal{V}}(e_\alpha\cdot v)(e_\beta\cdot v)b_{x,\beta}(v), \quad \alpha,\beta = 1,\dots,d.
\end{aligned} \tag{2.18}$$

It is convenient to introduce also the symmetric and antisymmetric part of the currents

$$w^{(s),\alpha}_{x,\beta} = \frac{w^{\alpha}_{x,\beta} + w^{*,\alpha}_{x,\beta}}{2}, \qquad w^{(a),\alpha}_{x,\beta} = \frac{w^{\alpha}_{x,\beta} - w^{*,\alpha}_{x,\beta}}{2}. \tag{2.19}$$

We will need to compute averages w.r.t. distributions of the form

$$\mu_{n,L} = Z^{-1} \prod_{x \in \Lambda_L} \exp\{n_0(x) I_0(\eta_x) + \sum_{\alpha=1}^{d} n_\alpha(x) I_\alpha(\eta_x)\}. \tag{2.20}$$

Define the velocity distribution at x as

$$f(x,v,n) = E^{\mu_n}[\eta(x,v)] = \frac{e^{n_0(x)+\underline{n}(x)\cdot v}}{1 + e^{n_0(x)+\underline{n}(x)\cdot v}},$$

with $\underline{n} = (n_1, \ldots, n_d)$. The average of the currents $w^{\alpha}_{x,\beta}$ w.r.t. a measure of this form are given by

$$\begin{aligned}
&E^{\mu_n}[w^0_{x,\beta}] = \sum_{v\in\mathcal{V}} (e_\beta \cdot v) E^{\mu_n}[b_{x,\beta}(v)] + \chi \nabla_\beta E^{\mu_n}[I_0] \\
&= \sum_{v\in\mathcal{V}} (v\cdot e_\beta)\big[f(x,v,n) f(x+e_\beta,v,n) - \frac{1}{2}(f(x,v,n) + f(x+e_\beta,v,n))\big] + \chi\nabla_\beta E^{\mu_n}[I_0], \\
&E^{\mu_n}[w^\alpha_{x,\beta}] = \sum_{v\in\mathcal{V}} (e_\alpha\cdot v)(e_\beta \cdot v) E^{\mu_n}[b_{x,\beta}(v)] + \chi \nabla_\beta E^{\mu_n}[I_\alpha] \\
&= \sum_{v\in\mathcal{V}} (e_\alpha\cdot v)(e_\beta\cdot v)\big[f(x,v,n) f(x+e_\beta,v,n) - \frac{1}{2}(f(x,v,n) + f(x+e_\beta,v,n)\big] \\
&+ \chi\nabla_\beta E^{\mu_n}[I_\alpha].
\end{aligned} \tag{2.21}$$

Let us give now a heuristic derivation of the hydrodynamic equations.

We start with the Euler scale. We choose as initial state for the system the local equilibrium measure (2.20) with chemical potentials $\tilde{n}(\varepsilon x, 0)$ slowly varying in space. Then we look at the system on the Euler time scale, i.e. at times of order ε^{-1}, that corresponds to replace the generator $\mathcal{L}$ with $\varepsilon^{-1}\mathcal{L}$. Then the hydrodynamic equations are recovered from the conservation laws

$$\frac{d}{dt}\varepsilon^d \sum_x J(\varepsilon x) E^{f_t\mu_r}[I_\alpha(\eta_x)] = \varepsilon^{d-1} \sum_x \sum_{\beta=1}^{d} J(\varepsilon x) \nabla^-_\beta E^{f_t\mu_r}[w^\alpha_{x,\beta}] \tag{2.22}$$

where $f_t\mu_r$ denotes the distribution of the process at time t.

Assuming that f_t is still well approximated by a local equilibrium with chemical potentials $\tilde{n}(\varepsilon x, t)$, by (2.21) and (2.22), in the limit $\varepsilon \to 0$ we get the Euler equations

$$\begin{aligned} &\frac{\partial}{\partial t}\rho + \sum_{\alpha=1}^{d} \partial_{z_\alpha} j_a = 0, \\ &\frac{\partial}{\partial t}u_\alpha + \sum_{\beta=1}^{d} \partial_{z_\beta} \pi^{\alpha\beta} = 0, \end{aligned} \tag{2.23}$$

where ∂_{z_α} denotes the partial derivative w.r.t. the macroscopic coordinate z_α,

$$\rho(\varepsilon x, t) = E^{\mu_{\tilde{n}}}[I_0(\eta_x)] = \sum_{v\in\mathcal{V}} f(x, v, \tilde{n}). \tag{2.24}$$

is the mass density,

$$\bar{u}_\alpha(\varepsilon x, t) = E^{\mu_{\tilde{n}}}[I_\alpha(\eta_x)] = \sum_{v\in\mathcal{V}} (v\cdot e_\alpha) f(x, v, \tilde{n}) \tag{2.25}$$

is the momentum density,

$$j_\alpha(\varepsilon x, t) = \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)(f(x, v, \tilde{n}) - f(x, v, \tilde{n})^2)$$

is the mass current and

$$\begin{aligned} \pi^{\alpha\beta}(\varepsilon x, t) &= E^{\mu_{\tilde{n}}}\Big[(I_\alpha(\eta_x) - E^{\mu_{\tilde{n}}}[I_\alpha(\eta_x)])(I_\beta(\eta_x) - E^{\mu_{\tilde{n}}}[I_\beta(\eta_x)])\Big] \\ &= \sum_{v\in\mathcal{V}} (e_\alpha\cdot v)(e_\beta\cdot v)[f(x, v, \tilde{n}) - f(x, v, \tilde{n})^2], \end{aligned} \tag{2.26}$$

is the stress tensor.

A more explicit expression for ρ, $\bar{u}_\alpha$, j_α and $\pi^{\alpha\beta}$ is obtained from the symmetry of $\mathcal{V}$ in the exchange $v \to -v$. We have:

$$\begin{aligned} \rho &= \sum_{v\in\mathcal{V}} \frac{\mathrm{e}^{2\tilde{n}_0} + \mathrm{e}^{\tilde{n}_0}\cosh(\tilde{n}\cdot v)}{1 + \mathrm{e}^{2\tilde{n}_0} + 2\mathrm{e}^{\tilde{n}_0}\cosh\tilde{(n\cdot v)}}, \\ \bar{u}_\alpha &= \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)\frac{\mathrm{e}^{\tilde{n}_0}\sinh(\tilde{n}\cdot v)}{1 + \mathrm{e}^{2\tilde{n}_0} + 2\mathrm{e}^{\tilde{n}_0}\cosh\tilde{(n\cdot v)}}, \\ j_\alpha &= \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)\frac{\mathrm{e}^{\tilde{n}_0}\sinh(\tilde{n}\cdot v)(1 - \mathrm{e}^{2\tilde{n}_0})}{(1 + \mathrm{e}^{2\tilde{n}_0} + 2\mathrm{e}^{\tilde{n}_0}\cosh\tilde{(n\cdot v)})^2}, \\ \pi^{\alpha\beta} &= \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)(v\cdot e_\beta)\frac{2\mathrm{e}^{\tilde{n}_0}\cosh(\tilde{n}\cdot v)(1 + \mathrm{e}^{2\tilde{n}_0}) + 4\mathrm{e}^{2\tilde{n}_0}}{(1 + \mathrm{e}^{2\tilde{n}_0} + 2\mathrm{e}^{\tilde{n}_0}\cosh\tilde{(n\cdot v)})^2}. \end{aligned}$$

Next, we consider the incompressible limit; we follow the same strategy. We choose as initial state for the system the local equilibrium measure (2.20) with chemical potentials $\tilde{n}(\varepsilon x, t)$ given by

$$\tilde{n}_0 = r + \varepsilon^2 \lambda_0(\varepsilon x), \quad \tilde{n}_\alpha = \varepsilon \lambda_\alpha(\varepsilon x), \tag{2.27}$$

such that

$$\operatorname{div} \lambda := \sum_{\alpha=1}^{d} \partial_{z_\alpha} \lambda_\alpha = 0.$$

The the conservation laws are the same as (2.22) except a change of scaling is needed, i.e. we replace $\mathcal{L}$ by $\varepsilon^{-2}\mathcal{L}$:

$$\frac{\partial}{\partial t} \varepsilon^d \sum_x J(\varepsilon x) E^{f_l \mu_r}[I_\alpha(\eta_x)] = \varepsilon^{d-2} \sum_x \sum_{\beta=1}^{d} J(\varepsilon x) \nabla_\beta^- E^{f_l \mu_r}[w_{x,\beta}^\alpha]. \tag{2.28}$$

Assume that a local equilibrium measure with chemical potentials like in (2.27) describes the system also at time t. With this choice of the local equilibrium, we have

$$\rho = N\theta + O(\varepsilon^2), \qquad \theta = \frac{e^r}{1 + e^r}, \tag{2.29}$$

$$E^{\mu_{\tilde{n}}}[\eta(x, v) + \eta(x, -v)] = 2\theta + O(\varepsilon^2) \tag{2.30}$$

Moreover, the assumption on $\mathcal{V}$ imply that

$$\sum_{v \in \mathcal{V}} (e_\alpha \cdot v)(e_\beta \cdot v) = K \delta_{\alpha,\beta},$$

$$\sum_{v \in \mathcal{V}} (e_\alpha \cdot v)(e_\beta \cdot v)(\lambda \cdot v)^2 = \delta_{\alpha,\beta} A \lambda^2 + \delta_{\alpha,\beta} B \lambda_\alpha^2 + C \lambda_\alpha \lambda_\beta$$

where A, B, C and K are constants independent of λ.

Therefore,

$$\bar{u}_\alpha = \varepsilon t(\theta) \lambda_\alpha + O(\varepsilon^3), \tag{2.31}$$

with

$$t(\theta) = K\theta(1 - \theta), \tag{2.32}$$

We have used the following identities:

$$\begin{aligned}
&\frac{\exp[a + \varepsilon b + \varepsilon^2 c]}{1 + \exp[a + \varepsilon b + \varepsilon^2 c]} = \theta + \varepsilon \theta(1-\theta) b + \frac{\varepsilon^2}{2} \theta(1-\theta) \Big[2c + (1 - 2\theta) b^2 \Big] + O(\varepsilon^3) \\
&\frac{\exp[a + \varepsilon b + \varepsilon^2 c]}{1 + \exp[a + \varepsilon b + \varepsilon^2 c]} - \left\{ \frac{\exp[a + \varepsilon b + \varepsilon^2 c]}{1 + \exp[a + \varepsilon b + \varepsilon^2 c]} \right\}^2 \\
&= \theta(1-\theta) + \varepsilon \theta(1-\theta)(1-2\theta) b + \frac{\varepsilon^2}{2} \theta(1-\theta) \Big[2(1-2\theta)c + (1 - 6(1-\theta)\theta) b^2 \Big] + O(\varepsilon^3),
\end{aligned} \tag{2.33}$$

where $\theta = e^a/(1+e^a)$. From this identities and (2.27),

$$
\begin{aligned}
f(x,v,n) &= \theta + \varepsilon\theta(1-\theta)(\lambda\cdot v) + \frac{\varepsilon^2}{2}\theta(1-\theta)\Big[2\lambda_0 + (1-2\theta)(\lambda\cdot v)^2\Big] + O(\varepsilon^3) \\
f(x,v,n)^2 - f(x,v,n) &= -\theta(1-\theta) - \varepsilon\theta(1-\theta)(1-2\theta)(\lambda\cdot v) \\
&\quad - \frac{\varepsilon^2}{2}\theta(1-\theta)\Big[2(1-2\theta)\lambda_0 + (1-6(1-\theta)\theta)(\lambda\cdot v)^2\Big] + O(\varepsilon^3).
\end{aligned}
\tag{2.34}
$$

The equations (2.29)-(2.31) are direct consequences of the last identities.

From (2.21) and (2.34) one can check that

$$
\varepsilon^{d-2}\sum_x\sum_{\beta=1}^d \nabla_\beta^- E^{\mu_{\tilde n}}[w^0_{x\beta}]J(\varepsilon x) = \varepsilon^d \sum_x\sum_{\beta=1}^d J(\varepsilon x)\Big\{(2\theta-1)t(\theta)\partial_{z_\beta}\lambda_\beta + \chi\partial^2_{z_\beta z_\beta}\rho\Big\} + O(\varepsilon^2),
\tag{2.35}
$$

because

$$
\nabla_\beta^-[\lambda(\varepsilon(x+e_\beta)) + \lambda(\varepsilon x)] = 2\varepsilon\partial_{z_\beta}\lambda(\varepsilon x) + O(\varepsilon^3).
$$

From (2.29) the average of the l.h.s. of (2.28) with $\alpha = 0$ is $O(\varepsilon^2)$. Moreover, the last term on the right side of (2.35) is of order ε^2. Hence we have to assume div $\lambda = 0$ to assure the continuity equation. Also, the leading order correction to (2.35) is of order ε^2, hence it is consistent with the assumption that the density correction is of order ε^2 (2.27). We remark that, if there is a first order correction to the density, which can be achieved by adding a term $\varepsilon\lambda_1(\varepsilon x,t)$ in the first equation of (2.27), then the leading contribution to the last term in (2.35) is

$$
\chi\varepsilon^{d+1}\sum_x\sum_{\beta=1}^d \partial^2_{z_\beta z_\beta}J(\varepsilon x)\rho_1
$$

where ρ_1 is the corresponding first order correction to the density (i.e., $\rho = \rho_0 + \varepsilon\rho_1$). Note that this term appears because of the stochastic nature of our dynamics (more precisely, because we use asymmetric simple exclusions to replace the deterministic dynamics).

The averages of the momentum currents in (2.28) can be computed as

$$
\begin{aligned}
\varepsilon^{-3}\sum_{\beta=1}^d E^{\mu_{\tilde n}}\nabla_\beta^-[w^\alpha_{x\beta}] &= -t(\theta)(1-2\theta)\sum_{\beta=1}^d \partial_{z_\alpha}\lambda_0 + \chi t(\theta)\sum_{\beta=1}^d \partial^2_{z_\beta z_\beta}\lambda_\alpha \\
&\quad - \frac{\theta}{2}(1-\theta)(1-6\theta(1-\theta))\sum_{v\in\mathcal{V}}\sum_{\beta=1}^d (e_\alpha\cdot v)(e_\beta\cdot v)\partial_{z_\beta}(\lambda\cdot v)^2 + O(\varepsilon).
\end{aligned}
\tag{2.36}
$$

Recall that $\mathcal{V}$ is invariant under the reflection w.r.t each coordinate axis. Hence, by (2.36), the equations for λ_α are of the form

$$
\frac{\partial}{\partial t}\lambda_\alpha = -\frac{1}{t(\theta)}\partial_{z_\alpha}p - \frac{Bh(\theta)}{t(\theta)}\partial_{z_\alpha}\lambda_\alpha^2 - \frac{Ch(\theta)}{t(\theta)}\sum_{\beta=1}^d \lambda_\beta\partial_{z_\beta}\lambda_\alpha + \chi\sum_{\beta=1}^d \partial^2_{z_\beta z_\beta}\lambda_\alpha,
\tag{2.37}
$$

where

$$p(\theta,\lambda) = t(\theta)(1-2\theta)\lambda_0 + Ah(\theta)\lambda^2, \quad h(\theta) = \frac{\theta}{2}(1-\theta)(1-6\theta(1-\theta)).$$

We obtain the usual Navier-Stokes equation (up to a scale factor if the coefficient $B = 0$. An example of $\mathcal{V}$ such that this is true will be given in the Model II below.

This heuristic derivation is not entirely correct. The diffusion coefficient appearing in (2.37) comes completely from the stochastic fluctuation, namely, the fluctuation associated with the asymmetric simple exclusion. Therefore, if (2.37) were correct, there would be no contribution to the viscosity from the "deterministic part" of the dynamics. We shall prove that (2.37) still holds but the diffusion coefficient is not given by χ. The true diffusion coefficient is always bigger than χ and this accounts for the viscosity from the "deterministic part" of the dynamics.

We now choose the space $\mathcal{V}$. First notice that, because of the symmetry properties of $\mathcal{V}$, the constants A, B, C and D have the more explicit expressions

$$A = \sum_{v\in\mathcal{V}} v_1^2 v_2^2 = C/2, \quad B = \sum_{v\in\mathcal{V}} [v_1^4 - 3v_1^2 v_2^2], \quad K = \sum_{v\in\mathcal{V}} v_1^2. \tag{2.38}$$

Model I

The simplest choice is the following: Let

$$\mathcal{V} = \mathcal{E}.$$

With this choice, the only possible collisions are those $q = (v, w, v, w')$ such that $v + w = 0$ and $v' + w' = 0$.

For this model, one can easily check that

$$A = C = 0, \quad B = 2, \quad K = 2 \tag{2.39}$$

Model II

Let $d = 3$ and σ denote any element of $\mathcal{J}$, the permutation group of $\{1, 2, 3\}$. Let

$$\mathcal{V} = \{v : \sigma v = (\pm 1, \pm 1, \pm\varpi) \text{ for some } \sigma \in \mathcal{J}\}$$

where ϖ is the positive solution of

$$\varpi^4 - 6\varpi^2 - 1 = 0.$$

$\mathcal{V}$ is invariant under reflections w.r.t. any coordinate plane and permutations of coordinate axes. Moreover, with the above choice of ϖ, one gets:

$$B = 0$$

and

$$A = 8(1 + 2\varpi^2), \quad C = 16(1 + 2\varpi^2), \quad K = 8(2 + \varpi^2) \tag{2.40}$$

The proof that the only collision invariants are total mass and total momentum is part of the ergodic theorem proven in next section.

Now we are ready to state our convergence result. Let f_t the distribution density, w.r.t. μ_r, of the process on the diffusive time scale, namely the solution of the forward equation

$$\frac{\partial}{\partial t} f_t = \varepsilon^{-2} \mathcal{L}^* f_t. \tag{2.41}$$

We consider the initial datum $f_{t=0} = \psi_0$ given by

$$\psi_0 = Z_\varepsilon^{-1} \exp \Big[\varepsilon \sum_{x \in \Lambda_L} \sum_{\alpha=1}^{d} \lambda_\alpha(\varepsilon x) I_\alpha(x) \Big] \tag{2.42}$$

with λ_α smooth periodic functions. Recall the definition of the specific relative entropy of two densities f and g w.r.t. μ_r:

$$s(f \mid g) = \varepsilon^d \big\langle f \log \frac{f}{g} \big\rangle \tag{2.43}$$

and let

$$s(f) \equiv s(f \mid 1).$$

The rest of this paper is devoted to prove the following

Theorem 2.1. *Assume $d \geq 3$ and $\mathcal{V}$ as in Model I or Model II. Let f_t the density w.r.t. the measure μ_r, defined in (2.10), at time t. Then there are $d \times d$ diagonal matrices D^α, $\alpha = 1, \dots, d$ such that $D^\alpha \geq D_s = \chi\, \mathbb{I}$ (as matrices) and the following holds: given a smooth solution $p(x,t), u(x,t)$ of the incompressible Navier-Stokes equations*

$$\text{div } u = 0, \tag{2.44}$$

$$\frac{\partial}{\partial t} u_\alpha + \frac{h(\theta)}{t(\theta)^2} \big[B \partial_{z_\alpha} u_\alpha^2 + C \sum_{\beta=1}^{d} u_\beta \partial_{z_\beta} u_\alpha \big] = -\partial_{z_\alpha} p + \sum_{\beta,\gamma=1}^{d} D^\alpha_{\beta,\gamma} \partial_{z_\beta} \partial_{z_\gamma} u_\alpha, \tag{2.45}$$

we define the following density Ψ_t w.r.t. μ_r

$$\Psi_t = Z_t^{-1} \exp \Big[\varepsilon \sum_{x \in \Lambda_L} \sum_{\alpha=1}^{d} \lambda_\alpha(\varepsilon x, t) I_\alpha(x) \Big], \quad \lambda_\alpha = t(\theta)^{-1} u_\alpha. \tag{2.46}$$

Then:

1)
$$\lim_{\varepsilon\to 0}\varepsilon^{-2}s(f_t|\Psi_t)=0. \tag{2.47}$$

2) Defining the empirical fields ν_α^ε for $\alpha=0,\dots,d$, as

$$\nu_0^\varepsilon(x,t)=\varepsilon^d\sum_{x\in\Lambda_L}\delta(z-\varepsilon x)I_0(\eta_x(t)) \tag{2.48}$$

$$\nu_\alpha^\varepsilon(x,t)=\varepsilon^{d-1}\sum_{x\in\Lambda_L}\delta(z-\varepsilon x)I_\alpha(\eta_x(t)),\quad \alpha=1,\dots,d, \tag{2.49}$$

we have that, for $\alpha=1,\dots,d$, ν_α^ε converges weakly in probability to u_α, solution of the Navier-Stokes equation, and ν_0^ε converges to the constant $N\theta$.

