A COMPARISON PRINCIPLE FOR A SOBOLEV GRADIENT SEMI-FLOW

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Abstract. We consider gradient descent equations for energy functionals of the type $S(u) = \frac{1}{2}\langle u(x), A(x)u(x)\rangle_{L^2} + \int_\Omega V(x,u)\,dx$, where A is a uniformly elliptic operator of order 2, with smooth coefficients. The gradient descent equation for such a functional depends on the metric under consideration.

We consider the steepest descent equation for S where the gradient is an element of the Sobolev space H^β, $\beta \in (0,1)$, with a metric that depends on A and a positive number $\gamma > \sup|V_{22}|$. We prove a weak comparison principle for such a gradient flow.

We extend our methods to the case where A is a fractional power of an elliptic operator.

We provide an application to the Aubry-Mather theory for partial differential equations and pseudo-differential equations.

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1. Introduction

In this paper we prove a comparison principle for steepest descent equations, in the Sobolev gradient direction (see (3)). When one is interested in minimizing functionals of the type

$$S(u) = \frac{1}{2}\langle u(x), A(x)u(x)\rangle_{L^2} + \int_\Omega V(x,u)\,dx, \tag{1}$$

where A is an elliptic operator, it is natural to consider the gradient descent equation $\partial_t u = -\nabla S(u)$. The gradient of S depends on the metric under consideration. The metrics and Sobolev gradients we consider are explained in Sections 1.2 and 1.4. See also [Neu97].

We are interested in comparison principles for gradient descent equations. One possible application is the use of comparison in establishing a priori bounds on the critical points of S in the context of the Aubry-Mather theory for partial differential equations and pseudo-differential equations (see [dlLV09]), but we develop the general methods in detail apart from applications. This will be discussed further in Section 6.

More concretely, we consider a self-adjoint, uniformly elliptic operator A given by the formula

$$Au = -\sum_{i,j=1}^{d} \partial_{x_j}\left(a^{ij}(x)\partial_{x_i} u\right) = -\mathrm{div}(a(x)\nabla u), \tag{2}$$

where the coefficient functions, $a^{ij} \in C^\infty(\mathbb{R}^d)$ are symmetric in i,j and we have positive constants Λ_1, Λ_2 such that for every$x \in \mathbb{R}^d$

$$\Lambda_1|\xi|^2 \le \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \le \Lambda_2|\xi|^2, \quad \forall \xi \in \mathbb{R}^d.$$

Then, for a suitably large constant γ, and for $\beta \in (0,1)$, we will show a comparison principle for the flow defined by the evolution equation

$$\partial_t u = -(\gamma + A)^{1-\beta} u + (\gamma + A)^{-\beta} \left(\gamma u - V_2(x,u)\right), \tag{3}$$

where $V(x,y) \in C^r(\mathbb{R}^d \times \mathbb{R})$, $r \geq 2$, and V_2 denotes the derivative of V with respect to its last argument. The fractional powers of $\gamma + A$ that appear in (3) will be defined in Section 3.2. As we shall show over the next few sections of this introduction, equation (3) is the steepest descent equation for S in the Sobolev space H^β with inner product $\langle u, v\rangle_{H^\beta} = \langle (\gamma + A)^\beta u, v\rangle_{L^2}$, as explained in Section 1.2. The domain and boundary considerations for equation (3) are discussed in the following section. A sufficient lower bound for the constant γ will be given in Section 4 and will depend on the nonlinear term V.

1.1. **Boundary Conditions.**

Our main results apply to Dirichlet boundary conditions for domains $\Omega \subset \mathbb{R}^d$ that are compact with smooth boundary, as well as for periodic boundary conditions (i.e. $\Omega = \mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d$).

In the application in Section 6 we will also consider $\Omega = N\mathbb{T}^d \cong \mathbb{R}^d/N\mathbb{Z}^d$, for which the reasoning regarding $N = 1$ applies. More succinctly, we pose (3) as an initial-boundary value problem, with $u(0,x) = u_0(x) \in L^\infty(\Omega)$ and one of the following two cases:

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{4}$$

$$u(x+e,t) = u(x,t) \qquad \forall e \in \mathbb{Z}^d \tag{5}$$

In the periodic setting we will require the functions $a^{ij}(x)$ and $V(x,y)$ to have period one in all variables.

Since our results rely on arguments that apply to both the Dirichlet and periodic settings, we will not distinguish between the different cases when stating the results.

1.2. **Sobolev Spaces.**

There are several equivalent definitions of the Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}$. For the integer case, we take $H^m(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \forall |\alpha| \leq m\}$, then the intermediate spaces may be defined by interpolation methods [Lun95], [Tay96]. Alternatively, one can use the Fourier transform to define $H^s(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : (1 + |\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^d)\}$ and then $H^s(\Omega) = \{u|_\Omega : u \in H^s(\mathbb{R}^d)\}$. The case where $\Omega = \mathbb{T}^d$ is handled simply by replacing the Fourier transform with the Fourier series. The factor $(1 + |\xi|^2)^{s/2}$ makes it clear that the operator $I - \Delta$ is the foundation of these spaces. Indeed, $u \in H^s(\mathbb{R}^d)$ if and only if $(I - \Delta)^{s/2} u \in L^2(\mathbb{R}^d)$, where $(I - \Delta)^{-s/2}$ is a particular case of the general definition of the power of an elliptic operator given in equation (6), and $(I - \Delta)^{s/2}$ is the inverse of $(I - \Delta)^{-s/2}$.

In fact, because $\gamma + A$ is an (order 2) elliptic, self-adjoint operator, we can define the Sobolev space $H^s_{\gamma,A}$ in the same manner as above, but replacing $I - \Delta$ with $\gamma + A$, for $\gamma > 0$. The inner product on $H^s_{\gamma,A}$ is given by $\langle u, v\rangle_{s,\gamma,A} = \langle (\gamma + A)^s u, v\rangle_0 \equiv \langle (\gamma + A)^s u, v\rangle_{L^2}$. In [Shu01], page 57 it is shown that the topology on $H^s_{\gamma,A}$ generated by the norm obtained from the above inner product is identical to the standard topology on H^s (i.e. the topology generated by $\langle u, v\rangle_{H^s} \equiv \langle (I - \Delta)^s u, v\rangle_{L^2}$). Thus, we henceforth omit the subscripts γ and A when referring to $H^s_{\gamma,A}$ and $\langle\cdot,\cdot\rangle_{s,\gamma,A}$. Will will write $\langle \cdot, \cdot\rangle_s$ for the inner product on H^s and $\|\cdot\|_s$ for the norm on H^s. Though the topologies are equivalent, the gradient of S depends on the chosen inner product. Thus, the gradient flow and therefore the comparison principle depend on the choice of inner product.

1.3. **Fractional Powers of Elliptic Operators.** For $s > 0$, the operator $(\gamma + A)^{-s}$ is self-adjoint, bounded, linear, invertible from H^r to H^{r+2s}, and is defined by

$$(\gamma + A)^{-s} = \frac{1}{2\pi i}\int_{\Gamma} z^{-s}(\gamma + A - z)^{-1}\,dz, \tag{6}$$

where Γ is a rectifiable curve in the resolvent set $\rho(\gamma + A) \subset \mathbb{C}$, avoiding $(-\infty, 0]$. Here z^s is taken to be positive for positive real values of z (see [Paz83] page 69, [Shu01] pages 83, 94). Positive powers are defined as $(\gamma+A)^s = (\gamma+A)^k(\gamma+A)^{s-k}$ where $k \in \mathbb{N}$ and $s < k$. It can be shown that $(\gamma + A)^s(\gamma + A)^r = (\gamma + A)^{s+r}$ for $s,r \in \mathbb{R}$, and that if $s \in \mathbb{Z}$ our definition coincides with the usual definition of integer powers of $(\gamma + A)$, see [Shu01]. In particular we have $(\gamma + A)^s = ((\gamma + A)^{-s})^{-1}$.

We will be interested in powers $0 < \alpha < 1$, for which the integral in (6) is equivalent to

$$(\gamma + A)^{-\alpha} = \frac{\sin \pi\alpha}{\pi}\int_0^{\infty} t^{-\alpha}(t + \gamma + A)^{-1}\,dt. \tag{7}$$

as shown in [Paz83], Section 2.6. This fact will be needed in Section 3.2 when we discuss the semigroup theory related to $(\gamma + A)^{1-\beta}$ and $(\gamma + A)^{-\beta}$. We will use repeatedly in Section 3.5 that the operator $(\gamma+A)^{-\beta} \in \mathcal{L}(H^s, H^{s+2\beta})$ is smoothing. We denote by $\mathcal{L}(H_1, H_2)$ the space of bounded linear operators from the Hilbert space H_1 to the Hilbert space H_2. For notational convenience we will sometimes use λ in place of $1 - \beta$, in particular when describing the domain $H^{s+2\lambda}$ of $(\gamma + A)^{\lambda} = (\gamma + A)^{1-\beta}$.

