HIERARCHICAL INTERFACES IN RANDOM MEDIA II: THE GIBBS MEASURES*

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Abstract: We continue the analysis of hierarchical interfaces in random media started in [BoP] and [BoK]. We show that from the estimates on the renormalized random variables established in these references, it faollows that these models possess unique Gibbs states describing mostly flat interfaces in dimension $D > 3$, if the disorder is weak and the temperature low enough. In the course of the proof we also present very explicit formulas for expectations of local observables.

Key Words: Disordered systems, interfaces, hierarchical model, Gibbs states, renormalization of stochastic sequences

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I. Introduction

In this paper we continue the analysis of hierarchical models for interfaces started in references [BoP] and [BoK]. It should really be seen as a companion paper to [BoK], and we will assume that the reader is familiar with that paper. In [BoK] we have proven a bound on the expected height of the surface at the origin for these models in $D > 3$ and for weak disorder. In this paper, we will show that from the estimates obtained there, one can actually prove existence and uniqueness of the Gibbs measure for these models and, moreover, to give a closed expression for the expectation values of any local observable with respect to this measure. We believe that this is, at least, of some pedagogical interest.

Although our models have been defined in [BoK], we would like to repeat it here in a slightly different wording, setting up, in particular, the precise probabilistic framework needed for the analysis of the Gibbs measures. This will be done in the remainder of this section where the main theorem will also be stated. In section II we will prove this theorem, making use of the estimates from [BoK].

Recall that our models describe d-dimensional solid-on-solid surfaces embedded in a $D = d+1$ lattice $\mathbb{Z}^D$. They will be constructed from collections of towers and wells with bases formed by squares of sidelength L^n, and of arbitrary height, fitting together in a hierarchical way that will be described formally as follows.

Let L be a positive integer that for simplicity we may choose to be odd, $L = 2k + 1$. We introduce the sets $Y^{(n)}$ of labels for the blocks of the n-th hierarchy. As for the moment we work in infinite volume, each of these sets is a copy of $\mathbb{Z}^d$. Let us define the map $\mathcal{L}^{-1} : Y^{(n)} \to Y^{(n+1)}$ by

$$\left(\mathcal{L}^{-1}y\right)_i \equiv \left[\frac{y_i + k}{L}\right] \tag{1.1}$$

where $[x]$ denotes the largest integer less than or equal to x. We also denote, for $y \in Y^{(n+1)}$, by $\mathcal{L}y$ the set of points $x \in Y^{(n)}$ such that $\mathcal{L}^{-1}x = y$. Note that thus the set $\mathcal{L}^n y$ is the cube of sidelength L^n centered at $L^n y$ in the underlying lattice $Y^{(0)} = \mathbb{Z}^d$. To each such cube we associate a 'tower', i.e. an integer valued height $h_y^{(n)}$. The collection of these heights describes the surfaces. In particular, its actual height, H_x, above a point $x \in Y^{(0)}$ is computed as

$$H_x = \sum_{n=0}^{\infty} h^{(n)}_{\mathcal{L}^{-n}x} \tag{1.2}$$

Notice that for (1.2) to make sense we must require that for each $x \in Y^{(0)}$, the sequence $h^{(n)}_{\mathcal{L}^{-n}}$ is summable, i.e. that only a finite number of its members be different from zero. Let us introduce also the 'coarse-grained' heights $H_y^{(N)}$ above the cube $y \in Y^{(N)}$ in the N-th hierarchy as

$$H_y^{(N)} = \sum_{n=0}^{\infty} h_{\mathcal{L}^{-n} y}^{(N+n)} \tag{1.3}$$

Note that the collection of the $H_y^{(N)}$, $N \in \mathbb{Z}_+$, $y \in \mathbb{Z}^d$ provides an alternative representation of our surfaces.

We denote the state space of all surfaces described in this way by Ω, i.e.

$$\Omega \equiv \left\{ \{\omega_y^{(n)}\}_{y \in Y^{(n)}, n \in \mathbb{Z}_+}, \omega_y^{(n)} \in \mathbb{Z} \middle| \sum_{n=0}^{\infty} |\omega_{\mathcal{L}^{-n} x}^{(n)}| < \infty, \forall x \in \mathbb{Z}^d \right\} \tag{1.4}$$

Ω is a dense subspace of $\tilde{\Omega} \equiv \left(\mathbb{Z}^{\mathbb{Z}^d}\right)^{\mathbb{Z}} = \mathbb{Z}^{\mathbb{Z}^d \times \mathbb{Z}}$ equipped with the product topology of the discrete topology of $\mathbb{Z}$. We will denote by Σ the trace of the Borel sigma-algebra in $\mathbb{Z}^{\mathbb{Z}^d \times \mathbb{Z}}$ in Ω, i.e.

$$\Sigma \equiv \left(\otimes_{n \in \mathbb{Z}_+} \otimes_{y \in Y^{(n)}} \mathcal{P}(\mathbb{Z})\right) \cap \Omega \tag{1.6}$$

Let $h_y^{(n)} : \Omega \to \mathbb{Z}$ be the projection given by

$$h_y^{(n)}(\omega) = \omega_y^{(n)} \tag{1.6}$$

Note that this notation implies that $H_y^{(N)}$, too, can be considered as functions on Ω in a natural way. The reader may find the summability condition imposed on the $\omega_{\mathcal{L}^{-n} x}^{(n)}$ in the definition of Ω puzzling. But notice that they are necessary to ensure that the heights $H_y^{(N)}(\omega)$ exist and are finite for all $\omega \in \Omega$. Since they furnish the actual physical description of the surfaces, this is a necessary requirement.