3) The matrices $D^\alpha\ge D_s$ are diagonal and have the form

$$D^\alpha_{\beta,\gamma}=\delta_{\beta,\gamma}[D_1\delta_{\alpha,\beta}+D_2(1-\delta_{\alpha,\beta})]$$

Moreover
$$D^\alpha\neq D_s,\quad \alpha=1,\dots,d.$$

The proof of this theorem is given in the Sect.3 and it is based on the Lemmas and Theorems of Sections 4–6.

Remarks.

(i) The matrix D_s is the diffusion matrix which appears in the limiting equations when one consider only the symmetric part of the generator $\mathcal{L}_s=1/2(\mathcal{L}+\mathcal{L}^*)$. The true diffusion matrices D^α are bigger than D_s, as stated in the last part of Theorem 2.1 because of the contribution of the *deterministic* motion.

(ii) The second part of the theorem is a consequence of the relative entropy estimate by means of the following entropy inequality. If f and g are normalized densities w.r.t. $d\mu$, then
$$E^f[X]\le\gamma^{-1}\varepsilon^{-d}s(f\mid g)+\gamma^{-1}\log E^g\left[\exp\{\gamma X\}\right] \tag{2.50}$$
for any positive γ and for any random variable X. See [23, 10] and lemma 2.2 below for details.

(iii) The factor $t(\theta)$ is due to the relation between the momentum density $\bar u$ and its conjugate chemical potential λ at the first order as in (2.31). Here $\bar u=\varepsilon u+O(\varepsilon^2)$.

(iv) We need the specific form of $\mathcal{V}$ only in the proof of the ergodic theorem, Theorem 3.3 in next section. The rest of the proof is valid under quite general conditions. Therefore Theorem 2.1 holds for any model in $d\ge 3$, with $\mathcal{V}$ a finite set of vectors of given length, invariant under reflections and permutation of axes, for which the ergodic Theorem 3.3 is valid.

To conclude, we recall a Lemma from [8]

Lemma 2.2. *Let f be a density satisfying*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2} s(f|\psi) = 0$$

where ψ is a density of a product measure of the type (2.46). Then for any J and any bounded local function F one has

$$\lim_{\varepsilon\to 0}[E^f - E^\psi][\varepsilon^{d-1}\sum_x J(\varepsilon x)\tau_x F] = 0.$$

If instead of $\varepsilon^{-2}s(f|\psi)\to 0$ one has $\varepsilon^{-2}s(f|\psi)\le$ const. then the following bound holds

$$|[E^f - E^\psi][\varepsilon^{d-1}\sum_x J(\varepsilon x)\tau_x F]| \le \text{const.}$$

In particular, ψ can be the equilibrium measure μ_r.

Finally, a direct computation shows that

$$s(\psi) = \varepsilon^d E^{\mu_r}[\psi\log\psi] \le \text{const.}\varepsilon^2$$

so that the previous lemma assures that $|[E^\psi - E^{\mu_r}][\varepsilon^{d-1}\sum_x J(\varepsilon x)\tau_x F]| \le \text{const.}$

3. The relative entropy estimate

The strategy of the proof of Theorem 2.1 is similar to the one used in [8]. The first step is to choose a local equilibrium density, $\tilde\Psi_t$, (and suitable corrections) to compare to the density f_t of the actual time evolution. This density will be chosen in such a way that its parameters to the first order satisfy the hydrodynamic equations, while the second order terms take into account the contribution of the fast modes. More precisely, we put

$$\begin{aligned}\tilde\Psi_t = \tilde Z_t^{-1}\exp\Bigg[&\varepsilon\sum_{x\in\Lambda_L}\sum_{\alpha=1}^{3}(\lambda_\alpha * \hat\omega)(\varepsilon x,t)I_\alpha(\eta_x)\\ &+\varepsilon^2\Bigg(\sum_{x\in\Lambda_L}(\lambda_0 * \hat\omega)(\varepsilon x,t)I_0(\eta_x) + \Phi(\eta)\Bigg)\Bigg]\end{aligned}\quad . \tag{3.1}$$

Here $\tilde Z_t$ is the normalization, $*$ is the convolution product on $\mathbb{Z}^d$, and $\lambda_\alpha(z,t)$, $\hat\omega$ and Φ are chosen as follows.

(3.A) Let ℓ and $k = \ell\epsilon^{-2/d}$ be integers, $\bar\ell = \ell^{d+2}$ and suppose that Λ_k is divided in disjoint cubes of size $(2\bar\ell+1)$, with centers $\sigma\in(2\bar\ell+1)\mathbb{Z}^d$, $|\sigma|\le k$. Let $\bar\ell_1 = \bar\ell - \ell^{1/d^2}$ and consider the cubes $\Lambda_{\bar\ell_1,\sigma}$ and $\tilde\Lambda_k = \bigcup_{|\sigma|\le k}\Lambda_{\bar\ell_1,\sigma}$ is the region Λ_k without corridors of width $2\ell^{1/d^2}$. Define $\hat\omega$ and ω to be the normalized characteristic functions

$$\hat\omega(x) = |\tilde\Lambda_k|^{-1}\mathbb{1}(x\in\tilde\Lambda_k), \quad \omega(x) = (2k+1)^{-d}\mathbb{1}(x\in\Lambda_k).$$

(3.B) The chemical potential λ_α, $\alpha = 1,\dots,d$, are solutions of

$$\begin{aligned} \operatorname{div}\lambda \equiv \sum_{\alpha=1}^d \partial_{z_\alpha}\lambda_\alpha = 0 \\ \frac{\partial}{\partial t}\lambda_\alpha = -\frac{1}{t(\theta)}\partial_{z_\alpha}p - \frac{h(\theta)}{t(\theta)}\Big\{B\partial_{z_\alpha}\lambda_\alpha^2 + C\sum_{\beta=1}^d\lambda_\beta\partial_{z_\beta}\lambda_\alpha\Big\} + \sum_{\beta,\gamma=1}^d D^\alpha_{\beta\gamma}\partial^2_{z_\beta z_\gamma}\lambda_\alpha \end{aligned} \tag{3.2}$$

with $D^\alpha_{\beta\gamma}$ given as in Theorem 2.1,

$$p(\theta,\lambda_0,\lambda_\alpha) = t(\theta)(1-2\theta)\lambda_0 + Ah(\theta)\lambda^2,$$

$$t(\theta) = K\theta(1-\theta), \quad h(\theta) = \frac{\theta}{2}(1-\theta)(1-6\theta(1-\theta)).$$

(3.C) Choice of Φ: $\Phi(\eta) = -\sum_{x\in\Lambda_L}\sum_{\alpha,\beta=1}^d \partial_\alpha\lambda_\beta(\varepsilon x,t)(\hat\omega * \tau_y F_\alpha^\beta)$ where F_α^β are local functions satisfying the conditions in Definition 4.2 of the next section.

The choice of $\hat\omega$ is for the convenience that no boundary terms arise in the multiscale analysis; one can simply use ω but a few steps are needed to bound the boundary terms. The multi-scale analysis is an important tool in proving the Theorem 4.6 of [8]. Its extension to the present setup will be used extensively here.

For the convenience of later reference, recall for our models the currents are given by

$$w^{*,\alpha}_{x,\beta} = w^{(s)\ \alpha}_{x,\beta} - w^{(a)\ \alpha}_{x,\beta}, \qquad w^{(s)\ \alpha}_{x,\beta} = \chi\nabla_\beta I_\alpha \tag{3.3}$$

and

$$\begin{aligned} w^{(a)\ \alpha}_{x,\beta} &= \sum_{v\in\mathcal{V}}(e_\alpha\cdot v)(e_\beta\cdot v)b_{x,\beta}(v), \\ w^{(a)\ 0}_{x,\beta} &= \sum_{v\in\mathcal{V}_+}(e_\beta\cdot v)b_{x,\beta}(v) \end{aligned} \tag{3.4}$$

where

$$b_{x,\beta}(v) = \eta(x+e_\beta,v)\eta(x,v) - \frac{1}{2}\Big[\eta(x+e_\beta,v)+\eta(x,v)\Big]$$

Furthermore, the density, velocity, stress tensor and the pressure are related by

$$
\begin{aligned}
\rho = N\theta + O(\varepsilon^2), \quad \theta = \frac{e^r}{1+e^r}, \quad u_\alpha = \varepsilon t(\theta)\lambda_\alpha + O(\varepsilon^3) \\
\pi^{\alpha\beta} = \big(t(\theta) + \varepsilon^2 p(\theta,\lambda)\big)\delta_{\alpha,\beta} + \varepsilon^2 h(\theta)\Big(B\lambda_\alpha^2\delta_{\alpha,\beta} + C\lambda_\alpha\lambda_\beta\Big).
\end{aligned}
\tag{3.5}
$$

We return to the proof Theorem 2.1. The density $\tilde\Psi_t$ differ from Ψ_t only at the second order in ε. The following Lemma shows that the difference of $s(f|\Psi)$ and $s(f|\tilde\Psi)$ is small.

Lemma 3.1. *Suppose that f is a probability measure on Λ_L. Let*

$$\phi = Z^{-1}\exp\Big[\sum_{x\in\Lambda_L}\sum_{\alpha=1}^{d}\varepsilon\ S_\alpha(\varepsilon x,t)I_\alpha(\eta_x)\Big]$$

$$\phi' = Z'^{-1}\exp\Big[\sum_{x\in\Lambda_L}\sum_{\alpha=1}^{d}\big(\varepsilon\ S_\alpha(\varepsilon x,t)I_\alpha(\eta_x) + \varepsilon^2 t_\alpha(\varepsilon x,t)\tau_x F_\alpha\big)\Big]$$

two densities w.r.t. μ_r where S_α and t_α are smooth functions of (z,t) and F_α local functions of the configuration such that $E[F_\alpha] = 0$. If $\varepsilon^{-2}s(f|\phi)\to 0$, then

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}\Big[s(f|\phi) - [s(f|\phi')\Big] = 0 \tag{3.6}$$

The proof is an obvious extension of the one given in [8].

From this remark, to prove Theorem 2.1 we shall only prove

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}s(f_t \mid \tilde\Psi_t) = 0. \tag{3.7}$$

To state the main Theorem of this section, we need the following definition.

Definition 3.2. *Let G be a local function such that $E^{\mu_r}[G] = 0$. For any vector $m = \{m_\alpha\}_{\alpha=0}^d$ we define the "variance" $V_\ell(G,m)$ by*

$$V_\ell(G,m) = (2\ell_1+1)^{-d}\big\langle[\sum_{|x|\le\ell_1}(\tau_x G - \alpha_\ell(G))](-\mathcal{L}_{s,\ell})^{-1}[\sum_{|x|\le\ell_1}(\tau_x G - \alpha_\ell(G))]\big\rangle_{\mu_{\ell,m}} \tag{3.8}$$

where $\ell_1 = \ell - \ell^{1/d^2}$ and $\mu_{\ell,m}$ is the canonical Gibbs state of $(2\ell+1)^d$ sites with parameters such that the mean of I_α, $\alpha\ge 0$, on the block Λ_ℓ are equal to m_α, namely,

$$\mu_{\ell,m} = \Xi_{\ell,m}^{-1}\prod_{\alpha=0}^{d}\delta(\bar I_{\alpha,\ell} - m_\alpha)\mu_\ell.$$

Here μ_ℓ denotes the counting measure on the configurations in Λ_ℓ, $\Xi_{\ell,m}$ is the normalization and

$$\alpha_\ell(G) = E^{\mu_r}[G|\bar{I}_\ell^+]$$

where $E^{\mu_r}[G|\bar{I}_\ell^+]$ is the conditional expectation given the averages

$$\bar{I}_{\alpha,\ell} = \frac{1}{(2\ell+1)^d} \sum_{|y|\leq\ell} I_\alpha(\eta_y) \qquad \alpha = 0, \ldots, d \tag{3.9}$$

The generator $\mathcal{L}_{s,\ell}$ is the symmetric part of the generator restricted to Λ_ℓ, in the sense that the sums in (2.2) and (2.4) are restricted to the sites and bonds contained in Λ_ℓ. We define also the "variance" of G by

$$V(G,r) = \limsup_{\ell\to\infty} E^{\mu_r}[V_\ell(G,\bar{I}_\ell)].$$

In order to make above definition meaningful, one needs to give sense to $\mathcal{L}_{s,\ell}^{-1}$. This is consequence of the finite volume ergodicity of the generator, summarized in the next Theorem 3.3. We will use the following notation: given a configuration η on a box Λ_ℓ of size ℓ, we put

$$N_\alpha(\eta) = \sum_{x\in\Lambda_\ell} I_\alpha(\eta_x), \;\; \alpha = 0, \ldots, d.$$

Moreover, for any choice of m_α, we denote by $\Omega_{\ell,m}$ the set of configurations η in Λ_ℓ such that $N_\alpha(\eta) = m_\alpha(2\ell+1)^d$ for $\alpha = 0, \ldots, d$. Note that there are many choices of the m_α's such that $\Omega_{\ell,m}$ is empty. We consider only the non trivial choices.

We have the following

Theorem 3.3. *The process generated by $\mathcal{L}_{s,\ell}$ on $\Omega_{\ell,m}$ endowed with the measure $\mu_{\ell,m}$ is ergodic.*

Proof: Since the process is a finite Markov chain, it is sufficient to show that it is transitive, in the sense that, given any couple of configurations η and $\bar{\eta}$ in $\Omega_{\ell,m}$, it is possible to find a sequence of jumps and collisions which transform η into $\bar{\eta}$.

We first consider Model I.

Let

$$N_\alpha^-(\eta) = \sum_{\xi\in\Lambda_\ell} [\eta(x,e_\alpha) + \eta(x,-e_\alpha)], \;\; \alpha = 1, \ldots, d, \quad \sum_{\alpha=1}^{d} N_\alpha^-(\eta) = N_0(\eta)$$

Step 1: If $\eta, \bar{\eta} \in \Omega_{\ell,m}$, then $N_\alpha^-(\eta) - N_\alpha^-(\bar{\eta})$ is even.

Proof of step 1: If $N_\alpha^-(\eta) - N_\alpha^-(\bar{\eta})$ were odd, then there is at least $x \in \Lambda_\ell$ such that $\eta(x,\varepsilon_\alpha) - \bar{\eta}(x,\varepsilon_\alpha) = \pm 1$ and $\eta(x,-\varepsilon_\alpha) - \bar{\eta}(x,-\varepsilon_\alpha) = 0$ or viceversa. Hence $[\eta(x,\varepsilon_\alpha) - \eta(x,-e_\alpha)] - [\bar{\eta}(x,\varepsilon_\alpha) - \bar{\eta}(x,-e_\alpha)] = \pm 1$. But $N_\alpha(\eta) = N_\alpha(\bar{\eta})$, so for each such x there is an $y \in \Lambda_\ell$ such that $[\eta(y,\varepsilon_\alpha)-\eta(y,-e_\alpha)]-[\bar\eta(y,\varepsilon_\alpha)-\bar\eta(y,-e_\alpha)] = \mp 1$, to compensate the discrepancy of momenta in x. But this means that $N_\alpha^-(\eta) - N_\alpha^-(\bar\eta)$ is not odd.

Step 2: Let $N_\alpha^-(\eta) - N_\alpha^-(\bar\eta) = 2k_\alpha$. If $k_\alpha > 0$ for some $\alpha = 1,\dots,d$, then there is a sequence of jumps and collisions which transform η into $\bar\eta$.

Proof of step 2: To simplify the argument, let $d = 3$. We have $N_0(\eta) = N_0(\bar\eta)$, hence $\sum_{\alpha=1}^3 k_\alpha = 0$. Without loss of generality we can assume $k_1 < 0$ and $k_2 \geq 0$, $k_3 \geq 0$. Therefore, with a suitable sequence of jumps, we can transform η into a configuration η_1 with $N_\alpha(\eta_1) = N_\alpha(\eta)$ and $N_\alpha^-(\eta_1) = N_\alpha^-(\eta)$ such that there are at least $-k_1$ sites $x_1,\dots,x_{-k_1}$ where there is no particle both with velocity e_1 and with velocity $-e_1$. In the same way, one can construct a configuration η_2 such that $N_\alpha(\eta_2) = N_\alpha(\bar\eta)$ and $N_\alpha^-(\eta_2) = N_\alpha^-(\bar\eta)$ such that there are k_2 sites $y_1,\dots,y_{k_2}$ without particles with velocity e_2 and $-e_2$ and k_3 sites $y_{k_2+1},\dots,y_{k_2+k_3}$ without particles with velocities e_3 and $-e_3$. Since $k_2+k_3 = -k_1$, there is no loss of generality assuming that the two sets of point coincide, because otherwise extra jumps can be done to reduce to this situation. Then we are in position to make a collision in each of the sites x_i, transforming η_1 into η_2.

Step 3: If $k_\alpha = 0$ for each $\alpha = 1,\dots,3$ then a sequence of jumps is sufficient to transform η into $\bar\eta$.

Proof of step 3: In fact, if $k_\alpha = 0$, $\sum_{x\in\Lambda_\ell}\eta(x,v) = \sum_{x\in\Lambda_\ell}\bar\eta(x,v)$ for each $v \in \mathcal{V}$. Therefore jumps are enough to transform η into $\bar\eta$.

As for Model II, we sketch the proof. Define $\eta \sim \zeta$ iff there exist a collision $q \in \mathcal{Q}$ and a site $y \in \Lambda_L$ such that $Q_y^q\eta = \zeta$. This defines a equivalence relation and we say η and ζ are connected if they are in the same equivalence class. If η is a configuration in 0, we will also denote by η the subset of $\mathcal{V}$ such that $\eta(v) = 1$.

Lemma 3.4 *For any two configurations η and ζ at 0, η and ζ are connected iff $I^+(\eta) = I^+(\zeta)$.*

Proof: *Step 1* We claim first the same result holds on the space $\mathcal{W} = \{(\pm 1, \pm 1)\}$. This can be checked easily.

Step 2 Next, we claim Lemma 3.4 holds if we restrict ourselves to the subspace $\mathcal{V}_2^{(3)} = \{(\pm 1, \pm 1, \pm\varpi)\}$. This can be checked easily too. We give a brief sketch. From the previous claim, the projection of η and ζ onto the space $\mathcal{W}$ are connected. Let $\eta_3^\pm$ = the number of $\{v \in \eta$ with $v_3 = \pm\varpi\}$. Since the momenta of η and ζ are the same, $\eta_3^\pm = \zeta_3^\pm$. Hence we only have to prove the following:

Suppose $\{u, v\} \subset \eta$ with $u \neq v$. Let $\zeta = (\eta \setminus \{u,v\}) \cup \{(u_1,u_2,v_3),(v_1,v_2,u_3)\}$. Then ζ and η are connected.

But $u + v = (u_1,u_2,v_3)+(v_1,v_2,u_3)$, hence we have proved the claim. This proposition is obvious and will be used repeatedly in the following proof. It basically means that the third component can be changed freely in $\mathcal{V}_2^{(3)}$.

Step 3 Suppose η and ζ are two configurations in $\mathcal{V}_2^{(3)}$ with the same momentum but not the same number of particles. Suppose $I_0(\eta) > I_0(\zeta)$. We claim that η is connected to $\zeta \cup A$ for some A satisfying that $A \cap \zeta = \phi$ and A is reflexive, i.e., if $v \in A$ then $-v \in A$. We first claim that $I_0(\eta) - I_0(\zeta) = 2n$ for some $n \in \mathcal{N}$ since otherwise $I_3(\eta) \neq I_3(\zeta)$. We now follows previous arguments by first considering the projection onto the space $\mathcal{W}$ and then adjusting the third components. In more details, we first prove our claim in the two dimensional space $\mathcal{W}$. This can be checked directly. We can then use the remark from the step 2 to adjust the third components. This proves the claim.

Step 4 Finally, the general case. Let $\eta^{(i)} = \{v \in \eta : v_i = \pm\varpi\}$. Clearly, $I_\alpha(\eta^{(i)}) = I_\alpha(\zeta^{(i)})$ for $\alpha > 0$. Hence $\eta^{(i)}$ and $\zeta^{(i)}$ differ only for pairs of opposite vectors. Since any pair of opposite vectors can be brought into any different pair of opposite vectors by a collision, it is not hard to check η and ζ are connected.

