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On a Model for Quantum Friction I
Fermi's Golden Rule and Dynamics at Zero Temperature

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Abstract. We investigate the dynamics of a quantum particle in a confining potential linearly coupled to a bosonic field at temperature zero. For a massive field we show, by employing complex deformation techniques, that the Markovian semi-group which approximates the particle dynamics on the time scale $\tau = \lambda^2 t$ (λ strength of the coupling) is determined by the resonances of the full energy operator. We also show that Markovian master equation technique leads to the right prediction for the life-time of resonances. We discuss the dissipation of the particle into its ground state both in the time mean and on the above time scale.

1. Introduction

Friction, as a notion in classical physics, has been set on solid mathematical grounds by the theory of Ornstein-Uhlenbeck processes. The dynamics of a particle experiencing frictional forces is governed by the Langevin equation, a second order non-linear stochastic differential equation in the time variable. This equation has been widely investigated beginning with the classical paper [12]. The long-standing conjecture that the Langevin equation (an equation in macroscopic variables) can be derived from microscopic considerations, has been positively answered in 1965 by Ford, Kac, and Mazur [8]. Starting from the model of a particle linearly coupled to a chain of harmonic oscillators

$$H = H_{\mathrm{p}} + H_{\mathrm{osc}} + \lambda H_{\mathrm{I}} = p^2 + V(q) + \sum_{k=1}^{N} \left[\frac{1}{2}(p_k^2 + \omega_k^2 q_k^2 - \omega_k) + \lambda \alpha_k q q_k \right], \tag{1.1}$$

(λ = friction constant) they derive the Langevin equation as follows: They first integrate the oscillator variables, under the assumption that their initial positions and momenta are given by the Gibbs distribution. Then they take the limit $N \to \infty$ assuming $\alpha_k = \alpha(\omega_k)/\sqrt{N}$, where the function α imposes an ultraviolet cutoff on the interaction. The equation thus obtained has a memory term which is further removed by taking the ultraviolet cutoff to infinity and letting α become linear: $\alpha(\omega) = \omega$. One then recovers the Langevin equation as contracted description of the particle interacting with the environment.

The importance of Ford-Kac-Mazur work goes beyond classical physics, it opened the way to the quantization of friction. One can introduce the quantum Ornstein-Uhlenbeck process and derive the corresponding quantum Langevin equation by a procedure formally analogous to the above one. Starting from the FKM seminal work, a vast number of papers has been devoted to the quantum Langevin equation (QLE). In contrast to its classical counterpart, QLE is a singular object which is difficult to study [17]. To our knowledge there are no rigorous results beyond bounded perturbations of a quadratic model.

In this paper, the first in a series devoted to the quantum (and classical [10]) notion of friction, we study some zero temperature models (we investigate finite temperature models in [9]). The explicit Hilbert space formalism leads to the possibility of studying the model using spectral theory, and allows us to relate the values obtained for the life-time of excited states (Fermi Golden Rule) with the ones obtained from Davies' master equation technique. We adopt the line of thought initiated by Davies [5], namely we keep the ultraviolet cutoff in the interaction, and study the dynamics on a time scale $\tau = \lambda^2 t$. The loss of memory in standard discussions of Langevin equation is obtained by removing the ultraviolet cutoff. This leads to strong singularities in the quantum case: We lose memory by passing to the above time scale, on which the dynamics is Markovian in the limit $\lambda \to 0$. We study this

Markovian evolution and show that it is determined by the resonances of the full energy operator.

To specify the model (for extensions see the remarks below), we take the quantum analog of (1.1), namely a particle interacting with bosonic field (usually referred to as the "bath")

$$H_\lambda = H_{\mathrm{p}} + H_{\mathrm{bos}} + \lambda x \otimes \phi(\alpha), \tag{1.2}$$

acting on $\mathcal{H} = L^2(\mathbf{R}) \otimes \mathcal{H}_{\mathrm{fock}}$, where $\mathcal{H}_{\mathrm{fock}}$ is the symmetric (bosonic) Fock space, constructed from $L^2(\mathbf{R})$. H_{bos}, the energy operator of free bosons, is the second quantization of the function $\omega(k) = |k| + m_0$ (the argument easily extends to the case when $\omega(k) = \sqrt{m_0^2 + k^2}$, see remark 2 below). In the sequel we will be mainly interested in the technically more accessible case of massive bosons ($m_0 > 0$). For $\alpha \in L^2(\mathbf{R})$

$$\phi(\alpha) = \frac{1}{\sqrt{2}}(a(\alpha) + a^*(\overline{\alpha})),$$

is a time-zero field, and $a(\alpha)$, $a^*(\alpha)$ are the boson annihilation and creation operators. The Hamiltonian of the particle is given by

$$H_{\mathrm{p}} = -\Delta + V(x),$$

and we suppose that $V(x) \geq 0$ and $V(x) \to \infty$ as $x \to \infty$. Formally, in (1.2), one should write $H_{\mathrm{p}} \otimes I$ and $I \otimes H_{\mathrm{bos}}$. However, whenever it is clear within the context, we will write A for $A \otimes I$ or $I \otimes A$. The operator H_{p} has purely discrete spectrum $\{E_j\}$ (enumerated in increasing order) and we denote the corresponding set of normalized eigenvectors by $\{\psi_j\}$. The precise technical conditions on H_{p} and α will be set below. For future reference, we note that if $\lambda = 0$ the system decouples, and

$$\begin{aligned}
\sigma_{\mathrm{ac}}(H_0) &= [m_0 + E_0, \infty[, \\
\sigma_{\mathrm{sc}}(H_0) &= \emptyset, \\
\sigma_{\mathrm{pp}}(H_0) &= \{E_j : j = 0, 1, \cdots\},
\end{aligned}$$

with corresponding eigenvectors $\{\psi_j \otimes \Omega\}$, where Ω is the vacuum on $\mathcal{H}_{\mathrm{fock}}$. To avoid discussion of some pathological cases (see [3] and remark 3 below), we will assume throughout the paper that for $i \geq 0$,

$$E_{i+1} - E_i > m_0. \tag{1.3}$$

Spectral properties of the Hamiltonian (1.2) have been rarely investigated. To the best of our knowledge there are no results except on the explicitly diagonalizable quadratic model [1], [2], [3]. The technical problems are obvious: the perturbation $H_{\mathrm{I}} = x \otimes \phi(\alpha)$ is not relatively compact. Even worse, in the mass-less model ($m_0 = 0$), the embedded eigenvalues of the decoupled system ($\lambda = 0$) are thresholds. However, one still expects (at least in the generic sense) that the eigenvalues $\{E_j : j \geq 1\}$ of the decoupled system will dissolve into resonances after turning on the "small" perturbation. The resonances determine the life-time of the excited states, namely we expect that