The energy of a surface was defined *formally* as

$$E_J = \sum_{n=0}^{\infty} \sum_{y \in Y^{(n)}} |h_y^{(n)}| L^{(d-1)n} + \sum_{x \in Y^{(0)}} J_x(H_x) \tag{1.7}$$

where the $J_x(H)$ are random surface energies that describe the random medium (see [BoK].

Our aim is now to turn (Ω, Σ) into a probability space with a probability measure given by a Gibbs measure associated to E_J. To do this, we have to construct first the corresponding *local specifications* (see e.g. [Ge,Si]), which is done as follows. Let Λ be a finite subset of $Y^{(0)} \equiv \mathbb{Z}^d$ (Note that later we will be interested in sequences Λ_n of such volumes that are increasing and absorbing (i.e. for any finite subset, $X \subset \mathbb{Z}^d$, there exists an n_0, such that for all $n \geq n_0$, $X \subset \Lambda_n$). We define

$$Y^{(n)}(\Lambda) \equiv \left\{ y \in Y^{(n)} \middle| \mathcal{L}^n y \subset \Lambda \right\} \tag{1.8}$$

Note that, for given Λ, there exists a maximal value of n such that $Y^{(n)}(\Lambda) \neq \emptyset$. We denote this value by $n_i(\Lambda)$. We then set

$$Y(\Lambda) \equiv \bigcup_{n=0}^{n_i(\Lambda)} Y^{(n)}(\Lambda) \tag{1.9}$$

Correspondingly, we denote the finite volume state spaces

$$\begin{aligned} \Omega_\Lambda^{(n)} &\equiv \left\{ \{\omega_y^{(n)}\}_{y\in Y^{(n)}(\Lambda)} \right\} \equiv \mathbb{Z}^{Y^{(n)}(\Lambda)} \quad \text{and} \\ \Omega_\Lambda &\equiv \left\{ \{\omega_y^{(n)}\}_{y\in Y^{(n)}(\Lambda), n=0,\dots,n_i(\Lambda)} \right\} \equiv \mathbb{Z}^{Y(\Lambda)} \end{aligned} \tag{1.10}$$

We denote by $\Sigma_\Lambda \equiv \sigma\left(\{h_y^{(n)}\}_{y\in Y^{(n)}(\Lambda), n=0,\dots,n_i(\Lambda)}\right) \subset \Sigma$ the sigma-algebra generated by the canonical projection from Ω to Ω_Λ. We will also write $\Sigma_{\Lambda^c} \equiv \sigma\left(\{h_y^{(n)}\}_{y\in Y^{(n)}(\Lambda)^c, n\in\mathbb{Z}_+}\right)$ and

$$\Omega_{\Lambda^c} \equiv \left\{ \{\omega_y^{(n)}\}_{y\in Y^{(n)c}, n\in\mathbb{Z}_+} \middle| (0_\Lambda, \omega_{\Lambda^c}) \in \Omega \right\} \tag{1.11}$$

The local specifications $\mu_{\Lambda,\beta,J}^\eta$ are probability kernels on Ω, such that for given configuration $\eta = (\eta_\Lambda, \eta_{\Lambda^c}) \in \Omega$ and any Σ-measurable function f,

$$\mu_{\Lambda,\beta,J}^\eta(f) \equiv \frac{1}{Z_{\Lambda,\beta,J}^\eta} \int_{\Omega_\Lambda} f(\omega_\Lambda, \eta_{\Lambda^c}) e^{-\beta E_{\Lambda,J}(\omega_\Lambda,\eta_{\Lambda^c})} d\omega_\Lambda \tag{1.12}$$

where

$$E_{\Lambda,J}(\omega_\Lambda, \eta_{\Lambda^c}) \equiv \sum_{n=0}^{n_i(\Lambda)} \sum_{y\in Y^{(n)}(\Lambda)} |h_y^{(n)}(\omega_\Lambda)| L^{(d-1)n} + \sum_{x\in\Lambda} J_x(H_x(\omega_\Lambda, \eta_{\Lambda^c})) \tag{1.13}$$

$Z_{\Lambda,\beta,J}^\eta$ is as usual the normalization constant ('partition function') which turns the $\mu_{\Lambda,\beta,J}^\eta$ into probability kernels on Ω. Note that $\mu_{\Lambda,\beta,J}^\eta(f)$ is Σ_{Λ^c}-measurable, $\mu_{\Lambda,\beta,J}^\eta(fg) = f(\eta)\mu_{\Lambda,\beta,J}^\eta(g)$, if f is Σ_{Λ^c}-measurable, and for $\Lambda_1 \subset \Lambda_2$ these kernels satisfy the compatibility condition

$$\mu_{\Lambda_2,\beta,J}^\eta \, \mu_{\Lambda_1,\beta,J}^{\cdot} = \mu_{\Lambda_2,\beta,J}^\eta \tag{1.14}$$

A probability measure μ on (Ω, Σ) is called a Gibbs measure for the local specification $\mu_{\Lambda,\beta,J}^\eta$, if it satisfies the Dobrushin-Landford-Ruelle (DLR) equations [Ge,Si]

$$\mu \, \mu_{\Lambda,\beta,J}^{\cdot} = \mu \quad \text{for all finite} \quad \Lambda \tag{1.15}$$

We recall now the assumptions that were made on the family of random variables describing the disorder. Let $(\Gamma, \mathcal{F}, I\!P)$ be an abstract probability space, and let $J \equiv \{J_x(H)\}_{H\in\mathbb{Z}, x\in\mathbb{Z}^d}$ be a family of random variables defined on this space, and denote by

$$D_x(H, H') \equiv J_x(H) - J_x(H') \tag{1.16}$$

the associated difference variables. We assume that the following properties hold:

(i) For fixed x, the stochastic processes $\{D_x(H,H')\}_{H,H'\in\mathbb{Z}}$ are stationary under the simultaneous shift $(H,H')\to(H+k,H'+k)$, for $k\in\mathbb{Z}$.