1.4. **The Sobolev Gradient.** The motivation for equation (3) is the desire to solve the semilinear elliptic equation

$$-Au = V_2(x, u) \quad x \in \Omega \tag{8}$$

subject to one of the boundary conditions (4) or (5). Equation (8) has an associated variational principle. In fact, it is the Euler-Lagrange equation for the functional

$$S(u) = \int_{\Omega} \frac{1}{2}\left(a(x)\nabla u(x) \cdot \nabla u(x)\right) + V(x, u(x))\,dx. \tag{9}$$

To minimize S, and therefore find a solution to (8), we could consider the steepest descent equation

$$\partial_t u = -Au - V_2(x, u). \tag{10}$$

The motivation for equation (10) is that the right-hand side is the unique element g of L^2 so that $DS(u)\eta = \langle g, \eta\rangle_{L^2}$, where $DS(u)$ is the Fréchet derivative of S at u. This element $g \in L^2$ is called the L^2 gradient of S with respect to the inner product $\langle\cdot,\cdot\rangle_{L^2}$. Instead of the standard L^2 inner product, if we used a different inner product, we would obtain a different gradient for S.

We consider the Sobolev space H^{β} with inner product $\langle u, v\rangle_{\beta} = \langle(\gamma+A)^{\beta}u, v\rangle_{L^2}$, and look for the gradient of S with respect to this space and inner product. That is, the unique element $g \in H^{\beta}$ such that $DS(u)\eta = \langle g, \eta\rangle_{\beta}$. We refer to g as the Sobolev gradient of S and write $g = \nabla_{\beta} S(u)$ (see. [Neu97]). As noted at the end of Section 1.2, this gradient depends not only on β but also on our choice of inner product, which was determined by A and γ.

We note that in each case, Dirichlet or periodic boundary conditions, we are able to use the integration by parts formula

$$-\int_{\Omega} \operatorname{div}(a(x)\nabla u)v\,dx = \int_{\Omega} a(x)\nabla u \cdot \nabla v\,dx, \tag{11}$$

and we calculate the H^β-gradient as follows:

$$
\begin{aligned}
DS(u)\eta &= \int_\Omega a(x)\nabla u\cdot\nabla\eta + V_2(x,u)\eta\,dx \\
&= \langle -\mathrm{div}(a(x)\nabla u) + V_2(x,u),\eta\rangle_{L^2} = \langle Au + V_2(x,u),\eta\rangle_{L^2} \\
&= \big\langle(\gamma+A)^\beta(\gamma+A)^{-\beta}(Au+V_2(x,u)),\eta\big\rangle_{L^2} \\
&= \big\langle(\gamma+A)^{-\beta}(\gamma u + Au - \gamma u + V_2(x,u)),\eta\big\rangle_\beta \\
&= \big\langle(\gamma+A)^{1-\beta}u - (\gamma+A)^{-\beta}(\gamma u - V_2(x,u)),\eta\big\rangle_\beta .
\end{aligned}
$$

Thus, our steepest descent equation in H^β, $\partial_t u = -\nabla_\beta S(u)$, becomes

$$\partial_t u = -(\gamma+A)^{1-\beta}u + (\gamma+A)^{-\beta}(\gamma u - V_2(x,u)),$$

which is identical to (3). If the solution $u(x,t)$ of (3) approaches a critical point, that is $u(x,t)\to u_c(x)$ as $t\to\infty$, then u_c will solve $(\gamma+A)^{1-\beta}u_c = (\gamma+A)^{-\beta}(\gamma u_c - V_2(x,u_c))$, which reduces to (8).

2. Main Result

We now wish to formulate our main theorem, which is a comparison principle for the flow defined by (3). The theorem is actually two theorems, one for each type of boundary condition. Thus, in the statement of the theorem, the space H^s may refer to either of the two types of Sobolev spaces H^s_0 (Dirichlet boundary conditions), or H^s_P (periodic boundary conditions). We will write Ω to represent the space domain, whether it is $\mathbb{T}^d$ or a smooth, bounded subset of $\mathbb{R}^d$ with Dirichlet boundary conditions.

Theorem 2.1. *Let $V \in C^r(\Omega\times\mathbb{R},\mathbb{R})$, $r\geq 2$, and choose γ such that $\gamma > \sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}}|V_{22}|$. Let $T>0$, and let $u(x,t)$ and $v(x,t)$ be solutions of (3) for $t\in[0,T]$ with initial conditions $u(x,0)=u_0\in L^\infty$ and $v(x,0)=v_0\in L^\infty$, where the exponent β in (3) is taken in the range $\beta\in(0,1)$. If $u_0\geq v_0$ for almost every $x\in\Omega$, then $u(x,t)\geq v(x,t)$ $\forall t\in[0,T]$ and almost every $x\in\Omega$.*

Remark. The regularity of V will be the only limit for the regularity of the solution u. We show in Proposition 3.6 that $u(t,\cdot)\in H^{r+\delta}$, for any $\delta<2$. In particular, if $r>d/2-1$ then for $u_0\in L^\infty$ we have that the solution $u(t,\cdot)\in C^0(\Omega)$.

The proof of existence will show that in our case, we have $T=\infty$, which allows this semi-flow to be used for finding critical points of S.

We use the notation $L := -(\gamma+A)^{1-\beta}$ and $X(u) := (\gamma+A)^{-\beta}(\gamma u - V_2(x,u))$ so that we can rewrite (3) as

$$\partial_t u = Lu + X(u), \tag{12}$$

and easily refer to the linear and nonlinear components of the equation as L and X. We now briefly outline the strategy for the proof of Theorem 2.1.

We will show that L generates a semigroup and this semigroup satisfies a comparison principle. The theory of semigroups will also allow us to show that, for large enough γ, the nonlinear operator X will also satisfy a comparison principle. We will then show that solutions to equation (12) exist for all time, and can be expressed via the integral formula

$$u(x,t) = e^{tL}u_0(x) + \int_0^t e^{(t-s)L}X(u(s,x))\,ds, \tag{13}$$

commonly referred to as Duhamel's formula (see [Tay97], page 272). This formula and the comparison principles for e^{tL} and X will allow us to show that the solutions to (12) must also satisfy a comparison principle.

3. Preliminaries

3.1. **Moser Estimates.** The composition $V(x,u)$ will be controlled by the use of Moser estimates for the composition of functions in H^s, $s \in \mathbb{N}$. If $f \in C^s(\mathbb{R}^d \times \mathbb{R})$ and $\phi \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ then

$$\|f(x,\phi)\|_s \leq c_s |f|_{C^s}(1 + \|\phi\|_s). \tag{14}$$

We also have that if $f \in C^{s+1}$ and if $\phi, \psi \in H^s$ are bounded with $s \in \mathbb{N}$, then

$$\|f(x,\phi) - f(x,\psi)\|_s \leq c_s |f|_{C^{s+1}}(1 + \|\phi\|_s + \|\psi\|_s)\|\phi - \psi\|_s. \tag{15}$$

The constant c_s depends on the supremum of ϕ and the diameter of Ω, see [Mos88], [Mos66]. When proving the existence of solutions to equation (3) we will first show that they exist in L^∞ for all time $t > 0$, and then that they are in fact continuous in the domain Ω and differentiable in time.

3.2. **Semigroups and Fractional Powers.** We will show that the operator $L = -(\gamma + A)^{1-\beta}$ generates a semigroup and that this semigroup has many nice properties, including a comparison principle. Most of these facts will be derived from properties of the semigroup generated by $-(\gamma + A)$. To prove that $-(\gamma + A)$ does, in fact, generate a semigroup, and to begin the investigation of some of its properties, we start with the following

Definition. If H is a Hilbert space and $D \subset H$ is a dense linear subspace of H, then a linear operator $B : D \to H$ is called *m-accretive* if

$$\begin{aligned} &\langle -Bu, u\rangle \geq 0 \quad \forall u \in D, \text{ and} \\ &(-B + I)D = H \end{aligned} \tag{16}$$

The Lumer-Phillips theorem (see [Paz83]) states that if B is m-accretive, then it generates a strongly continuous semigroup of contractions, e^{tB}. That is, $e^{tB} \in C([0,\infty), H) \cap C^1((0,\infty), H)$, there exists $c \geq 0$ such that $\|e^{tB}\|_{\mathcal{L}(H)} \leq e^{-ct}$, and if $u_0 \in H$, then $u(t,x) := e^{tB}u_0(x)$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= Bu \\ u(0,x) &= u_0(x). \end{aligned}$$

It is not hard to see that if $\gamma \geq 0$ then $\gamma + A$ is m-accretive on the Hilbert space H^s (with either periodic or Dirichlet boundary conditions). Let u be an element of the dense subspace $H^{s+2} \subset H^s$, then

$$\langle (\gamma + A)u, u\rangle_s = \langle (\gamma + A)^{s+1}u, u\rangle_{L^2} = \|u\|_{s+1}^2 \geq 0.$$

Here we have used the inner product on H^s as described in Section 1.2.