$$|(\psi_j \otimes \Omega, \exp(-itH_\lambda)\psi_j \otimes \Omega)| \sim \exp(-\lambda^2 t\Gamma_j), \tag{1.4}$$

where Γ_j is given by the Fermi Golden Rule. The physical mechanism behind (1.4) is simple: A particle interacting with the boson field at zero temperature dissipates exponentially fast (on the time scale $\tau = \lambda^2 t$) to its ground state, radiating energy into the "bath". Our goal is to evaluate Γ_j using complex deformation techniques, and to show that the Davies master equations approach leads to the right prediction for Γ_j. To avoid problems with thresholds, we restrict ourselves to the massive model ($m_0 > 0$). To specify the simplest set of conditions for which our results apply, we introduce some notation. For $0 < \delta < \pi/2$, let $H^2(\delta)$ be the Hardy class of the cone $\mathcal{A}(\delta) = \{k : |\mathrm{Im}(k)| < \tan(\delta)|\mathrm{Re}(k)|\}$. It consists of all functions $f(k)$ which are analytic in $\mathcal{A}(\delta)$ and satisfy

$$\|f\|^2_{H^2(\delta)} = \sup_{|\theta|<\delta} \int_{-\infty}^{\infty} |f(\exp(i\theta)k)|^2 dk < \infty.$$

By $\tau(H_0)$ we denote the threshold set of the the operator H_0,

$$\tau(H_0) = \{E_j + nm_0 : j = 0, 1, \cdots; n = 1, 2, \cdots\}. \tag{1.5}$$

Let $u(\theta)$ be the dilation unitary group on $L^2(\mathbf{R})$, defined in the momentum representation by

$$(u(\theta)f)(k) = \exp(\theta/2)f(\exp(\theta)k), \tag{1.6}$$

and let $U(\theta) = \Gamma(u(\theta))$ be its second quantization.

We set the hypotheses:

(H1) α *is a real, even function on* $\mathbf{R}$, *and* $\alpha \in H^2(\delta)$ *for some* $\delta > 0$. *Furthermore, the bounded operator-valued function*

$$U(\theta)\, x \otimes \phi(\alpha)\, U(-\theta)(H_0 + i)^{-1} = x \otimes \phi(u(\theta)\alpha)(H_0 + i)^{-1}, \tag{1.7}$$

has an analytic continuation into the strip $\mathcal{S}(\delta) = \{\theta : |\mathrm{Im}(\theta)| < \delta\}$.

(H2) x^2 *is* H_{p} *bounded.*

The hypotheses (H1) and (H2) ensure that for small λ, H_λ is an essentially self-adjoint operator on $D_0 = C_0^\infty(\mathbf{R}) \otimes F$, where F is a set of finite particle vectors. The analyticity assumptions of the hypothesis (H1) are a variant of the dilation analyticity requirement for two-body potentials (see [15]). They are satisfied, for example, if $\alpha(k) = (1 + |k|^n)^{-1}$, for some integer $n > 0$. We define the function

$$\alpha_m(k) = \begin{cases} \alpha(k-m) & \text{if } k \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

Our main result is:

Theorem 1.1. *Suppose that (H1) and (H2) are satisfied, and that for some $j \geq 1$ one has $E_j \notin \tau(H_0)$. Then, if λ is small enough, there exists a dense set of vectors $D \subset \mathcal{H}$ such that, for $\Phi, \Psi \in D$, the matrix elements*

$$(\Phi, (H_\lambda - z)^{-1}\Psi), \tag{1.8}$$

have a meromorphic continuation from the upper half plane onto $O_j = \{z : |z - E_j| < C_j\}$. On O_j, the functions (1.8) are analytic except for a simple pole (independent of Φ, Ψ) at

$$E_j(\lambda) = E_j + a_j^{(2)}\lambda^2 + \cdots. \tag{1.9}$$

Furthermore

$$\mathrm{Im}(a_j^{(2)}) = -\Gamma_j = -\pi \sum_{i<j} |(\psi_i, x\psi_j)|^2 \left(\alpha_{m_0}(E_j - E_i)\right)^2, \tag{1.10}$$

and

$$\mathrm{Re}(a_j^{(2)}) = \Lambda_j = -\sum_i |(\psi_i, x\psi_j)|^2 \,\mathrm{PV}\!\int_0^\infty \frac{\alpha(k)^2}{\omega(k) - (E_j - E_i)} dk, \tag{1.11}$$

where PV stands for the principal value integral.

The following theorem is concerned with the dynamics of the system, and is a rigorous version of (1.4).

Theorem 1.2. *Let us introduce the time scale $\tau = \lambda^2 t$. Then under the conditions of Theorem 1.1, for any two fixed constants $b > a > 0$, one has*

$$\lim_{\lambda \to 0} \sup_{\tau \in (a,b)} |(\psi_i \otimes \Omega, \exp(-it(H_\lambda - E_j))\psi_j \otimes \Omega) - \delta_{ij} \exp(-\tau(\Gamma_j + i\Lambda_j))| = 0.$$

In particular, if τ is kept fixed,

$$\lim_{\lambda \to 0} |(\psi_i \otimes \Omega, \exp(-itH_\lambda)\psi_j \otimes \Omega)| = \delta_{ij} \exp(-\tau\Gamma_j).$$

Remark 1: The result can be extended to a more general model where $L^2(\mathbf{R})$ is replaced by an arbitrary Hilbert space $\mathcal{H}_p$, and H_λ is of the form

$$H_\lambda = H_{\rm p} + H_{\rm bos} + \lambda Q \otimes \phi(\alpha). \tag{1.12}$$

Here $H_{\rm p}$ is a semi-bounded self-adjoint operator with discrete spectrum, and Q a self-adjoint perturbation satisfying

$$\|Q^2\psi\| \leq C(\|H_{\rm p}\psi\| + \|\psi\|), \tag{1.13}$$

for some $C > 0$ and for all $\psi \in D(H_{\rm p})$. If (H1) and (1.13) are satisfied, and if E_j is a simple eigenvalue of $H_{\rm p}$ such that $E_j \notin \tau(H_0)$, then both theorems remain valid.