We now consider the general cases:

Lemma 3.5 *For any two configurations η and ζ on Λ_ℓ, η and ζ are connected iff $I^+(\eta) = I^+(\zeta)$.*

Proof. The proof follows almost exactly the same arguments as in the previous lemma except some explanations are needed for the use of the simple exclusion. Let us illustrate it by proving the step 1 carefully. We are now in the setting of two dimensional configurations η and ζ with the same momentum and number of particles. We now bring particles with opposite velocities to the same sites as much as possible. We can then decompose η as union of two configurations η^{pairs}, consisting of pairs of opposite vectors at the same sites, and the rest, $\eta^{singles}$. Since we have only four directions and no pairs are allowed, we can assume, without loss of generality, with obvious changes to cover the different situations, that $\eta^{singles}$ contains α particles in the direction $(1,1)$ and β particles in the direction $(1,-1)$ and nothing else. Similarly, we can decompose ζ in the same way. Since $(1,1)$ and $(1,-1)$ are linearly independent, we can check that $\zeta^{singles}$ contains exactly the same number of particles in the directions $(1,1)$ and $(1,-1)$ and nothing else. Hence η^{pairs} and ζ^{pairs} contains the same number of pairs because the total number of particles of η and ζ are the same. We can connect η^{pairs} and ζ^{pairs} by simple exclusions and collisions and we can also connect $\eta^{singles}$ and $\zeta^{singles}$ by simple exclusions. This concludes the proof of step 1.

Step 2 is obvious. We now prove the claim in step 3. First we check if instead of having the same number of particles and momentum in step 1, η and ζ only have the same momentum (note they are now configurations on $\mathcal{W}$). From previous arguments we still have that $\eta^{singles}$ and $\zeta^{singles}$ have the same number of particles. Hence η and ζ can be brought into configurations η' and ζ' such that $\eta' \setminus \zeta'$ consists of pairs (assuming $I_0(\eta) \geq I_0(\zeta)$). With this comment, we can prove step 3 easily. Finally, step 4 is by now straightforward.

The main Theorem of this section is

Theorem 3.6 *Let $\tilde{\Psi}_t$ be defined as above, $k = \ell\varepsilon^{-2/d}$ and f_t as in (2.41). Then, for $d \geq 3$, and for any $\gamma > 0$*

$$\begin{aligned} &\lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \varepsilon^{-2} s(f_T|\tilde{\Psi}_T) \\ &\leq C \lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \int_0^T \varepsilon^{-2} s(f_t|\tilde{\Psi}_t)dt + \frac{CT}{\gamma}\sum_{\alpha,\beta=1}^d V(H_\alpha^\beta, r) + \text{ const.}\gamma \end{aligned} \tag{3.10}$$

Here C is a positive constant, V is defined in (3.8) and

$$H_\alpha^\beta \equiv H_{0,\alpha}^\beta. \qquad H_{x,\alpha}^\beta = w_{x,\alpha}^{(a)\ \beta} - \sum_{\gamma=1}^{d} \tilde{D}_{\alpha\gamma}^\beta \partial_\gamma I_\beta(\eta_x) - \mathcal{L}^* \tau_x F_\alpha^\beta \tag{3.11}$$

where

$$\tilde{D}_{\alpha\gamma}^\beta = D_{\alpha\gamma}^\alpha - \chi\delta_{\alpha,\gamma}.$$

We need the following lemmas. The first one is a large deviation bound; the other two are bounds on the entropy and the Dirichlet form.

Lemma 3.7. *Let ψ be the density given by (2.46) and let, for $\alpha = 0, \ldots, d$,*

$$\overline{m}_{x,k}^\alpha = E^\psi[\bar{I}_{\alpha,k}(x)]; \quad \bar{I}_{\alpha,k}(x) = (2k+1)^{-d} \sum_{|x-y|\le k} I_\alpha(\eta_x)$$

$$\zeta_{x,k}^\alpha = E^\psi[\bar{I}_{\alpha,k}(x); \bar{I}_{\alpha,k}(x)] \equiv E^\psi[(\bar{I}_{\alpha,k}(x) - \overline{m}_{x,k}^\alpha)^2]$$

where $E[A;B] = E[AB] - E[A]E[B]$.
Then, for $k = \ell\varepsilon^{-2/d}$ and for any $q < q_0$ with q_0 a small fixed constant,

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \varepsilon^{d-2} \log E^\psi \left[\exp\left\{ q \sum_x \left[(\bar{I}_{\alpha,k}(x) - \overline{m}_{x,k}^\alpha(v))^2 - \zeta_{x,k}^\alpha \right] \right\} \right] = 0 \tag{3.12}$$

In general, if $G_x = G_x(y_0, \cdots, y_d)$ is a family of bounded smooth functions such that

$$G_x(\overline{m}_{x,k}) = \frac{\partial G_x}{\partial y_\alpha}\Big|_{y=\overline{m}_{x,k}} = 0, \quad \alpha = 0, \ldots, d,$$

then for any smooth function J, there is q_0 such that for any $q < q_0$

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \varepsilon^{d-2} \log E^\psi \left[\exp\left\{ q \sum_x J(\varepsilon x) \left\{ G_x(\bar{I}_{\alpha,k}(x)) - E^\psi[G_x(\bar{I}_{\alpha,k}(x))] \right\} \right\} \right] = 0. \tag{3.13}$$

This proof of Lemma 3.7 is similar to the one for the corresponding lemma in [8]. We omit the details.

Lemma 3.8 *The density f_t satisfies the following bounds:*

(i)

$$\frac{d}{dt} s(f_t) \le -\text{const.}\varepsilon^{-2+d} D_{\Lambda_L}(\sqrt{f_t}),$$

where $D_{\Lambda_L}(g) = (D^{ex}_{\Lambda_L} + D^c_{\Lambda_L})(g)$ *is the Dirichlet form. Here*

$$D^{ex}_{\Lambda_L}(g) = \sum_{x\in\Lambda_L}\sum_{v\in\mathcal{V}}\sum_{\alpha=1}^{d}\chi\int[\nabla^v_{x,\alpha}g(\eta)]^2 d\mu_r(\eta), \tag{3.14}$$

with

$$\nabla^v_{x,\alpha}g(\eta) = g(\eta^{x,x+e_\alpha,v}) - g(\eta)$$

and $D^c_{\Lambda_L}(g)$ *is given by*

$$D^c_{\Lambda_L}(g) = \sum_{x\in\Lambda_L}\sum_{q\in\mathcal{Q}}\int[(Q^q_x g)(\eta)]^2 d\mu_r(\eta).$$

(ii)] For any $t \geq 0$, $\varepsilon^{-2}s(f_t) \leq \text{const.}$
(iii) For any local function F *and any* J,

$$|[E^{f_t} - E^{\mu_r}][\varepsilon^{d-1}\sum_x J(\varepsilon x)\tau_x F]| \leq \text{const.}$$

Lemma 3.9. *The relative entropy* $s(f_t|\tilde{\Psi}_t)$ *satisfies the bound*
(i)

$$\frac{d}{dt}s(f_t|\tilde{\Psi}_t) \leq \varepsilon^d \int f_t\tilde{\Psi}_t^{-1}(\varepsilon^{-2}\mathcal{L}^* - \frac{\partial}{\partial t})\tilde{\Psi}_t d\mu_r.$$

(ii) There is a constant c_t *independent of* f_t *such that*

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}[\frac{d}{dt}s(f_t|\tilde{\Psi}_t) - \varepsilon^d\int f_t(\varepsilon^{-2}\mathcal{L}^* - \frac{\partial}{\partial t})\log\tilde{\Psi}_t d\mu_r] - c_t \leq 0.$$

(iii)

$$\int \mathcal{L}^*\tilde{\Psi}_t d\mu_\beta = 0 = \int\frac{\partial}{\partial t}\tilde{\Psi}_t d\mu_r.$$

Lemma 3.8 is obtained by adapting arguments from [23]. The first equation of Lemma 3.9 is proven in [2, 10]. The last equation is obvious. The second equation differs from the first one only because the term $\mathcal{L}^*\tilde{\Psi}_t$ is replaced by $\mathcal{L}^*\log\tilde{\Psi}_t$. From Lemma 2.2, their difference is small. Details were given in [8].

*From now on we use the notation "*const.*" for any constant independent of* f_t *and of the configuration (but may depend on the initial state and the parameters* λ*).* We shall not compute these constants at each step but will use equation (iii) of Lemma 3.9 to determine the final constant.

By Lemma 3.9 to estimate the derivative of the relative entropy it is enough to compute the quantity

$$\mathcal{I} \equiv \varepsilon^{d-2} E^{f_t}[(\varepsilon^{-2}\mathcal{L}^* - \frac{\partial}{\partial t}) \log \tilde{\Psi}_t d\mu_r].$$

We first compute $\varepsilon^{d-4}\mathcal{L}^* \log \tilde{\Psi}_t$:

$$\begin{aligned}
\varepsilon^{d-4}\mathcal{L}^* \log \tilde{\Psi}_t =& \varepsilon^{d-4}(\mathcal{L}^{ex*}(\log \tilde{\Psi}_t - \varepsilon^2 \Phi) + \varepsilon^{d-2}\mathcal{L}^*\Phi \\
=& \varepsilon^{d-3} \sum_x \sum_{\alpha,\beta=1}^d (\lambda_\alpha * \hat{\omega})(\varepsilon x, t) \nabla_\beta^- w_{x,\beta}^{*\alpha} \\
&+ \varepsilon^{d-2} \sum_x \sum_{\alpha=1}^d (\lambda_0 * \hat{\omega})(\varepsilon x, t) \nabla_\beta^- w_{x\alpha}^{*0} + \varepsilon^{d-2}\mathcal{L}^*\Phi = A_1 + A_2 + A_3
\end{aligned} \tag{3.15}$$

The term A_2 is easier than A_1 because of the factor ε^{d-2}. We first sum by parts. The difference operator is thus moved to act on $\lambda_0 * \hat{\omega}$. Recall the expression of the current given by (3.3). We can separate the current into the symmetric part and the asymmetric part. Let $A_2 = B_1 + B_2$ with B_1 (B_2 resp.) related to the symmetric (asymmetric resp.) part. Hence

$$B_1 = \chi \varepsilon^{d-2} \sum_x \sum_{\beta=1}^d \nabla_\beta^- \nabla_\beta \lambda_0 [\hat{\omega} * I_0]_x$$

$$B_2 = \varepsilon^{d-2} \sum_x \sum_{\alpha=1}^d \nabla_\alpha^- \lambda_0(\varepsilon x, t)(\hat{\omega} * w_{x\alpha}^{(a)\ 0})$$

where $[\hat{\omega} * I_\alpha]_x = \sum_y \hat{\omega}(x-y) I_\alpha(\eta_y)$. From the Taylor expansion we can approximate $\nabla_\beta^- \nabla_\beta$ by $\varepsilon^2 \partial_{z_\beta} \partial_{z_\beta}$ with a negligible lower order term.

Since our density f_t is very close to the equilibrium, we can compute B_1 to great accuracy. Because $E^{\mu_r}[B_1] = 0$, we have from (ii) and (iii) of Lemma 3.4

$$|E^f(B_1)| = |E^f(B_1) - E^{\mu_r}[B_1]| \le \varepsilon.$$

Hence B_1 is negligible.

We now examine the difficult term A_1.

$$\begin{aligned}
A_1 =& -\varepsilon^{d-3} \sum_x \sum_{\alpha,\beta=1}^d (\nabla_\beta \lambda_\alpha)(\varepsilon x, t)[\hat{\omega} * w_{y,\alpha}^{*\beta}]_x \\
=& -\chi \varepsilon^{d-3} \sum_x \sum_{\alpha,\beta=1}^d \nabla_\beta^- \nabla_\beta \lambda_\alpha (\hat{\omega} * I_\alpha)_x - \varepsilon^{d-3} \sum_x \sum_{\alpha,\beta=1}^d \nabla_\beta \lambda_\alpha [\hat{\omega} * w_{x,\alpha}^{(a)\ \beta}]_x
\end{aligned}$$

As remarked after (3.15), we can replace $\varepsilon^{-2}\nabla_\alpha^-\nabla_\alpha$ by $\partial_{z_\alpha}\partial_{z_\alpha}$ up to a negligible error term.

To deal with the currents appearing in A_1 and in B_2, we need to look at their expectations w.r.t. suitable grandcanonical measures modeled according to empirical averages of the conserved quantities. To this end, for any $Y = (Y_0, \dots, Y_d)$, let $\mu_{1,Y}$ be the grandcanonical measure such that

$$E^{\mu_{1,Y}}[I_\alpha] = Y_\alpha, \quad \alpha = 0, \cdots, d \tag{3.16}$$

The measure $\mu_{1,Y}$ is uniquely defined because it is required to be an invariant measure of the dynamics (exclusion + collision). Explicitly, $\mu_{1,w}$ is the product measure with marginal at one point point given by

$$\mu_{1,Y}(\eta) = Z_{r,n}^{-1} \exp\{\sum_{\alpha=0}^{d} n_\alpha^{(Y)} I_\alpha(\eta)\}\mu_r(\eta), \tag{3.17}$$

for $\eta = \{\eta(v), v \in \mathcal{V}\}$. The chemical potentials $n_\alpha^{(Y)}$ are suitably chosen to give the correct averages (3.16). The subscript 1 is used because we will need the grandcanonical measures for the symmetric simple exclusion which will be introduced later on and carries a subscript 2.

Given an integer $k > 0$ for any configuration η in Λ_L, we consider the local empirical averages $\bar{I}_{\alpha,k}(x)$, $\alpha = 0, 1, \dots, d$, defined in (3.9). The measure $\mu_{1,\bar{I}_k^+(x)}$ is defined according to (3.17), with $Y_\alpha = \bar{I}_{\alpha,k}(x)$. $n_\alpha^{(Y)} = n_\alpha^{(\bar{I}_k(x))}$, $\alpha = 0, \dots, d$ are the corresponding chemical potentials; here $\bar{I}_k^+(x) = \{\bar{I}_{\alpha,k}(x), \alpha = 0, \dots, d\}$. Moreover we put

$$f(v, n^{(Y)}) := \frac{e^{n_0^{(Y)} + \underline{n}^{(Y)} \cdot v}}{1 + e^{n_0^{(Y)} + \underline{n}^{(Y)} \cdot v}}. \tag{3.18}$$

With above notations, we have, for $Y_\alpha = \bar{I}_{\alpha,k}(x)$

$$\begin{aligned} \bar{I}_{0,k}(x) &= E^{\mu_{1,\bar{I}_k(x)}}[I_0] = \sum_{v \in \mathcal{V}} f(v, n^{(Y)}), \\ \bar{I}_{\alpha,k}(x) &= E^{\mu_{1,\bar{I}_k(x)}}[I_\alpha] = \sum_{v \in \mathcal{V}} (v \cdot e_\alpha) f(v, n^{(Y)}) \end{aligned} \tag{3.19}$$

We can now define

$$\begin{aligned} g_{x,\beta}^\alpha &= (\hat{\omega} * w_{y,\beta}^{(a)\ \alpha})_x - E^{\mu_{1,\bar{I}_k(x)}}[(\hat{\omega} * w_{y,\alpha}^{(a)})_x]; \\ g_{x,\alpha}^0 &= (\hat{\omega} * w_{y,\alpha}^{(a)\ 0})_x - E^{\mu_{1,\bar{I}_k(x)}}[(\hat{\omega} * w_{y,\alpha}^{(a)\ 0})_x] \quad, \alpha = 1, \cdots, d \end{aligned} \tag{3.20}$$

From previous definitions we have,

$$\begin{aligned}
E^{\mu_{1,Y}}[(\hat{\omega} * w^{(a)\ \alpha}_{y,\beta})_x] &= \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)(v\cdot e_\beta)\Big\{[f(v,n^{(Y)})]^2 - f(v,n^{(Y)})\Big\} \\
&:= -\pi^{\alpha\beta}(Y) \\
E^{\mu_{1,Y}}[(\hat{\omega} * w^{(a)\ 0}_{y,\alpha})_x] &= \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)\Big\{[f(v,n^{(Y)})]^2 - f(v,n^{(Y)})\Big\} \\
&:= -\pi^0_\alpha(Y)
\end{aligned} \tag{3.21}$$

The equations (3.21) are not explicit in terms of the conserved quantities. To obtain an explicit expression one has to solve (3.19). For our purpose, some approximation will be sufficient. This will be carried out later on in this section.

Summarizing, we have

$$\begin{aligned}
A_1 = \chi\varepsilon^{d-1}\sum_x \sum_{\alpha,\beta=1}^d \partial^2_{z_\alpha z_\alpha}\lambda_\beta[\hat{\omega} * I_\beta]_x - \varepsilon^{d-3}\sum_x\sum_{\alpha,\beta=1}^d (\nabla_\alpha\lambda_\beta)(\varepsilon x,t)\pi^{\alpha\beta}(\bar{I}^+_k(x)) \\
+ \varepsilon^{d-3}\sum_x\sum_{\alpha,\beta=1}^d (\nabla_\alpha\lambda_\beta)(\varepsilon x,t) g^\beta_{x,\alpha} + o(1).
\end{aligned}$$

Let

$$D^\beta_{\alpha,\gamma} = \tilde{D}^\beta_{\alpha,\gamma} + \delta_{\alpha,\gamma}\chi. \tag{3.22}$$

We can rewrite

$$\begin{aligned}
A_1 &= \varepsilon^{d-1}\sum_x\sum_{\alpha,\beta,\gamma=1}^d D^\beta_{\alpha,\gamma}(\partial_{z_\alpha}\partial_{z_\gamma}\lambda_\beta)(\varepsilon x,t)[\hat{\omega} * I_\beta]_x \\
&+ \varepsilon^{d-3}\sum_{x,\alpha,\beta=1}^d (\partial_{z_\alpha}\lambda_\beta)(\varepsilon x,t)\Big[g^\beta_{x\alpha} - \sum_{\gamma=1}^d \tilde{D}^\beta_{\alpha,\gamma}\Big(\hat{\omega} * \nabla_\gamma I_\beta(\eta_y)\Big)_x\Big] \\
&- \varepsilon^{d-3}\sum_x\sum_{\alpha,\beta=1}^d (\nabla_\alpha\lambda_\beta)(\varepsilon x,t)\pi^{\alpha\beta}(\bar{I}^+_k(x)) + o(1).
\end{aligned} \tag{3.23}$$

Similarly, $B_2 = C_3 + B_3$ where

$$\begin{aligned}
C_3 &= \varepsilon^{d-1}\sum_{x,\alpha}\partial_{z_\alpha}\lambda_0 g^0_{x,\alpha} \\
B_3 &= \varepsilon^{d-2}\sum_{x,\alpha}\nabla_\alpha\lambda_0\pi^0_\alpha(\bar{I}^+_k(x))
\end{aligned}$$

From (3.15), (3.23) we rearrange the remaining terms as

$$A_1 + A_2 + A_3 = C_1 + C_2 + C_3 + B_3 + o(1)$$

where

$$C_1 = \varepsilon^{d-3} \sum_x \sum_{\alpha,\beta=1}^{d} (\partial_{z_\alpha} \lambda_\beta)(\varepsilon x, t) \Big[g^\beta_{x\alpha} - \sum_{\gamma=1}^{d} \tilde{D}^\beta_{\alpha,\gamma} \Big(\hat{\omega} * \nabla_\gamma I_\beta(\eta_y) \Big)_x - \Big(\hat{\omega} * \mathcal{L}^*(\tau_y F^\beta_\alpha) \Big)_x \Big], \tag{3.24}$$

and

$$C_2 = \sum_x \sum_{\alpha,\beta=1}^{d} \Big[\varepsilon^{d-1} \sum_{\gamma=1}^{d} D^\beta_{\alpha,\gamma} (\partial_{z_\alpha} \partial_{z_\gamma} \lambda_\beta)(\varepsilon x, t) \bar{I}_{\beta,k}(x) - \varepsilon^{d-3} (\nabla_\alpha \lambda_\beta)(\varepsilon x, t) \pi^{\alpha\beta}(\bar{I}^+_k(x)) \Big] \tag{3.25}$$

We also need to compute

$$\begin{aligned} & - \varepsilon^{d-2} E^f \{ \frac{\partial}{\partial t} \log \tilde{\Psi}_t \} \\ & = -E^f \big[\varepsilon^{d-1} \sum_x \sum_{\alpha=1}^{d} \frac{\partial}{\partial t} \lambda_\alpha(\varepsilon x.t) \bar{I}_{\alpha,k}(x) + \varepsilon^d \sum_x \frac{\partial}{\partial t} \lambda_0(\varepsilon x, t) \bar{I}_{0,k}(x) \big] + \text{const.} \end{aligned}$$

Using an argument similar to the one explained in bounding B_1, one can prove that the second term on the right side is negligible up to a constant.

