Analyticity for the Kuramoto-Sivashinsky Equation

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Abstract. We study the analyticity properties of solutions of the Kuramoto-Sivashinsky equation

$$\partial_t U(x,t) \;=\; -(\partial_x^2+\partial_x^4)U(x,t) - U(x,t)\partial_x U(x,t) \;,$$

for initial data which are periodic with period L. Numerical experiments are presented which show that the solutions of the KS-equation are analytic in a strip around the real axis whose width is independent of L. A rigorous lower bound $\mathcal{O}(L^{-16/25})$ is given for this width.

1. Introduction

The Kuramoto-Sivashinsky (KS) equation in one space dimension

$$\partial_t U \;=\; -\partial_x^2 U - \partial_x^4 U - U \partial_x U \;, \tag{1.1}$$

serves as a prototype of a system with "self-generated" chaos. By this we mean that the linear part of the equation is (in momentum space) of the form $k^2 - k^4$. It is therefore unstable for $0 < |k| < 1$, leading to growing modes. These growing modes are coupled through the nonlinearity to the stable modes of $|k| > 1$, and it is the "equilibrium" between these modes which leads to global bounds for the KS-equation, see below.

A rigorous study of this equation was initiated by [NST], who considered the equation in a finite volume L, with periodic boundary conditions. In terms of the Fourier coefficients

$$U(x,t) \;=\; \sum_{n\in\mathbf{Z}} e^{iqnx} u_n(t) \;, \tag{1.2}$$

with $q = 2\pi/L$, the Eq.(1.1) takes the form

$$\frac{d}{dt} u_n(t) \;=\; ((qn)^2 - (qn)^4) u_n(t) - i\frac{qn}{2} \sum_{n'+n''=n} u_{n'}(t) u_{n''}(t) \;. \tag{1.3}$$

The momentum k equals qn. It seems that nothing rigorous is known for the problem on the infinite line. But even with periodic boundary conditions, the KS-equation is still a very interesting problem, since, for large enough L, enough unstable Fourier modes exist to make the system chaotic. In terms of rigorous results even a global bound on the solution was found only recently for general periodic functions, in two papers [I], [CEES], with the following result:

Theorem 1.1. *[CEES]. If the initial condition U_0 of the KS-equation is periodic with period L, then the same is true for the solution at time t. There is a constant C_0 such that*

$$\limsup_{t\to\infty} \int_{-L/2}^{L/2} dx \; |U(x,t)|^2 \;\le\; R_L^2 \;\equiv\; C_0 L^{16/5} \;, \tag{1.4}$$

and furthermore, if $\int_{-L/2}^{L/2} dx |U(x,t)| \;\le\; R_L^2$ for some t, then the same is true for all $t' > t$.

The purpose of the present paper is to show a bound on the *space derivatives* of U. More precisely, we prove

Theorem 1.2. *For large t, the function $U(x,t)$ is analytic in x in a strip of width*

$$\beta_L \;\ge\; C L^{-16/25} \;, \tag{1.5}$$

around the real axis.

In other words, we show that the high momentum part of the Fourier transform satisfies

$$|u_n(t)| \approx \mathcal{O}(e^{-\mathrm{const}.L^{-16/25} q|n|}) . \qquad (1.6)$$

This is analogous to statements about the high momentum behavior in hydrodynamics [M1, p.353, Fig.11], the so-called dissipative range. In the next section, we present some conjectures concerning this high momentum behavior, and in particular that *the width of the domain of analyticity seems to be independent of* L. This conjecture is based on numerical experiments. This observation is in line with a series of other (unproven) observations [M1], [M2], which suggest that the KS-equation shows *extensive* behavior in the parameter L, i.e., it behaves essentially like a large union of smaller systems, (see, however [P], [L] for some deviations from extensivity for time-dependent phenomena). In Section 3 we prove Theorem 1.2.

2. A numerical study of the domain of analyticity

In this section, we report some conjectures related to the large k behavior of the solutions of the KS-equation.

Conjecture 2.1. *There is a* $\beta > 0$, *independent of* L, *such that the solutions* U *of the* KS*-equation with space period* L *satisfy*

$$\limsup_{t\to\infty} \sum_{n\in\mathbf{Z}} \left| e^{\beta q|n|} u_n(t) \right|^2 \leq \mathrm{const}. \qquad (2.1)$$

Since $q = 2\pi/L$ this means that the decay is uniform in L as a function of the wavevector $k = qn$. We can reformulate the Conjecture 2.1 in the following form:

Conjecture 2.2. *There is a* $\beta > 0$, *independent of* L, *such that the solutions* $U(\cdot, t)$ *of the* KS*-equation with space period* L *are analytic in* x *in the strip* $|\mathrm{Im}\, x| < \beta$, *for sufficiently large* t.