Remark 2: One can replace condition (1.13) with

$$|(\psi, Q^2\psi)| \leq C(\psi, (H_{\rm p} + 1)\psi), \tag{1.14}$$

for any $\psi \in \mathcal{Q}(H_{\rm p})$, the form domain of $H_{\rm p}$. The passage to quadratic forms includes a number of additional technicalities: the family of operators (1.7) has to be replaced with

$$(H_0 + 1)^{-1/2} Q \otimes \phi(u(\theta)\alpha)(H_0 + 1)^{-1/2}. \tag{1.15}$$

In any case, if (H1) (with (1.15)) and (1.14) are satisfied, and if E_j is a simple eigenvalue of $H_{\rm p}$ such that $E_j \notin \tau(H_0)$, then again both theorems remain valid. In addition, the results can be easily extended to the bosonic field of an arbitrary dimension, and to a class of rotationally invariant ω's which includes $\omega(k) = \sqrt{m_0^2 + |k|^2}$. For a list of conditions which such ω's has to satisfy, we refer to [3].

Remark 3: Although one can conjecture that under the hypotheses (H1)-(H2) and some reasonable genericity assumptions,

$$\begin{aligned} \sigma_{\rm ac}(H_\lambda) &= [C_2(\lambda), \infty[, \\ \sigma_{\rm sc}(H_\lambda) &= \emptyset, \\ \sigma_{\rm pp}(H_\lambda) &= \{C_1(\lambda)\}, \end{aligned} \tag{1.16}$$

holds for small λ with $C_1(\lambda) < C_2(\lambda)$, we cannot prove this. The above result has been obtained for quadratic potentials by Arai [3] and it is certainly a very interesting problem to extend it to a more general class of potentials. In [3] Arai also discovered that, in a massive model, it can happen that all eigenvalues survive after turning on the small perturbation! He considers the case when

$$H_{\rm p} = \frac{1}{2}(p^2 + \omega_0^2 q^2), \tag{1.17}$$

and $m_0 > \omega_0$, and shows (essentially under the hypothesis (H1)) that *all* embedded eigenvalues of H_λ survive, independently of how small λ is. This is consistent with our result since for the model (1.17) one easily calculates that $\Gamma_j = 0$. The expression (1.10) yields that this result is certainly not "generically" true.

That on the the time scale $\tau = \lambda^2 t$ the dynamics of a particle interacting with the environment is Markovian has been known since early seventies. Davies [5], [6], for a very general class of models, explicitly constructed the generator of the semi-group which approximates the dynamics of the particle subsystem on the time scale given by τ. To apply his construction (see Section 2.5 for details), one has to regularize the Hamiltonian H_λ. Let

$$H_\lambda(L) = H_{\mathrm{p}} + H_{\mathrm{bos}} + \lambda x_L \otimes \phi(\alpha),$$

where $x_L = x$, if $|x| \le L$, and $x_L = 0$ if $|x| > L$. Davies theory gives a bounded operator K_L on $\mathcal{H}_p$, such that for any $\psi \in \mathcal{H}_p$,

$$\lim_{\lambda\to 0} \sup_{\tau\in(a,b)} \| \exp(itH_0(L)) \exp(-itH_\lambda(L))\psi \otimes \Omega - \exp(-\tau K_L)\psi \otimes \Omega\| = 0.$$

We will show that the operator K_L is diagonal in the basis ψ_j with eigenvalues which are, as $L \to \infty$, given by $\{0\} \cup \{\Gamma_j + i\Lambda_j \,:\, j \ge 1\}$. One advantage of Davies' construction is that it applies without any changes to the case of a mass-less field. We refer to Section 2.6 for a comparison of the two approaches to the dynamics of non-isolated systems.

The consequence of Theorem 1.2 is that for any $\psi \in \mathcal{H}_p$ (keeping in the first limit $\tau = \lambda^2 t$ fixed),

$$\lim_{\tau\to\infty} \lim_{\lambda\to 0} |(\psi \otimes \Omega, \exp(-itH_\lambda)\psi \otimes \Omega)| = |(\psi, \psi_G)|^2, \tag{1.18}$$

where $\psi_G = \psi_0$ is the ground state of H_{p}. Physically, (1.18) stands for "*on the time scale* $\lambda^2 t$ *the particle dissipates into its ground state, radiating energy into the bath*". It is an interesting question, raised in [5], [7], whether or not one can improve (1.18), for example by showing that it remains valid if we first let $t \to \infty$ and then $\lambda \to 0$. To answer that question in our model, some information on the nature of the spectrum of H_λ is needed. For example, if (1.16) is valid, it follows from the RAGE theorem [4] that for any $\psi, \phi \in \mathcal{H}_p$,

$$\lim_{\lambda\to 0} \lim_{T\to\infty} \frac{1}{T} \int_0^T |(\psi \otimes \Omega, \exp(-itH_\lambda)\phi \otimes \Omega)|^2 dt = |(\psi, \psi_G)|^2 \, |(\phi, \psi_G)|^2. \tag{1.19}$$

We cannot prove (1.16), but the spectral information which we obtained as a by-product of Theorem 1.1 actually suffices for (1.19). We have

Theorem 1.3. *Suppose that (H1) and (H2) are satisfied. If* $E_j \notin \tau(H_0)$ *and* $\Gamma_j \neq 0$ *for all* $j \ge 1$, *then (1.19) holds.*

The remarks after Theorem 1.2 are also valid for Theorem 1.3.

When this work was finished, we learned from V. Bach that in reference [11] T. Okamoto and K. Yajima investigated resonances of a QED model related to (1.2). Using similar techniques, they proved a result analogous to the Theorem 1.1 above. In addition, they investigated the case when eigenvalues $\{E_j\}$ have multiplicities higher then 1. Their physical motivation was different and they did not pursue the study of the dynamical aspects of the model.

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2. Resonances and Fermi Golden Rule

2. 1. Complex Deformation

Throughout this chapter we assume that the hypotheses (H1) and (H2) are satisfied. Let $u(\theta)$ be given by (1.6), and $U(\theta) = \Gamma(u(\theta))$. Defining $\alpha_\theta = u(\theta)\alpha$ and $\omega_\theta(k) = \omega(\exp(\theta)k)$, we get that

$$\begin{aligned} U(\theta)a^*(\alpha)U(\theta)^{-1} &= a^*(\alpha_\theta), \\ U(\theta)a(\alpha)U(\theta)^{-1} &= a(\alpha_\theta), \\ U(\theta)H_{\text{bos}}U(\theta)^{-1} &= \mathrm{d}\Gamma(\omega_\theta). \end{aligned}$$

For $\theta \in S(\delta)$, the operator

$$H_\lambda(\theta) = H_{\text{p}} + \mathrm{d}\Gamma(\omega_\theta) + \lambda x \otimes \phi(\alpha_\theta),$$

is well defined on $C_0^\infty(\mathbf{R}) \otimes F$, where F is the subspace of finite particle vectors. We start by showing that $H_{\text{I}}(\theta) = x \otimes \phi(\alpha_\theta)$ is a relatively bounded perturbation of H_0.