(ii) The stochastic processes $\{J_x(H)\}_{H\in\mathbb{Z}}$ are independent for different x, and

(iii) $\mathbb{E}D_x(H,H')=0$, for all $x\in\mathbb{Z}^d$.

We are now in a position to announce our main result

THEOREM 1: *Let $d>2$, $J\equiv\{J_x(H)\}_{H\in\mathbb{Z},x\in\mathbb{Z}^d}$ be such that (i), (ii) and (iii) hold. Assume in addition that, for some fixed $0\le r\le\frac{1}{2}$ the associated difference variables satisfy for all $\delta>0$, for all $x\in\mathbb{Z}^d$ and all $H\neq H'\in\mathbb{Z}$*

$$\mathbb{P}\left[D_x(H,H')>\delta\right]\le\exp\left(-\frac{\delta^2}{2\sigma^2|H-H'|^{2r}}\right)+\exp\left(-\frac{\delta^{2-2r}}{\sigma^2}\right) \tag{1.17}$$

Then there exist $\beta_0<\infty$, $\sigma_0^2>0$ and $L_0<\infty$ and a set $\tilde\Gamma\subset\Gamma$, $\mathbb{P}(\tilde\Gamma)=1$, such that for all $\beta\ge\beta_0$, $\sigma^2\le\sigma_0^2$, $L\ge L_0$, and $J\in\tilde\Gamma$ the local specification $\mu^{\cdot}_{\Lambda,\beta,J}$ has a unique Gibbs measure $\mu_{\beta,J}$.

Remark 1: We will give an explicit construction of $\mu_{\beta,J}$ for local observables.

Remark 2: Note that the condition (1.17) is in fact slightly more general than the conditions in [BoP] and [BoK], which correspond to the extreme cases $r=0$ and $r=\frac{1}{2}$, respectively. It is however very easy to extend the estimates from [BoK] to this entire range (details can be found in [K]).

Remark 3: It may appear surprising that we prove the existence of one unique Gibbs state. In fact, this is an artefact of the hierarchical structure of the model. The nontrivial statement of the theorem is that of the existence of the Gibbs measure.

Remark 4: As explained in [BoK], the condition $d>2$ is crucial for our proof, and the result is not expected to hold if $d=2$ (at least for the non-hierarchical model. The minimal value of L for which we can proof the result depends on d and tends to infinity as $d\downarrow 2$. One may expect that in $d=3$ the theorem should hold with $L_0=2$ (or 3), however, for technical reasons due to the method of proof reasons we need a larger value. This is explained in [BoK].

II. Construction of the Gibbs measure

In this section we will derive an explicit expression for the local specifications introduced in the last section that reflects in a particularly transparent manner the hierarchical structure of our models, and that will turn out to be crucial in constructing the actual infinite volume Gibbs measures.

Let us begin by introducing some notation. Let us denote, for any $\beta > 0$, by $\mathcal{S}_\beta \subset I\!R^{\mathbb{Z}}$, the set

$$\mathcal{S}_\beta \equiv \left\{ \{a_n\}_{n\in\mathbb{Z}} \,\middle|\, \sum_{n\in\mathbb{Z}} e^{-\beta(|n|+a_n)} < \infty \right\} \tag{2.1}$$

and let $\mathcal{S}$ be the intersection of all these sets, i.e.

$$\mathcal{S} \equiv \bigcap_{\beta>0} \mathcal{S}_\beta \tag{2.2}$$

For any sequence $\{a_n\}_{n\in\mathbb{Z}} \in \mathcal{S}$, we define measures $\tilde\rho_\beta\left(\{a_n\}_{n\in\mathbb{Z}}\right)$ on $\mathbb{Z}$ by

$$\tilde\rho_\beta\left(\{a_n\}_{n\in\mathbb{Z}}\right)(\{m\}) \equiv e^{-\beta(|m|+a_m)} \tag{2.3}$$

We denote by $\zeta_\beta\left(\{a_n\}_{n\in\mathbb{Z}}\right)$ the corresponding 'partition functions',

$$\zeta_\beta\left(\{a_n\}_{n\in\mathbb{Z}}\right) \equiv \sum_{n\in\mathbb{Z}} e^{-\beta(|n|+a_n)} \tag{2.4}$$

and by $\rho_\beta\left(\{a_n\}_{n\in\mathbb{Z}}\right)$ the probability measures obtained by normalizing $\tilde\rho_\beta\left(\{a_n\}_{n\in\mathbb{Z}}\right)$. Note that these measures are really Gibbs measures on $\mathbb{Z}$. The measures are the same for sequences that differ only by a constant, and we will view ρ_β as probability measure valued functions on $\mathcal{S}/I\!R$. An object of particular interest will be seen to be the 'free energy', Φ_β, associated to these measures which we will consider as a real valued function on $\mathcal{S}$, defined by

$$\Phi_\beta\left(\{a_n\}_{n\in\mathbb{Z}}\right) \equiv -\frac{1}{\beta}\ln \sum_{n\in\mathbb{Z}} e^{-\beta(|n|+a_n)}, \tag{2.5}$$

(Note that we have changed the notation here slightly from that of [BoK] (see eq. (2.4) there) for later convenience). Finally, we have to consider the action of the translation group $\mathbb{Z}$ on our objects. We will denote by the generic name T_k the lift of the translation operator on sequences, i.e.

$$T_k\rho_\beta\left(\{a_n\}_{n\in\mathbb{Z}}\right) \equiv \rho_\beta\left(\{a_{n+k}\}_{n\in\mathbb{Z}}\right) \tag{2.6}$$

$$T_k\Phi_\beta\left(\{a_n\}_{n\in\mathbb{Z}}\right) \equiv \Phi_\beta\left(\{a_{n+k}\}_{n\in\mathbb{Z}}\right) \tag{2.7}$$

etc.