The conjectures are based on a series of numerical experiments which we now describe. We integrate the KS-equation in k-space, using the Eq.(1.3). We do this using a pseudo-spectral code with an integrator for stiff differential equations [HW, Radau5]. We compute solutions to absolute precision 10^{-9} and to relative precision 10^{-8}. We choose the discretization in such a way that u_n is computed for $|n| < n_{\max}$ and $n_{\max} q \geq 12$. At a given time t the graph of $\log |u_n|$ as a function of $k = qn$ looks as in Fig. 1. Beyond k about 12, the precision of the

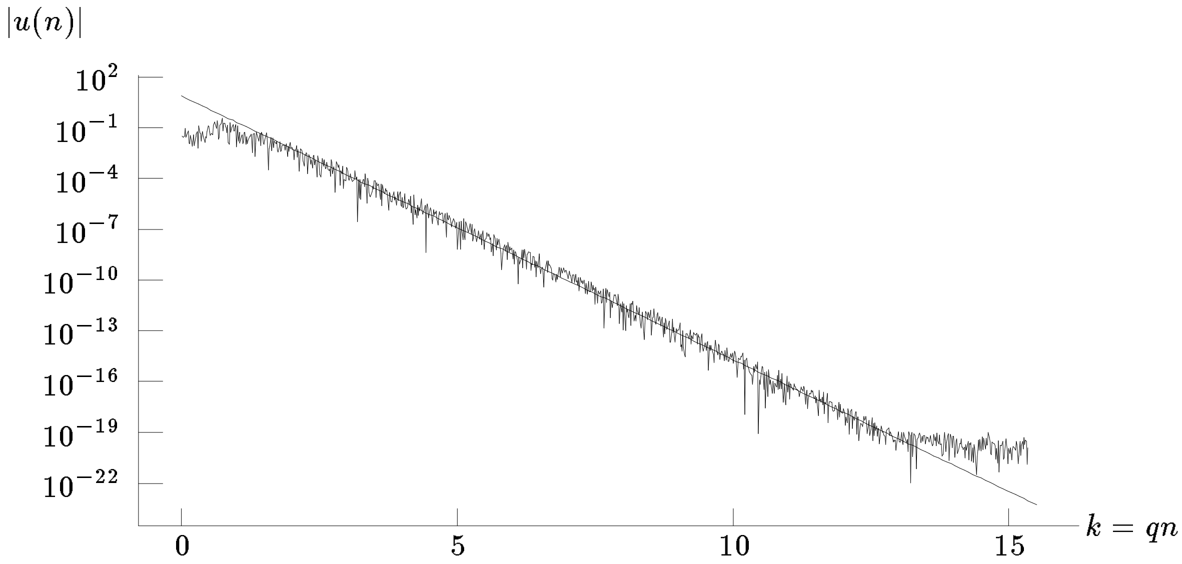

Fig. 1: Plot of $\log|u_n|$ versus $k = qn$ for $q = 0.015$, with $n_{\max} = 1024$, $k_{\max} = 15.36$.