Lemma 2.1. *There exists a constant $C > 0$ such that, for all $\theta \in \mathcal{S}(\delta)$ and $\Phi \in D(H_0)$,*

$$\|x \otimes \phi(\alpha_\theta)\Phi\| \le C\left(\|H_0\Phi\| + \|\Phi\|\right). \tag{2.1}$$

Proof: Let

$$C_1 = \|\alpha/\sqrt{\omega}\|_{H^2(\delta)}, \qquad C_2 = \|\alpha\|_{H^2(\delta)},$$

then, for $\Phi \in D(H_{\text{bos}}^{1/2})$ (see e.g. [1])

$$\begin{aligned} \|a^*(\alpha_\theta)\Phi\| &\le C_1\|H_{\text{bos}}^{1/2}\Phi\| + C_2\|\Phi\|, \\ \|a(\alpha_\theta)\Phi\| &\le C_1\|H_{\text{bos}}^{1/2}\Phi\|. \end{aligned} \tag{2.2}$$

Consequently, there exist constant C_3 such that, in the operator sense,

$$|\phi(\alpha_\theta)|^2 \le C_3(H_{\text{bos}} + 1). \tag{2.3}$$

Let C_4 be such that

$$x^2 \le C_4(H_{\text{p}} + 1). \tag{2.4}$$

Then, for any $\Phi \in D(H_{\text{p}}) \otimes D(H_{\text{bos}})$,

$$\begin{aligned} \|x \otimes \phi(\alpha_\theta)\Phi\|^2 &= \left(\Phi, x^2 \otimes |\phi(\alpha_\theta)|^2 \,\Phi\right) \\ &\le C_3C_4 \left(\Phi, (H_{\text{p}} + 1) \otimes (H_{\text{bos}} + 1)\Phi\right) \\ &\le \frac{1}{2}C_3C_4 \left(\Phi, \left((H_{\text{p}} + 1)^2 + (H_{\text{bos}} + 1)^2\right) \Phi\right) \\ &\le C_3C_4 \left(\Phi, \left(H_{\text{p}} + H_{\text{bos}} + 1\right)^2 \Phi\right). \end{aligned}$$

The proof is complete. ∎

Remark: A consequence of the above argument is that if x^2 is infinitesimally small with respect to H_{p}, then so is $H_{\text{I}}(\theta)$ with respect to H_0.

Remark: Arguing as above, one can prove a form version of the above results, namely: If x^2 is relatively form bounded with respect to H_{p} then so is $H_{\text{I}}(\theta)$ with respect to H_0. Again, if x^2 is infinitesimally form bounded with respect to H_{p}, so is $H_{\text{I}}(\theta)$ with respect to H_0.

Before stating our main technical lemma, we introduce some additional notation:

$$\begin{aligned} B(\varepsilon) &= \{\lambda \in \mathbf{C} : |\lambda| \le \varepsilon\}, \\ \mathcal{S}(a, b, \theta) &= \{s + \exp(\theta)t : t \in \mathbf{R},\, a \le s \le b\}. \end{aligned}$$

We enumerate the thresholds $\tau_k \in \tau(H_0)$ in increasing order:

$$E_0 + m_0 = \tau_1 < \tau_2 < \cdots.$$

Lemma 2.2. *There exists $\varepsilon > 0$ such that:*

(i) For real $\lambda \in B(\varepsilon)$, the operator H_λ is essentially self-adjoint on $C_0^\infty(\mathbf{R}) \otimes F$.

(ii) On the set $B(\varepsilon) \times \mathcal{S}(\delta)$, $H_\lambda(\theta)$ is an analytic family of type A in each variable separately.

(iii) If $\operatorname{Im}(\theta) \ne 0$, then for any $j > 0$ there exists $\varepsilon_j(\lambda) > 0$ such that

$$\lim_{\lambda \to 0} \varepsilon_j(\lambda) = 0, \qquad \sigma_{\text{ess}}(H_\lambda(\theta)) \cap \mathcal{S}(\tau_j + \varepsilon_j(\lambda), \tau_{j+1} - \varepsilon_j(\lambda), \theta) = \emptyset.$$

Proof: (i) follows immediately from Kato-Rellich Theorem. To prove (ii), note that the estimate (2.1) yields that $H_\lambda(\theta)$ is a closed operator with $D(H_\lambda(\theta)) = D(H_0)$. Since

$$\sigma_{\mathrm{pp}}(H_0(\theta)) = \{E_j \ : j = 0, 1, \cdots\},$$
$$\sigma_{\mathrm{ess}}(H_0(\theta)) = \{\tau_j + t\exp(\theta) \ : j = 1, 2, \cdots ; t \geq 0\},$$

we have (see [14]) that for any $\theta \in \mathcal{S}(\delta)$, $H_\lambda(\theta)$ is an analytic family of type A in λ for $\lambda \in B(1/C)$, where C is given in (2.1). That $H_\lambda(\theta)$ is an analytic family of type A in θ follows from the hypothesis (H1). Part (iii) is a consequence of regular perturbation theory (see, for example, proof of Theorem XII.9 in [14]). ∎

Parts (i) and (iii) of the above lemma are also valid in the quadratic form model (1.14). Part (ii) is altered since $H_\lambda(\theta)$ is now an analytic family of type B in each variable separately.

2. 2. Fermi Golden Rule

The discussion of resonances in the model (1.2) follows closely that of the Auger states in helium atom [14], [15]. In the sequel

$$H_{\mathrm{I}}(\theta) = x \otimes \phi(\alpha_\theta), \qquad H_{\mathrm{I}}(0) = H_{\mathrm{I}}, \qquad \Phi_j = \psi_j \otimes \Omega. \tag{2.5}$$

