Subcritical inhomogeneous percolation on general graphs

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Abstract

We study an inhomogeneous bond percolation process on graphs with general degree sequences. The edges of the host graph G are occupied independently with different probabilities. We show that the percolation phenomenon will not occur if the occupation probabilities are less than $(1-\varepsilon)/(\sigma\triangle+\tilde{d})$. Here, σ, $\triangle$ and $\tilde{d}$ are the spectral gap of the normalized Laplacian, maximum degree and second order average degree of the host graph G, respectively.

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1. Introduction

The bond percolation process on a host graph, typically the cubic lattice $\mathbb{Z}^d$ or a connected finite graph G, can be sketched as follows [13]. Each edge of the host graph is occupied with a uniform probability p, called occupation probability, and empty with probability $1-p$ independently from edge to edge. A percolation cluster is a set of vertices connected by occupied edges in the host graph. When the value p of the occupation probability is larger than a critical value p_c, the percolation transition, the size of the largest cluster goes from some lower order to the same order of the host graph, occurs. The phase before percolation transition is called subcritical and that after transition is referred to as supercritical. Analogously, the site percolation process corresponds to the pond percolation on the line graph (or conjugate lattice) of the host graph, and gives similar results.

Recently, percolation theory [13] has been introduced in the field of complex networks, and percolation processes on some finite host graphs have been investigated. The simplest host graph is the complete graph K_n, and the corresponding percolation model becomes the well-known Erdős-Rényi random graph $G(n,p)$. The percolation threshold $p_c = 1/n$ is shown in their seminal paper [11]. For dense host graphs with degrees of order $\Theta(n)$, it is shown in [3] that $p_c = 1/\rho$ where ρ is the largest eigenvalue of the adjacency matrix. Extending earlier results on Cayley graphs [14] and hypercubes [4], the host graph is taken to be a d-regular graph in the work [12]. In this case, the critical probability is shown to be near $1/d$. Similar situations have been treated for other host graphs with good algebraic properties, such as expander graphs (with bounded degrees) [1, 2] and transitive graphs [5, 15]. Host graphs with more general degree distributions have been analyzed by [8] and the critical probability is shown to close to $1/\tilde{d}$, where $\tilde{d}$ is the second order average degree. However, only homogeneous occupation probability is considered in the above works.

The functionality of real networks depends on the ability of the nodes and the edges in bearing loads of energy, information and other physical quantities. Therefore, inhomogeneous percolation models are highly desirable. Existing related literatures mainly concern the case where low dimensional regular lattices, such as $\mathbb{Z}^2$, are taken to be the host graphs [6, 16, 17]. Others tackle the inhomogeneity by using mean-field approximation in statistical physics, see e.g. [9] and references therein.

In this brief paper, we study an inhomogeneous percolation process with different occupation probabilities on graphs with general degree sequences. The subcritical regime of the percolation, i.e. the phase where no giant clusters appear, is defined via Laplacian spectral gap of the host graph.

The rest of the paper is organized as follows. Some necessary preliminaries on our percolation model and Laplacian spectrum are introduced in Section 2. We present the main result in Section 3.

2. Preliminaries

Let $G = (V,E)$ be a connected graph on n vertices with vertex set V and edge set E. Suppose $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is the adjacency matrix of G, where $a_{ij} = a_{ji} = 1$ if $(i,j) \in E$, and $a_{ij} = 0$ otherwise. Let $d_{v_i} = \sum_{j=1}^n a_{ij}$ be the degree of vertex $v_i \in V$. Denote by $\triangle$ the maximum degree of G. For each vertex set $S \subset V$, we define the volume and 2-th volume of S to be

$$\mathrm{vol}(S) = \sum_{v\in S} d_v \quad \text{and} \quad \mathrm{vol}_2(S) = \sum_{v\in S} d_v^2,$$

respectively. For $S = V$ we simply denote $\mathrm{vol}(G) = \mathrm{vol}(V)$ and $\mathrm{vol}_2(G) = \mathrm{vol}_2(V)$, respectively. We define the average degree $d = \mathrm{vol}(G)/n$ and the second order average degree $\tilde{d} = \mathrm{vol}_2(G)/\mathrm{vol}(G)$.

Now let $D = diag(d_{v_1}, \cdots, d_{v_n}) \in \mathbb{R}^{n\times n}$ be the diagonal degree matrix of G and I be the n-dimensional identity matrix. Let $\mathbf{1} \in \mathbb{R}^{n\times 1}$ denote the column vector with all entries one and $\mathbf{d} = D\mathbf{1}$ be the column vector of degrees. Then, the normalized Laplacian matrix [7] of G is defined as

$$L = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}.$$

The spectrum of L is its eigenvalues sorted in increasing order $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$. We have $\lambda_2 > 0$ since G is connected and $\lambda_n \leq 2$ with equality holding only if G is a bipartite graph. Let $\sigma = \max_{i\neq 1}\{|1-\lambda_i|\}$ be the spectral gap of normalized Laplacian matrix L. If G is connected and non-bipartite, then we have $\sigma < 1$.