Summarizing these computations, we have

$$\mathcal{I} = E^f [C_1 + C_3 + C_4] + \text{const.} + o(1) \tag{3.26}$$

where

$$C_4 = \varepsilon^{d-2} \sum_x \Gamma_x(\bar{I}^+_k(x)) \tag{3.27}$$

and

$$\begin{aligned} \Gamma_x(Y) = \varepsilon \sum_{\alpha=1}^{d} \Big\{ \Big(-\frac{\partial}{\partial t} \lambda_\alpha + \sum_{\beta,\gamma=1}^{d} D^\beta_{\alpha\gamma} \partial_{z_\beta} \partial_{z_\gamma} \lambda_\alpha \Big)(\varepsilon x, t) Y_\alpha \\ - \varepsilon^{-2} \sum_{\beta=1}^{d} \nabla_\alpha \lambda_\beta(\varepsilon x, t) \pi^{\alpha\beta}(Y) - \varepsilon^{-1} \nabla_\alpha \lambda_0(\varepsilon x, t) \pi^0(Y) \Big\} \end{aligned} \tag{3.28}$$

The terms $E^f[C_1]$ and $E^f[C_3]$ are dealt with by using the following theorem.

Theorem 3.10. *Suppose h is a local function, J is a smooth function and ψ is a local Gibbs state defined as in (3.1). For any $\gamma > 0$*

$$
\begin{aligned}
\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\Bigg\{ & \varepsilon^{d-2}\int f \sum_{x\in\Lambda_L} J(\varepsilon x)\{(\hat\omega * h)(x) - E^{\mu_{1,\bar{l}_k^+(x)}}[h]\}d\mu_\beta \\
& - \text{const. } \gamma^{-1}\int J(z)^2 dz V(h,r) - \gamma\varepsilon^{d-4}D_{\Lambda_L}(\sqrt{f}) - \delta^{-1}\varepsilon^{-2}s(f|\psi)\Bigg\} \le 0 \ ,
\end{aligned}
\tag{3.29}
$$

provided δ is small enough. Here $\bar\ell = \ell^{d+2}$, $k = \ell\varepsilon^{-2/d}$ and V is defined in (3.8).

Corollary 3.11. *For any $\gamma > 0$, any f and any Gibbs state ψ such that $\varepsilon^{-2}s(f|\psi)$ is bounded,*

$$
\begin{aligned}
& \lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\varepsilon^{d-2}\int \sum_{x\in\Lambda_L} J(\varepsilon x)\left\{\hat\omega * (\tau_y h - E^{\mu_{1,\bar{l}_k^+(x)}}[h]\right\}(x) f d\mu_r \\
& \le \frac{1}{2\gamma}\int J^2(z)dz\ V(h,r) + \gamma\lim_{\varepsilon\to 0}\varepsilon^{d-4}D_{\Lambda_L}(\sqrt{f}) + \delta^{-1}\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\varepsilon^{-2}s(f|\psi).
\end{aligned}
$$

In particular, if f_t is the density in Theorem 2.1 then for any $\gamma > 0$,

$$
\begin{aligned}
& \lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\int_0^T dt\Big\{\varepsilon^{d-2}\int \sum_{x\in\Lambda_L}\left\{J(\varepsilon x)\hat\omega * (\tau_y g - E^{\mu_{1,\bar{l}_k^+(x)}}[g])\right\}(x) f_t d\mu_r\Big\} \\
& \le \gamma^{-1}CTV(h,r) + \delta^{-1}\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\int_0^T dt\varepsilon^{-2}s(f_t|\tilde\Psi_t) + \text{const. }\gamma
\end{aligned}
\tag{3.30}
$$

where C is some constant depending on J and γ.

The proofs of both the theorem and its corollary are postponed to the last section.

From Corollary 3.11 we can bound $E^f[C_1]$ by a variance term and the Dirichlet form of the density f. Because we are interested in a time integrated inequality in Theorem 3.6, the Dirichlet form will be integrated in time and we can use (3.30). Hence

$$
E^f[C_1] \le \text{const.}\gamma^{-1}\sum_{\alpha,\beta=1}^{d} V(H_\alpha^\beta, r) + \delta^{-1}\varepsilon^{-2}s(f_t|\tilde\Psi_t) + \text{const.}\gamma + \text{const.} + o(1)
$$

where the $o(1)$ denotes an error negligible after time integration.

Similarly, we can bound C_3. This time we have an extra ε factor and the variance term vanishes in the limit $\varepsilon \to 0$.

Summarizing,

$$\mathcal{I} \leq \text{const.}\gamma^{-1} \sum_{\alpha,\beta=1}^{d} V(H_\alpha^\beta, r) + \delta^{-1}\varepsilon^{-2} s(f_t|\tilde{\Psi}_t) + \text{const.}\gamma + E^f[C_4] + \text{const.} + o(1) \quad (3.31)$$

The constant is determined by the condition

$$E^{\tilde{\Psi}_t}\Big[\varepsilon^{d-2}\sum_x \Gamma_x(\bar{I}_k^+(x)) + \text{const.}\Big] = o(1)$$

By Lemma 2.2 one can replace $\tilde{\Psi}_t$ by Ψ_t. Let $\bar{m}_{x,k}^\alpha = E^{\Psi_t}[\bar{I}_{\alpha,k}(x)]$, $\alpha = 0,\dots,d$. From the central limit theorem for the product measure Ψ_t, $\bar{m}_{x,k}^\alpha = m_x^\alpha + O(k^{-d})$, with $m_x^\alpha = E^{\Psi_t}[I(x)]$. Hence the constant in (3.31) equals to

$$E^{\tilde{\Psi}_t}\Big[\varepsilon^{d-2}\sum_x \Gamma_x(\bar{I}_k^+(x))\Big] + o(1) = \varepsilon^{d-2}\sum_x E^{\Psi_t}[\Gamma_x(m_x^\alpha)] + o(1)$$

Using the entropy inequality (2.50) with $\gamma = \varepsilon^{2-d}$ we get

$$\begin{aligned}\mathcal{I} \leq& \varepsilon^{-2}s(f_t \mid \tilde{\Psi}_t) + \text{const.}\gamma^{-1}\sum_{\alpha,\beta=1}^{d} V(H_\alpha^\beta, r) + \delta^{-1}\varepsilon^{-2}s(f_t|\tilde{\Psi}_t) + \text{const.}\gamma \\ &+\varepsilon^{d-2}\log E^{\Psi_t}\Big[\exp\Big\{\sum_x\Big[\Gamma_x(\bar{I}_k^+(x)) - \Gamma_x^\alpha(m_x^\alpha)\Big]\Big\}\Big] + o(1)\end{aligned} \quad (3.32)$$

We will use the large deviations Lemma 3.2 to bound last term in (3.32). To apply it, we have to check that the function Γ_x satisfies the following condition

$$\frac{\partial\Gamma_x}{\partial Y_\alpha}\Big|_{Y_\alpha = m_x^\alpha} = o(\varepsilon), \quad \alpha = 0,\cdots,d \quad (3.33)$$

To check this we have to evaluate derivatives of $\pi^{\alpha\beta}(Y)$, defined in (3.21). To this end we compute $\pi^{\alpha\beta}(Y+\delta Y) - \pi^{\alpha\beta}(Y)$ to the first order in δY. We use the notation $\delta Y = (\delta Y_0, \delta\underline{Y})$. From (3.18) and (2.33) (and a simple computation), to the first order in δY we have

$$\begin{aligned}f(v,n+\delta n) - f(v,n+\delta n)^2 &- (f(v,n) - f(v,n)^2) \\ &= (\delta n_0 + \delta\underline{n}\cdot v)\Big[f(v,n)(1-f(v,n))(1-2f(v,n))\Big]\end{aligned}$$

with $\delta n = n^{(Y+\delta Y)} - n^{(Y)}$. Hence

$$
\begin{aligned}
&\pi^{\alpha\beta}(Y+\delta Y) - \pi^{\alpha\beta}(Y) \\
&= \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)(v\cdot e_\beta)\Big\{ f(v,n+\delta n) - f(v,n+\delta n)^2 - (f(v,n) - f(v,n)^2)\Big\} \\
&= \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)(v\cdot e_\beta)[\delta n_0 + (\delta \underline{n}\cdot v)] f(v,n)(1-f(v,n))(1-2f(v,n)) + O(\delta n^2)
\end{aligned}
\tag{3.34}
$$

Recall (2.34). Hence for $Y_\alpha = m_\alpha$ we have

$$
\begin{aligned}
&\pi^{\alpha\beta}(Y+\delta Y) - \pi^{\alpha\beta}(Y) \\
&= \sum_{v\in\mathcal{V}} [\delta n_0 + (\delta\underline{n}\cdot v)](v\cdot e_\alpha)(v\cdot e_\beta) f(v,n)(1-f(v,n))(1-2f(v,n)) \\
&= [S + O(\varepsilon^2)]\delta n_0 \delta_{\alpha,\beta} + \varepsilon T \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)(v\cdot e_\beta)(\lambda\cdot v)(\delta\underline{n}\cdot v) \\
&= S\delta n_0\delta_{\alpha,\beta} + 2\varepsilon h(\theta)\Big[B\lambda_\alpha \delta n_\alpha \delta_{\alpha,\beta} + C\frac{\lambda_\alpha\delta n_\beta + \lambda_\beta \delta n_\alpha}{2}\Big] + O(\varepsilon^2)
\end{aligned}
\tag{3.35}
$$

where S and T are constants depending on θ and can be computed explicitly. The analogous computation for π^0_α is much simpler because an extra factor ε appears before π^0_α. We have

$$
\begin{aligned}
\pi^0_\alpha(Y+\delta Y) - \pi^0_\alpha(Y) &= \sum_{v\in\mathcal{V}} [\delta n_0 + (\delta\underline{n}\cdot v)](v\cdot e_\alpha) f(v,n)(1-f(v,n))(1-2f(v,n)) + o(1) \\
&= t(\theta)(1-2\theta)\delta n_\alpha + o(1).
\end{aligned}
\tag{3.36}
$$

Moreover,

$$
\begin{aligned}
\delta Y_\alpha &= t(\theta)\delta n_\alpha + o(1), \\
\delta Y_0 &= t(\theta)\delta n_0 + o(1)
\end{aligned}
$$

Therefore, up to order $o(\varepsilon)$ we have

$$
\begin{aligned}
&\frac{\partial \Gamma_x}{\partial Y_\alpha}\Big|_{Y_\alpha = m^\alpha_x} = \varepsilon\Big(-\frac{\partial}{\partial t}\lambda_\alpha(z,t) + \sum_{\beta,\gamma=1}^{d} (\partial_{z_\beta}\partial_{z_\gamma}\lambda_\alpha)(z,t) D^\alpha_{\beta\gamma} \\
&- \frac{1}{t(\theta)}\partial_{z_\alpha} p(z,t) - \frac{h(\theta)}{t(\theta)}\big[B\partial_{z_\alpha}\lambda_\alpha(z,t)^2 + C\sum_{\beta=1}^{d}\lambda_\beta(z,t)\partial_{z_\beta}\lambda_\alpha(z,t)\big]\Big) + o(\varepsilon), \qquad (3.37) \\
&\frac{\partial \Gamma_x}{\partial Y_0}\Big|_{Y_\alpha = m^\alpha_x} = \text{const.}\varepsilon \sum_{\alpha=1}^{d} \partial_{z_\alpha}\lambda_\alpha(z,t) + o(\varepsilon)
\end{aligned}
$$

The vanishingness of the right sides of (3.37) is exactly the incompressible Navier-Stokes equations.

Therefore, the last term in (3.32) vanishes in the limit as $\ell \to \infty$. Applying Lemma 3.9 integrated on time we get

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\varepsilon^{-2}s(f_T \mid \tilde{\Psi}_T) \le \lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\int_0^T \Big[(\text{const.}+\delta^{-1})\varepsilon^{-2}s(f_t \mid \tilde{\Psi}_t)dt +$$

$$\frac{CT}{\gamma}\sum_{\alpha,\beta=1}^{d} V(H_\alpha^\beta, r) + \text{const.}\gamma)\Big]$$

The proof of Theorem 3.4 is then achieved by the use of Gronwall lemma. We shall prove that there exists a positive diffusion matrix D such that

$$\inf_F \sum_{\alpha,\beta=1}^{d} V(H_\alpha^\beta, r) = 0, \tag{3.38}$$

with H_α^β defined in (3.11). Hence Theorem 2.1 follows from Theorem 3.4. The rest of the paper is devoted to prove that (3.38) holds with F_α in a suitable class of local functions.

4. The structure theorem.

In this section we define the spaces of functions we will work with, and characterize their elements. We use several results of [8] for the simple exclusion. We first introduce some notations. Since the reference density is fixed, we shall use the symbol E^μ instead of E^{μ_r}.

Fixed any $x \in \Lambda_L$ consider the following alternative description of the configurations in x. Recall that $\#(\mathcal{V}) = N$. We introduce N^2 numbers $c_\alpha(v)$ with $\alpha = -N+d+1,\dots,0,1,\dots,d$ and $v \in \mathcal{V}$. We put

$$I_\alpha(\eta_x) = \sum_{v\in\mathcal{V}} c_\alpha(v)\eta(x,v). \tag{4.1}$$

To be consistent with the previous definition of the conserved quantities $I_\alpha(\eta_x)$ for $\alpha = 0,\dots,d$, we put

$$c_0(v) = a, \quad c_\alpha(v) = be_\alpha \cdot v, \quad \alpha = 1,\dots,d$$

with the constants a and b specified by the normalization conditions below. The constants $c_\alpha(v)$ for $\alpha < 0$ are chosen so that the mapping $\eta_x \to \{I_\alpha(\eta_x)\}$ be invertible and the orthogonality conditions below are satisfied. Define

$$\tilde{\eta}(\cdot,v) = \eta(\cdot,v) - \theta, \quad \theta := \big\langle \eta(\cdot,v)\big\rangle = \frac{e^r}{1+e^r}.$$

For any f and g functions of η_x define the scalar product

$$(f,g) := E^\mu[f(\eta_x); g(\eta_x)] = E^\mu[f(\eta_x)g(\eta_x)] - E^\mu[f(\eta_x)]E^\mu[g(\eta_x)].$$

Then we have the orthogonality relations:

$$(\eta(\cdot,v),\eta(\cdot,v')) = \delta_{v,v'}\theta(1-\theta)$$

We also require the conditions

$$(I_\alpha(\eta_x), I_\beta(\eta_x)) = \delta_{\alpha,\beta}, \quad \alpha,\beta = -N+d+1,\dots,0,1,\dots,d. \tag{4.2}$$

For Model I for example we have $N = 2d$. Let

$$I_\alpha(\eta_x) = [2\theta(1-\theta)]^{-1/2}\sum_{v\in\mathcal{V}}(e_\alpha\cdot v)\eta(x,v), \alpha = 1,\dots,d$$

$$I_0(\eta_x) = [2d\theta(1-\theta)]^{-1/2}\sum_{v\in\mathcal{V}}\eta(x,v)$$

and define $I_\alpha(\eta_x)$, $\alpha = -d+1,\cdots,-1$ by

$$I_\alpha := \sum_{\beta=-d+1}^{-1} A_{\alpha\beta}[\eta(x,e_\beta)+\eta(x,-e_\beta)],$$

where $A_{\alpha\beta}$ are chosen so that

$$E^\mu[I_\alpha(\eta_x); I_\beta(\eta_x)] = \delta_{\alpha,\beta}, \quad \alpha,\beta = -d+1,\cdots,d \tag{4.3}$$

The relation (4.3) for $\alpha = 0,\cdots,d$ holds because of the definition of $I_\alpha(\eta_x)$, $\alpha = 0,\cdots,d$. It remains to choose $A_{\alpha\beta}$ so that (4.3) holds for $\alpha < 0$. This can be done easily for $d \geq 2$. In particular, for $d = 3$ one can choose

$$A_{1\beta} = [2\theta(1-\theta)]^{-1/2}(1,-1,0); \quad A_{2\beta} = [8\theta(1-\theta)]^{-1/2}(1,1,-2)$$

In general, we will use the notation I^-, I^+ to mean the sets $\{I_\alpha, \alpha < 0\}$, $\{I_\alpha, \alpha \geq 0\}$ and $\tilde{I}_\alpha$ will denote the centered variable

$$\tilde{I}_\alpha = I_\alpha - E^\mu[I_\alpha]. \tag{4.4}$$

We now introduce two different spaces $\mathcal{G}^{ex}$ and $\tilde{\mathcal{G}}^{ex}$ which are suitable to describe the exclusion part only. The first one is constructed specifying the local densities of particles of each color, while the second one involves infinite volume grandcanonical averages. The space $\tilde{\mathcal{G}}^{ex}$ was introduced in [8]; the space $\mathcal{G}^{ex}$ was introduced in [24].

Definition 4.1 *Let*

$$\mathcal{G}^{ex} = \{g \text{ local function of } \eta : E^{\mu}[g \mid \bar{I}_k] = 0 \text{ for some } k > 0 \text{ and all values of } \bar{I}_k\}.$$

Here $E^{\mu}[\,\cdot \mid \bar{I}_k]$ denotes the expectation conditioned on fixed values of

$$\bar{I}_{\alpha,k} = Av_{x\in\Lambda_k} I_\alpha(\eta_x), \quad \textit{for } \alpha = -N+d+1, \dots, d.$$

The conditional expectation in principle depends on the measure μ. Suppose μ is a measure on Λ given by

$$\mu(\eta) = Z_{\Lambda,p}^{-1} \prod_{x\in\Lambda} \prod_{\alpha=-N+d+1}^{d} \exp[p_\alpha I_\alpha(\eta_x)]. \tag{4.5}$$

When it is conditioned to $\bar{I}_k$, it is the same as the *uniform* distribution of $\eta(x,v)$ on Λ_k conditioned on $\bar{I}_k$. Therefore the choice of the measure μ is not important as long as it is in the class of measure satisfying (4.5).

Definition 4.2 *Let*

$$\tilde{\mathcal{G}}^{ex} = \{g \text{ local function of } \eta \text{ satisfying (4.6)}\}$$

$$E^{\mu}[g] = 0, \quad \sum_x E^{\mu}[g; I_\alpha(\eta_x)] = 0, \quad \textit{for } \alpha = -N+d+1, \dots, d. \tag{4.6}$$

Let $\hat{\mu}$ be the measure such that $E^{\hat{\mu}}[I_\alpha] = m_\alpha$ for $\alpha = -N+d+1, \dots, d$ and $\hat{g}(m) = E^{\hat{\mu}}[g]$. Then the second condition is equivalent to

$$\frac{\partial \hat{g}(m)}{\partial m_\alpha}\Big|_{m=\bar{m}} = 0 \textit{ for } \alpha = -N+d+1, \dots, d, \tag{4.7}$$

where m is the vector $\{m_\alpha, \alpha = -N+d+1, \dots, d\}$ and $\bar{m}_\alpha = E^{\mu_r}[I_\alpha]$, $\alpha = -N+(d+1), \dots, d$, are the values corresponding to the equilibrium measure μ_r.

We introduce in $\mathcal{G}^{ex}$ the following bilinear form and the associated seminorm, based on the symmetric simple exclusion generator

$$\ll g, h \gg_{ex} = \frac{1}{4}\big(V^{ex}(g+h) - V^{ex}(g-h)\big], \quad \| g \|_{-1,ex}^2 = \ll g, g \gg_{ex}, \tag{4.8}$$

with $V^{ex}(g)$ defined similarly to $V(g,r)$ in Definition 3.1 of the previous section, but with the generator $\mathcal{L}_s$ replaced by $\mathcal{L}_s^{ex}$.