Standard results from the theory of elliptic boundary value problems also give the existence of a solution $u \in H^{s+2}$ to the equation $((1+\gamma)I + A)u = f$, subject to either of the two boundary conditions. Thus $(1+\gamma)I + A$ is a surjection from H^{s+2} to H^s (this was also discussed in Section 1.2). This establishes the second condition in (16). Hence, by the Lumer-Phillips Theorem, we have established

Proposition 3.1. *For each $s \geq 0$, the operator $-(\gamma + A)$ is m-accretive with respect to H^s, and therefore generates a semigroup $e^{-(\gamma+A)t} \in C([0,\infty), H^s) \cap C^1((0,\infty), H^s)$.*

The fractional power $(\gamma + A)^{-\alpha}$ for $\alpha \in (0,1)$ as defined in (7) can be expressed in terms of the semigroup $e^{-(\gamma+A)t}$. We have the formula

$$(\gamma + A)^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t(\gamma+A)} f \, dt, \tag{17}$$

as shown in [Paz83], [Vra03]. The above integral converges in H^s for $\alpha \in (0,\infty)$ because of the bound $\|e^{-t(\gamma+A)}\|_{\mathcal{L}(H^s)} \le e^{-c_s t}$, which follows from the fact that $e^{-t(\gamma+A)}$ is a semigroup of contractions on H^s. However, (17) is equivalent to (7) only for $\alpha \in (0,1)$. The representation above will be useful in proving a comparison principle for the operator $(\gamma+A)^{-\beta}$ in the next section.

A theorem in [Kat60] gives

Proposition 3.2. *For each $s \ge 0$, the operator $L := -(\gamma+A)^{1-\beta}$ is m-accretive in H^s, and therefore generates the semigroup $e^{tL} \in C([0,\infty), H^s) \cap C^1((0,\infty), H^s)$.*

Instead of referring to Kato, one can see this fact in the self-adjoint case by an argument similar to the one for $-(\gamma+A)$. In particular, $((1+\gamma)I + A)^\alpha$ maps $H^{s+2\alpha}$ onto H^s, as discussed in Section 1.2. And for $u \in H^{s+\alpha}$ we have

$$\langle(\gamma+A)^\alpha u, u\rangle_s = \langle(\gamma+A)^{s+\alpha} u, u\rangle_{L^2} = \|u\|^2_{s+\alpha} \ge 0.$$

3.3. **Further Estimates for e^{tL}.** If H is a Hilbert space and B is self-adjoint and m-accretive on H (this implies B is regular m-accretive, as defined on page 22 of [Sho97]), then for any integer $n \ge 1$ and any $u \in H$, we have $e^{tB}u \in D(B^n)$ and that

$$\|B^n e^{tB}\|_{\mathcal{L}(H)} \le \left(\frac{n}{\sqrt{2}t}\right)^n, \tag{18}$$

see page 29 of [Sho97]. This result applies to fractional powers of $-B$. In fact, if B is as above, and $\alpha \in (0,1)$ then there exists a constant $C_{\alpha,T}$ such that for any $t \in (0,T]$ one has

$$\|(-B)^\alpha e^{tB}\|_{\mathcal{L}(H)} \le C_{\alpha,T}\left(\frac{\alpha}{t}\right)^\alpha. \tag{19}$$

Furthermore, if we set $Y = D((-B)^\alpha)$, the domain of $(-B)^\alpha$, endowed with the graph norm $\|u\|_Y = \|u\|_H + \|(-B)^\alpha u\|_H$, then

$$\begin{aligned} &\|e^{tB}\|_{\mathcal{L}(H,Y)} \le C_{\alpha,T}\left(\frac{\alpha}{t}\right)^\alpha, \text{ and} \\ &\|e^{tB} - I\|_{\mathcal{L}(Y,H)} \le C'_{\alpha,T} t^\alpha. \end{aligned} \tag{20}$$

For further details on (18), (19), and (20) see Section 4.1 of [dlLV09].

We can use these facts to show that e^{tL} is a smoothing operator and to establish operator bounds on e^{tL} and $e^{tL} - I$. We say that e^{tL} is smoothing if it increases the regularity of a function as measured by membership in different spaces. That is, an operator is smoothing if it maps a space of functions into another space of smoother functions. In our case, if $u \in L^2$, then for any $s > 0$, $e^{tL} \in H^s$. To see this, we first note that because L is self-adjoint and m-accretive on H^s for $s \ge 0$, estimate (18) gives

$$\|L^n e^{tL}\|_{\mathcal{L}(H^s)} \le \left(\frac{n}{\sqrt{2}t}\right)^n. \tag{21}$$

Recall, as we established in Section 1.2, that $L : H^{s+2\lambda} \to H^s$ and that the inner product on H^s is given by $\langle u, v\rangle_s = \langle(\gamma+A)^s u, v\rangle_{L^2} = \langle(-L)^{s/\lambda} u, v\rangle_{L^2}$, where we set $\lambda = 1-\beta$. Note that $\lambda > 0$. If $u \in H^s$, we compute

$$\begin{aligned} \|e^{tL}u\|^2_{s+2n\lambda} &= \langle e^{tL}u, e^{tL}u\rangle_{s+2n\lambda} = \langle(-L)^{2n\lambda/\lambda} e^{tL}u, e^{tL}\rangle_s \\ &= \langle(-L)^n e^{tL}u, (-L)^n e^{tL}\rangle_s = (-1)^{2n}\|L^n e^{tL}u\|^2_s \le \left(\frac{n}{\sqrt{2}t}\right)^{2n} \|u\|^2_s. \end{aligned}$$

Thus, e^{tL} is smoothing in the sense described above, and we have the bound

$$\|e^{tL}\|_{\mathcal{L}(H^s, H^{s+2n\lambda})} \le \left(\frac{n}{\sqrt{2}t}\right)^n. \tag{22}$$

To apply (20) in the case $B = L$ and $H = H^s$, we have that $Y = H^{s+2\alpha\lambda}$. Then estimates (19) and (20) yield

$$\|(-L)^\alpha e^{tL}\|_{\mathcal{L}(H^s)} \le C_{\alpha,T}\left(\frac{\alpha}{t}\right)^\alpha, \tag{23}$$

$$\|e^{tL}\|_{\mathcal{L}(H^s,H^{s+2\alpha\lambda})} \le C_{\alpha,T}\left(\frac{\alpha}{t}\right)^\alpha, \tag{24}$$

$$\|e^{tL} - I\|_{\mathcal{L}(H^{s+2\alpha\lambda},H^s)} \le C'_{\alpha,T} t^\alpha. \tag{25}$$

In fact, (24) can be obtained from a calculation similar to that above for $\|e^{tL}u\|^2_{s+2n\lambda}$. In particular

$$\|e^{tL}u\|^2_{s+2\alpha\lambda} = \langle(-L)^\alpha e^{tL}u, (-L)^\alpha e^{tL}u\rangle_s \le C^2_{\alpha,T}\left(\frac{\alpha}{t}\right)^{2\alpha}\|u\|_s.$$

Thus, for $\alpha \in (0,1)$, $\|e^{tL}\|_{\mathcal{L}(H^s,H^{s+2\alpha\lambda})} \le C_{\alpha,T}\left(\frac{\alpha}{t}\right)^\alpha$.