Our percolation model is performed on the host graph G with occupation probability matrix $P = (p_{ij}) \in \mathbb{R}^{n\times n}$. Each edge $(i,j) \in E$ is occupied with probability p_{ij} independently with other edges, and we let $p_{ij} = 0$ if $(i,j) \notin E$. For any matrix $M = (m_{ij}) \in \mathbb{R}^{n\times n}$, let the max norm of M be $\|M\|_{\max} = \max_{1\leq i,j\leq n}\{|m_{ij}|\}$ and the spectral norm be $\|M\|_2 = (\text{maximum eigenvalue of } M^T M)^{1/2}$. Here, M^T denotes the transpose of M. It is easy to verify that $\|M\|_{\max} \leq \|M\|_2 \leq n\|M\|_{\max}$, see e.g. [10].

3. The result

In this section, we present the subcritical percolation result with a proof in similar spirit of [8]. The following spectral lemma is useful.

Lemma 1.([8]) *For the host graph G, we have*

$$\big|\|A\|_2 - \tilde{d}\big| \leq \sigma\triangle.$$

We will establish the following result.

Theorem 1. *For the inhomogeneous percolation process on G, if*

$$\|P\|_{\max} \leq \frac{1-\varepsilon}{\sigma\triangle + \tilde{d}},$$

for some $\varepsilon > 0$, then almost surely every percolation cluster has volume less than $\omega(n)$, for any $\omega(n) \gg \sqrt{\mathrm{vol}_2(G)}$, as $n \to \infty$.

Proof. Let $p(n)$ be the probability that there is a cluster, denoted by C, having volume greater than $\omega(n)$. We want to show that $p(n) \to 0$ as $n \to \infty$.

Now we choose two random vertices with the probability of being chosen proportional to their degrees in G. Condition on the event that the cluster C appears, the probability of each vertex being chosen in C is at least $\omega(n)/\mathrm{vol}(G)$. Hence, by multiplication formula, the probability that both of the random vertices are chosen in C is at least

$$p(n)\Big(\frac{\omega(n)}{\mathrm{vol}(G)}\Big)^2. \tag{1}$$

Next, for any fixed pair of vertices u and v, the probability that u and v are connected by a given path of length k in G is at most $\|P\|_{\max}^k$. The number of paths from u to v having length k is less than $\mathbf{1}_u^T A^k \mathbf{1}_v$, where $\mathbf{1}_u$ is the indicative column vector of vertex u. The probabilities that u and v are chosen are $d_v/\mathrm{vol}(G)$ and $d_u/\mathrm{vol}(G)$, respectively. Therefore, the probability that the random pair of vertices are in the same cluster is less than

$$\sum_{u,v\in G} \frac{d_u d_v}{(\mathrm{vol}(G))^2} \sum_{k=0}^{n} \|P\|_{\max}^k \mathbf{1}_u^T A^k \mathbf{1}_v = \sum_{k=0}^{n} \frac{1}{(\mathrm{vol}(G))^2} \|P\|_{\max}^k \mathbf{d}^T A^k \mathbf{d}. \tag{2}$$

From Lemma 1, we have $\|A\|_2 \le \sigma\triangle + \tilde{d}$. By our assumption, $\|P\|_{\max}\|A\|_2 \le 1 - \varepsilon$. The right-hand side of (2) is upper bounded by

$$\begin{aligned}
\Big\| \sum_{k=0}^{n} \frac{1}{(\mathrm{vol}(G))^2} \|P\|_{\max}^k \mathbf{d}^T A^k \mathbf{d} \Big\|_2 &\le \sum_{k=0}^{n} \frac{\|P\|_{\max}^k \|A\|_2^k \|\mathbf{d}\|_2^2}{(\mathrm{vol}(G))^2} \\
&= \frac{\mathrm{vol}_2(G)}{(\mathrm{vol}(G))^2} \sum_{k=0}^{n} \|P\|_{\max}^k \|A\|_2^k \\
&\le \frac{\tilde{d}}{\mathrm{vol}(G)\cdot(1 - \|P\|_{\max}\|A\|_2)} \\
&\le \frac{\tilde{d}}{\mathrm{vol}(G)\cdot\varepsilon}. \qquad (3)
\end{aligned}$$

Combining (1) with (3), we obtain

$$p(n)\Big(\frac{\omega(n)}{\mathrm{vol}(G)}\Big)^2 \le \frac{\tilde{d}}{\mathrm{vol}(G)\cdot\varepsilon}.$$

Consequently, we have

$$p(n) \le \frac{\tilde{d}\mathrm{vol}(G)}{\omega(n)^2\varepsilon} = \frac{\mathrm{vol}_2(G)}{\omega(n)^2\varepsilon},$$

which tends to 0 as $n \to \infty$, since $\omega(n) \gg \sqrt{\mathrm{vol}_2(G)}$. The proof is complete. □

For future research, the supercritical phase of the inhomogeneous percolation process should be studied and a unified transition phenomenon as a whole is desired.

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