More precisely, let

$$V_\ell^{ex}(g,m) = (2\ell_1+1)^{-d} \Big\langle \sum_{|x|\le\ell_1} \tau_x g (-\mathcal{L}_{s,\ell}^{ex})^{-1} \sum_{|x|\le\ell_1} \tau_x g \Big\rangle_{\mu_{\ell,m}}$$

with $\ell_1 = \ell - \ell^{1/d^2}$. Here $\mu_{\ell,m}$ is the canonical Gibbs state of $(2\ell+1)^d$ sites with $\bar{I}_{\alpha,\ell} = m_\alpha$, $\alpha = -N+d+1, \dots, d$. Explicitly

$$\mu_{\ell,m} = Z_{\ell,m}^{-1} \prod_{\alpha=-N+d+1}^{d} \delta(\bar{I}_{\alpha,\ell} - m_\alpha)\mu_\ell.$$

Here μ_ℓ denotes the counting measure on the configurations in Λ_ℓ and $Z_{\ell,m}$ is the normalization. With these notations,

$$V^{ex}(g,r) = \limsup_{\ell\to\infty} E^{\mu_r}[V_\ell^{ex}(g,\bar{I}_\ell)].$$

We omit the label r because it is fixed through this section.

Let

$$\mathcal{G}_0^{ex} = \{\sum_\alpha \sum_{\beta=1}^{d} a_{\alpha,\beta}\nabla_\beta I_\alpha(\eta_0), a_{\alpha,\beta}\in\mathbb{R}\} = \{\sum_\alpha \sum_{\beta=1}^{d} a_{\alpha,\beta}\nabla_\beta \tilde{I}_\alpha(\eta_0), a_{\alpha,\beta}\in\mathbb{R}\} \tag{4.9}$$

be the space of the gradients. We used the shorthand $\sum_\alpha$ instead of $\sum_{\alpha=-N+(d+1)}^{d}$. From [8], we have the following theorem.

Theorem 4.3 *The bilinear form $\ll \cdot,\cdot \gg_{ex}$ is a scalar product and it can be extended to $\tilde{\mathcal{G}}^{ex}$. Let $\bar{\mathcal{G}}^{ex}$ be the closure of $\mathcal{G}^{ex}$ under the norm (4.8). One has*

$$\bar{\mathcal{G}}^{ex} = \overline{\tilde{\mathcal{G}}^{ex}} = \overline{(\mathcal{L}_s^{ex}\mathcal{G}^{ex}) \oplus \mathcal{G}_0^{ex}} \tag{4.10}$$

Previous definitions are natural when one disregards the collisions, so that all the I_α's are conserved. With collisions one needs the analogous of definitions 4.1 and 4.2 involving only the conserved quantities I_α, for $\alpha = 0, \dots d$.

Definition 4.4 *Let*

$$\mathcal{G} = \{g \text{ local function of } \eta : E[g \mid \bar{I}_k^+] = 0 \text{ for some } k > 0 \text{ and for all values of } \bar{I}_k, \alpha \geq 0\}.$$

Here $E^\mu[\cdot \mid \bar{I}_k^+]$ means expectation conditioned to fixed values of $Av_{x\in\Lambda_k} I_\alpha(\eta_x)$ for $\alpha = 0, \dots, d$.

Definition 4.5 *Let*

$$\tilde{\mathcal{G}} = \{g \text{ local function of } \eta \text{ satisfying (4.11)}\},$$

$$E^\mu[g] = 0, \quad \sum_x E^\mu[g; I_\alpha(\eta_x)] = 0, \quad \alpha = 0, \dots, d. \tag{4.11}$$

Let ν be the product measure such that $E^{\nu}[I_\alpha] = m_\alpha$ for $\alpha \geq 0$ and $\hat{g}(m) = E^{\nu_m}[g]$. Then as before the second condition is equivalent to

$$\frac{\partial \hat{g}(m)}{\partial m_\alpha}\Big|_{m=\bar{m}} = 0, \quad \alpha = 0, \ldots, d. \tag{4.12}$$

where m is the vector $\{m_\alpha, \alpha = 0, \ldots, d\}$ and $\bar{m}_\alpha = E^{\mu_r}[I_\alpha]$, $\alpha = 0, \ldots, d$, are, as before, the values corresponding to the equilibrium measure μ_r. Clearly, for any function g,

$$g - \sum_{\alpha \geq 0} \frac{\partial \hat{g}(m)}{\partial m_\alpha}\Big|_{m=\bar{m}} \tilde{I}_\alpha(\eta_0) \in \tilde{\mathcal{G}}. \tag{4.13}$$

Let

$$\| \cdot \|_{-1}^2 = V(\cdot)$$

with V defined in definition (3.1). Up to this point we have not proved that $\| g \|_{-1} < \infty$ for all $g \in \mathcal{G}$. The only information we have is that for all $g \in \mathcal{G}^{ex}$ (recall $\mathcal{G}^{ex} \subset \mathcal{G}$)

$$\| g \|_{-1} \leq \| g \|_{-1,ex} . \tag{4.14}$$

Let

$$\mathcal{G}_0 = \{\sum_{\alpha=0}^{d} \sum_{\beta=1}^{d} a_{\alpha,\beta} \nabla_\beta \tilde{I}_\alpha(\eta_0), a_{\alpha,\beta} \in \mathbb{R}\}$$

be the space of the gradients of the conserved quantities. The main result of this section is the following theorem.

Theorem 4.6 *For all $g \in \tilde{\mathcal{G}}$, one has $\| g \|_{-1} < \infty$. Furthermore, let $\bar{\mathcal{G}}$ denote the closure of $\mathcal{G}$ under $\| \cdot \|_{-1}$ seminorm. Then $\bar{\mathcal{G}}$ can be decomposed as*

$$\bar{\mathcal{G}} = \overline{\tilde{\mathcal{G}}} = \overline{\mathcal{L}_s \mathcal{G} \oplus \mathcal{G}_0} := \mathcal{H}, \tag{4.15}$$

where the closure are taken w.r.t. $\| \cdot \|_{-1}$. Furthermore, let

$$\ll g, h \gg = \frac{1}{4}[\, \| g + h \|_{-1} + \| g - h \|_{-1} \,]. \tag{4.16}$$

Then $\ll \cdot, \cdot \gg$ is an inner product and $\bar{\mathcal{G}}$ equipped with this inner product is a Hilbert space.

Remark 1 Though the seminorm $\| \cdot \|_{-1}$ has not been proved to be finite outside the space $\mathcal{G}^{ex}$, one can easily check that it is finite on the space

$$\mathcal{H}_0 \equiv \mathcal{L}_s \mathcal{G} \oplus \mathcal{G}_0$$

Indeed, it is not hard to check that $\mathcal{L}_s\mathcal{G}$ and $\mathcal{G}_0$ are orthogonal w.r.t. $\ll \cdot, \cdot \gg$ and $\ll \cdot,\cdot \gg$ define an inner product such that $\mathcal{H}_0$ is a pre-Hilbert space. Hence the space $\overline{\mathcal{L}_s\mathcal{G} \oplus \mathcal{G}_0} = \mathcal{H}$ is well defined. This and some more properties of $\ll \cdot,\cdot \gg$ will be proved in Theorem 5.1. Note that in $\mathcal{H}$ one has

$$\mathcal{L}_s\mathcal{G} \cap \mathcal{G}_0 = \{0\}.$$

Remark 2 We emphasize that the boundedness of $\| \cdot \|_{-1}$ on $\tilde{\mathcal{G}}$ is a consequence of the theorem.

Proof. We outline the proof first. The detailed proofs of Lemmas will be presented at the end of this section.

Step 1: Let

$$\mathcal{G}_- = \{\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0), a_\alpha \in \mathbb{R}\}. \tag{4.17}$$

Then we have the following lemma.
Lemma 4.7

$$\tilde{\mathcal{G}} = \tilde{\mathcal{G}}^{ex} + \{\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0), a_\alpha \in \mathbb{R}\}, \tag{4.18}$$

In other words, one has

$$\tilde{\mathcal{G}}^{ex} + \{\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0), a_\alpha \in \mathbb{R}\} \subset \tilde{\mathcal{G}}. \tag{4.19}$$

and for any function $g \in \tilde{\mathcal{G}}$ one can find constants $a_\alpha, \alpha = -N + (d+1), \ldots, -1$ such that

$$g - \sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) \in \tilde{\mathcal{G}}^{ex} \tag{4.20}$$

In fact, it is enough to observe that for any function $g \in \tilde{\mathcal{G}}$ one can find constants $a_\alpha, \alpha = -N + (d+1), \ldots, -1$, such that the extra conditions in (4.6) (with $\alpha < 0$) are satisfied by the combination (4.20). This proves (4.20). On the other hand, (4.19) follows from (4.3). This proves Lemma 4.7.

Step 2: We first state the following Lemma.
Lemma 4.8. *Put $Qf \equiv \sum_{q\in\mathcal{Q}} Q_x^q f$. Then, given $\tilde{I}_\alpha(\eta_x)$, for $\alpha < 0$, there exist a local function h and a function $g \in \tilde{\mathcal{G}}^{ex}$ such that*

$$\tilde{I}_\alpha(\eta_x) = (Q_x h)(\eta_x) + g(\eta_x), \tag{4.21}$$

Therefore $\{\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0)\} \subset \tilde{\mathcal{G}}^{ex} + \mathcal{G}$.
Corollary 4.9 *For any constant $a_\alpha \in \mathbb{R}$*

$$\| \sum_{\alpha<0} a_\alpha I_\alpha] \|_{-1} < \infty \tag{4.22}$$

Hence for any constant $c_{\alpha,\beta} \in \mathbb{R}$

$$\| \sum_{\alpha<0} \sum_{\beta=1}^{d} c_{\alpha,\beta} \nabla_\beta I_\alpha \|_{-1} = 0 \tag{4.23}$$

Lemma 4.8 will be proved at the end of this section. We now prove Corollary 4.9.
Proof of Corollary 4.9 From the definition of the norm $\| \cdot \|_{-1}$, it is easy to prove that (4.23) follows from (4.22). We now prove (4.22). Clearly, $\| Qh \|_{-1}$ is bounded for any bounded local function h, since $Qh = \mathcal{L}_s h - \mathcal{L}_s^{ex} h$ and $\mathcal{L}_s^{ex} h \in \mathcal{G}^{ex}$ so that $||\mathcal{L}_s^{ex} h||_{-1} < \infty$ by (4.14). Also, for $g \in \tilde{\mathcal{G}}^{ex}$ one has $\| g \|_{-1} \leq \| g \|_{-1,ex} < \infty$. Together with Lemma 4.8, this proves the boundedness of $\| \tilde{I}_{-\alpha}(\eta_x) \|_{-1}$ and thus concludes Corollary 4.9.

Eq. (4.22) and Lemma 4.7 imply that $\| \cdot \|_{-1}$ is finite on the space $\tilde{\mathcal{G}}$.

Step 3: Since we are interested in the operator $\mathcal{L}_s$ rather than $\mathcal{L}_s^{ex}$, we now replace the subspace $\mathcal{L}_s^{ex}\mathcal{G}^{ex}$ in the structure theorem of $\mathcal{G}^{ex}$ by $\mathcal{L}_s\mathcal{G}^{ex}$. The subspace $\mathcal{L}_s\mathcal{G}^{ex}$, however, may not be a subset of $\mathcal{G}^{ex}$ because the condition (4.6) may not hold for all elements in $\mathcal{L}_s\mathcal{G}^{ex}$. On the other hand, from the definition (4.3) we have that $\mathcal{L}_s\mathcal{G}^{ex} \subset \mathcal{G} \subset \tilde{\mathcal{G}}$. Hence from Lemma 4.3 for any $g \in \mathcal{G}^{ex}$ there exists $\{a_\alpha\}$ such that

$$\mathcal{L}_s g - \sum_{\alpha<0} a_\alpha \tilde{I}_\alpha \in \tilde{\mathcal{G}}^{ex}.$$

We denote by $\big(\mathcal{L}_s\mathcal{G}^{ex} + \mathcal{G}_-\big) \cap \tilde{\mathcal{G}}^{ex}$ the subspace of $\tilde{\mathcal{G}}^{ex}$ so obtained. The following structure theorem involves the operator $\mathcal{L}_s$ rather than $\mathcal{L}_s^{ex}$.

Lemma 4.10

$$\tilde{\mathcal{G}}^{ex} \subset \overline{\big(\mathcal{L}_s\mathcal{G}^{ex} + \mathcal{G}_-\big) \cap \tilde{\mathcal{G}}^{ex} + \mathcal{G}_0^{ex}} \tag{4.24}$$

The closure on the right side is taken w.r.t. the seminorm $\| \cdot \|_{-1,ex}$. *On the other hand,*

$$\tilde{\mathcal{G}} \subset \overline{\mathcal{L}_s\mathcal{G}^{ex} + \mathcal{G}_- + \mathcal{G}_0} \tag{4.25}$$

if the closure is taken w.r.t the seminorm $\| \cdot \|_{-1}$.

Combining (4.18) and (4.25), we have prove that

$$\tilde{\mathcal{G}} \subset \overline{\mathcal{L}_s\mathcal{G}^{ex} + \mathcal{G}_- + \mathcal{G}_0} \tag{4.26}$$

Step 4: To conclude the identity (4.15) we only have to prove that

$$\mathcal{G}_- = \{\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0), a_\alpha \in \mathbb{R}\} \subset \overline{\mathcal{L}_s\mathcal{G} \oplus \mathcal{G}_0}$$

This will be proved in the following Lemma 4.11. The rest of theorem 4.6 follows from the remark after Theorem 4.6 and Theorem 5.1.

Lemma 4.11

$$\mathcal{G}_- = \{\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0), a_\alpha \in \mathbb{R}\} \subset \overline{\mathcal{L}_s\mathcal{G} \oplus \mathcal{G}_0} \tag{4.27}$$

The closure on the right side is taken w.r.t. the seminorm $\|\cdot\|_{-1}$

Proof of Lemma 4.8. Fix a site x. Let us introduce the linear spaces $V_\pm$ defined by

$$V_+ = \{\sum_{\alpha\geq 0} a_\alpha \tilde{I}_\alpha(\eta_x), a_\alpha \in \mathbb{R}\}, \quad V_- = \{\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_x), a_\alpha \in \mathbb{R}\}.$$

Let $\pi_3 = E^\mu[\cdot \mid I^+]$ be the expectation conditioned to fixed value of $I_\alpha(\eta_x)$ for $\alpha = 0, \dots, d$, where μ is a measure on the set of the configurations of the site x. The projection π_3 in principle depends on the measure μ. When conditioned on $I_\alpha(\eta_x)$, μ is the same as the *uniform* distribution of $\eta(x, v)$ conditioned on $\{I_\alpha(\eta_x), \alpha \geq 0\}$ and hence independent of μ.

Let $\mathcal{S}_x$ denote the Hilbert space of square integrable functions (w.r.t. the expectation μ) depending only on configurations on the site x and with expectation 0 w.r.t. μ. For the rest of this section, orthogonality and norm are all taken w.r.t. the natural L_2 inner product of this Hilbert space. We claim that for any function $u \in \mathcal{S}_x$

$$u - \pi_3 u = Qh \tag{4.28}$$

for some $h \in \mathcal{S}_x$. In the following, we shall drop the subscript x. To prove this, observe that the null space of Q is the space of all the functions depending on η_x only through the functions $I_\alpha(\eta_x)$, for $\alpha \geq 0$, because they are the only conserved quantities during collisions. Hence (4.28) is equivalent to

$$E[(u - \pi_3 u) f(I_\alpha)] = 0, \alpha \geq 0$$

for any function f. But this is true just by the definition of π_3 as the conditional expectation given the $\{I_\alpha(\eta_x)$'s, $\alpha \geq 0\}$.

Let π_i be the orthogonal projection from S onto the space V_i, $i = -, +$ and let $V_3 = \pi_3\mathcal{S}$. Consider the map $\tau = \mathbb{I} - \pi_-\pi_3$ from V_- to V_-. We claim that τ is surjective. Indeed, we have $\| \pi_-\pi_3 \| < 1$, therefore τ has a trivial null space in V_- and its surjectiveness is a consequence of the bound. To prove the bound, it suffices to prove that the intersection $V_- \cap V_3 = \{0\}$, for otherwise we would have $\| \pi_-\pi_3 \| = 1$. Let $u \in V_- \cap V_3$. Then $u = \pi_3 u$ is a function depending only on I_α, α nonnegative since $u \in V_3$. On the other hand $u = \sum_{\alpha<0} a_\alpha I_\alpha(\eta_x)$ because $u \in V_-$. This proves u is constant and hence 0.

Since τ is surjective, for any $v \in V_-$ there is a $w \in V_-$ such that $v = w - \pi_-\pi_3 w$. Rewrite v as

$$v = (w - \pi_3 w) + ((\pi_3 w) - \pi_-(\pi_3 w))$$

It follows, from the remark above, that there is an $h \in S$ such that $(w - \pi_3 w) = Qh$. On the other hand, for any function ϕ, one has that $\phi - \pi_-\phi - \pi_+\phi$ has no linear part because by definition $\pi_-\phi + \pi_+\phi$ is the linear part of ϕ. Hence by definition of $\tilde{\mathcal{G}}^{ex}$ we have

$$(\phi - \pi_-\phi - \pi_+\phi) \in \tilde{\mathcal{G}}^{ex}.$$

Hence there is $g \in \tilde{\mathcal{G}}^{ex}$ s.t.

$$v - Qh = \pi_3 w - \pi_- \pi_3 w = g + \pi_+ \pi_3 w, \qquad \text{for some } g \in \tilde{\mathcal{G}}^{ex}$$

Since v, Qh and g have no linear part in $\{I_\alpha,\ \alpha \geq 0\}$ we have $\pi_+ \pi_3 w = 0$. This implies $v = Qh + g$. We have thus proved that if $v \in V_-$ then

$$v = Qh + g, \qquad \text{for some } g \in \tilde{\mathcal{G}}^{ex}.$$

This proves Lemma 4.8.

Proof of Lemma 4.10 We first prove that (4.25) follows from (4.24). From (4.18), it suffices to check that the right side of (4.24) is a subset of the right side of (4.25). From (4.23) , $\mathcal{G}_0^{ex} = \mathcal{G}_0$ under the seminorm $\| \cdot \|_{-1}$. This concludes the proof.

We now prove (4.24). From the structure theorem for $\bar{\mathcal{G}}^{ex}$ (4.10), we only have to prove the surjectiveness of the projection (under the inner product $\ll, \cdot, \cdot \gg_{ex}$) from $\big(\mathcal{L}_s \mathcal{G}^{ex} + \mathcal{G}_-\big) \cap \tilde{\mathcal{G}}^{ex}$ to the space $\overline{\mathcal{L}_s^{ex} \mathcal{G}^{ex}}$. Suppose that, for some $h \in \mathcal{L}_s^{ex} \mathcal{G}^{ex}$, one has $\ll \mathcal{L}_s g - \sum_{\alpha < 0} a_\alpha \tilde{I}_\alpha(\eta_0), \mathcal{L}_s^{ex} h \gg_{ex} = 0$ for all $g \in \mathcal{G}$. In particular choose $g = h$. Hence

$$\ll \mathcal{L}_s h - \sum_{\alpha < 0} a_\alpha \tilde{I}_\alpha(\eta_0), \mathcal{L}_s^{ex} h \gg_{ex} = \big\langle h, \mathcal{L}_s h \big\rangle_0 \geq \big\langle h, \mathcal{L}_s^{ex} h \big\rangle_0 = \ll \mathcal{L}_s^{ex} h, \mathcal{L}_s^{ex} h \gg_{ex} \quad (4.29)$$

where

$$\big\langle f, g \big\rangle_0 = \sum_x \big\langle f \tau_x g \big\rangle \qquad (4.30).$$

The first equality follows from the fact that for $g \in \mathcal{G}^{ex}$ one has $\big\langle \tilde{I}_\alpha, g \big\rangle_0 = 0$. The inequality is due to the presence in the definition of $\mathcal{L}_s$ of the collision part which gives a positive contribution to the Dirichlet form $\big\langle h, \mathcal{L}_s h \big\rangle_0$.

Hence $\ll \mathcal{L}_s h - \sum_{\alpha < 0} a_\alpha \tilde{I}_\alpha, \mathcal{L}_s^{ex} h \gg_{ex} = 0$ and this implies $\mathcal{L}_s^{ex} h = 0$ in $\bar{\mathcal{G}}^{ex}$. This proves surjectiveness of the projection and concludes Lemma 4.10.