Using this terminology, the local specifications (1.12) can be written as follows:

$$\begin{aligned}\mu^{\eta}_{\Lambda,\beta,J}(f) = \frac{1}{Z^{\eta}_{\Lambda,\beta,J}} & \int_{\Omega_{\Lambda}^{(n_i(\Lambda))}} \int_{\Omega_{\Lambda}^{(n_i(\Lambda)-1)}} \cdots \int_{\Omega_{\Lambda}^{(0)}} \\ & \prod_{y_{n_i}\in Y^{(n_i)}(\Lambda)} \tilde{\rho}_{\beta^{(n_i)}} \left(\{0\}_{n\in\mathbb{Z}}\right)(d\omega^{(n_i)}_{y_{n_i}}) \\ & \prod_{y_{n_i-1}\in Y^{(n_i-1)}(\Lambda)} \tilde{\rho}_{\beta^{(n_i-1)}} \left(\{0\}_{n\in\mathbb{Z}}\right)(d\omega^{(n_i-1)}_{y_{n_i-1}}) \dots \\ & \dots \\ & \dots \\ & \prod_{y_1\in Y^{(1)}(\Lambda)} \tilde{\rho}_{\beta^{(1)}} \left(\{0\}_{n\in\mathbb{Z}}\right)(d\omega^{(1)}_{y_1}) \\ & \prod_{y_0\in Y^{(0)}(\Lambda)} T_{H^{(1)}_{\mathcal{L}^{-1}y_0}(\omega_\Lambda,\eta_{\Lambda^c})} \tilde{\rho}_{\beta} \left(\{J_{y_0}(n)\}_{n\in\mathbb{Z}}\right)(d\omega^{(0)}_{y_0}) f(\omega_\Lambda,\eta_{\Lambda^c})\end{aligned} \tag{2.8}$$

where we have set $\beta^{(n)} \equiv L^{(d-1)n}\beta$. Notice that $H^{(1)}_{y_1}$ does not depend on variables in $\Omega^{(0)}_{\Lambda}$. Our strategy is to replace the measures $\tilde{\rho}_\beta$ in the last line by the corresponding normalized measures ρ_β, using that

$$\begin{aligned} & \prod_{y_1\in Y^{(1)}(\Lambda)} \tilde{\rho}_{\beta^{(1)}} \left(\{0\}_{n\in\mathbb{Z}}\right)(d\omega^{(1)}_{y_1}) \prod_{y_0\in Y^{(0)}(\Lambda)} T_{H^{(1)}_{\mathcal{L}^{-1}y_0}(\omega_\Lambda,\eta_{\Lambda^c})} \tilde{\rho}_{\beta} \left(\{J_{y_0}(n)\}_{n\in\mathbb{Z}}\right)(d\omega^{(0)}_{y_0}) f(\omega_\Lambda,\eta_{\Lambda^c}) \\ & = \prod_{y_1\in Y^{(1)}(\Lambda)} \tilde{\rho}_{\beta^{(1)}} \left(\{0\}_{n\in\mathbb{Z}}\right)(d\omega^{(1)}_{y_1}) T_{H^{(1)}_{y_1}(\omega_\Lambda,\eta_{\Lambda^c})} \prod_{y_0\in\mathcal{L}y_1} \zeta_\beta\left(\{J_{y_0}(n)\}_{n\in\mathbb{Z}}\right) \\ & \prod_{y_0\in Y^{(0)}(\Lambda)} T_{H^{(1)}_{\mathcal{L}^{-1}y_0}(\omega_\Lambda,\eta_{\Lambda^c})} \rho_{\beta} \left(\{J_{y_0}(n)\}_{n\in\mathbb{Z}}\right)(d\omega^{(0)}_{y_0}) \\ & \prod_{y_0\in Y^{(0)}(\Lambda)\backslash\mathcal{L}Y^{(1)}(\Lambda)} T_{H^{(1)}_{\mathcal{L}^{-1}y_0}(\eta_{\Lambda^c})} \zeta_\beta\left(\{J_{y_0}(n)\}_{n\in\mathbb{Z}}\right) f(\omega_\Lambda,\eta_{\Lambda^c}) \\ & = \prod_{y_1\in Y^{(1)}(\Lambda)} T_{H^{(2)}_{\mathcal{L}^{-1}y_1}(\omega_\Lambda,\eta_{\Lambda^c})} \tilde{\rho}_{\beta^{(1)}} \left(\left\{J^{(1)}_{y_1}(n)\right\}_{n\in\mathbb{Z}}\right)(d\omega^{(1)}_{y_1}) \\ & \prod_{y_0\in Y^{(0)}(\Lambda)} T_{H^{(1)}_{\mathcal{L}^{-1}y_0}(\omega_\Lambda,\eta_{\Lambda^c})} \rho_{\beta} \left(\{J_{y_0}(n)\}_{n\in\mathbb{Z}}\right)(d\omega^{(0)}_{y_0}) \\ & \prod_{y_0\in Y^{(0)}(\Lambda)\backslash\mathcal{L}Y^{(1)}(\Lambda)} T_{H^{(1)}_{\mathcal{L}^{-1}y_0}(\eta_{\Lambda^c})} \zeta_\beta\left(\{J_{y_0}(n)\}_{n\in\mathbb{Z}}\right) f(\omega_\Lambda,\eta_{\Lambda^c})\end{aligned} \tag{2.9}$$