If $\mathrm{Im}(\theta) < 0$, part (iii) of Lemma 2.2 and regular perturbation theory yield that the spectrum of $H_\lambda(\theta)$ in $\mathcal{S}(\tau_k + \varepsilon_k(\lambda), \tau_{k+1} - \varepsilon_k(\lambda), \theta)$ consists of at most one isolated and non-degenerate eigenvalue $E_j(\lambda)$, given by a convergent power series in λ

$$E_j(\lambda) = a_j^{(0)} + a_j^{(1)}\lambda + a_j^{(2)}\lambda^2 + \cdots .$$

The coefficients $a_j^{(k)}$ (and thus $E_j(\lambda)$ itself) do not depend on θ (if $\mathrm{Im}(\theta) < 0$) since they are analytic in θ for fixed λ, and since $H_\lambda(\theta)$ and $H_\lambda(\theta')$ are unitarily equivalent for $\theta - \theta'$ real. Obviously $a_j^{(0)} = E_j$, and since

$$(\Phi_j, H_{\mathrm{I}}(\theta)\Phi_j) = 0, \tag{2.6}$$

we have that $a_j^{(1)} = 0$. In proving (1.10), (1.11) we follow the argument in [14]. For $\mathrm{Im}(\theta) < 0$, the coefficient $a_j^{(2)}$ is given by

$$a_j^{(2)} = \frac{1}{2\pi i} \oint_{|E_j - z| = \delta} \left(\Phi_j, \left[H_{\mathrm{I}}(\theta)(H_0(\theta) - z)^{-1}\right]^2 \Phi_j\right) dz, \tag{2.7}$$

for δ and λ small enough. We rewrite (2.8) as

$$a_j^{(2)} = \frac{1}{2\pi i} \oint_{|E_j - z| = \delta} \left(\Phi_j, H_{\mathrm{I}}(\theta)(H_0(\theta) - z)^{-1} H_{\mathrm{I}}(\theta)\Phi_j \right) \frac{dz}{E_j - z}.$$

Introducing

$$F(\theta, z) = \left(\Phi_j, H_{\mathrm{I}}(\theta)(H_0(\theta) - z)^{-1} H_{\mathrm{I}}(\theta)\Phi_j \right), \tag{2.8}$$

the Cauchy integral theorem yields that

$$a_j^{(2)} = -\lim_{\varepsilon \to 0} F(\theta, E_j + i\varepsilon) = -\lim_{\varepsilon \to 0} F(0, E_j + i\varepsilon).$$

As in [14], we used that $F(\theta, z)$ is defined for θ real as long as $\mathrm{Im}(z) > 0$, and is actually independent of θ, as long as $\mathrm{Im}(\theta) \leq 0$. The explicit calculation yields

$$F(0, E_j + i\varepsilon) = \sum_{i=0}^{\infty} |(\psi_i, x\psi_j)|^2 \int_0^{\infty} \frac{\alpha(k)^2}{\omega(k) - (E_j - E_i) - i\varepsilon} dk.$$

The above expression is finite, since (H2) yield that $x\psi_j \in L^2(\mathbf{R})$. Using the well-known formula

$$\lim_{\varepsilon \to 0} \frac{1}{x - x_0 - i\varepsilon} = i\pi\delta(x - x_0) + \mathrm{PV}\frac{1}{x - x_0},$$

(or just by residue calculus), we obtain (1.10), (1.11).

To finish the proof of Theorem 1.1 it suffices to show that, for a dense set of vectors D, the matrix elements

$$R(z) = \left(\Phi, (H_\lambda - z)^{-1} \Psi \right), \qquad \Phi,\ \Psi \in D, \tag{2.9}$$

have a meromorphic continuation from the upper half-plane onto the region

$$\mathcal{S}(\tau_k + \varepsilon_k(\lambda), \tau_{k+1} - \varepsilon_k(\lambda), \theta_0), \qquad \mathrm{Im}(\theta_0) < 0.$$

Let D_δ be the set of all vectors $\Phi \in \mathcal{H}$ such that $U(\theta)\Phi$ has an extension to analytic vector-valued function in the strip $\mathcal{S}(\delta)$. Then, for real θ and for any $\Phi, \Psi \in D_\delta$

$$R(z) = \left(U(\theta)\Phi, (H_\lambda(\theta) - z)^{-1} U(\theta)\Psi \right),$$

and by analytic continuation, this formula holds (if $\mathrm{Im}(z) > 0$) for $\mathrm{Im}(\theta) < \delta$. The result follows, and the proof of Theorem 1.1 is complete. ∎

From the above discussion, some spectral informations can be deduced. In the sequel we will make use of the following result which is an easy consequence of the above argument.

Theorem 2.3. *Suppose that for some $k \geq 1$ one has:*

$$\Gamma_j > 0 \quad \textit{for all} \quad j \geq 1 \quad \textit{such that} \quad E_j \in]\tau_k, \tau_{k+1}[.$$

Then there exists $\varepsilon_k(\lambda) > 0$ with $\varepsilon_k(\lambda) \to 0$ as $\lambda \to 0$, such that, for $\lambda \neq 0$, H_λ has purely absolutely continuous spectrum on $[\tau_k + \varepsilon_k(\lambda), \tau_{k+1} - \varepsilon_k(\lambda)]$.

The argument of this section extends without changes to the model (1.14).

2. 3. Dynamics of the system

In this section we prove Theorem 1.2. By the Stone formula,

$$(\Phi_j, \exp(-itH_\lambda)\Phi_j) = \frac{1}{2\pi i} \int_\Gamma \exp(-itz) \left(\Phi_j, (H_\lambda - z)^{-1}\Phi_j\right) dz,$$

where the contour Γ is the straight line $\mathrm{Im}(z) = \lambda^2$. Let Γ_1 be the contour constructed as follows: it coincides with Γ, except in a neighborhood of the point E_j where it goes down to enclose the resonance $E_j(\lambda)$ in such a way as to remain within the set $E_j + B(C_j)$. Since in this set

$$\left(\Phi_j, (H_\lambda - z)^{-1}\Phi_j\right) = \left(\Phi_j, (H_\lambda(\theta) - z)^{-1}\Phi_j\right), \tag{2.10}$$

the function $f(z) = (\Phi_j, (H_\lambda - z)^{-1}\Phi_j)$ is analytic in the simply connected domain bounded by the curves Γ and Γ_1, except for a simple pole at the resonance eigenvalue. Thus

$$\int_\Gamma \exp(-itz) f(z) \frac{dz}{2\pi i} = \exp(-itE_j(\lambda)) \left(\Phi_j, P_j(\lambda)\Phi_j\right) + \int_{\Gamma_1} \exp(-itz) f(z) \frac{dz}{2\pi i},$$

where $P_j(\lambda)$ is the projection on the resonance eigenfunction. To control the remaining integral over Γ_1 we proceed as follows. Since

$$\int_{\Gamma_1} \exp(-itz) \left(\Phi_j, (H_0 - z)^{-1}\Phi_j\right) dz = \int_{\Gamma_1} \exp(-itz) \frac{dz}{E_j - z} = 0,$$

we can apply the resolvent identity to get

$$\int_{\Gamma_1} \exp(-itz) f(z)\, dz = \lambda \int_{\Gamma_1} \exp(-itz) \left(\Phi_j, (H_\lambda - z)^{-1} H_I \Phi_j\right) \frac{dz}{E_j - z}.$$