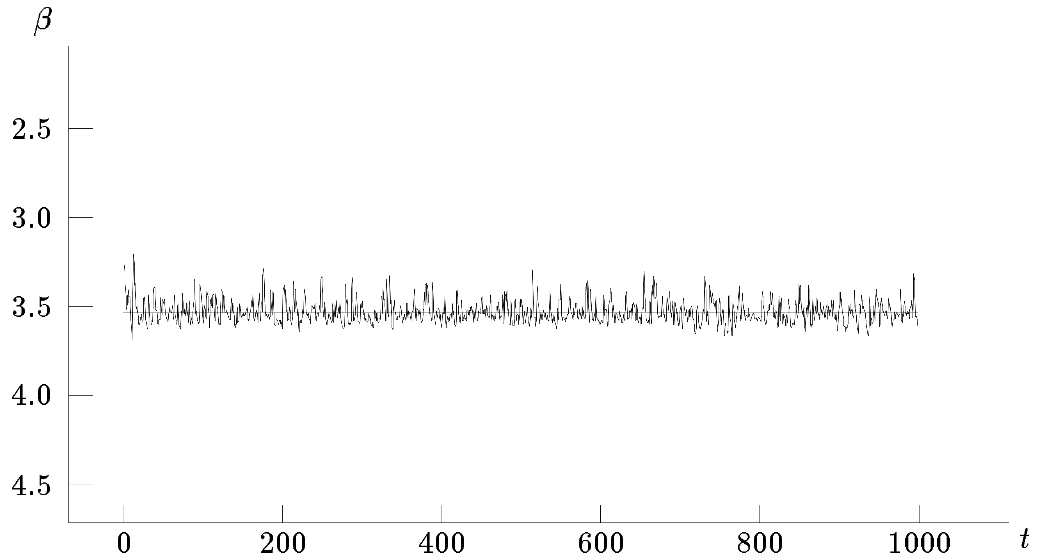

Fig. 2: The slope β of $-\log|u_n|$ versus $k = qn$ as measured in Fig. 1 as a function of time t, for $q = 0.015$, with $n_{\max} = 1024$. The horizontal line is at $\beta = 3.531$ and is obtained as the least squares fit for this value of q, from $t = 40$ to $t = 1000$.

data representation starts to matter and we get irrelevant results (in fact, for those values of n, we have $|u_n| < 10^{-20}$). We determine, for each time $t = 0.5 \cdot j$, $j = 1, \ldots, 2000$, the best fitting slope between $k = 2$ and $k = 12$, and plot this, together with the average of these slopes (over $t \in [40, 1000]$) in Fig. 2. We find that $\beta \approx 3.5$ *independently* of the value of q, for q

between 0.004 and 0.18, Fig. 3. This leads to the Conjecture 2.1.

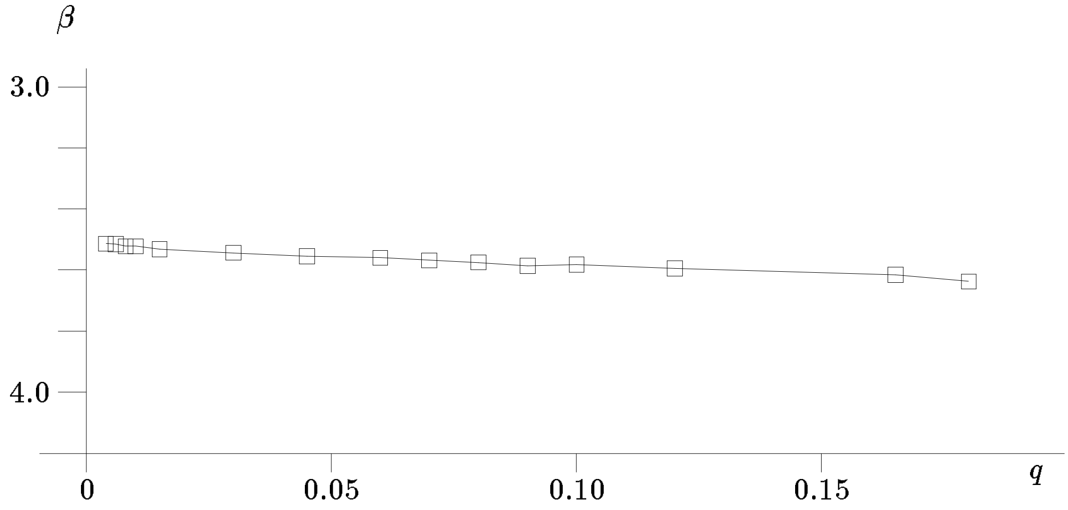

Fig. 3: The slope β as a function of $q = 2\pi/L$. This quantity is essentially independent of q and the limit value for $q = 0$ is about 3.5.

Together with G.Goren, we have also explored the modified KS-equation

$$\dot{u} \;=\; -\partial_x^2 u - |\partial_x^{2m}| u - uu' \;, \tag{2.2}$$

for several values of $m > 1$. One finds, with $\beta^{(m)}$ defined in the same spirit as β in the foregoing discussion.

Conjecture 2.3. *For every $m > 1$, there is a $\beta^{(m)}$ such that for all L one has for the solutions of Eq.(2.2):*

$$\limsup_{t\to\infty} \sum_{n\in\mathbf{Z}} \left| e^{\beta^{(m)} q|n|} u_n(t) \right|^2 \;\le\; \mathrm{const.} \tag{2.3}$$

In particular, $\beta^{(m)} \approx 1.75m$.