Here we have defined

$$J^{(1)}_{y_1}(H) \equiv \frac{1}{L^{d-1}} \sum_{y\in\mathcal{L}y_1} T_H \Phi_\beta\left(\{J_y(h)\}_{h\in\mathbb{Z}}\right) \tag{2.10}$$

Note that we have written $H^{(1)}_{\mathcal{L}^{-1}y_0}(\eta_{\Lambda^c})$ in the last line of (2.9) to make explicit that this variable depends only on the external configuration. Thus the product in the last line is just a Σ_{Λ^c}-measurable constant. It is now evident that this procedure can be iterated, resulting in the following expression:

$$
\begin{aligned}
\mu^{\eta}_{\Lambda,\beta,J}(f) = &\int_{\Omega_\Lambda^{(n_i(\Lambda))}} \int_{\Omega_\Lambda^{(n_i(\Lambda)-1)}} \dots \int_{\Omega_\Lambda^{(0)}} \\
&\prod_{y_{n_i} \in Y^{(n_i)}(\Lambda)} T_{H^{(n_i+1)}_{\mathcal{L}^{-1} y_{n_i}}(\eta_{\Lambda^c})} \rho_{\beta^{(n_i)}} \left(J^{(n_i)}_{y_{n_i}}\right) (d\omega^{(n_i)}_{y_{n_i}}) \\
&\prod_{y_{n_i-1} \in Y^{(n_i-1)}(\Lambda)} T_{H^{(n_i)}_{\mathcal{L}^{-1} y_{n_i-1}}(\omega_\Lambda,\eta_{\Lambda^c})} \rho_{\beta^{(n_i-1)}} \left(J^{(n_i-1)}_{y_{n_i-1}}\right) (d\omega^{(n_i-1)}_{y_{n_i-1}}) \\
&\dots \\
&\dots \\
&\prod_{y_1 \in Y^{(1)}(\Lambda)} T_{H^{(2)}_{\mathcal{L}^{-1} y_1}(\omega_\Lambda,\eta_{\Lambda^c})} \rho_{\beta^{(1)}} \left(J^{(1)}_{y_1}\right) (d\omega^{(1)}_{y_1}) \\
&\prod_{y_0 \in Y^{(0)}(\Lambda)} T_{H^{(1)}_{\mathcal{L}^{-1} y_0}(\omega_\Lambda,\eta_{\Lambda^c})} \rho_{\beta^{(0)}} \left(J^{(0)}_{y_0}\right) (d\omega^{(0)}_{y_0}) f(\omega_\Lambda, \eta_{\Lambda^c})
\end{aligned}
\tag{2.11}
$$

Here we use, for notational convenience, the abbreviations $J^{(n)}_y \equiv \left\{J^{(n)}_y(h)\right\}_{h\in\mathbb{Z}}$. The sequences $J^{(n)}_y$ are recursively defined by

$$
\begin{aligned}
J^{(0)}_y(H) &\equiv J_y(H) \\
J^{(n)}_y(H) &\equiv \frac{1}{L^{d-1}} \sum_{x\in\mathcal{L}y} T_H \Phi_{\beta^{(n-1)}}\left(J^{(n-1)}_x\right)
\end{aligned}
\tag{2.12}
$$

These are of course just the renormalization group equations derived and analysed already in [BoP] and [BoK]. Note that the constants that were still present in (2.9) have cancelled against the partition function, so that in (2.12) we have achieved our goal to express the local specification entirely in terms of the probability measures ρ_β.

Let us now consider a function f, that is measurable with respect to the σ-algebra Σ_Λ, which we will call 'local function'. (Note that each cylinder function is a local function in this sense.)

Define now for any $\Lambda \subset \mathbb{Z}^d$, the sets

$$
\tilde{Y}^{(n)}(\Lambda) \equiv \left\{ y \in Y^{(n)} \,\middle|\, \mathcal{L}^n y \cap \Lambda \neq \emptyset \right\}
\tag{2.13}
$$

Notice that for any finite Λ, for n large enough $\tilde{Y}^{(n)}(\Lambda)$ will eventually consist of just the point zero. We define

$$
n_a(\Lambda) \equiv \inf \left\{ n \,\middle|\, \tilde{Y}^{(n)} = \{0\} \right\}
\tag{2.14}
$$

If now we consider an absorbing and increasing sequence of volumes Λ_k, then for any given Λ, there exists some $k_0 < \infty$ such that $n^0_i(\Lambda_{k_0}) \equiv \sup \left\{ n \,\middle|\, \mathcal{L}^n 0 \subset \Lambda_{k_0} \right\} \geq n_a(\Lambda)$. The expectation of

f with respect to the local specifications corresponding to Λ_k, $k \geq k_0$ can then be written (with $n_i^0 \equiv n_i^0(\Lambda_k)$) as