3.4. **Subordination and Comparison.** The operator $\gamma + A$ is uniformly elliptic so the maximum principle for parabolic equations applies to

$$\frac{\partial u}{\partial t} = -(\gamma + A)u, \quad u(0,x) = u_0(x). \tag{26}$$

Thus, a solution to (26) on the interval $[0,T]$ obtains is maximum on the boundary of $\Omega \times (0,T]$, but not at $\Omega \times \{T\}$, (see [PW67]). Therefore, if $u \ge 0$, then for $t > 0$ we have $e^{-t(\gamma+A)}u \ge 0$. We will use this fact to establish comparison principles for X and e^{tL}. We begin with

Proposition 3.3. *Let $\gamma > \sup_{x,y}\{V_{22}(x,y)\}$. Then X satisfies a comparison principle. That is, if $u \ge v$ a.e., then $X(u) = (\gamma + A)^{-\beta}(\gamma u - V_2(x,u)) \ge (\gamma + A)^{-\beta}(\gamma u - V_2(x,u)) = X(v)$ a.e.*

Proof. Since $\gamma > \sup_{x,y}\{V_{22}(x,y)\}$, then $\gamma u - V_2(x,u)$ is increasing in u. Thus, $u \ge v$ implies $\gamma u - V_2(x,u) \ge \gamma v - V_2(x,v)$. Then, as mentioned above, the maximum principle for parabolic equations implies

$$e^{-t(\gamma+A)}(\gamma u - V_2(x,u)) \ge e^{-t(\gamma+A)}(\gamma v - V_2(x,v)).$$

Hence, for $t \ge 0$,

$$t^{\beta-1}e^{-t(\gamma+A)}(\gamma u - V_2(x,u)) \ge t^{\beta-1}e^{-t(\gamma+A)}(\gamma v - V_2(x,v)),$$

thus the representation of $(\gamma + A)^{-\beta}$ in equation (17) implies that $(\gamma + A)^{-\beta}(\gamma u - V_2(x,u)) \ge (\gamma + A)^{-\beta}(\gamma v - V_2(x,v))$, and we conclude that $X(u) \ge X(v)$. □

To show a comparison principle for e^{tL}, we again use the semigroup generated by $\gamma + A$. However, we will need a different representation of e^{tL}, which is given by the subordination identity of Bochner [Boc49]. For $\sigma > 0$, $t > 0$, $\tau \ge 0$, and $0 < \alpha < 1$ we define

$$\phi_{t,\alpha}(\tau) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\tau z - tz^\alpha}\,dz$$

and $\phi_{t,\alpha}(\tau) = 0$ if $\tau < 0$. Additionally, $\phi_{t,\alpha}(\tau) \ge 0$ for all $\tau > 0$. Then, for B m-accretive, we can represent $e^{-t(-B)^\alpha}$ as

$$e^{-t(-B)^\alpha} = \int_0^\infty e^{\tau B}\phi_{t,\alpha}(\tau)\,d\tau, \quad t > 0, \tag{27}$$

as is shown in [Yos74] Section IX.11. This provides a representation of e^{tL} as

$$e^{tL} = e^{-t(\gamma+A)^\lambda} = \int_0^\infty e^{-\tau(\gamma+A)}\phi_{t,\lambda}(\tau)\,d\tau, \quad \forall t > 0, \tag{28}$$

with $\phi_{t,\lambda}(\tau) \ge 0$, which will be key tool in establishing

Proposition 3.4. *If $u \geq v$ a.e. in Ω, then $e^{tL}u \geq e^{tL}v$ in Ω.*

Proof. If $u \geq v$ a.e. then for each $t > 0$ and $\tau > 0$, we have

$$e^{-t(\gamma+A)}\phi_{t,\lambda}(\tau)u \geq e^{-t(\gamma+A)}\phi_{t,\lambda}(\tau)v$$

because $\phi_{t,\lambda} \geq 0$ and $e^{-t(\gamma+A)}$ satisfies a comparison principle as explained above. Integrating both sides of the inequality yields $e^{tL}u \geq e^{tL}v$. The fact that e^{tL} is smoothing, in the sense of inequality (24), guarantees that $e^{tL}u$ and $e^{tL}v$ are continuous from Ω to $\mathbb{R}$, and therefore $e^{tL}u \geq e^{tL}v$ for all $x \in \Omega$. $\square$

In preparation for the proof of existence and uniqueness of solutions to (12), which will require L^∞ estimates on X and e^{tL}, we will show that X and e^{tL} are, in fact, locally bounded maps from L^∞ to itself. This is clear for e^{tL} by the remark at the end of Section 3.2 because for $t > 0$, $e^{tL}u \in H^s$ for arbitrarily large s, and therefore it is in L^∞ by the Sobolev embedding theorem. However, we can use the comparison principles for X and e^{tL} to provide explicit bounds.

Proposition 3.5. *$X : L^\infty \to L^\infty$ is locally bounded with $\|X(u)\|_{L^\infty} \leq \gamma^\lambda \|u\|_{L^\infty} + \gamma^{-\beta}\|V_2\|_{L^\infty}$. Additionally, for each $t > 0$, $e^{tL} : L^\infty \to L^\infty$ is a bounded linear map and $\|e^{tL}\|_{\mathcal{L}(L^\infty)} \leq e^{-\gamma^\lambda t}$.*

Proof. Let $u \in L^\infty$, and set $\overline{u} = \|u\|_{L^\infty}$, so that $u \leq \overline{u}$. Then the comparison principles established in Propositions 3.3 and 3.4 imply that $X(u) \leq X(\overline{u})$ and $e^{tL}u \leq e^{tL}\overline{u}$. Hence, if X and e^{tL} are bounded on constant functions then they are bounded on L^∞.

Consider first X, and set $C_V = \gamma\overline{u} + \|V_2\|_{L^\infty}$. We see that the boundedness of V_2 gives $\gamma\overline{u} - V_2(x, \overline{u}) \leq C_V$. The integral representation of $(\gamma + A)^{-\beta}$ in equation (17) and the same reasoning as in the proof for Proposition 3.3 imply $X(\overline{u}) = (\gamma + A)^{-\beta}(\gamma\overline{u} - V_2(x, \overline{u})) \leq (\gamma + A)^{-\beta}C_V$. Thus, to establish bounds on X, we only need to understand how $(\gamma + A)^{-\beta}$ acts on constants.

A consequence of Proposition 10.3 from [Shu01], page 93, is that if ψ is an eigenfunction of $\gamma+A$ with eigenvalue μ, then ψ is also an eigenfunction of $(\gamma+A)^{-\beta}$ with eigenvalue $\mu^{-\beta}$. But $(\gamma+A)C_V = \gamma C_V$ because A is a second-order differential operator. Hence $(\gamma + A)^{-\beta}C_V = \gamma^{-\beta}C_V$ and

$$X(u) \leq X(\overline{u}) \leq \gamma^{-\beta}(\gamma\overline{u} + \|V_2\|_{L^\infty}) = \gamma^\lambda \|u\|_{L^\infty} + \gamma^{-\beta}\|V_2\|_{L^\infty},$$

establishing the first claim in Proposition 3.5.

To bound e^{tL} we examine how L acts on constants. Using $(\gamma + A)^{-\beta}\overline{u} = \gamma^{-\beta}\overline{u}$, we calculate

$$L\overline{u} = -(\gamma + A)^\lambda \overline{u} = -(\gamma + A)(\gamma + A)^{-\beta}\overline{u} = (\gamma + A)\gamma^{-\beta}\overline{u} = -\gamma^{1-\beta}\overline{u} = -\gamma^\lambda \overline{u}.$$

Now e^{tL} is the semigroup generated by L, so we know $e^{tL}\overline{u}$ solves $\partial_t u = Lu$, $u(0) = \overline{u}$. But $Le^{tL} = e^{tL}L$, hence

$$\partial_t e^{tL}\overline{u} = e^{tL}L\overline{u} = e^{tL}(-\gamma^\lambda \overline{u}) = -\gamma^\lambda e^{tL}\overline{u}.$$

Thus $e^{tL}\overline{u}$ solves $\partial_t u = -\gamma^\lambda u$, $u(0) = \overline{u}$. Hence $e^{tL}\overline{u} = e^{-\gamma^\lambda t}\overline{u}$, and we have $e^{tL}u \leq e^{tL}\overline{u} = e^{-\gamma^\lambda t}\overline{u}$. $\square$

3.5. **Existence and Uniqueness of Solutions to Equation (12).** The comparison result in Theorem 2.1 requires only the existence of the flow generated by (12) for some short time $T > 0$. However, the motivation for studying this flow is to find critical points of the functional (9), for which the flow must be defined on all of $(0, \infty)$. In this section we establish the following

Proposition 3.6. *If the potential $V \in C^{r+1}(\Omega \times \mathbb{R}, \mathbb{R})$, $r \geq 1$, and $u_0 \in L^\infty(\Omega)$, then for every $\delta < 2$ there exists a unique solution, $u(x,t) \in C((0,\infty), H^{r+\delta} \cap L^\infty)$, to (12) with $u(x,0) = u_0(x)$. If $r \geq 2$ then $u(x,t) \in C^1((0,\infty), H^{r-2\lambda})$.*

We begin by showing existence and uniqueness of a mild solution in L^∞, and use the smoothing properties of the flow to obtain the desired regularity.