3. Analyticity

In this section, we prove Theorem 1.2, using techniques which have been developed in [FT]. Exploiting the fact that the trilinear term vanishes for equal arguments (see below) we get a slight improvement over the standard techniques. Instead of Theorem 1.2 we really prove the stronger

Theorem 3.1. *If the initial condition U_0 for the* KS*-equation with L-periodic boundary conditions satisfies $\int_{-L/2}^{L/2} dx U_0^2(x) \leq R_L^2$, then the solution $U(x,t)$ satisfies the bound*

$$\left\| e^{\alpha_L \min(t,t_L)\mathbf{A}} U(\cdot,t) \right\|_2 \leq 2R_L , \tag{3.1}$$

where

$$\mathbf{A} = \sqrt{-\partial_x^2} , \quad \alpha_L = \alpha_0 R_L^{6/5} , \quad t_L = t_0 R_L^{-8/5} .$$

The constants α_0 and t_0 are independent of L.

Proof of Theorem 1.2. By Theorem 1.1, we know that $\|U(\cdot,t')\|_2 < R_L$ for sufficiently large t'. We now apply Theorem 3.1, shifting the origin of time to t'. Since $R_L = \mathcal{O}(L^{8/5})$, the assertion of Theorem 1.2 follows at once.

Proof of Theorem 3.1. We consider again the Fourier transform Eq.(1.2). We shall always denote the x-space functions by upper case letters and their Fourier transforms by lower case letters. Since U is real we have $u_{-n} = u_n^*$ and since $\int U = 0$ we have $u_0 = 0$. The operator $\mathbf{A}$ acts by multiplication on the Fourier coefficients:

$$(\mathbf{A}U)_n = q|n|u_n , \quad n \in \mathbf{Z} .$$

We define $V = e^{\alpha t \mathbf{A}} U$ and therefore the Fourier coefficients of V are given by

$$v_n = u_n e^{\alpha t q|n|} .$$

The following computations and estimates leading to the bound of the theorem are formal, but they can be made rigorous by considering a Galerkin approximation for the Fourier coefficients u_n, i.e., by considering a finite number of Fourier coefficients and showing that the bounds are independent of the cutoff. Multiplying the Eq.(1.3) by $v_{-n}e^{\alpha t q|n|}$ and summing over n, we obtain

$$\tfrac{1}{2}\frac{d}{dt}\sum_{n\in\mathbf{Z}} |v_n|^2 = \alpha \sum_{n\in\mathbf{Z}} q|n||v_n|^2 + \sum_{n\in\mathbf{Z}} \mathcal{L}_n |v_n|^2 - \frac{1}{L} b(U,U,e^{\alpha t\mathbf{A}}V) , \tag{3.2}$$

where $\mathcal{L}_n = (qn)^2 - (qn)^4$ and b is defined by

$$b(U,V,W) = iLq \sum_{n\in\mathbf{Z}} \sum_{m\in\mathbf{Z}} (n-m) u_m v_{n-m} w_{-n} = \int U(\partial_x V) W .$$

One has the following estimate for the nonlinear term:

Lemma 3.2. *There is a constant C such that*

$$|b(U,U,e^{\alpha t\mathbf{A}}V)| \;\leq\; C\sqrt{\alpha t}\|U\|_2\|\mathbf{A}V\|_2^2\,. \tag{3.3}$$

Proof. Using $b(U,U,W)=-\frac{1}{2}b(U,W,U)$, and in particular, $b(U,U,U)=0$, we get

$$\begin{aligned} b(U,U,e^{\alpha t\mathbf{A}}V) &= -\tfrac{1}{2}b(U,e^{\alpha t\mathbf{A}}V,U) \\ &= -\tfrac{1}{2}\left[b(e^{-\alpha t\mathbf{A}}V,e^{\alpha t\mathbf{A}}V,e^{-\alpha t\mathbf{A}}V)-b(V,V,V)\right] \\ &= \frac{iL}{2}\sum_{n\in\mathbf{Z}}\sum_{m\in\mathbf{Z}} qnv_m v_{n-m}v_{-n}\left[e^{\alpha tq(|n|-|m|-|n-m|)}-1\right]\,. \end{aligned}$$

Note now that

$$|n|-|m|-|n-m| = \begin{cases} 0, & \text{if } n\geq m>0 \text{ or } n\leq m<0 \\ -2|m|, & \text{if } n>0>m \text{ or } n<0<m \\ -2|n-m|, & \text{if } m>n>0 \text{ or } m<n<0 \end{cases}\,.$$