$$
\begin{aligned}
\mu^{\eta}_{\Lambda_k,\beta,J}(f) = &\int_{\mathbb{Z}} T_{H_0^{(n_i^0+1)}(\eta_{\Lambda_k^c})} \rho_{\beta^{(n_i^0)}} \left(J_0^{(n_i^0)}\right) (d\omega_0^{(n_i^0)}) \\
&\int_{\mathbb{Z}} T_{H_0^{(n_i^0)}(\omega_{\Lambda_k},\eta_{\Lambda_k^c})} \rho_{\beta^{(n_i^0-1)}} \left(J_0^{(n_i^0-1)}\right) (d\omega_0^{(n_i^0-1)}) \\
&\dots \\
&\dots \\
&\int_{\mathbb{Z}} T_{H_0^{(n_a(\Lambda)+1)}(\omega_{\Lambda_k},\eta_{\Lambda_k^c})} \rho_{\beta^{(n_a(\Lambda))}} \left(J_0^{(n_a(\Lambda))}\right) (d\omega_0^{(n_a(\Lambda))}) \\
&\prod_{y_{n_a-1}\in \tilde{Y}^{(n_a-1)}(\Lambda)} \int_{\mathbb{Z}} T_{H^{(n_a(\Lambda))}_{\mathcal{L}^{-1}y_{n_a-1}}(\omega_{\Lambda_k},\eta_{\Lambda_k^c})} \rho_{\beta^{(n_a-1)}} \left(J_{y_{n_a-1}}^{(n_a-1)}\right) (d\omega_{y_{n_a-1}}^{(n_a-1)}) \\
&\dots \\
&\dots \\
&\prod_{y_0\in \tilde{Y}^{(0)}(\Lambda)} \int_{\mathbb{Z}} T_{H^{(1)}_{\mathcal{L}^{-1}y_0}(\omega_{\Lambda_k},\eta_{\Lambda_k^c})} \rho_{\beta^{(0)}} \left(J_{y_0}^{(0)}\right) (d\omega_{y_0}^{(0)}) f(\omega_{\Lambda_k})
\end{aligned}
\tag{2.15}
$$

Notice that the dependence on $\eta_{\Lambda_k^c}$ only enters through $H_0^{(n_i^0+1)}(\eta_{\Lambda_k^c})$ and that the integral in the last four lines is just a function $\tilde{f}$, depending only on $\omega_0^{(n_a^{(0)}(\Lambda))} + \ldots + \omega_0^{(n_i^0)} + H_0^{(n_i^0+1)}(\eta_{\Lambda_k^c})$. Thus (2.15) can be written in the form

$$
\begin{aligned}
\mu^{\eta}_{\Lambda_k,\beta,J}(f) = &\int_{\mathbb{Z}} T_{H_0^{(n_i^0+1)}(\eta_{\Lambda_k^c})} \rho_{\beta^{(n_i^0)}} \left(J_0^{(n_i^0)}\right) (d\omega_0^{(n_i^0)}) \\
&\int_{\mathbb{Z}} T_{H_0^{(n_i^0+1)}(\eta_{\Lambda_k^c})+\omega_0^{(n_i^0)}} \rho_{\beta^{(n_i^0-1)}} \left(J_0^{(n_i^0-1)}\right) (d\omega_0^{(n_i^0-1)}) \\
&\dots \\
&\dots \\
&\int_{\mathbb{Z}} T_{H_0^{(n_i^0+1)}(\eta_{\Lambda_k^c})+\omega_0^{(n_i^0)}+\ldots+\omega_0^{(n_a(\Lambda)+1)}} \rho_{\beta^{(n_a(\Lambda))}} \left(J_0^{(n_a(\Lambda))}\right) (d\omega_0^{(n_a(\Lambda))}) \\
&\tilde{f}(\omega_0^{(n_a^{(0)}(\Lambda))} + \ldots + \omega_0^{(n_i^0)} + H_0^{(n_i^0+1)}(\eta_{\Lambda_k^c})) \\
\equiv &\int_{\mathbb{Z}} \nu_{n_i^0(\Lambda_k),n_a(\Lambda)}^{H_0^{n_i^0+1}(\eta_{\Lambda_k^c})}(dH) \tilde{f}(H)
\end{aligned}
\tag{2.16}
$$

In the last equation, only the measures $\nu_{n_i^0(\Lambda_k),n_a(\Lambda)}^{H_0^{n_i^0+1}(\eta_{\Lambda_k^c})}$ depend on the finite volumes Λ_k and the boundary conditions $\eta_{\Lambda_k^c}$. Moreover, the dependence on these boundary conditions is only through the sequence of integers $H_0^{n_i^0(\Lambda_k)+1}(\eta_{\Lambda_k^c})$, that is the corresponding infinite volume limits depend only on the limit of this sequence. Now due to the definition of Ω (recall (1.4)), these sequences converge to zero as $\Lambda_k \uparrow \mathbb{Z}^d$, for all $\eta \in \Omega$. In this way we see that the hierarchical model admits effectively only one 'boundary condition at infinity'. To construct the infinite volume Gibbs measures, we are thus left to analyse the convergence of the measures on the integers $\nu_{n_i^\circ(\Lambda_k),n_a(\Lambda)} \equiv \nu^0_{n_i^\circ(\Lambda_k),n_a(\Lambda)}$.

LEMMA 2.1: *Suppose that $J_0^{(n)}$ are such that*

$$\sum_{n=0}^{\infty} \rho_{\beta^{(n)}}\left(J_0^{(n)}\right)\left(\omega^{(n)} \neq 0\right) < \infty \tag{2.17}$$

Let ν_{n,n_0} be the measures defined below eq. (2.16). Then

$$w\text{-}\lim_{n\uparrow\infty} \nu_{n,n_0} \equiv \nu_{n_0} \tag{2.18}$$

exists.