Definition. We say u is a *mild solution* of the equation $\partial_t u = Lu + X(u)$, with $u(0,x) = u_0(x)$ if u satisfies

$$u(t,x) = e^{tL}u_0(x) + \int_0^t e^{(t-\tau)L} X(u(\tau,x))\, d\tau.$$

We consider the map Ψ, on $C([0,T], L^\infty) : u(0,x) = u_0 \in L^\infty\}$ given by

$$\Psi u(t) = e^{tL}u_0 + \int_0^t e^{(t-\tau)L} X(u(\tau))\, d\tau.$$

The mild solution, $u(x,t)$, of (12) will be the fixed point of Ψ.

Lemma 3.7. *For $T \in \mathbb{R}$, define $W_T = \{u \in C([0,T], L^\infty) : u(0,x) = u_0 \in L^\infty\}$. Then for small enough $T > 0$, Ψ is a contraction on W_T. The size of T is independent of u_0.*

Proof. The norm on W will be defined as $\|u\|_{\infty,T} = \sup_{\tau\in[0,T]} \|u(\tau)\|_{L^\infty}$. For any $t \in [0,T]$, it is clear that $\|u\|_{\infty,t} \leq \|u\|_{\infty,T}$. Let $u \in W_T$, then $\Psi u(0) = u_0$, and we have

$$\begin{aligned}
\|\Psi u(t)\|_{L^\infty} &\leq \|u_0\|_{L^\infty} + \int_0^t e^{-\gamma^\lambda(t-\tau)}(\gamma^\lambda \|u(\tau)\|_{L^\infty} + \gamma^{-\beta}\|V_2(x,u(x,\tau))\|_{L^\infty})\, d\tau \\
&\leq \|u_0\|_{L^\infty} + (1 - e^{-\gamma^\lambda t})(\gamma^\lambda \|u\|_{\infty,T} + \gamma^{-\beta}\|V_2\|_{L^\infty}) \leq C_T,
\end{aligned}$$

where C_T is finite for finite T. This follows directly from Proposition 3.5.

Note that by the differentiability assumptions on V, we know that for any $x \in \Omega$ and any $y_1, y_2 \in \mathbb{R}$, $|V_2(x,y_1) - V_2(x,y_2)| \leq |V_2|_{C^1}|y_1 - y_2|$. To see that Ψ is a contraction, we compute

$$\begin{aligned}
\|\Psi u(t) - \Psi v(t)\|_{L^\infty} &\leq \int_0^t \|e^{(t-\tau)L}(X(u) - X(v))\|_{L^\infty}\, d\tau \\
&\leq \int_0^t e^{-\gamma^\lambda(t-\tau)}\|\gamma^\lambda(u-v) + \gamma^{-\beta}(V_2(x,v) - V_2(x,u))\|_{L^\infty}\, d\tau \\
&\leq (1 - e^{-\gamma^\lambda t})(\gamma^\lambda + \gamma^{-\beta}|V_2|_{C^1})\|u - v\|_{\infty,T} = Ct\|u - v\|_{\infty,T},
\end{aligned}$$

where $C = \gamma^\lambda(\gamma^\lambda + \gamma^{-\beta}|V_2|_{C^1})$ depends only on γ, β, and V. Here we have used only for convenience the fact that $1 - e^{-\gamma^\lambda t} \leq \gamma^\lambda t$ for $t \geq 0$. Choosing $T = \frac{1}{2C}$ ensures that $\|\Psi u - \Psi v\|_{\infty,T} \leq \frac{1}{2}\|u - v\|_{\infty,T}$, and thus Ψ is a contraction on W_T. This choice of T depends depends only on γ, β, and V. In particular, T is independent of the initial condition $u_0 \in L^\infty$. □

Proof of Proposition 3.6. Lemma 3.7 ensures that, for T sufficiently small, Ψ has a unique fixed point $u \in C([0,T], L^\infty)$ satisfying

$$u(t) = e^{tL}u_0 + \int_0^t e^{(t-\tau)L} X(u(\tau))\, d\tau,$$

establishing the existence and uniqueness of a mild solution to (12) in L^∞. Because T was chosen independently of u_0, we will also have existence on $[0,T]$ with initial condition $u(x,T)$, which gives existence on the interval $[0,2T]$ for initial condition u_0. This can be repeated indefinitely and we have L^∞ existence on $[0,\infty)$.

The operator $(\gamma + A)^{-\beta}$ is smoothing in the sense that it maps H^s into $H^{s+2\beta}$. Specifically, for $w \in H^s$,

$$\begin{aligned} \|(\gamma+A)^{-\beta}w\|_{s+2\beta}^2 &= \langle(\gamma+A)^{-\beta}w,(\gamma+A)^{-\beta}w\rangle_{s+2\beta} \\ &= \langle(\gamma+A)^{-2\beta}w,w\rangle_{s+2\beta} = \langle w,w\rangle_s = \|w\|_s^2, \end{aligned}$$

which gives $\|(\gamma+A)^{-\beta}\|_{\mathcal{L}(H^s,H^{s+2\beta})} = 1$. This implies that X will have a smoothing property too. However, this will be limited by the regularity of V. Recall that Ω is bounded, so $L^\infty \subset L^2 = H^0$, hence $(\gamma + A)^{-\beta}$ maps L^∞ into $H^{2\beta}$, and for any $u \in L^\infty$, and

$$\|X(u)\|_{2\beta} \le \|\gamma u+V_2(x,u)\|_0 \le \gamma\|u\|_{L^2}+\|V_2(x,u)\|_{L^2} \le |\Omega|^{1/2}(\gamma\|u\|_{L^\infty}+\|V_2\|_{L^\infty}).$$

Thus $X(u(t))$ is bounded in $H^{2\beta}$ as long as $u(t)$ is bounded in L^∞. Note that for any fixed $t \in [0,\infty)$ we know that $\|u(\tau)\|_{\infty,t}$ is bounded, where the norm $\|\cdot\|_{\infty,t}$ is the same notation as in the proof of Lemma (3.7). To see that $u(t)$ is actually a solution in $H^{2\beta}$, we simply compute, for a fixed $t > 0$,

$$\begin{aligned} \|u(t)\|_{2\beta} &\le \|e^{tL}u_0\|_{2\beta} + \int_0^t \|e^{(t-\tau)L}X(u(\tau))\|_{2\beta}\,d\tau \\ &\le \|e^{tL}u_0\|_{2\beta} + |\Omega|^{1/2}\int_0^t e^{-\gamma^\lambda(t-\tau)}\left(\gamma\|u(\tau)\|_{L^\infty}+\|V_2\|_{L^\infty}\right)d\tau \\ &\le \|e^{tL}u_0\|_{2\beta} + |\Omega|^{1/2}\frac{1}{\gamma^\lambda}\left(1-e^{-\gamma^\lambda t}\right)\left(\gamma\|u\|_{\infty,t}+\|V_2\|_{L^\infty}\right) < \infty. \end{aligned}$$

From here we wish to repeat this process to show that $u(t)$ is actually a solution in $H^{4\beta}$. However, this will require a bound on the composition $V(x,u(x,t))$ in the space $H^{2\beta}$. For this we will need to employ the Moser estimates (14), but these estimates only apply for H^n, $n \in \mathbb{N}$. This difficulty can be handled by splitting our analysis into two cases, first for $\beta \in [1/2,1)$ and then for $\beta \in (0,1/2)$. Recall that e^{tL} is smoothing in the sense of estimate (22), so that if $u_0 \in L^\infty \subset L^2$ then for all $t > 0$, $e^{tL}u_0 \in H^s$ for any $s \ge 0$. Thus, the term $\|e^{tL}u_0\|_{n+1}$ that appears in the estimates below will not impede to our regularity-building scheme.