Proof of Lemma 4.11. Observe that $\mathcal{L}_s \mathcal{G}_- \subset \mathcal{G}$. Let $v = \sum_{\alpha < 0} a_\alpha \tilde{I}_\alpha$ be any element of $\mathcal{G}_-$. By (4.25) there are $g \in \mathcal{G}^{ex}$ and constants b_α and $c_{\alpha,\beta}$ such that

$$\| \mathcal{L}_s v - \sum_{\alpha < 0} b_\alpha \tilde{I}_\alpha - \mathcal{L}_s g - \sum_{\alpha=0}^{d} \sum_{\beta=1}^{d} c_{\alpha,\beta} \nabla_\beta I_\alpha \|_{-1} = 0 \qquad (4.31)$$

This defines a linear map of $\mathcal{G}_-$ into itself and hence a linear map L associating to any $A = \{a_\alpha, \alpha = -N + (d+1), \ldots, -1\}$ a vector $B = \{b_\alpha, \alpha = -N + (d+1), \ldots, -1\}$, so that (4.31) is satisfied. Strictly speaking, (4.31) is not correct. The correct statement is that for every $\epsilon > 0$ there exist $g \in \mathcal{G}^{ex}$ and constants b_α and $c_{\alpha,\beta}$ such that the left side of (4.31) is bounded by ϵ. Let us first pretend that (4.31) holds as it is. To prove Lemma 4.11, it suffices to prove the surjectiveness of L. We prove this statement by contradiction.

Suppose that for some choice of a_α one has $b_\alpha = 0$

$$\| \mathcal{L}_s \big[\sum_{\alpha<0} a_{-\alpha} \tilde{I}_\alpha(\eta_0) - g \big] - \sum_{\alpha=0}^{d} \sum_{\beta=1}^{d} c_{\alpha,\beta} \nabla_\beta \tilde{I}_\alpha(\eta_0) \|_{-1} = 0. \tag{4.32}$$

From Remark 1, we have $\mathcal{L}_s \mathcal{G} \cap \mathcal{G}_0 = \{0\}$.

Together with (4.32), this implies that

$$\| \mathcal{L}_s \big[\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g \big] \|_{-1}^2 = \big\langle [\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g], \mathcal{L}_s [\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g] \big\rangle_0 = 0. \tag{4.33}$$

Since also for $\alpha < 0$, $\mathcal{L}^{ex} I_\alpha = 0$, eq. (4.33) implies $\big\langle g, \mathcal{L}_s^{ex} g \big\rangle_0 = 0$ and

$$\big\langle [\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g], \mathcal{L}^c [\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g] \big\rangle_0 = 0$$

where $\big\langle \cdot, \cdot \big\rangle_0$ is defined in (4.30). From the last identity we obtain

$$\sum_{q\in\mathcal{Q}} \big\langle Q_0^q [\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g], Q_0^q [\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g] \big\rangle_0 = 0. \tag{4.34}$$

Next we show that (4.34) and $g \in \bar{\mathcal{G}}^{ex}$ imply $a_\alpha = 0$. To this end, recall $\mathcal{S}_x$ is the space of functions depending only on the configuration at the site x with vanishing μ-expectation. Note that S_x is equivalent to S_y for any pairs (x, y) w.r.t. the inner product $\big\langle \cdot, \cdot \big\rangle_0$. Let π_S be the orthogonal projection onto the space S_0. Clearly, Q_0^q commutes with π_S. In particular,

$$\big\langle Q_0^q [\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g_1], Q_0^q [\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g_1] \big\rangle_0 = 0 \tag{4.35}$$

where $g_1 = \pi_S g$. Since g_1 depends only on the configuration at the origin, one can omit the sum on the translations in the scalar product in (4.35) and obtains

$$E^\mu \Big\{ \big(Q_0^q [\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g_1]^2 \big) \Big\} = 0 \tag{4.36}$$

Hence $Q_0^q[\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g_1] = 0$ pointwise, for all $q \in \mathcal{Q}$. The null space of Q_0^q consists of functions depending only on I_α, $\alpha \geq 0$. Since g is in $\bar{\mathcal{G}}^{ex}$ and $g_1 = \pi_S g$, g_1 has no linear part in $I_\alpha, \alpha < 0$. Thus the linear combination $\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0)$ cannot be compensated by g_1 and it has to vanish. Therefore $a_\alpha = 0$ for $\alpha = -N + (d+1), \dots, -1$ and the map L is surjective.

As remarked before, in general (4.24) holds only up to an arbitrarily small ε. This requires only standard modifications as follows: Equations (4.32),…,(4.36) hold if one replaces the 0 on their r.h.s. by ε. This means that $Q_0^q[\sum_{\alpha<0} a_\alpha \tilde{I}_\alpha(\eta_0) - g_1]$ is pointwise arbitrarily small and hence a_α have to be arbitrarily small. This is enough to conclude the surjectivity of L.

5. The diffusion coefficient.

Recall that the space $\bar{\mathcal{G}}$, defined in Sect 4 (see definition 4.4) with the scalar product (4.16), is a Hilbert space and it can be decomposed into $\mathcal{G}_0$ and $\mathcal{L}\mathcal{G}$ (4.15). The structure theorem as well as the scalar product $\ll \cdot,\cdot \gg$ are based on the symmetric part of the generator, $\mathcal{L}_s$. The strategy of the non-gradient method, introduced for the first time by Varadhan in [15] is to decompose the currents into the sum of a gradient term and a term of the form $\mathcal{L}\phi$. We want to adopt a similar strategy. To do this we need a new structure theorem based on the complete operator $\mathcal{L}$. We get such a theorem following the procedure already used in [8]. As a consequence we identify the diffusion coefficient and characterize it by a variational principle. Throughout this section, we shall use indices $\alpha, \beta, \gamma = 0, \cdots, d$ to denote the "color" indices and use $i, j, k = 1, \cdots, d$ to denote the d coordinate directions.

We use the notations:

$$\nabla_{v,j} g(\eta) = g(\eta^{0,e_j,v}) - g(\eta),$$

$$\nabla_j g(\eta) = g(\tau_{e_j}\eta) - g(\eta),$$

where $(\tau_x\eta)(y,v) = \eta(y+x,v)$ and $(\tau_x f)(\eta) = f(\tau_x\eta)$. For any local function g, let

$$t_j^\alpha(g) = \frac{1}{\chi} \lim_{\ell\to\infty} E^\mu[g \sum_{x\in\Lambda_\ell} (e_j \cdot x)\tilde{I}_\alpha(\eta_x)]. \tag{5.1}$$

Here $\tilde{I}$ and $\tilde{\eta}$ denote the centered variables introduced in (4.4). Note that the right side is independent of ℓ, for ℓ large enough, since $E^\mu[g\, I_\alpha(\eta_x)] = 0$ if g does not depend on η_x. Recall that for any functions f and g,

$$\langle f, g \rangle_0 = \sum_x \langle f \tau_x g \rangle, \quad \langle \nabla_j I_\alpha(\eta_0), g \rangle_0 = 0, \quad \alpha \geq 0,\ j = 1, \cdots, d. \tag{5.2}$$

The following lemma characterize the scalar product $\ll \cdot,\cdot \gg$.

Lemma 5.1. . *The bilinear form $\ll \cdot,\cdot \gg$ defined by (4.14) is a inner product on $\mathcal{G}$ such that for any $g, h \in \mathcal{G}$:*

$$\ll \mathcal{L}_s g, h \gg = -\langle h, g \rangle_0 \tag{5.3}$$

$$\mathcal{G}^{(0)} \perp \mathcal{L}_s\mathcal{G} \tag{5.4}$$

$$\begin{aligned} \ll \mathcal{L}_s g, \mathcal{L}_s h \gg = &\frac{1}{2} E^\mu\Big[\sum_{v\in\mathcal{V}} \sum_{j=1}^d \chi(\nabla_{v,j} \sum_x \tau_x g)(\nabla_{v,j} \sum_x \tau_x h) \\ &+ \sum_{q\in\mathcal{Q}} (Q_0^q \sum_x \tau_x g)(Q_0^q \sum_x \tau_x h)\Big] \end{aligned} \tag{5.5}$$

$$\ll \nabla_j I_\alpha(\eta_0), g \gg = t_j^\alpha(g) \tag{5.6}$$

$$\ll \nabla_i I_\alpha(\eta), \nabla_j I_\beta(\eta_0) \gg = t_i^\alpha(\nabla_j I_\beta(\eta)) = \frac{1}{2\chi}\delta_{i,j}\delta_{\alpha,\beta}E^\mu[(\nabla_i I_\alpha(\eta_0))^2] \tag{5.7}$$

Proof. (5.3) follows from the definition of the scalar product and one gets the orthogonality relation (5.4) by using (5.3) and (5.2)

$$\ll \mathcal{L}_s g, \nabla_j I_\alpha(\eta_0) \gg = -\langle \nabla_j I_\alpha(\eta_0), g\rangle_0 = 0,\ \alpha \geq 0$$

Still using the definition of the scalar product and the structure of $\mathcal{L}_s$ we get (5.5). Clearly (5.6) and (5.7) are consequence of the relation

$$\mathcal{L}_s \sum_x (e_j \cdot x) I_\alpha(\eta_x) = -\chi \nabla_j I_\alpha(\eta_0).$$

Finally, (5.7) follows from explicit computation.

The scalar product defined in Lemma 5.1 extends obviously to the space $\bar{\mathcal{G}}$. Using the structure theorem 4.6 for $\bar{\mathcal{G}}$, the space $\bar{\mathcal{G}}$ endowed with the scalar product $\ll \cdot,\cdot \gg$ is a Hilbert space. Hence we have the following variational formula for the variance V.

Theorem 5.2 *The variance $V(g) := \ll g, g \gg$ satisfies the variational principle*

$$\begin{aligned}\frac{1}{2}V(g) = \sup_{c_j^\alpha \in \mathbb{R}, h\in\mathcal{G}} \Big[\sum_{\alpha\geq 0, j\geq 1} c_j^\alpha t_j^\alpha(g) + \langle g, h\rangle_0 - \frac{1}{4}\Big\{ \sum_{\alpha\geq 0, j\geq 1} (c_j^\alpha)^2 V(\nabla_j I_\alpha(\eta)) \\ + E^\mu\Big[\sum_{v\in\mathcal{V}, j\geq 1} \chi(\nabla_{v,j}\sum_x \tau_x h)^2 + \sum_{q\in\mathcal{Q}} (Q_0^q \sum_x \tau_x h)^2\Big]\Big\}.\end{aligned} \tag{5.8}$$

Proof. For $g \in \bar{\mathcal{G}}$ the following identity holds

$$\frac{1}{2} \ll g, g \gg = \sup_{u\in\bar{\mathcal{G}}} \Big\{ \ll g, u \gg - \frac{1}{2} \ll u, u \gg \Big\} \qquad (5.9).$$

By the structure theorem 4.6, Theorem 4.2, there exist a set $\{c_j^\alpha\}$ and $h \in \bar{G}$ such that

$$u = \sum_{\alpha\geq 0, j\geq 1} c_j^\alpha \nabla_j I_\alpha + \mathcal{L}_s h\ .$$

Hence

$$\begin{aligned}\frac{1}{2} \ll g, g \gg = \sup_{c_j^\alpha, h\in\bar{G}} \Big\{ \sum_{\alpha\geq 0, j\geq 1} c_j^\alpha \ll g, \nabla_j I_\alpha \gg + \ll g, \mathcal{L}_s h \gg \\ -\frac{1}{2}\Big[\sum_{\alpha\geq 0, j\geq 1}\sum_{\beta\geq 0, i\geq 1} c_j^\alpha c_i^\beta \ll \nabla_j I_\alpha, \nabla_i I_\beta \gg + \ll \mathcal{L}_s h, \mathcal{L}_s h \gg \Big]\Big\}\end{aligned} \tag{5.10}$$

Finally, using Lemma 5.1 in (5.10) we get the result.

We need another structure theorem for $\bar{\mathcal{G}}$ so that its elements can be decomposed in terms of a gradient and $\mathcal{L}\mathcal{G}$. Theorem 5.4 below provides such a decomposition. First of all note that the following inclusion relations hold

$$\mathcal{L}\mathcal{G} \subset \mathcal{G}\,,\ \mathcal{L}^*\mathcal{G} \subset \mathcal{G}.$$

Recall the expressions (2.15) and (2.20) of the currents w_α^β (we set $x = 0$ and omit the position label) given by

$$\begin{aligned} w_j^0 &= \sum_{v\in\mathcal{V}} w_j(v) = w_j^{(s)0} + \sum_{v\in\mathcal{V}} (e_j\cdot v) b_j(v) \\ w_j^\alpha &= \sum_{v\in\mathcal{V}} (e_\alpha\cdot v) w_j(v) = w_j^{(s)\alpha} + \sum_{v\in\mathcal{V}} (e_\alpha\cdot v)(e_j\cdot v) b_j(v), \quad \alpha = 1,\ldots,d. \end{aligned} \tag{5.11}$$

with

$$w_j^{(s)\alpha} = \chi\nabla_j I_\alpha(\eta_0), \quad b_j(v) = \eta(e_j,v)\eta(0,v) - \frac{1}{2}\Big[\eta(e_j,v) + \eta(0,v)\Big] \tag{5.12}$$

Note that $w_j^\alpha \notin \tilde{\mathcal{G}}$. From (4.13), there exist constants $c_i^{\alpha,\beta}$ such that

$$\sigma_i^\alpha = w_i^\alpha - \sum_{\beta\geq 0} c_i^{\alpha,\beta}\tilde{I}_\beta(\eta_0) \in \tilde{\mathcal{G}}.$$

The explicit form of $c_i^{\alpha,\beta}$ is not important and all computations in this section are independent of $c_i^{\alpha,\beta}$. In the same way, one can define $\sigma_i^{*\alpha} = w_i^{*\alpha} - \sum_{\beta\geq 0} c_i^{*\alpha,\beta}\tilde{I}_\beta(\eta_0) \in \tilde{\mathcal{G}}$. Note that from (5.3), for $g\in\mathcal{G}$, $\big\langle g, w_i^\beta\big\rangle_0 = \big\langle g,\sigma_i^\beta\big\rangle_0$ and $\big\langle g, w_i^{*\ \beta}\big\rangle_0 = \big\langle g, \sigma_i^{*\ \beta}\big\rangle_0$.

Lemma 5.3. *Suppose $g\in\mathcal{G}$. Then for $\alpha = 0,\ldots,d$ and $j = 1,\ldots,d$*

$$\chi t_j^\alpha(\mathcal{L}g) = -\big\langle g, w_j^{*\ \alpha}\big\rangle_0 = -\big\langle g, \sigma_j^{*\ \alpha}\big\rangle_0, \tag{5.13}$$

$$\chi t_j^\alpha(\mathcal{L}^* g) = -\big\langle g, w_j^\alpha\big\rangle_0 = -\big\langle g,\sigma_j^\alpha\big\rangle_0 \tag{5.14}$$

$$\ll \sigma_j^\beta, \nabla_i I_\alpha(\eta) \gg = t_i^\alpha(\sigma_j^\beta) = \delta_{i,j}\delta_{\alpha,\beta}\chi V(\nabla_i I_\alpha), \qquad \alpha,\beta,i,j = 1,\cdots,d. \tag{5.15}$$

$$\ll \sigma_i^0, \nabla_j I_0(\eta) \gg = \delta_{i,j}\chi V(\nabla_i I_0), i,j = 1,\cdots,d.$$

Proof. The relations (5.13) and (5.14) follow from the identities

$$\mathcal{L}^*\sum_x (x\cdot e_j)\tilde{I}_\alpha(\eta(x)) = -\sum_x w_{j,x}^{*\alpha}, \quad \mathcal{L}\sum_x (x\cdot e_j)\tilde{I}_\alpha(\eta(x)) = -\sum_x w_{j,x}^\alpha \tag{5.16}$$

and the same argument used in the proof of Lemma 5.7 in [8]. To prove (5.15), from (5.1) and (5.16) and the definition of σ_j^β we get

$$\begin{aligned}
\chi t_i^\alpha(\sigma_j^\beta) &= \lim_{\ell\to\infty} E^\mu[\sigma_j^\beta \sum_{|x|\le\ell}(x\cdot e_i)\tilde{I}_\alpha(\eta(x))] \\
&= \lim_{\ell\to\infty} E^\mu[w_j^\beta; \sum_{|x|\le\ell}(x\cdot e_i)\tilde{I}_\alpha(\eta(x))] \\
&= \lim_{\ell\to\infty} \ell^{-d} E^\mu[\mathcal{L}\sum_x (x\cdot e_j)\tilde{I}_\alpha(\eta(x)); \sum_{|x|\le\ell}(x\cdot e_i)\tilde{I}_\alpha(\eta(x))]
\end{aligned}$$

Clearly, if $j\neq i$ or $\alpha\neq\beta$ then $t_i^\alpha(\sigma_j^\beta)=0$ by the reflection symmetry. Hence we assume $i=j$ and $\alpha=\beta$. But under this assumption we can replace $\mathcal{L}$ by $\mathcal{L}_s$. On the other hand, from (5.6) and (5.7)

$$\lim_{\ell\to\infty} \ell^{-d} E^\mu[\mathcal{L}\sum_x (x\cdot e_i)\tilde{I}_\alpha(\eta(x)); \sum_{|x|\le\ell}(x\cdot e_i)\tilde{I}_\alpha(\eta(x))] = \chi^2 t_i^\alpha(\nabla_i I_\alpha) = \chi^2 V(\nabla_i I_\alpha)$$

This proves Lemma 5.3

Next we state the second structure theorem for $\bar{\mathcal{G}}$.

Theorem 5.4.

(i) $\overline{\mathcal{L}\mathcal{G}+\mathcal{G}_0} = \overline{\mathcal{G}} = \overline{\mathcal{L}^*\mathcal{G}+\mathcal{G}_0}$

(ii) Let $\mathcal{G}_w = \left\{\sum_{\alpha\ge 0, j\ge 1} b_j^\alpha\sigma_j^\alpha\right\}$. *Then* $\overline{\mathcal{G}_w+\mathcal{L}\mathcal{G}} = \overline{\mathcal{G}} = \overline{\mathcal{G}_w+\mathcal{L}_s\mathcal{G}}$. *Furthermore,*

$$V(\sum_{\alpha\ge 0,j\ge 1} b_j^\alpha\sigma_j^\alpha + \mathcal{L}g) \ge V(\sum_{\alpha\ge 0,j\ge 1} b_j^\alpha w_j^{(s)\alpha} + \mathcal{L}_s g) \tag{5.17}$$

Similarly, $\overline{\mathcal{G}_{w^*}+\mathcal{L}^*\mathcal{G}} = \overline{\mathcal{G}}$ *and*

$$V(\sum_{\alpha\ge 0,j\ge 1} b_j^\alpha\sigma_j^{*\ \alpha} + \mathcal{L}^* g) \ge V(\sum_{\alpha\ge 0,j\ge 1} b_j^\alpha w_j^{(s)\alpha} + \mathcal{L}_s g) \tag{5.18}$$

Proof of (i). Clearly, one only has to check that the orthogonal projection from $\overline{\mathcal{L}\mathcal{G}}$ to $\overline{\mathcal{L}_s\mathcal{G}}$ is onto. Suppose that, for some $h\in\mathcal{G}$, $\ll \mathcal{L}g, \mathcal{L}_s h\gg = 0$ for all $g\in\mathcal{G}$. In particular, $\ll \mathcal{L}h,\mathcal{L}_s h\gg = 0$. By Lemma 5.1,

$$\ll \mathcal{L}h,\mathcal{L}_s h\gg = -\langle \mathcal{L}h, h\rangle_0 = -\langle \mathcal{L}_s h, h\rangle_0 = \ll \mathcal{L}_s h,\mathcal{L}_s h\gg$$

This proves $\mathcal{L}_s h = 0$ in $\mathcal{G}$ and hence (i).