Proof: Let us define measures μ_{n,n_0} on $\mathbb{Z}^{n-n_0+1}$ through

$$\begin{aligned}\mu_{n,n_0}(g) = &\int_{\mathbb{Z}} \rho_{\beta^{(n)}}\left(J_0^{(n)}\right)(d\omega^{(n)})\\ &\int_{\mathbb{Z}} T_{\omega^{(n)}}\rho_{\beta^{(n-1)}}\left(J_0^{(n-1)}\right)(d\omega^{(n-1)})\\ &\dots\\ &\dots\\ &\int_{\mathbb{Z}} T_{\omega^{(n)}+\ldots+\omega^{(n_0)}}\rho_{\beta^{(n_0)}}\left(J_0^{(n_0)}\right)(d\omega^{(n_0)})\\ &g(\omega^{(n_0+1)},\ldots,\omega^{(n)})\end{aligned} \tag{2.19}$$

Of course the measures ν_{n,n_0} are just the measures on the sum-variables $\omega^{(n)}+\ldots+\omega^{(n_0)}$ induced by the μ_{n,n_0}. To prove the weak convergence of those measures, we first show that for all $k \in \mathbb{Z}$ the sequences $\nu_{n,n_0}(\{k\})$ are Cauchy-sequences (w.r.t. n).

For integers $n_2 \geq n_1 \geq n_0$ we have

$$\begin{aligned}\nu_{n_2,n_0}(\{k\}) =&\mu_{n_2,n_0}\left(\omega^{(n_0)}+\ldots+\omega^{(n_1)} = k \wedge (\omega^{(n_1+1)},\ldots,\omega^{(n_2)}) = 0\right)\\ &+ \mu_{n_2,n_0}\left(\omega^{(n_0)}+\ldots+\omega^{(n_2)} = k \wedge (\omega^{(n_1+1)},\ldots,\omega^{(n_2)}) \neq 0\right)\end{aligned} \tag{2.20}$$

For the first term we use

$$\begin{aligned}&\mu_{n_2,n_0}\left(\omega^{(n_0)}+\ldots+\omega^{(n_1)} = k \,\middle|\, (\omega^{(n_1+1)},\ldots,\omega^{(n_2)}) = 0\right)\\ =&\mu_{n_1,n_0}\left(\omega^{(n_0)}+\ldots+\omega^{(n_1)} = k\right)\end{aligned} \tag{2.21}$$

to write

$$
\begin{aligned}
&\mu_{n_2,n_0}\left(\omega^{(n_0)}+\ldots+\omega^{(n_1)}=k\wedge(\omega^{(n_1+1)},\ldots,\omega^{(n_2)})=0\right)\\
=&\mu_{n_2,n_0}\left(\omega^{(n_0)}+\ldots+\omega^{(n_1)}=k\,\middle|\,(\omega^{(n_1+1)},\ldots,\omega^{(n_2)})=0\right)\\
&\times\mu_{n_2,n_0}\left((\omega^{(n_1+1)},\ldots,\omega^{(n_2)})=0\right)\\
=&\nu_{n_1,n_0}(\{k\})\left(1-\mu_{n_2,n_0}\left((\omega^{(n_1+1)},\ldots,\omega^{(n_2)})\neq 0\right)\right)
\end{aligned}
\tag{2.22}
$$

We estimate the second term in (2.20) by

$$
\begin{aligned}
&\mu_{n_2,n_0}\left(\omega^{(n_0)}+\ldots+\omega^{(n_2)}=k\wedge(\omega^{(n_1+1)},\ldots,\omega^{(n_2)})\neq 0\right)\\
&\leq\mu_{n_2,n_0}\left((\omega^{(n_1+1)},\ldots,\omega^{(n_2)})\neq 0\right)
\end{aligned}
\tag{2.23}
$$

Hence

$$
\begin{aligned}
&|\nu_{n_2,n_0}(\{k\})-\nu_{n_1,n_0}(\{k\})|\\
&\leq\mu_{n_2,n_0}\left((\omega^{(n_1+1)},\ldots,\omega^{(n_2)})\neq 0\right)(1+\nu_{n_1,n_0}(\{k\}))\\
&\leq 2\mu_{n_2,n_0}\left((\omega^{(n_1+1)},\ldots,\omega^{(n_2)})\neq 0\right)\\
&=2\sum_{m=n_1+1}^{n_2}\mu_{n_2,n_0}\left(\omega^{(m)}\neq 0\wedge(\omega^{(m+1)},\ldots,\omega^{(n_2)})=0\right)\\
&=2\sum_{m=n_1+1}^{n_2}\mu_{m,n_0}\left(\omega^{(m)}\neq 0\right)\mu_{n_2,n_0}\left((\omega^{(m+1)},\ldots,\omega^{(n_2)})=0\right)\\
&\leq 2\sum_{m=n_1+1}^{n_2}\rho_{\beta^{(m)}}\left(J_0^{(m)}\right)\left(\omega^{(m)}\neq 0\right)
\end{aligned}
\tag{2.24}
$$

But by our assumption, $\rho_{\beta^{(m)}}\left(J_0^{(m)}\right)\left(\omega^{(m)}\neq 0\right)$ is summable and thus the last line in (2.24) converges to zero as $n_1,n_2\uparrow\infty$, as desired. Hence we define $\nu_{n_0}(\{k\})\equiv\lim_{n\uparrow\infty}\nu_{n,n_0}(\{k\})$. To end the proof, we have to verify that this defines a probability measure ν_{n_0} on $\mathbb{Z}$.