Suppose $\beta \in [1/2,1)$, then the solution $u(t) \in L^\infty$ of (12) is bounded in $H^{2\beta}$, but $2\beta \ge 1$, so $\|u(t)\|_1 \le \|u\|_{2\beta}$. Thus, $u(t)$ is an H^1-solution of (12). To show that $u \in H^n$, we use induction on n, with $n = 1$ just established. Now assume $u \in H^n$ with $n \le r$, then by (14) we have $\|V_2(x,u)\|_n \le c_V(1+\|u\|_n)$, where c_V depends on $\|u\|_{\infty,t}$ and V. Because $2\beta \ge 1$ we have, for any $w \in H^n$, $\|(\gamma+A)^{-\beta}w\|_{n+1} \le \|w\|_{n+1-2\beta} \le \|w\|_n$. Using these facts, we compute

$$\begin{aligned} \|u(t)\|_{n+1} &\le \|e^{tL}u_0\|_{n+1} + \int_0^t \|e^{(t-\tau)L}X(u(\tau))\|_{n+1}d\tau \\ &\le \|e^{tL}u_0\|_{n+1} + \int_0^t \|(\gamma+A)^{-\beta}(\gamma u(\tau)-V_2(x,u(\tau)))\|_{n+1}d\tau \\ &\le \|e^{tL}u_0\|_{n+1} + \int_0^t \|\gamma u(\tau)-V_2(x,u(\tau))\|_n\,d\tau \\ &\le \|e^{tL}u_0\|_{n+1} + \int_0^t \gamma\|u(\tau)\|_n + c_V(1+\|u(\tau)\|_n)\,d\tau \\ &\le \|e^{tL}u_0\|_{n+1} + Ct\sup_{\tau\in[0,t]}(1+\|u(\tau)\|_n) < \infty. \end{aligned}$$

Thus, $u(t) \in H^{n+1}$, for any $n \le r$. Improved regularity to $u(t) \in H^{r+\delta}$ for any $\delta < 2$ will be shown after the case $\beta < 1/2$.

Between lines one and two of the above computation we have used the fact that e^{tL} is a contraction semigroup on H^{n+1}, i.e. $\|e^{(t-\tau)L}\|_{\mathcal{L}(H^{n+1})} \le 1$. We did not, however, take advantage of the smoothing property of e^{tL}. This is something we cannot afford to waste when $\beta \in (0, 1/2)$.

Suppose $\beta \in (0, 1/2)$, then $\lambda = 1 - \beta \in (1/2, 1)$. Let $u(t)$ be the L^∞-solution of (12). The previous method of using the smoothing properties of $(\gamma + A)^{-\beta}$ fails because the smoothing factor of 2β is too small. Fortunately, this is precisely when estimate (24) provides a large smoothing factor from e^{tL}. Applying (24) with $\alpha = 1/2\lambda$ gives

$$\|e^{tL}\|_{\mathcal{L}(H^s, H^{s+1})} \le C_{\lambda,T} t^{-1/2\lambda}.$$

The cost of the smoothing is the factor of $t^{-1/2\lambda}$. However, $2\lambda > 1$ so this is integrable on $[0, T]$, so we can proceed with a similar argument to the case for $\beta \in [1/2, 1)$. Just as we did not use the smoothing properties of e^{tL} in the previous case, we do not need the smoothing properties of $(\gamma + A)^{-\beta}$ in this case. To begin induction on n, we note that $u(t) \in L^\infty \subset H^0$. If $u(t) \in H^n$ with $n \le r$, then

$$\begin{aligned}
\|u(t)\|_{n+1} &\le \|e^{tL}u_0\|_{n+1} + \int_0^t \|e^{(t-\tau)L} X(u(\tau))\|_{n+1} d\tau \\
&\le \|e^{tL}u_0\|_{n+1} + \int_0^t C_{\lambda,T} t^{-1/2\lambda} \|(\gamma u(\tau) - V_2(x, u(\tau))\|_n d\tau \\
&\le \|e^{tL}u_0\|_{n+1} + \tilde{C} \sup_{\tau \in [0,t]} (1 + \|u(\tau)\|_n)\, t^{1-1/2\lambda} < \infty,
\end{aligned}$$

where $\tilde{C}$ depends on c_V, γ, λ, and T. Thus, $u(t) \in H^n$ for all $n \le r + 1$.

In either of the two cases, $\beta \ge 1/2$ and $\beta < 1/2$, we have established that $u \in H^r$. From here we can easily improve to $u \in H^{r+2\beta}$ because

$$\|u(t)\|_{r+2\beta} \le \|e^{tL}u_0\|_{r+2\beta} + \int_0^t \gamma\|u(\tau)\|_r + c_V(1 + \|u(\tau)\|_r) d\tau,$$

regardless of the size of β. Now we can push a little further and consider $\|u(t)\|_{r+2\beta+\epsilon}$ for some $\epsilon > 0$. We can use estimate (24) to obtain

$$\|e^{(t-\tau)L} X(u(\tau))\|_{H^{r+2\beta+2\lambda\alpha}} \le C^*(t - \tau)^{-\alpha}(1 + \|u(\tau)\|_r),$$

provided $\alpha \in (0, 1)$. The right side of the inequality is integrable on $[0, t]$. Thus $u(t) \in H^{r+2\beta+2\lambda\alpha}$ for any $\alpha < 1$, which gives $u(t) \in H^{r+\delta}$ for any $\delta < 2\beta + 2\lambda = 2\beta + 2(1 - \beta) = 2$. Hence $u \in C([0, T], H^{r+\delta} \cap L^\infty)$ for any $\delta < 2$.

To finish the proof of Proposition 3.6, we must show that the derivative, u_t, of u exists in $H^{r-2\lambda}$ so that u is actually C^1 and not just a mild solution. To see this, set $p = r - 2\lambda$, and consider $R_h = \frac{1}{h}\|u(t + h) - u(t) - hu_t\|_p$, where

$$u_t := Le^{tL}u_0 + \int_0^t Le^{(t-\tau)L} X(u(\tau))\, d\tau + X(u(t)),$$

Showing $R_h \to 0$ as $h \to 0$ will prove the desired result. We have

$$\begin{aligned}
R_h &\le \|\frac{1}{h}(e^{hL} - I)e^{tL}u_0 - Le^{tL}u_0\|_p + \\
&+ \|\int_0^t \left(\frac{1}{h}(e^{hL} - I) - L\right) e^{(t-\tau)L} X(u(\tau))\, d\tau\|_p + \\
&+ \|\frac{1}{h}\int_t^{t+h} e^{(t+h-\tau)L} X(u(\tau))\, d\tau - X(u(t))\|_p
\end{aligned}$$

For brevity, we will refer to the three terms on the right-hand side of the inequality as I_1, I_2, and I_3. I_1 goes to zero with h because L is the generator of the C_0-semigroup e^{tL} on H^p and the fact that $e^{tL}u_0 \in D(L)$ for all $t > 0$. In I_2, we can replace $(e^{hL}-I)$ by $\int_0^h Le^{\sigma L} d\sigma$ and we see that

$$\begin{aligned} I_2 &= \| \int_0^t e^{(t-\tau)L} \left(\frac{1}{h} \int_0^h Le^{\sigma L}\, d\sigma - L \right) X(u(\tau))\, d\tau \|_p \\ &= \| \int_0^t e^{(t-\tau)L} \frac{1}{h} \int_0^h \left(e^{\sigma L} - I \right) LX(u(\tau))\, d\sigma d\tau \|_p \\ &\le \int_0^t \frac{1}{h} \int_0^h \| e^{\sigma L} - I \|_{\mathcal{L}(H^{p+2\beta\lambda}, H^p)} \| LX(u(\tau)) \|_{p+2\beta\lambda}\, d\sigma d\tau \\ &\le \int_0^t \frac{1}{h} \int_0^h C_2 \sigma^\beta \, d\sigma \| LX(u(\tau)) \|_{p+2\beta\lambda}\, d\tau \\ &\le C_2' h^\beta \int_0^t \left(\|(u(\tau))\|_r + c_V(\|u(\tau)\|_\infty)(1 + \|u(\tau)\|_r) \right) d\tau. \end{aligned}$$

We have used $\|e^{(t-\tau)L}\|_{\mathcal{L}(H^p)} \le 1$ and, between lines three and four, applied (25) with $\alpha = \beta$. The estimate in the final line of the calculation follows from the bound $\|(\gamma + A)^{1-2\beta} w\|_s \le \|w\|_{s+2(1-2\beta)}$, which applies to the operator $LX(u)$. Hence, the fact that $p + 2\beta\lambda + 2(1-2\beta) = r - 2\beta^2 < r$ allows the use of (14). We know that for any fixed $t > 0$, $\|u(\tau)\|_r$ and $\|u(\tau)\|_\infty$ are bounded on $\tau \in [0,t]$, so I_2 can be made arbitrarily small for a suitable choice of h.