After rearranging the summation indices, we therefore find

$$b(U,U,e^{\alpha t\mathbf{A}}V) \;=\; 2L\,\mathrm{Im}\sum_{k>m>0} q(k-m)v_{-m}v_k v_{-(k-m)}\left(1-e^{-2\alpha tq|m|}\right)\,.$$

We apply the Cauchy-Schwarz and the Minkowski inequalities and get the following estimates:

$$\begin{aligned} \left|b(U,U,e^{\alpha t\mathbf{A}}V)\right| &\leq 2L\left(\sum_{k>0}|v_k|^2\right)^{1/2}\left(\sum_{m>0}q^2m^2|v_m|^2\right)^{1/2} \\ &\quad\cdot\left(\sum_{k>m>0}q^2(k-m)^2|v_{k-m}|^2\Big(\frac{1-e^{-2\alpha tqm}}{qm}\Big)^2\right)^{1/2} \\ &\leq 2L\left(\sum_{k>0}|v_k|^2\right)^{1/2}\cdot\sum_{m>0}q^2m^2|v_m|^2 \\ &\quad\cdot\left(\frac{2\alpha t}{q}\int_0^\infty dx\,\Big(\frac{1-e^{-x}}{x}\Big)^2\right)^{1/2}\,. \end{aligned}$$

Reexpressing the sums in terms of the norms $\|\cdot\|_2$ in x-space, the Lemma 3.2 follows with

$$C^2 \;=\; \frac{1}{2\pi}\int_0^\infty dx\,\Big(\frac{1-e^{-x}}{x}\Big)^2\,.$$

We now continue the proof of Theorem 3.1. Starting from Eq.(3.2) and applying Lemma 3.2, we obtain the following differential inequality:

$$\tfrac{1}{2}\frac{d}{dt}\|V\|_2^2 \le \alpha\|\mathbf{A}^{1/2}V\|_2^2 + \|\mathbf{A}V\|_2^2 - \|\mathbf{A}^2V\|_2^2 + C\sqrt{\alpha t}\|V\|_2\|\mathbf{A}V\|_2^2 \,. \tag{3.4}$$

We use now the inequalities

$$\|\mathbf{A}V\|_2^2 \le \|V\|_2\|\mathbf{A}^2V\|_2 \,, \quad \|\mathbf{A}^{1/2}V\|_2^2 \le \|V\|_2^{3/2}\|\mathbf{A}^2V\|_2^{1/2} \,,$$

and we define the function $\Phi(t) \equiv \|V\|_2^2$. After polarization of the products using Young's inequality, we get from Eq.(3.4),

$$\frac{d}{dt}\Phi(t) \le (C_1 + C_2\alpha^{4/3})\Phi(t) + C_3\alpha t\Phi^2(t) \,, \tag{3.5}$$

where C_1, C_2, C_3 are constants which are independent of $\Phi(0)$ and of L. By the assumption of Theorem 3.1, we have $\Phi(0) \le R_L^2$. As long as $\Phi(t) \le 4R_L^2$, the Eq.(3.5) implies

$$\frac{d}{dt}\Phi(t) \le \big(C_1 + C_2\alpha^{4/3} + 4C_3R_L^2\alpha t\big)\Phi(t) \,, \tag{3.6}$$

which leads to

$$\Phi(t) \le R_L^2 \exp\big((C_1 + C_2\alpha^{4/3})\,t + 2C_3R_L^2\alpha t^2\big) \,.$$

Therefore, $\Phi(t) \le 4R_L^2$ as long as

$$(C_1 + C_2\alpha^{4/3})\,t + 2C_3R_L^2\alpha t^2 \le \log 4 \,. \tag{3.7}$$

Choosing $\alpha = \alpha_L = \alpha_0 R_L^{6/5}$ the Eq.(3.7) holds true for $t \le t_L = t_0 R_L^{-8/5}$, where α_0 and t_0 are suitable constants. This proves the Theorem 3.1 for $t \le t_L$. Since, by Theorem 1.1 the solution stays in the ball of radius R_L once it has entered it, we can choose the initial time in Theorem 3.1 as any time $t'' > 0$ and then Theorem 3.1 follows also for $t > t_L$. The proof of Theorem 3.1 is complete.

Acknowledgments. This research was partially supported by the Fonds National Suisse. JPE has profited from the kind hospitality of the IHES and of the Weizmann Institute.

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