Proof of (ii). We shall only prove the bound on V, namely (5.17). By Theorem 5.2 the left side of (5.17) is equal to

$$
\begin{aligned}
&2\sup_{\tilde b,h}\Big\{\sum_{\alpha\geq 0,i\geq 1}\tilde b_i^\alpha t_i^\alpha\big(\sum_{\beta\geq 0,j\geq 1} b_j^\beta\sigma_j^\beta+\mathcal{L}g\big)+\big\langle\sum_{\beta\geq 0,j\geq 1}^{d} b_k^\beta\sigma_k^\beta+\mathcal{L}g,h\big\rangle_0\\
&-\frac14 E^\mu\Big[\sum_{\alpha\geq 0,i\geq 1}\big[\frac{\tilde b_i^\alpha}{\chi}(\nabla_i I_\alpha(\eta))^2+\sum_{v\in\mathcal{V},i\geq 1}\chi\big(\nabla_{v,i}\sum_x\tau_x h\big)^2\\
&+\sum_{q\in\mathcal{Q}}\big(Q_0^q\sum_x\tau_x h\big)^2\Big]\Big\}
\end{aligned}
$$

In particular, let $h=-g$ and $\tilde b_i^\alpha=b_i^\alpha\ \chi$. Hence from Lemma 5.1 we can bound the last expression from below by

$$
\begin{aligned}
&2\Big\{\sum_{\beta\geq 0,i\geq 1}\chi b_i^\beta t_i^\beta\big(\sum_{j\geq 1,\gamma\geq 0} b_j^\gamma\sigma_j^\gamma+\mathcal{L}g\big)\\
&-\big\langle\sum_{j\geq 1,\gamma\geq 0} b_j^\gamma\sigma_j^\gamma+\mathcal{L}g,g\big\rangle_0-\frac12 V\big(\sum_{j\geq 1,\gamma\geq 0} b_j^\gamma\nabla_j I_\gamma(\eta)+\mathcal{L}_s g\big)\Big\}
\end{aligned}
$$

By definition of $\mathcal{L}$, $\langle\mathcal{L}g,g\rangle_0=\langle\mathcal{L}_s g,g\rangle_0$. Also, from (5.13) $t_j^\gamma(\mathcal{L}g)=-\langle g,w_j^{*\ \gamma}\rangle_0\chi$. To summarize, we have

$$
\begin{aligned}
\frac12 V\big(\sum_{j\geq 1,\gamma\geq 0} b_j^\gamma\sigma_j^\gamma+\mathcal{L}g\big)&\geq\sum_{\beta\geq 0,i\geq 1}\chi\big[b_i^\beta t_i^\beta\big(\sum_{j\geq 1,\gamma\geq 0} b_j^\gamma\sigma_j^\gamma\big)-\langle g,w_i^{*\ \beta}\rangle_0\big]\\
&-\big\langle\sum_{j\geq 1,\gamma\geq 0} b_j^\gamma\sigma_j^\gamma,g\big\rangle_0-\langle\mathcal{L}_s g,g\rangle_0-\frac12 V\big(\sum_{j\geq 1,\gamma\geq 0} b_j^\gamma\nabla_j I_\gamma(\eta)+\mathcal{L}_s g\big)
\end{aligned}
\tag{5.19}
$$

Since $g\in\mathcal{G}$, $\langle\sigma_j^\beta,g\rangle_0=\langle w_j^\beta,g\rangle_0$ by (5.14). By definition $w_j^\beta+w_j^{\beta\ *}=2w_j^{(s)\beta}$ and hence $\langle g,w_j^{\beta\ *}+w_j^\beta\rangle_0=0$. Together with (5.15)

$$
\begin{aligned}
&\frac12 V\big(\sum_{j\geq 1,\beta\geq 0} b_j^\beta\sigma_j^\beta+\mathcal{L}g\big)\\
&\geq V\big(\sum_{j\geq 1,\beta\geq 0} b_j^\beta w_j^{(s)\beta}\big)-\langle\mathcal{L}_s g,g\rangle_0-\frac12 V\big(\sum_{j\geq 1,\beta\geq 0} b_j^\beta w_j^{(s)\beta}+\mathcal{L}_s g\big)\\
&=\frac12 V\big(\sum_{j\geq 1,\beta\geq 0} b_j^\beta w_j^{(s)\beta}+\mathcal{L}_s g\big).
\end{aligned}
\tag{5.20}
$$

This proves the bound on V and concludes Theorem 5.4.

To use at the best this decomposition of the Hilbert space, one should introduce a scalar product based on the inverse generator $\mathcal{L}^{-1}$ instead of using $\ll \cdot,\cdot \gg$ in which only the symmetric part appears. Unfortunately, we do not know how to give sense to $\mathcal{L}^{-1}$, so the arguments based on it are only formal and we have to introduce a suitable maps T below to make the computation rigorous. At formal level one can think of the map T as $T = \mathcal{L}_s \mathcal{L}^{-1}$, so that formally $\ll g, Th \gg = \langle g, \mathcal{L}^{-1} h \rangle_0$. With this in mind, it is easy to understand the statements of Theorems 5.5 and 5.6 below and the calculus of the diffusion matrix.

Theorem 5.5.

(i) Let

$$u_j^\beta = \chi \nabla_j I_\beta \tag{5.21}$$

Define T to be the linear transformation from $\mathcal{G}$ to $\mathcal{G}$ s.t.

$$T\Big(\sum_{j\geq 1,\beta\geq 0} b_j^\beta \sigma_j^\beta + \mathcal{L}g\Big) = \sum_{j\geq 1,\beta\geq 0} b_j^\beta u_j^\beta + \mathcal{L}_s g. \tag{5.22}$$

Then T is bounded above by 1 and

$$\begin{aligned} &(a) \quad T\nabla_j I_\alpha(\eta) \perp \mathcal{L}^*\mathcal{G} \qquad \textit{for } j\geq 1, \alpha \geq 0 \\ &(b) \quad \ll T\nabla_i I_\alpha(\eta)\,,\, \sigma_j^{*\,\beta} \gg = \chi \delta_{i,j}\delta_{\alpha,\beta} V(\nabla_i I_\alpha) \end{aligned}$$

(ii) Let M be the matrix (with double indices (α,i) and (β,j))

$$M_{i,j}^{\alpha,\beta} = \ll \nabla_i I_\alpha(\eta), T\nabla_j I_\beta(\eta) \gg .$$

Then the diffusion matrices D is given by

$$D_{i,j}^{\alpha,\beta} = \chi (M^{-1})_{i,j}^{\alpha,\beta} V(\nabla_i I_\alpha) \tag{5.23}$$

As a quadratic form, $D \geq D_s$ and $D \neq D_s$, where $(D_s)_{i,j}^{\alpha,\beta} = \chi\delta_{\alpha,\beta}\delta_{i,j}$. Furthermore, there exist constants D_1 and D_2 such that

$$D_{i,j}^{\alpha,\beta} = \delta_{\alpha,\beta}\delta_{i,j} D_1 + [\delta_{\alpha,i}\delta_{\beta,j}\mathbb{1}(\alpha\neq\beta) + \delta_{\alpha,\beta}\delta_{i,j}\mathbb{1}(\alpha\neq i)] D_2 \tag{5.24}$$

(iii) Since T is one-to-one and the image is dense, its inverse T^{-1} is well-defined on a dense subset of $\overline{\mathcal{G}}$. For any ξ in the domain of T^{-1}

$$\ll \xi, (T^{-1})_s \xi \gg = \ll \xi, T^{-1}\xi \gg = \ll \xi, \xi \gg$$

A similar statement holds with T replaced by T^*.

Proof of (i). The inequality (5.20) is precisely the statement that the map T in (i) is bounded by 1. To prove (a), recall that $\nabla_i I_\alpha$ as an element of $\mathcal{G}$ can be written as $\sum_{\beta\geq 0, j\geq 1} a^{\alpha\beta}_{i,j}\sigma^\beta_j + \mathcal{L}g$ with $g \in \mathcal{G}$ (approximation arguments are needed here and in the following). Then, by the definition of the map T

$$\ll T\nabla_i I_\alpha, \mathcal{L}^* h \gg = \sum_{\beta\geq 0, j\geq 1} a^{\alpha\beta}_{i,j} \ll u^\beta_j, \mathcal{L}^* h \gg + \ll \mathcal{L}_s g, \mathcal{L}^* h \gg$$

From Lemma 5.1 and (5.14) we get

$$\ll \mathcal{L}_s g, \mathcal{L}^* h \gg = -\big\langle \mathcal{L} g, h \big\rangle_0, \quad \ll u^\beta_j, \mathcal{L}^* h \gg = -\big\langle \sigma^\beta_j, h\big\rangle_0.$$

Then

$$\ll T\nabla_i I_\alpha, \mathcal{L}^* h \gg = \big\langle \nabla_i I_\alpha, h\big\rangle_0 = 0$$

To prove (b) of (i) we observe that, again by definition of T,

$$\ll T\nabla_i I_\alpha, \sigma^{*\gamma}_k \gg = \ll \sum_{\beta\geq 0, j\geq 1} a^{\alpha\beta}_{i,j} u^\beta_j, \sigma^{*\gamma}_k \gg + \ll \mathcal{L}_s g\ , \sigma^{*\gamma}_k \gg$$

By Lemma 5.1 and Lemma 5.3 we have,

$$\ll \mathcal{L}_s g\ , \sigma^{*\gamma}_k \gg = \big\langle g, \sigma^{*\gamma}_k\big\rangle_0 = E[\mathcal{L} g \sum_x (x\cdot e_k)\tilde{I}_\gamma(\eta(x))] = \ll \mathcal{L} g, u^\gamma_k \gg$$

Similarly

$$\ll u^\beta_j, \sigma^{*\gamma}_k \gg = \ll \sigma^\beta_j, u^\gamma_k \gg .$$

From the definition of u^γ_k (5.21) and the identity (5.7)

$$\ll T\nabla_i I_\alpha, \sigma^{*\gamma}_k \gg = \ll \sum_{\beta\geq 0, j\geq 1} a^{\alpha\beta}_{i,j}\sigma^\beta_j + \mathcal{L} g\ \ , u^\gamma_k \gg = \ll \nabla_i I_\alpha, u^\gamma_k \gg = \chi\delta_{i,k}\delta_{\alpha,\gamma} V(\nabla_i I_\alpha)$$

Hence we get the result.

Proof of (ii). The diffusion coefficient is a matrix $D^{\alpha,\beta}_{i,j}$ characterized by $\sigma^{*\ \alpha}_i - \sum_{j\geq 1,\beta\geq 0} D^{\alpha,\beta}_{ij}\ \nabla_j I_\beta \in \overline{\mathcal{L}^*\mathcal{G}}$, From (a) of (i),

$$\ll (\sigma^{*\ \alpha}_i - \sum_{j\geq 1,\beta\geq 0} D^{\alpha,\beta}_{i,j} \nabla_j I_\alpha), T\ \nabla_k I_\gamma \gg = 0 \tag{5.25}$$

for all k and γ. From (b) of (i),

$$\sum_{j\geq 1,\beta\geq 0} D^{\alpha,\beta}_{i,j} M^{\beta,\gamma}_{j,k} = \chi\delta_{i,k}\delta_{\alpha,\gamma} V(\nabla_i I_\alpha) \tag{5.26}$$

Hence

$$D_{i,j}^{\alpha,\beta} = \chi(M^{-1})_{i,j}^{\alpha,\beta} V(\nabla_i I_\alpha)$$

Since T is bounded above by 1, so is M (as a quadratic form). Hence as a quadratic form we have $D \geq D_s$. The proof of $D \neq D_s$ is similar to the one in [8]; in fact $D = D_s$ if and only if $\ll T\nabla_i I_\alpha, \nabla_j I_\beta \gg = \delta_{i,j}\delta_{\alpha,\beta} V(\nabla_i I_\alpha)$. This implies $T\nabla_i I_\alpha = \nabla_i I_\alpha + \mathcal{L}_s \phi$. But $\phi = 0$ because otherwise $\| T \| > 1$. Hence $T\nabla_i I_\alpha = \nabla_i I_\alpha$. By (a) of (i) it follows that $\nabla_j I_\alpha(\eta) \perp \mathcal{L}^* \mathcal{G}$ and this leads to contradiction.

Finally, we claim from symmetry arguments,

$$\ll \nabla_i I_\alpha, T\nabla_j I_\beta \gg = 0, \quad \text{if } \delta_{k,i} + \delta_{k,j} + \delta_{k,\alpha} + \delta_{k,\beta} = 1 \text{ or } 3 \text{ for some } 1 \leq k \leq d. \tag{5.27}$$

Consider the reflection symmetry

$$R_1(x) = (-x_1, x_2, x_3),\ R_1(v) = (-v_1, v_2, v_3),\ R_1(\eta(x,v)) = \eta(R_1 x, R_1 v), \qquad x \in \mathbb{Z},\ v \in \mathcal{V}.$$

Clearly, the commutator $[R_1, \mathcal{L}] = 0$. Hence $[R_1, T] = 0$. Furthermore, R_1 is an isometry on $\mathcal{G}$ with inner product $\ll \cdot, \cdot \gg$. Together with $[R_1, T] = 0$, we have

$$\ll R_1 g, TR_1 h \gg = \ll g, Th \gg .$$

By definition of R_1, if $i \neq 1$ then $R_1(\nabla_i I_1) = -\nabla_i I_1$. Similarly, if $\alpha \neq 1$ then $R_1(\nabla_1 I_\alpha) = -\nabla_1 I_\alpha$. On the other hand, if $\alpha \neq 1$ and $i \neq 1$ of if $\alpha = i = 1$ then $R_1(\nabla_i I_\alpha) = \nabla_\gamma I_\alpha$. Hence if the indices satisfy the condition of (5.27) for $k = 1$ then

$$\ll \nabla_i I_\alpha, T\nabla_j I_\beta \gg = \ll R_1(\nabla_i I_\alpha), R_1(T\nabla_j I_\beta) \gg = - \ll \nabla_i I_\alpha, T\nabla_j I_\beta \gg .$$

Hence $\ll \nabla_i I_\alpha, T\nabla_j I_\beta \gg = 0$. This proves (5.27).

Furthermore, there are permutation symmetries. Let $R_{k,\ell}$ $(k \neq \ell)$ denote the permutation of the coordinate axis $k \to \ell$. Then

$$\ll \nabla_i I_\alpha, T\nabla_j I_\beta \gg = \ll \nabla_{R_{k,\ell} i} I_{R_{k,\ell}\alpha}, T\nabla_{R_{k,\ell} j} I_{R_{k,\ell}\beta} \gg \quad (R_{k\ell}\alpha = \alpha \text{ if } \alpha = 0). \tag{5.28}$$

Since D inherits the symmetries of $\ll \nabla_i I_\alpha, T\nabla_j I_\beta \gg$, (5.24) holds. This proves (ii).

Proof of (iii). Suppose $\xi = \sum_{i\geq 1, \beta \geq 0} b_i^\beta u_i^\beta + \mathcal{L}_s g$. From (5.22),

$$\ll \xi, T^{-1}\xi \gg = \ll \sum_{i\geq 1,\beta\geq 0} b_i^\beta u_i^\beta + \mathcal{L}_s g, \sum_{i\geq 1,\beta\geq 0} b_i^\beta \sigma_i^\beta + \mathcal{L} g \gg$$

Hence (iii) follows from explicit computation using Lemma 5.2, 5.3 and the definition of u_i^β.

Remark. The inverse of T, R, was considered in [8]. The proof of its boundedness given there, based on open mapping principle, is incorrect. But the operator R was only used to derive a variational formula for the diffusion coefficient. The variation formula in [8] is indeed correct and only minor modifications are needed to prove it; the variational formula can also be casted in a somewhat simpler manner [25]. In next theorem, we derive a variational characterization of the diffusion coefficient similar to [25].

Theorem 5.6 : *The following variational representation for the diffusion matrix D holds: For any a_j^α,*

$$\left\{ \sum_{\alpha,\beta\geq 0; i,j\geq 1} a_i^\alpha (D^{-1})_{i,j}^{\alpha,\beta} a_j^\beta \right\}^{-1} = \inf V\Big(\sum_{\beta\geq 0, j\geq 1} b_j^\beta \sigma_j^\beta + \mathcal{L}g\Big) \tag{5.29}$$

where the infimum is taken over the set

$$\{b_j^\alpha \in \mathbb{R},\, g\in\mathcal{G} \text{ s.t. } \sum_{\alpha\geq 0, j\geq 1} b_j^\alpha a_j^\alpha \chi V(\nabla_j I_\alpha) = 1\} \tag{5.30}$$

Proof. Suppose $\xi = \sum_{\alpha\geq 0, j\geq 1} a_j^\alpha \nabla_j I_\alpha \in \mathcal{G}_0$. From (iii) of Theorem 5.5

$$\ll \xi, T\xi \gg = \ll T^*\xi, (T^*)^{-1}T^*\xi \gg = \ll T^*\xi, T^*\xi \gg$$

We can rewrite $\ll T^*\xi, T^*\xi \gg$ using variational principle as

$$\ll T^*\xi, T^*\xi \gg = \sup_\zeta [\ll T^*\xi, \zeta \gg - \frac{1}{2} \ll \zeta,\zeta \gg]$$

where

$$\zeta = \sum_{i\geq 1, \beta\geq 0} b_i^\beta \sigma_i^\beta + \mathcal{L}g$$

and the sup is over $b_i^\beta \in \mathbb{R}$ and $g \in \mathcal{G}$. By definition,

$$\ll T^*\xi, \zeta \gg = \ll \xi, T\zeta \gg = \ll \xi, \sum_{i\geq 1, \beta\geq 0} b_i^\beta u_i^\beta \gg = \sum_{\alpha\geq 0, j\geq 1} b_j^\alpha a_j^\alpha \chi V(\nabla_j I_\alpha)$$

Replacing b by λb and optimizing over λ, we have

$$\ll \xi, T\xi \gg^{-1} = \ll T^*\xi, T^*\xi \gg^{-1} = \inf \frac{V(\sum_{\beta\geq 0, j\geq 1} b_j^\beta \sigma_j^\beta + \mathcal{L}g)}{\sum_{\alpha\geq 0, j\geq 1} b_j^\alpha a_j^\alpha \chi V(\nabla_j I_\alpha)}$$

where the inf is over $b_i^\beta \in \mathbb{R}$ and $g \in \mathcal{G}$. We can remove the denominator by restricting the inf to the set given by (5.30). We have thus proved (5.29). This concludes Theorem 5.6.

We can combine Theorem 5.6 and Theorem 5.2 to have a variational formula of D^{-1} involving only $\langle\, , \rangle_0$ and expectations w.r.t. the equilibrium measure. Since this formula is quite complicated and its derivation from Theorem 5.6 and Theorem 5.2 is straightforward, we shall not present it here. Similar formula for asymmetric simple exclusion has been done in details [8]. Finally, we remark that the diffusion matrix $D^{\alpha,\beta}_{i,j}$ has components for $\alpha = 0$ or $\beta = 0$. These components are irrelevant for the problem considered in this paper because the total density I_0 is of lower order than I_α for $\alpha > 0$.

6. Eigenvalues Estimates

In this section we prove Theorem 3.10, which is parallel to Theorem 4.2 in [8] for symmetric simple exclusion. If one can establish the spectral gap and the log-Sobolev inequality for the full dynamics $\mathcal{L}_s + \mathcal{L}^c$ then a parallel proof will conclude Theorem 3.10. We do not follow this route because the spectral gap and log-Sobolev for $\mathcal{L}_s^{ex} + \mathcal{L}^c$ are hard to obtain. Instead, we shall control the fast modes I^-, which are conserved quantities for $\mathcal{L}_s^{ex}$ but not for $\mathcal{L}^c$, using the following lemma.

Lemma 6.1. *Put $Qf \equiv \sum_{q\in\mathcal{Q}} Q_0^q f$. Suppose that $I_\alpha, \alpha = -N+d+1, \cdots, -1$, defined as in (4.1), satisfy*

$$E^\mu\,[\ I_\alpha(\eta_x); I_\beta(\eta_x)\ ] = 0, \quad \alpha < 0, \beta < 0, \alpha \neq \beta. \tag{6.1}$$

where μ is any equilibrium measure for the whole dynamics, i.e.

$$\mu = Z_{r,n}^{-1} \exp\{\sum_{\alpha=0}^{d} n_\alpha I_\alpha(\eta)\}\mu_r\ . \tag{6.2}$$

Let $z_\alpha = E^\mu\,[\ I_\alpha(\eta_x)\ ]$. Then there exist a local function v and a function $g \in \tilde{\mathcal{G}}_z^{ex}$ such that

$$I_\alpha(\eta_x) - z_\alpha = (Qv)(\eta_x) + g(\eta_x), \quad \alpha < 0\ . \tag{6.3}$$

Here $\tilde{\mathcal{G}}_z^{ex}$ is the analog of $\tilde{\mathcal{G}}^{ex}$ with the measure μ_r replaced by μ given by (6.2) with chemical potentials n_α chosen so that $E^\mu\,[\ I_\alpha(\eta_x)\ ] = z_\alpha$, $\alpha = 0, \ldots, d$

Lemma 6.1 is exactly the same as Lemma 4.8 except that the measure is not restricted to μ_r. Note the orthogonality relation (6.1) is the same as (4.3) except that the reference measure was changed accordingly. The proof of Lemma 6.1 is exactly the same as Lemma 4.8 and we omit it.