Without loss of generality, we assume that $h < 1$ and we set $M_1 = \max_{t \le \tau \le t+1} \|u(\tau)\|_\infty$ and $M_2 = \max_{t \le \tau \le t+1} \|u(\tau)\|_p$. Let $\epsilon > 0$, by continuity there exists $h \in (0,1)$ such that $\|u(\tau) - u(t)\|_p < \epsilon$ for $|\tau - t| \le h$. Thus we have

$$\begin{aligned} I_3 &\le \frac{1}{h} \int_t^{t+h} \| e^{(t+h-\tau)L} \left(X(u(\tau)) - X(u(t)) \right) \|_p \, d\tau \\ &\le \frac{1}{h} \int_t^{t+h} \| \gamma(u(\tau) - u(t)) - (V_2(x, u(\tau)) - V_2(x, u(t))) \|_{p-2\beta} \, d\tau \\ &\le \frac{1}{h} \int_t^{t+h} C_V(M_1)(1 + 2M_2) \| u(\tau) - u(t) \|_{r-2} \\ &\le \frac{C_{V,M}}{h} \int_t^{t+h} \| u(\tau) - u(t) \|_{r-2} \le C'_{V,M} \epsilon \frac{1}{h} h \le \epsilon C'_{V,M}, \end{aligned}$$

so that I_3 can be made arbitrarily small. We have again used $\|e^{(t+h-\tau)L}\|_{\mathcal{L}(H^p)} \le 1$ and that $p - 2\beta = r - 2$. We have also used (15) between lines two and three above. Thus we have shown I_1, I_2, $I_3 \to 0$ as $h \to 0$, establishing $u \in C^1([0,\infty), H^{r-2\lambda})$, completing the proof of Proposition 3.6. □

4. Proof of Theorem 2.1

We have established comparison principles for the semigroup e^{tL} and the operator X, as well as the existence of solutions to (12). To emphasize the initial conditions, it will be convenient to write the solutions of (12) as $u(x,t) = \Phi_t u_0$. Hence, we aim to show that if u_0, $v_0 \in L^\infty$ and $u_0 \ge v_0$ then $\Phi_t u_0 \ge \Phi_t v_0$ on a short time interval $[0,T]$. This will follow from the iteration method below.

4.1. **Iteration Method.** For $u \in L^\infty$ we define $F_t^0 u = e^{tL} u$, and the jth iterate of u as

$$F_t^{j+1} u = e^{tL} u + \int_0^t e^{(t-\tau)L} X(F_\tau^j u)\, d\tau,$$

defined on some interval $[0,T]$. F_t^{j+1} is well defined because X and e^{tL} are both bounded maps from L^∞ to itself, as shown in Proposition 3.5.

Proposition 4.1. *Let $T > 0$. If $u \ge v$ then $F_t^j u \ge F_t^j v$ for all $t \in [0,T]$.*

Proof. Assume $u \geq v$. Then by Proposition 3.4, $F_t^0 u = e^{tL}u \geq e^{tL}v = F_t^0 v$. We assume that $F_t^j u \geq F_t^j v$ and proceed by induction on j. By Proposition 3.3, we have that $X(F_t^j u) - X(F_t^j v) \geq 0$. Once again invoking Proposition 3.4 we have $e^{(t-s)L}[X(F_s^j u) - X(F_s^j v)] \geq 0$. Hence

$$F_t^{j+1}u - F_t^{j+1}v = e^{tL}(u-v) + \int_0^t e^{(t-s)L}\left[X(F_s^j u) - X(F_s^j v)\right]\, ds \geq 0$$

because the integrand is positive and $e^{tL}(u-v) \geq 0$. □

We need to show that this iteration converges to the solution in Theorem 2.1, so we now focus on a single initial condition, u_0. For notational convenience we write $u^j(t)$ in place of $F_t^j u_0$, and $u^0(t) = e^{tL}u_0$. Thus for each $j \in \mathbb{N}$ we have

$$u^{j+1}(t) = e^{tL}u_0 + \int_0^t e^{(t-s)L}X(u^j(s))\, ds.$$

Proposition 4.2. *If $u_0 \in L^\infty$ then there exists a $T > 0$ such that for all $t \in (0,T]$, $u^j(t) \in L^\infty \cap H^s$, for all $0 \leq s \leq r+1$ and every $j \in \mathbb{N}$.*

Proof. A consequence of Proposition 3.5 is that e^{tL} and X are bounded on L^∞. We use this to compute

$$\begin{aligned}
\|u^{j+1}(t)\|_\infty &\leq \|e^{tL}u_0\|_\infty + \int_0^t \|e^{(t-s)L}X(u^j(s))\|_\infty\, ds \\
&\leq \|u_0\|_\infty + T\max_{0\leq s\leq T}\|u^j\|_\infty \\
&\leq \|u_0\|_\infty + T(\|u_0\|_\infty + T\max_{0\leq s\leq T}\|u^{j-1}\|_\infty) \leq \ldots \\
&\leq \|u_0\|_\infty T + \|u_0\|_\infty T^2 + \ldots + \|u_0\|_\infty T^j + T^{j+1}\|u^0\|_\infty \\
&\leq \|u_0\|_\infty \frac{1-T^{j+1}}{1-T} + T^{j+1}\|u_0\|_\infty \leq \|u_0\|_\infty \frac{1}{1-T} = C_0
\end{aligned}$$

Where, without loss of generality, we have assumed $T < 1$. So we know that each iterate u^j is contained in the ball with radius C_0 in L^∞ for any j and for all $t \in [0,T]$. The L^∞ bounds allow the use of the Moser estimates (14). For integer $k \leq r+1$ we have

$$\begin{aligned}
\|u^{j+1}(t)\|_k &\leq \|e^{tL}u_0\|_k + \int_0^t \|e^{(t-s)L}X(u^j(s))\|_k\, ds \\
&\leq \|e^{tL}u_0\|_k + C_0T\max_{0\leq s\leq T}(1 + \|u^j(s)\|_k) \\
&\leq \|e^{tL}u_0\|_k + C_0T(1 + \|u_0\|_k + C_0T\max_{0\leq s\leq T}(1+\|u^{j-1}(s)\|_k)) \leq \ldots \\
&\leq \|e^{tL}u_0\|_k + C_0T(1+\|u_0\|_k) + \ldots + (C_0T)^j(1+\|u_0\|_k) + (C_0T)^{j+1}(1+\|u_0\|_k) \\
&\leq \sup_{0\leq t\leq T}\|e^{tL}u_0\|_k + \frac{(1+\|u_0\|_k)}{1-C_0T} = C_k.
\end{aligned}$$

□

We have assumed that $TC_0 < 1$ and $T < 1$, which require only that $T < \frac{1}{1+\|u_0\|_\infty}$.

Proposition 4.3. *u^j converges in $C([0,T], H^r)$.*

Proof.

$$\begin{aligned}
\|u^{j+1}(t)-u^j(t)\|_r &\le \int_0^t \|e^{(t-s)L}[X(u^j(s))-X(u^{j-1}(s))]\|_r\,ds\\
&\le t\max_{0\le s\le t}\|X(u^j(s))-X(u^{j-1}(s))\|_r\\
&\le t\max_{0\le s\le t} C|V|_rC_0(1+\|u^j(s)\|_r+\|u^{j-1}(s)\|_r)\|u^j(s)-u^{j-1}(s)\|_r\\
&\le t\max_{0\le s\le t} C^*\|u^j(s)-u^{j-1}(s)\|_r,
\end{aligned}$$

where C^* depends on $|V|_r$, C_0, and C_r. So we have

$$\max_{0\le t\le T}\|u^{j+1}(t)-u^j(t)\|_r \le C^*T\max_{0\le t\le T}\|u^j(s)-u^{j-1}(s)\|_r.$$

We can set $T=\frac{1}{2C^*}$, which will ensure that the sequence $u^j(t)$ is Cauchy in H^r. The argument for this is the following. First, notice that if $\|u^{j+1}-u^j\|\le \frac{1}{2}\|u^j-u^{j-1}\|$, then $\|u^{j+1}-u^j\|\le(\frac{1}{2})^j\|u^1-u^0\|$. So for any $\epsilon>0$, choose N such that $(\frac{1}{2})^{N-1}\|u^1-u^0\|<\epsilon$. Then we have, for $m>n\ge N$,

$$\begin{aligned}
\|u^m-u^n\| &\le \frac{1}{2}\|u^m-u^{m-1}\|+\frac{1}{2}\|u^{m-1}-u^{m-2}\|+\ldots+\frac{1}{2}\|u^{n+1}-u^n\|\\
&\le \left[\left(\frac{1}{2}\right)^{m-1}+\left(\frac{1}{2}\right)^{m-2}+\ldots+\left(\frac{1}{2}\right)^{n}\right]\|u^1-u^0\|\\
&\le \left(\frac{1}{2}\right)^{m-1}\left[1+2+2^2+\ldots+2^{m-n-1}\right]\|u^1-u^0\|\\
&\le \left(\frac{1}{2}\right)^{m-1}2^{m-n}\|u^1-u^0\|\le\left(\frac{1}{2}\right)^{n-1}\|u^1-u^0\|\le\epsilon.
\end{aligned}$$