To do so, note that from the same reasoning as above we get

$$
\sup_{n_2\geq n_1}\sum_{l,|l|\geq k}\nu_{n_2,n_0}(\{l\})\leq 2\sum_{m=n_1+1}^{\infty}\rho_{\beta^{(m)}}\left(J_0^{(m)}\right)\left(\omega^{(m)}\neq 0\right)+\sum_{l,|l|\geq k}\nu_{n_1,n_0}(\{l\})
\tag{2.25}
$$

From this follows that

$$
\lim_{k\uparrow\infty}\limsup_{n\uparrow\infty}\sum_{l,|l|\geq k}\nu_{n,n_0}(\{l\})=0
\tag{2.26}
$$

From (2.26) then in fact follows, of course, that

$$
\sum_{l=-\infty}^{\infty}\nu_{n_0}(\{l\})=\lim_{k\uparrow\infty}\lim_{n\uparrow\infty}\sum_{l,|l|<k}\nu_{n,n_0}(\{l\})=1-\lim_{k\uparrow\infty}\lim_{n\uparrow\infty}\sum_{l,|l|\geq k}\nu_{n,n_0}(\{l\})=1
\tag{2.27}
$$

$\diamondsuit$

An immediate corollary of lemma 2.1 is the following proposition, which forms the central result of this section:

PROPOSITION 2.2: *Let $J_0^{(n)}$ satisfy the assumptions (2.17) of lemma 2.1. Then the infinite volume limit w-$\lim_{k\uparrow\infty}\mu^{\eta}_{\Lambda_k,\beta,J}\equiv\mu_{\beta,J}$ exists for all $\eta\in\Omega$ and is independent of the choice of η and of the sequence Λ_k. $\mu_{\beta,J}$ is the unique Gibbs measure for the local specification $\mu^{\cdot}_{\Lambda,\beta,J}$.*

Proof: To prove (i) just notice that, for any Σ_Λ-measurable function f, (2.16) together with lemma 2.1 implies that

$$\lim_{k\uparrow\infty}\mu^{\eta}_{\Lambda_k,\beta,J}(f)=\int_{\mathbb{Z}}\nu_{n_a(\Lambda)}(d\omega)\tilde{f}(\omega) \tag{2.28}$$

since $\lim_{k\uparrow\infty}H_0^{(n_i^0(\Lambda_k))}(\eta)=0$. Thus we take the right hand side of (2.28) as a definition of the finite dimensional marginals. These are easily checked to be compatible and hence give by Kolmogorov's theorem a unique infinite volume measure $\mu_{\beta,J}$. Since our specification is not continuous, we cannot directly conclude from the compatibility relations (1.14) by performing the infinite volume limit that $\mu_{\beta,J}$ satisfies the DLR-equation. However, using (2.28), this is easily verified by a direct calculation on finite dimensional cylinder events. Conversely it is well known that for all extremal Gibbs measures $\mu_{\beta,J}$ we have the $\mu_{\beta,J}$-a.s. convergence of $\mu^{\eta}_{\Lambda_k,\beta,J}(f)$ to $\mu_{\beta,J}(f)$ for all bounded measurable functions f (see [Geo p.122]). Since we have classified the weak limits of local specifications for *all* boundary conditions, we have in fact already obtained all Gibbs measures. $\diamondsuit$

We see that to prove theorem 1 from proposition 2.2, all what is left to do is to show that the assumption of lemma 2.1, eq. (2.17) is verified, with probability one, under our hypothesis on the random variables J. But *given* the estimates proven in [BoK], this is quite easy. Namely, from proposition 3.1 of [BoK], we have that

$$I\!P\left[J_0^{(n)}(H)\leq-\delta\right]\leq\exp\left(-\frac{\delta^2}{2\sigma_n^2|H|^{2r}}\right)+\exp\left(-\frac{\delta^{2-2r}}{\sigma_n^2}\right) \tag{2.29}$$

with $\sigma_n^2=c^n\sigma^2$, for some $c<1$ (for $r=0$, see [BoP] and for $0<r<\frac{1}{2}$, see [K]). Now the convergence of the series in (2.17) is garanteed if all but a finite number of its terms are dominated by the terms, c_n, of some convergent series. Thus the theorem is proven if

$$I\!P\left[\rho_{\beta^{(n)}}\left(J_0^{(n)}\right)\left(\omega^{(n)}\neq0\right)>c_n\text{ i.o.}\right]=0 \tag{2.30}$$

which follows from the Borel Cantelli lemma [S] and the fact that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} I\!P\left[\rho_{\beta^{(n)}}\left(J_0^{(n)}\right)\left(\omega^{(n)} \neq 0\right) > c_n\right] \\
&= \sum_{n=0}^{\infty} I\!P\left[\frac{\sum_{h\neq 0}\exp\left\{-\beta^{(n)}\left[|h| + J_0^{(n)}(h)\right]\right\}}{\sum_{h}\exp\left\{-\beta^{(n)}\left[|h| + J_0^{(n)}(h)\right]\right\}} > c_n\right] \\
&\leq \sum_{n=0}^{\infty} I\!P\left[\sum_{h\neq 0} e^{-\beta^{(n)}\left[|h|+J_0^{(n)}(h)\right]} > c_n\right] \\
&\leq \sum_{n=0}^{\infty}\sum_{h\neq 0} I\!P\left[J_0^{(n)}(h) < -|h| - \frac{1}{\beta^{(n)}}\ln(c_n\alpha_h)\right] < \infty
\end{aligned}
\tag{2.31}
$$

where $\alpha_h = \frac{e^{-|h|}}{1-e^{-1}}$, and c_n may be choosen e.g. equal to $(1/2)^n$. Thus theorem 1 is proven.

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