Since $(\Phi_j, (H_0 - z)^{-1} H_I \Phi_j) = 0$, another application of the resolvent identity yields

$$\int_{\Gamma_1} \exp(-itz) f(z)\, dz = \lambda^2 \int_{\Gamma_1} \exp(-itz) \left(H_I \Phi_j, (H_\lambda - z)^{-1} H_I \Phi_j \right) \frac{dz}{(z - E_j)^2}.$$

Let us divide the contour Γ_1 into two parts, $\Gamma_1^+ = \{z \in \Gamma_1 \; : \mathrm{Im}(z) = \lambda^2\}$ and $\Gamma_1^- = \Gamma_1 \setminus \Gamma_1^+$. We estimate the integral over Γ_1 by

$$\begin{aligned} &\left| \lambda^2 \int_{\Gamma_1^-} \exp(-itz) \left(H_I \Phi_j, (H_\lambda - z)^{-1} H_I \Phi_j \right) \frac{dz}{(z - E_j)^2} \right| \\ = &\left| \lambda^2 \int_{\Gamma_1^-} \exp(-itz) \left(H_I(\bar{\theta}) \Phi_j, (H_\lambda(\theta) - z)^{-1} H_I(\theta) \Phi_j \right) \frac{dz}{(z - E_j)^2} \right| = O(\lambda^2 \exp(\tau)). \end{aligned}$$

On the other hand we have

$$\left| \lambda^2 \int_{\Gamma_1^+} \exp(-itz) \left(H_I \Phi_j, (H_\lambda - z)^{-1} H_I \Phi_j \right) \frac{dz}{(z - E_j)^2} \right| \leq M(\lambda) \exp(\tau),$$

where

$$M(\lambda) = \int_{|s - E_j| > C_j} \left| \lambda^2 \left(H_I \Phi_j, (H_\lambda - s - i\lambda^2)^{-1} H_I \Phi_j \right) \right| \frac{ds}{(s - E_j)^2}. \tag{2.11}$$

The statement will follow if we show that

$$\lim_{\lambda \to 0} M(\lambda) = 0. \tag{2.12}$$

To this end let us introduce, for small $\varepsilon > 0$, the set

$$T(\varepsilon) = \bigcup_k \,]\tau_k - \varepsilon, \tau_k + \varepsilon[, \tag{2.13}$$

and, for large $E > 0$, the sets

$$\begin{aligned} S(\varepsilon, E) &= \{s \in \mathbf{R} : C_j < |s - E_j| < E\} \cap T(\varepsilon), \\ R(\varepsilon, E) &= \{s \in \mathbf{R} : C_j < |s - E_j| < E\} \setminus T(\varepsilon). \end{aligned}$$

For almost all $s \in R(\varepsilon, E)$, and for any $\Psi \in D_\delta$ (note that $H_I \Phi_j \in D_\delta$) we have

$$\lim_{\lambda \to 0} \lambda^2 \left(\Psi, (H_\lambda - s - i\lambda^2)^{-1} \Psi \right) = \lim_{\lambda \to 0} \lambda^2 \left(U(\bar{\theta}) \Psi, (H_\lambda(\theta) - s - i\lambda^2)^{-1} U(\theta) \Psi \right) = 0,$$

provided $-\delta < \mathrm{Im}\theta < 0$. Decomposing the integration in (2.11) into parts over $R(\varepsilon, E)$, $S(\varepsilon, E)$ and $\{|s - E_j| \geq E\}$, and using the simple estimate

$$\left|\lambda^2 \left(\Psi, (H_\lambda - s - i\lambda^2)^{-1}\Psi\right)\right| \leq \|\Psi\|^2,$$

we easily get

$$\limsup_{\lambda\to 0} M(\lambda) \leq M_1(E)\varepsilon + \frac{M_2}{E},$$

and (2.12) follows by letting first $\varepsilon \to 0$, and then $E \to \infty$.

A completely analogous argument yields that for $i \neq j$,

$$\lim_{\lambda\to 0} \sup_{\tau\in(a,b)} |(\psi_i \otimes \Omega, (\exp(-itH_\lambda)\psi_j \otimes \Omega)| = 0.$$

The proof is complete. ∎

The argument of this section applies without changes to the model (1.14).

2. 4. Proof of Theorem 1.3

For fixed large $E > 0$, let

$$P_E^-(\lambda) = \chi_{]-\infty,E]}(H_\lambda), \qquad P_E^+(\lambda) = \chi_{]E,\infty[}(H_\lambda), \tag{2.14}$$

where χ_A stands for a characteristic function of a set $A \subset \mathbf{R}$. If λ is small enough, and $\Phi, \Psi \in D(H_0)$ we have

$$\frac{1}{T}\int_0^T |(\Phi, \exp(-itH_\lambda)\Psi)|^2 dt = \frac{1}{T}\int_0^T |(\Phi, \exp(-itH_\lambda)P_E^-(\lambda)\Psi)|^2 dt + O(E^{-2}). \tag{2.15}$$

This follows from the simple observation

$$\begin{aligned}|(\Phi, P_E^+(\lambda)\exp(-itH_\lambda)\Psi)| &= |((H_\lambda + 1)\Phi, (H_\lambda + 1)^{-2}P_E^+(\lambda)\exp(-itH_\lambda)(H_\lambda + 1)\Psi)| \\ &\leq \frac{1}{(E+1)^2}\|(H_\lambda + 1)\Phi\|\|(H_\lambda + 1)\Psi\|.\end{aligned}$$

For small λ, the bottom of the spectrum of H_λ is an isolated, non-degenerate eigenvalue $E_0(\lambda)$. We denote the projection on the corresponding eigenvector by $P_G(\lambda)$. With $T(\varepsilon)$ given by (2.13), we set

$$S'(\varepsilon, E) =]-\infty, E\,] \cap T(\varepsilon), \qquad R'(\varepsilon, E) =]-\infty, E\,] \setminus T(\varepsilon),$$

and

$$P^-_{E,\varepsilon,1}(\lambda) = \chi_{S'(\varepsilon,E)}(H_\lambda), \qquad P^-_{E,\varepsilon,2}(\lambda) = \chi_{R'(\varepsilon,E)}(H_\lambda). \tag{2.16}$$

Projections (2.16) have the following two properties: For any $\Phi, \Psi \in \mathcal{H}$ we have that

$$\lim_{\varepsilon\to 0}\lim_{\lambda\to 0} \|P^-_{E,\varepsilon,1}(\lambda)\Phi\| = 0. \tag{2.17}$$

For sufficiently small ε, there exists $\eta(\varepsilon) > 0$ such that, for $|\lambda| < \eta(\varepsilon)$, one has

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |(\Phi, P^-_{E,\varepsilon,2}(\lambda)\exp(-itH_\lambda)\Psi)|^2 dt = \|P_G(\lambda)\Phi\|^2\,\|P_G(\lambda)\Psi\|^2. \tag{2.18}$$

To establish (2.17) we note that, for sufficiently small ε, the set $S'(\varepsilon, E)$ does not contain any singular spectrum of H_0. This implies both

$$\lim_{\lambda\to 0}\|P^-_{E,\varepsilon,1}(\lambda)\Phi\| = \|P^-_{E,\varepsilon,1}(0)\Phi\|, \qquad \lim_{\varepsilon\to 0}\|P^-_{E,\varepsilon,1}(0)\Phi\| = 0,$$

and hence (2.17). To establish (2.18) we infer from Theorem 2.3, for fixed ε and small $\lambda \neq 0$,

$$\sigma_{pp}(H_\lambda) \cap R'(\varepsilon, E) = \{E_0(\lambda)\}.$$

The claim follows from the RAGE Theorem (see e.g. [4], Theorem 5.8).