Recall the definitions of $\hat\omega$, ω and $\tilde\Lambda_k$ in (3.A). Denote $x \in \Lambda_{\bar\ell_1,\sigma}$ by $x \in \sigma$. and let $\bar I_\sigma^+ = \bar I^+_{\Lambda_{\bar\ell,\sigma}}$ For any local function h_x, let

$$\bar h_\sigma = E^{\mu_r}[Av_{x\in\sigma} h_x \mid \bar I_\sigma^+].$$

From the definition of $\hat\omega$ we have $(\hat\omega * h)(x) = Av_{|\sigma - x|\le k}\, \bar h_\sigma$ where $\sigma \in (2\bar\ell+1)\mathbb{Z}^d$. For the rest of this chapter, $Av_{|\sigma-x|\le k}$ takes this meaning and we shall not repeat it. The following theorem is a restatement of Theorem 4.6 in [8]. Indeed, Theorem 6.2 summarizes the result for the "case 1" in the proof of Theorem 4.6 in [8]. The proof given in [8] is somehow sketchy; details can be found in [24].

Theorem 6.2 *Suppose h is a local function. Then for any constant $\gamma > 0$ and smooth function J,*

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\Big\{\varepsilon^{d-2}\int f\sum_{x\in\Lambda_L} J(\varepsilon x)\,\big[\,Av_{|\sigma-x|<k}\,\bar h_\sigma - E^{\mu_r}[Av_{|\sigma-x|<k}\,\bar h_\sigma \mid \bar I_{k,x}]\,\big]\,d\mu_r \\ -\gamma\varepsilon^{d-4}D^{ex}_{\Lambda_L}(\sqrt{f})\Big\}\le 0 \tag{6.4}$$

Here $k=\ell\varepsilon^{-2/d}$ and $E^{\mu_r}[h \mid \bar I_{k,x}]$ has been defined in Section 4, $\bar I_k = \{\bar I_{\alpha,k}, \alpha = -N+d+1,\dots,d\}$. Finally D^{ex} is the Dirichlet form associated to $\mathcal{L}^{ex}_s$, defined in (3.14)

Proof of Theorem 3.10. *Step 1.* Consider the decomposition

$$Av_{x\in\sigma}h_x = (\ Av_{x\in\sigma}h_x - \bar h_\sigma\ ) + \bar h_\sigma := \tilde h_\sigma + \bar h_\sigma\ . \tag{6.5}$$

Since $E^{\mu_r}[\tilde h_\sigma \mid \bar I^+_\sigma]=0$, we can bound $\tilde h_\sigma$ with standard perturbation argument so that

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\ \{\varepsilon^{-2}\int fAv_x J(\varepsilon x)\Big[\ Av_{|\sigma-x|<k}\tilde h_\sigma\ \Big]\,d\mu_r \\ -\text{const.}\gamma^{-1}\varepsilon^d\sum_{x\in\Lambda_L}J^2(\varepsilon x)Av_{|\sigma-x|\le k}\int V_\ell(h,\bar I^+_\sigma)fd\mu_r - \gamma\varepsilon^{d-4}D_\Lambda(\sqrt{f})\}\le 0.$$

From the entropy bound and the fact that ψ is a local Gibbs state defined in (3.1), one can check that

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\left\{\varepsilon^d\sum_{x\in\Lambda_L}Av_{|\sigma-x|\le k}J(\varepsilon x)^2\int V_\ell(h,\bar I^+_\sigma)fd\mu_r - \varepsilon^{-1}s(f|\psi)\right\} \\ \le \text{const.}\int J(z)^2dz\ V(h,r)$$

Hence the second term converges to the second term in (3.29). This procedure was carried out in details in e.g., [15,8] and we omit the details.

Step 2. We now bound $\bar h_\sigma$. Applying Theorem 6.2, one has

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\Big\{\varepsilon^{d-2}\int f\sum_{x\in\Lambda_L} J(\varepsilon x)\,\big[\,Av_{|\sigma-x|<k}\,\bar h_\sigma - E^{\mu_r}[Av_{|\sigma-x|<k}\,\bar h_\sigma \mid \bar I_{k,x}]\,\big]\,d\mu_\beta \\ -\gamma\varepsilon^{d-4}D_{\Lambda_L}(\sqrt{f})\Big\}\le 0$$

Note that the conditional expectation is w.r.t. $\bar{I}_k$ on the block Λ_k and $\bar{I}_k = \{\bar{I}_{\alpha,k}, \alpha = -N+d+1,\dots,d\}$. We would like to replace the canonical expectation by the grand canonical expectation. More precisely, let $\mu_{2,w}$ be the grandcanonical measure such that

$$E^{\mu_{2,w}}[I_\alpha] = w_\alpha, \alpha = -N+d+1,\cdots,d;$$

explicitly $\mu_{2,w}$ is the product measure with marginal measure given by

$$\mu_{2,w} = Z_{r,n}^{-1}\exp\{\sum_{\alpha=-N+d+1}^{d} \lambda_\alpha I_\alpha(\eta)\}\mu_r. \tag{6.6}$$

Note that in the definition of $\mu_{2,w}$ we specify the averages of I_α also for $\alpha > 0$, while in the definition of $\mu_{1,w}$ in Section 3 we only specified the averages of I_α for $\alpha < 0$. Fixed the local empirical averages $\bar{I}_{\alpha,k}$ the corresponding grand canonical expectation is defined by

$$E^{\mu_{2,w}}[Av_{|\sigma-x|<k}\,\bar{h}_\sigma]$$

with $w = \bar{I}_{k,x}$. Let ϑ_x denote the difference

$$\vartheta_x = E^{\mu_{2,w}}[Av_{|\sigma-x|<k}\,\bar{h}_\sigma] - E^{\mu_r}[Av_{|\sigma-x|<k}\,\bar{h}_\sigma \mid \bar{I}_{k,x}]$$

We claim that this difference is negligible in the sense that for any $\gamma > 0$

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\left\{\varepsilon^{d-2}\int f\sum_{x\in\Lambda_L} J(\varepsilon x)\vartheta_x d\mu_r - \gamma\varepsilon^{-2}s(f)\right\} \le 0 \tag{6.7}$$

The strong equivalence of ensembles states that ϑ_x is small provided $\bar{I}_{k,x}$ is bounded away from 0 or 1. A precise statement is given in the Corollary 6.3 in [21]. We state a simplified version in our context: For some small constant γ,

$$|\vartheta_x| \le \begin{cases} k^{-d} & \text{if } k^{-\gamma} \le \bar{I}_{k,x} \le 1-k^{-\gamma} \\ k^{-d/2} & \text{otherwise.}\end{cases} \tag{6.8}$$

The second bound is proved in [21] only if $\bar{I}_{k,x}$ is bounded away from 0 or 1 in the sense that $k^{-d/2} \le \bar{I}_{k,x} \le 1-k^{-d/2}$. On the other hand, if $\bar{I}_{k,x} \le k^{-d/2}$ then the number of particles for each color is less than $k^{-d/2}$. Hence both $E^{\mu_{2,w}}[Av_{|\sigma-x|<k}\,\bar{h}_\sigma]$ and $E^{\mu_r}[Av_{|\sigma-x|<k}\,\bar{h}_\sigma \mid \bar{I}_{k,x}]$ are bounded by $Ck^{-d/2}$ and (6.8) holds trivially.

From (6.8) one has

$$\left|\varepsilon^d E^{\mu_r}\Big[\sum_{x\in\Lambda_L} J(\varepsilon x)\vartheta_x\Big]\right| \le Ck^{-d}$$

From the entropy inequality the left side of (6.7) is bounded by

$$\varepsilon^{d-2}\log E^{\mu_r}\left[\exp\left\{\sum_{x\in\Lambda_L} J(\varepsilon x)\vartheta_x\right\}\right]$$

It is now standard to bound the last term by

$$\varepsilon^{d-2}\log E^{\mu_r}\Big[\exp\Big\{\sum_{x\in\Lambda_L} J(\varepsilon x)\vartheta_x\Big\}\Big]$$
$$\le \varepsilon^{d-2}E^{\mu_r}\Big[\sum_{x\in\Lambda_L} J(\varepsilon x)\vartheta_x\Big] + C\varepsilon^{d-2}\sum_{x\in\Lambda_L} E^{\mu_r}\big[\ \vartheta_x^2\ \big] + C\varepsilon^{-2}k^{-d} \le C\varepsilon^{-2}k^{-d}$$

Taking the limit and remembering $k=\ell\varepsilon^{-2/d}$, we have proved

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\Big\{\varepsilon^{d-2}\int f\sum_{x\in\Lambda_L} J(\varepsilon x)\big[\ Av_{|\sigma-x|<k}\,\bar h_\sigma - E^{\mu_{1,\bar I_{k,x}}}[Av_{|\sigma-x|<k}\,\bar h_\sigma]\ \big]\,d\mu_r$$
$$-\gamma\varepsilon^{d-4}D_{\Lambda_L}(\sqrt{f})\Big\} - \delta^{-1}\varepsilon^{-2}s(f|\psi)\le 0$$

The change from $s(f)$ to $s(f|\psi)$ is not important in the previous argument.

Step 3. We would like to replace the expectation $E^{\mu_{2,w}}$ with the one w.r.t. $\mu_{1,Y}$ defined in section 3, with $y_\alpha = w_\alpha$ for $\alpha=0,\dots,d$. Let us consider the difference

$$u(w) = E^{\mu_{2,w}}\left[Av_{|\sigma-x|<k}\,\bar h_\sigma\right] - E^{\mu_{1,Y}}\left[Av_{|\sigma-x|<k}\,\bar h_\sigma\right] \tag{6.9}$$

To prove theorem 3.10, it suffices to prove for δ small enough,

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\left\{\varepsilon^{d-2}\int f\sum_{x\in\Lambda_L} J(\varepsilon x)\tau_x u\,d\mu_r - \gamma\varepsilon^{d-4}D_{\Lambda_L}(\sqrt{f}) - \delta^{-1}\varepsilon^{-2}s(f|\psi)\right\}\le 0\ , \tag{6.10}$$

Step 4. Since μ_i are product measures for $i=1,2$, we can drop the average over σ and u is simply

$$u(w) = E^{\mu_{2,w}}\left[E^{\mu_r}[h\mid \bar I_\ell^+]\right] - E^{\mu_{1,Y}}\left[E^{\mu_r}[h\mid \bar I_\ell^+]\right] \tag{6.11}$$

Note that from the definition of $\mu_i, i=1,2$, we have

$$E^{\mu_{2,w}}[I_\alpha] = w_\alpha = Y_\alpha = E^{\mu_{1,Y}}[I_\alpha],\quad \alpha=0,\dots,d \tag{6.12}$$

But this equality fails for $\alpha<0$. Furthermore, even though (6.12) holds, it fails if I_α is replaced by nonlinear functionals of I_α ($\alpha\ge 0$).

Define

$$\hat u_x(w) = \tau_x u(w) - \phi_x = \tau_x u(w) - \sum_{\alpha=-N+(d+1)}^{d} c_{\ell,\alpha,x}(I_{\alpha,x} - z_{\alpha,x}) \tag{6.13}$$

where

$$z_{\alpha,x} = E^{\psi}[I_{\alpha,x}],\quad c_{\ell,\alpha,x} = \frac{\partial u}{\partial w_\alpha}\Big|_{w=z_x}.$$

From the entropy inequality one has for $\delta > 0$,

$$\epsilon^{-2} \int f Av_{x\in\Lambda_L} J(\varepsilon x)\hat{u}_x d\mu \leq \delta^{-1}\epsilon^{-2+d} \log E^{\psi}\Big[\exp\{\delta \sum_x J(\varepsilon x)\hat{u}_x\}\Big] + \delta^{-1}\varepsilon^{-2} s(f|\psi) \tag{6.14}$$

By definition of $c_{\ell,\alpha,x}$, we can apply Lemma 3.7 to bound the first term on the right side of (6.14) provided δ is small enough. Hence one has

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \delta^{-1}\epsilon^{-2+d} \log E^{\psi}\Big[\exp\{\delta \sum_{x\in\Lambda_L} J(\varepsilon x)\hat{u}_x\}\Big] = 0 \tag{6.15}$$

provided δ is small enough. Therefore, to prove Theorem 3.10, we only have to bound ϕ_x defined by the second of (6.13), in the sense that for any $\gamma > 0$ and δ sufficiently small

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \Big\{\varepsilon^{d-2}\int f \sum_{x\in\Lambda_L} J(\varepsilon x)\phi_x d\mu_r - \gamma\varepsilon^{d-4} D_{\Lambda_L}(\sqrt{f}) - \delta^{-1}\varepsilon^{-2} s(f|\psi)\Big\} \leq 0\ . \tag{6.16}$$

Step 5. We claim the following estimates hold:

$$c_{\alpha,\ell,x} = 0, \quad \alpha \geq 0, \tag{6.17}$$

and

$$\lim_{\ell\to\infty} c_{\alpha,\ell,x} = 0, \quad \alpha < 0. \tag{6.18}$$

We shall prove these two estimates in step 6. Assuming these two estimates, we now prove (6.16).

From Lemma 6.1, one has, for all $\alpha < 0$, $I_{\alpha,x} - z_{\alpha,x} = Q_x v_x + \xi_{\alpha,x}$ with $\xi_{\alpha,x} \in \tilde{\mathcal{G}}^{ex}_{z_x}$. From the integration by parts,

$$\int f \sum_{x\in\Lambda_L} Q_x v_x d\mu \leq \text{const.} D_{\Lambda_L}(\sqrt{f}).$$

Hence (6.16) holds if we neglect the contribution of $\xi_{\alpha,x} \in \tilde{\mathcal{G}}^{ex}_{z_x}x$. On the other hand, from the entropy bound,

$$\begin{aligned}&\epsilon^{-2}\int f Av_{x\in\Lambda_L} J(\varepsilon x)\xi_{\alpha,x} d\mu \leq \\ &\delta^{-1}\epsilon^{-2+d}\log E^{\psi}\Big[\exp\{\delta\sum_{x\in\Lambda_L} J(\varepsilon x)\xi_{\alpha,x}\}\Big] + \delta^{-1}\varepsilon^{-2}s(f|\psi)\end{aligned} \tag{6.19}$$

Because $\xi_{\alpha,x} \in \tilde{\mathcal{G}}^{ex}_{z_x}$, we can apply Lemma 3.7 and this concludes the proof of (6.16).

Step 6. Finally we have to prove (6.17) and (6.18). From (6.12), the value of $u(w)$ does not change if we replace h in the definition (6.11) by

$$h_x := h - \zeta_x := h - \sum_{\alpha \geq 0} (\ e + \varepsilon a_{\alpha,x}\ )(\ I_{\alpha,x} - z_{\alpha,x}\ ).$$

for some constants e and $a_{\alpha,x}$.

Since ψ is a local Gibbs state given by (3.1), we can choose constants e and $a_{\alpha,x}$ such that $h_x \in \tilde{\mathcal{G}}_{z_x}$. Because the statements of (6.17) and (6.18) are local, we shall drop the subscript x and assume that, without loss of generality, $h_x \in \tilde{\mathcal{G}}_{z_x}$.

By definition one has the identity

$$\frac{\partial u}{\partial w_\alpha}\Big|_{w=z} = \sum_{\beta=-N+d+1}^{d} \sum_x \left\{ E^{\mu_{2,w}}\ [\ E^{\mu_r}[h|\bar{I}_\ell^+]; \tau_x I_\beta]\ \frac{\partial \lambda_\beta}{\partial w_\alpha} - E^{\mu_{1,Y}}\ [\ E^{\mu_r}[h|\bar{I}_\ell^+]; \tau_x I_\beta]\ \frac{\partial n_\beta}{\partial w_\alpha} \right\}_{w=z} \tag{6.20}$$

By definition, $\mu_{2,z} = \mu_{1,z} := \mu_z$ and

$$\frac{\partial n_\beta}{\partial w_\alpha}\Big|_{w=z} = 0, \quad \alpha < 0.$$

From the orthogonality relation $E^{\mu_z}[I_\alpha; I_\beta] = 0$ for $\alpha \neq \beta$, one has

$$\frac{\partial w_\alpha}{\partial \lambda_\beta}\Big|_{w=z} = 0, \quad \text{if } \alpha \neq \beta.$$

Hence

$$\frac{\partial \lambda_\beta}{\partial w_\alpha}\Big|_{w=z} = 0, \quad \text{if } \alpha \neq \beta.$$

Since $h \in \tilde{\mathcal{G}}_z$, we have $\sum_x E^{\mu_z}\ \big[\ E^{\mu_z}[h|\bar{I}_\ell^+]; \tau_x I_\beta\big] = 0$ if $\beta \geq 0$. Using these identities in (6.19), we have proved (6.17).

For $\alpha < 0$ we have

$$\frac{\partial u}{\partial w_\alpha}\Big|_{w=z} = \sum_x E^{\mu_z}\left[E^{\mu_z}[h \mid \bar{I}_\ell^+]; I_\alpha(\eta_x)\right] \frac{\partial \lambda_\alpha}{\partial w_\alpha}\Big|_{w=z} \tag{6.21}$$

Using the Schwartz inequality we get

$$\begin{aligned} \sum_x E^{\mu_z}\left[E^{\mu_z}[h \mid \bar{I}_\ell^+]; I_\alpha(\eta_x)\right] &= E^{\mu_z}\left[E^{\mu_z}[h \mid \bar{I}_\ell^+]; \sum_{|x|\leq \ell} I^-(\eta_x)\right] \\ &\leq E^{\mu_z}\left[E^{\mu_z}[h \mid \bar{I}_\ell^+]; E^{\mu_z}[h \mid \bar{I}_\ell^+]\right]^{1/2} E^{\mu_z}\Big[\sum_{|x|\leq\ell} I^-(\eta_x); \sum_{|x|\leq\ell} I^-(\eta_x)\Big]^{1/2} \\ &\leq \ell^{d/2} E^{\mu_z}\left[\ E^{\mu_z}[h \mid \bar{I}_\ell^+]; E^{\mu_z}[h \mid \bar{I}_\ell^+]\right]^{1/2} \end{aligned} \tag{6.22}$$

Next lemma shows that, since $h \in \tilde{\mathcal{G}}_z$, the last term approach to 0 as $\ell \to \infty$. This concludes (6.18).

Lemma 6.3. *Suppose* $h \in \tilde{\mathcal{G}}_z$. *Then*

$$\lim_{\ell\to\infty} \ell^d E^{\mu_z}\Big[\ E^{\mu_z}[h \mid \bar{I}^+_\ell]; E^{\mu_z}[h \mid \bar{I}^+_\ell]\Big] = 0 \tag{6.23}$$

Proof. Let $\hat{h}(Y) = E^{\mu_{1,Y}}[h]$, where $\mu_{1,Y}$ was defined in (3.16). It is not hard to check that equivalence of ensembles implies that for some $\gamma > 0$

$$E^{\mu_z}\Big[\big(E^{\mu_z}[h \mid \bar{I}^+_\ell] - \hat{h}(Y = \bar{I}^+_\ell)\big); \big(E^{\mu_z}[h \mid \bar{I}^+_\ell] - \hat{h}(Y = \bar{I}^+_\ell)\big)\Big] \le \text{const.}\ell^{-d-\gamma}. \tag{6.24}$$

From the assumption $h \in \tilde{\mathcal{G}}_r$ one has

$$\frac{\partial \hat{h}}{\partial Y_\alpha}\Big|_{Y=z} = 0, \quad \alpha \ge 0$$

where $z_\alpha = E^{\mu_z}[\ I_\alpha\ ]$. From the Taylor expansion we have

$$\hat{h}(Y) = A(Y - z) + \text{ higher order terms}$$

where A denote the quadratic terms. Assume that $\bar{I}^+_\ell$ has a Gaussian distribution with mean z. Then

$$E^{\mu_z}\Big[\ \hat{h}(\bar{I}^+_\ell); \hat{h}(\bar{I}^+_\ell)\Big] \sim E^{\mu_z}\Big[\ Q(\bar{I}^+_\ell - z); Q(\bar{I}^+_\ell - z)\Big] \sim \ell^{-2d} \tag{6.25}$$

and we have proved (6.23). The proof of (6.25) is elementary and involves only i.i.d. random variables. Since similar estimates were proved in [8] (Lemma 3.1) and in more details in [24] (Lemma 11.3), we omit the details.

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