Thus for fixed E we have

$$\begin{aligned}\limsup_{\lambda\to 0}\ \limsup_{T\to\infty}\Bigg|\frac{1}{T}\int_0^T &|(\Phi, \exp(-itH_\lambda)P^-_E(\lambda)\Psi)|^2 dt \\ &- \|P_G(0)\Phi\|^2\,\|P_G(0)\Psi\|^2\Bigg| = o(1),\end{aligned} \tag{2.19}$$

as $\varepsilon \to 0$. We obtain (1.19) for $\Phi, \Psi \in D(H_0)$ by letting first $\varepsilon \to 0$ in (2.19), and then $E \to \infty$ in (2.15). By density the statement extends to the entire space $\mathcal{H}$, and Theorem 1.3 is proven. ∎

Theorem 1.3 extends by continuity to the following setting:

Theorem 2.4.

(i) *If K and K' are Hilbert-Schmidt operators on $\mathcal{H}$, we have that*

$$\lim_{\lambda\to 0}\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{Tr}\left[K'\exp(-itH_\lambda)K\exp(itH_\lambda)\right] dt = (\Psi_G, K\Psi_G)(\Psi, K'\Psi_G),$$

where $\Psi_G = \psi_G \otimes \Omega$.

(ii) *If* K *is compact, and* $\Psi \in \mathcal{H}$,

$$\lim_{\lambda\to 0}\lim_{T\to\infty}\frac{1}{T}\int_0^T \|K\exp(-itH_\lambda)\Psi\|^2 dt = |(\Psi_G,\Psi)|^2\|K\Psi_G\|^2.$$

2. 5. Davies theory

In Chapter 2 we investigated the dynamics of the Hamiltonian (1.2) with the help of spectral theory. There exists, however, a completely different approach to the dynamics of the open systems, based on the use of master equations. The master equations technique, often heuristically used in the physics literature, has been raised to high art by Davies [5], [6]. We specify his general construction to the model at hand and show that it yields the right prediction for $a_j^{(2)}$, the second Rayleigh-Schrödinger coefficient in the expansion (1.9) for $E_j(\lambda)$. The usual technical assumption is that the interacting part of the Hamiltonian is a bounded operator, and consequently, in the literature, the fermionic field is more often considered then the bosonic one. Although technicalities imposed by the unboundedness of the interacting part of the Hamiltonian in the field variable can be resolved, we are forced to take a cutoff in x, namely to consider the Hamiltonian $H_\lambda(L)$ given by

$$H_\lambda(L) = H_{\mathrm{p}} + x\chi_{[-L,L]}(x)\otimes\phi(\alpha) + H_{\mathrm{bos}}. \tag{2.20}$$

Instead of the hypothesis (H1), throughout this section we will assume that α is even real function in $L^2(\mathbf{R})$, with a continuous Hilbert transform, such that for some $\mu > 0$

$$(1+|t|^\mu)|\widehat{\alpha}(t)| \in L^1(\mathbf{R}), \quad \text{where} \quad \widehat{\alpha}(t) = \int_0^\infty \alpha(k)^2\exp(-i\omega(k)t)dk. \tag{2.21}$$

Under the above assumptions,

Theorem 2.5. *There exists an operator* K_L*, acting on* $\mathcal{H}_p$*, such that for any* $b > a > 0$ *and* $\psi \in \mathcal{H}_p$,

$$\lim_{\lambda\to 0}\sup_{\tau\in(a,b)}\|\exp(itH_0)\exp(-itH_\lambda(L))\psi\otimes\Omega - \exp(-\tau K_L)\psi\otimes\Omega\| = 0.$$

Furthermore, the operator K_L *is diagonal in the basis* ψ_j*, with eigenvalues* $\Sigma_j(L)$ *such that*

$$\Sigma_0(L) = 0, \qquad \lim_{L\to\infty}\Sigma(L) = \Gamma_j + i\Lambda_j, \qquad j \geq 1,$$

where Γ_j and Λ_j are given by (1.10) and (1.11).

We remark that the argument below is fairly general, and applies also to the model (1.3), whenever Q is a bounded operator. It also accommodates without changes the *mass-less field* model, with Γ_j and Λ_j given by (2.23) below.

We now sketch the main steps of Davies theory in the framework of model (2.20). Let P be the orthogonal projection on $\mathcal{H}$ given by $P = I \otimes (\cdot, \Omega)\Omega$, and let $Q = I - P$. Since we are interested in the reduced dynamics, namely its restriction to the particle subspace, we integrate out the field variables by setting

$$U_\lambda(t) = P \exp(-itH_\lambda(L))P.$$

Defining

$$V_\lambda(t) = \exp(-it(H_0 + \lambda Q H_{\mathrm{I}} Q)),$$

we derive the integral form of NPRZ (Nakajima-Prigogine-Resibois-Zwanzig) equation as follows (see also [5], [7]): First, since PH_{I} is a bounded operator, we have a well-defined equation

$$U_\lambda(t) = V_\lambda(t)P - i\lambda \int_0^t V_\lambda(t-s)PH_{\mathrm{I}}Q \exp(-isH_\lambda(L))P ds.$$

Since HP_I is also bounded, we have

$$Q \exp(-itH_\lambda(L))P = -i\lambda \int_0^t V_\lambda(t-s)QH_{\mathrm{I}}PU_\lambda(s)ds.$$

Combining the above equations and using that $V_\lambda(t)P = U_0(t)$, we obtain the NPRZ equation:

$$U_\lambda(t) = U_0(t) - \lambda^2 \int_0^t ds \int_0^s ds' U_0(t-s)PH_{\mathrm{I}}QV_\lambda(s-s')QH_{\mathrm{I}}PU_\lambda(s'). \tag{2.22}$$

Introducing the new variables $\tau = \lambda^2 t$ and $\sigma = \lambda^2 s'$, we get

$$U_\lambda(\tau/\lambda^2) = U_0(\tau/\lambda^2) + \int_0^\tau U_0((\tau-\sigma)/\lambda^2)K(\lambda, \tau-\sigma)U_\lambda(\sigma/\lambda^2)d\sigma,$$

where

$$K(\lambda, \tau) = \int_0^{\tau/\lambda^2} U_0(-s)PH_{\mathrm{I}}QV_\lambda(s)QH_{\mathrm{I}}Pds.$$

The operator $K(\lambda, \tau)$ is bounded on $\mathcal{H}$. One can immediately extract the contribution to the dissipative part of the dynamics in second order perturbation theory:

$$K = \int_0^\infty U_0(-s)PH_{\mathrm{I}}QU_0(s)QH_{\mathrm{I}}P ds.$$

Let

$$K_{\mathrm{av}} = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \exp(itH_{\mathrm{p}})K\exp(-itH_{\mathrm{p}})Pdt = \sum_j P_j K P_j,$$

where P_j is the projection on $\psi_j \otimes \Omega$. The following two theorems form the core of Davies theory.

Theorem 2.6.

$$\lim_{\lambda\to 0} \sup_{\lambda^2 t\in(0,a)} \| \exp(-(itH_{\mathrm{p}} + \lambda^2 tK))\psi\otimes\Omega - \exp(-(itH_{\mathrm{p}} + \lambda^2 tK_{\mathrm{av}}))\psi\otimes\Omega\| = 0.$$

Theorem 2.7. *Suppose that for every $a > 0$ there exists a positive constant C_a such that*

$$\sup_{\tau\in(0,a)} \|K(\lambda,\tau)\| \le C_a.$$

Suppose also that, for any $c > b > 0$ fixed,

$$\lim_{\lambda\to 0} \sup_{\tau\in(b,c)} \|K(\lambda,\tau) - K\| = 0.$$

Then

$$\lim_{\lambda\to 0} \sup_{\lambda^2 t\in(0,a)} \|U_\lambda(s) - \exp(-(itH_{\mathrm{p}} + t\lambda^2 K))\| = 0.$$

The proofs of the above two theorems are somewhat technical [5]. It is even more involved to check that their conditions are satisfied in our model. Using condition (2.21), this can be done following line by line Davies argument in [5].

To finish the proof of Theorem 2.5, we need the following

Lemma 2.8. *The operator K_{av} restricted to the subspace generated by $\{\psi_j\otimes\Omega\}$ has a discrete spectrum, its eigenvalues and eigenvectors are given by*

$$K_{\mathrm{av}}\psi_j\otimes\Omega = (\Gamma_j(L) + i\Lambda_j(L))\psi_j\otimes\Omega,$$

$$\Gamma_0 = 0, \qquad \Gamma_j(L) = \pi \sum_{i=1}^{j-1} |(\psi_i, x_L \psi_j)|^2 \left(\alpha_m(E_j - E_i)\right)^2, \qquad j \geq 1,$$

$$\Lambda_j(L) = -\sum_{i>0} |(\psi_i, x_L\psi_j)|^2 \,\mathrm{PV} \int_0^\infty \frac{\alpha(k)^2}{\omega(k) - (E_j - E_i)} dk.$$

Proof: We have

$$(\psi_i \otimes \Omega, K_{\mathrm{av}} \psi_i \otimes \Omega) = (\psi_i \otimes \Omega, K \psi_i \otimes \Omega)$$

$$= \int_0^\infty \sum_{i>0} \exp(-it(E_i - E_j)) |(x_L \psi_i, \psi_j)|^2 \left[\int_0^\infty \exp(-i\omega(k)t)\alpha(k)^2 dk \right] dt,$$

and the result follows from the well known formula

$$\int_0^\infty \exp(i(\alpha - \beta)t)dt = \pi\delta(\alpha - \beta) + \mathrm{PV}\frac{i}{\alpha - \beta}.$$

Setting $K_{\mathrm{av}} = K_L$, we finish the proof of Theorem 2.5. ∎

2. 6. Some remarks

To obtain dissipation, both in the time mean and on the time scale τ, the hypothesis that $E_j \notin \tau(H_0)$ has been crucial. It is essentially saying that embedded eigenvalues of H_0 are not thresholds (whose presence invalidates regular perturbation theory). Intuitively, it is clearly "generically" satisfied. One can make that statement rigorous for a finite-dimensional system coupled to a heat bath.

When $m_0 = 0$, all embedded eigenvalues are thresholds, and our argument breaks down. Although one expects that embedded eigenvalues again dissolve into resonances after "turning on" small perturbations, the available techniques (even in much simpler models) are too crude to address such issues. One way of dealing with thresholds could be by adding small perturbations to the field, namely to consider

$$H_\lambda = H_{\mathrm{p}} + \lambda H_{\mathrm{I}} + H_{\mathrm{bos}} + m_0 P,$$

where $m_0 > 0$ is a small parameter, and P acts on $\mathcal{H}_{\mathrm{fock}}$ as $P = 1 - (\,\cdot\,, \Omega)\Omega$. One then can proceed as in previous sections. The expression for Fermi Golden Rule has a well-defined limit as $m_0 \to 0$, namely

$$\begin{aligned}\lim_{m_0 \to 0} \Gamma_j(m_0) &= -\pi \sum_{i<j} |(\psi_i, x\psi_j)|^2 \left(\alpha(E_j - E_i)\right)^2, \\ \lim_{m_0 \to 0} \Lambda_j(m_0) &= -\sum_{i>0} |(\psi_i, x\psi_j)|^2 \,\mathrm{PV} \int_0^\infty \frac{\alpha(k)^2}{k - (E_j - E_i)} dk.\end{aligned} \tag{2.23}$$

However, the analytic continuation of resolvent matrix elements takes place on a domain depending on m_0. Thus, in studying dynamics, one can at best let m_0 and λ tend to zero simultaneously, and formal exchange of limits (first m_0 and then λ), although probably yielding the right result, is not allowed.

It is perhaps worth mentioning that master equation approach, although very powerful, also requires renormalization of the Hamiltonian (even for the positive mass model!). One cannot exchange the limits $L \to \infty$ and $\lambda \to 0$ in Theorem 2.5. Again, these limits can at best be taken simultaneously (but only after considerable technical effort). Thus, for a positive mass model, our results on dynamics are stronger then the existing ones. For the mass-less model, all existing approaches require some regularization of the Hamiltonian. It remains as an open (and we believe hard) question to obtain results analogous to Theorems 1.1 and 1.2 for the model (1.2) with mass-less field.

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