□

Corollary 4.4. *There exists a $T>0$ such that Φ_t satisfies a comparison principle on the interval $[0,T]$.*

Proof. From Proposition 4.3 we have the existence of $u^\infty\in C([0,T],H^r)$ such that $u^j\to u^\infty$ in $C([0,T],H^r)$. This function $u^\infty(t,x)=\lim_{j\to\infty}F_t^ju_0(x)$ must satisfy

$$u^\infty(t,x)=e^{tL}u_0+\int_0^t e^{(t-s)L}X(u^\infty(s,x))\,ds,$$

and therefore, by Proposition 3.6, $u^\infty(t,x)=\Phi_tu_0$. By Proposition 4.1, if $u_0\ge v_0$ then $F_t^ju_0\ge F_t^jv_0$ and therefore $\Phi_tu_0\ge\Phi_tv_0$ on $[0,T]$. Thus we know Φ_t obeys a comparison principle on a small time interval $[0,T]$. □

This establishes the comparison principle on a finite time interval $[0,T]$ and therefore concludes the proof of Theorem 2.1. To see that this comparison holds for all time $t>0$, we have the following lemma.

Lemma 4.5. *If $u_0\ge v_0$ for a.e. $x\in\Omega$, and there exist a time $t_1>0$ for which $\Phi_{t_1}u_0(x)<\Phi_{t_1}v_0(x)$ on a set of positive measure then for every $t>0$, $\Phi_tu_0(x)<\Phi_tv_0(x)$ on a set of positive measure.*

Proof. Let $u_0(x)\ge v_0(x)$ for a.e. $x\in\Omega$. Suppose there is a first time t_1 such that We define the family of $\Phi_tu_0(x)<\Phi_tv_0(x)$ on a set of positive measure. Then on this set of positive measure

$$e^{t_1L}(u_0(x)-v_0(x))+\int_0^{t_1}e^{(t_1-\tau)L}(X(\Phi_\tau u_0(x))-X(\Phi_\tau v_0(x)))\,d\tau<0. \tag{29}$$

However, $e^{t_1 L}(u_0(x) - v_0(x)) \geq 0$ for all $x \in \Omega$ by Proposition 3.4. We have by assumption that for all $\tau \in [0, t_1)$ $\Phi_\tau u(x) \geq \Phi_\tau v(x)$ for a.e. $x \in \Omega$. Hence, by Proposition 3.3 for all $\tau \in [0, t_1)$ $X(\Phi_\tau u(y)) - X(\Phi_\tau v(y)) \geq 0$ for a.e. $x \in \Omega$. Again applying Proposition 3.4 we have for any, $x \in \Omega$, $e^{(t_1-\tau)L}(X(\Phi_\tau u(y)) - X(\Phi_\tau v(y))) \geq 0$ for all $\tau \in [0, t_1)$. Thus for each $\tau \in [0, t_1)$ the integrand in (29) is a non-negative function on Ω, and therefore the integral (which is a function in H^{r+1}) must be non-negative for a.e. $x \in \Omega$. Therefore the left side of inequality (29) is the sum of the two terms that are non-negative for a.e. x and cannot be strictly negative on a set of positive measure. □

Combining Corollary 4.4 and Lemma 4.5 we have that Φ_t satisfies a comparison principle on the interval $[0, T]$, for $T > 0$ and therefore Φ_t satisfies a comparison principle on $[0, \infty)$.

5. Fractional Elliptic Equations

We can extend the methods above to gradient descent equations for energy functionals of the form

$$S_\alpha(u) = \langle u, A^\alpha u\rangle_{L^2} + \int_\Omega V(x,u)dx,$$

where A is given by (2) and $\alpha \in (0,1)$. The Euler-Lagrange equation for S_α is $A^\alpha u + V_2(x,u) = 0$.

For a fixed $\alpha > 0$, we use the inner product for the Sobolev space H^α given by

$$\langle u, v\rangle^*_\alpha = \langle(\gamma + A^\alpha)u, v\rangle_{L^2}$$

we have the inner product on $H^{\alpha r}$ given by

$$\langle u, v\rangle^*_{\alpha s} = \langle(\gamma + A^\alpha)^r u, v\rangle_{L^2}.$$

Now consider $\alpha \in (0,1)$. To calculate the Sobolev gradient of S_α, we first note that the derivative of S_α is $DS_\alpha(u)\eta = \langle \eta, A^\alpha u + V_2(x,u)\rangle_{L^2}$. Thus, the Sobolev gradient of S_α in $H^{\alpha\beta}$, $\beta \in (0,1)$, is calculated as

$$\begin{aligned} DS_\alpha(u)\eta &= \langle \eta, A^\alpha u + V_2(x,u)\rangle_{L^2} \\ &= \langle \eta, (\gamma + A^\alpha)^\beta(\gamma + A^\alpha)^{-\beta}(A^\alpha u + \gamma u - \gamma u + V_2(x,u))\rangle_{L^2} \\ &= \langle \eta, (\gamma + A^\alpha)^{-\beta}(A^\alpha u + \gamma u - \gamma u + V_2(x,u))\rangle^*_{\alpha\beta} \\ &= \langle \eta, (\gamma + A^\alpha)^{1-\beta}u - (\gamma + A^\alpha)^{-\beta}(\gamma u - V_2(x,u))\rangle^*_{\alpha\beta}. \end{aligned}$$

Hence, the gradient descent equation is

$$\partial_t u = -(\gamma + A^\alpha)^{1-\beta}u + (\gamma + A^\alpha)^{-\beta}(\gamma u - V_2(x,u)). \tag{30}$$

More concisely, we write $\partial_t u = \tilde{L}u + \tilde{X}(u)$ with $\tilde{L}u := -(\gamma + A^\alpha)^{1-\beta}u$ and $\tilde{X}(u) := (\gamma + A^\alpha)^{-\beta}(\gamma u - V_2(x,u))$.

Recall that the maximum principle for parabolic equations ensures that the semigroup e^{-tA} satisfies a comparison principle. The Bochner subordination identity (27) allows us to write the semigroup generated by $-A^\alpha$ as

$$e^{-tA^\alpha} = \int_0^\infty e^{-\tau A}\phi_{t,\alpha}(\tau)\, d\tau, \quad t > 0. \tag{31}$$

Thus, by the same argument as in the proof of Proposition 3.4, the comparison principle for e^{tA} guarantees that e^{-tA^α} satisfies a comparison principle as well.

Analogously, $-(\gamma + A^\alpha)$ generates the semigroup $e^{-t(\gamma + A^\alpha)}$, which also satisfies a comparison principle. Just as in (17) we can define the real powers as

$$(\gamma + A^\alpha)^{-\beta} f = \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{-t(\gamma + A^\alpha)} f\, dt,$$

and positive powers are again the inverses of negative powers. Thus, because we have a comparison principle for $e^{-t(\gamma+A^\alpha)}$ we have that $(\gamma+A^\alpha)^{-\beta}$ also satisfies a comparison principle. An argument as in the proof of Proposition 3.3 shows that if $\gamma > \sup_{x,y}|V_{22}(x,y)|$ then $\tilde{X}$ will satisfy a comparison principle.

Employing the subordination identity once again, we have

$$e^{t\tilde{L}} = e^{-t(\gamma+A^\alpha)^{1-\beta}} = \int_0^\infty e^{-\tau(\gamma+A^\alpha)}\phi_{t,1-\beta}(\tau)\,d\tau, \quad t>0.$$

Hence, we see $e^{t\tilde{L}}$ satisfies a comparison principle because $\phi_{t,1-\beta}(\tau)>0$ for all $\tau>0$ and $e^{-\tau(\gamma+A^\alpha)}$ satisfies comparison principle, just as in the proof of Proposition 3.4.

Smoothing estimates like (24) and (25) follow for $e^{t\tilde{L}}$ just as they did for e^{tL} in Section 3.2. The existence of solutions for $\partial_t u = \tilde{L}u + \tilde{X}(u)$ follows from these smoothing estimates and the arguments from Section 3.5.

Finally, the proof for a comparison principle for the flow defined by $\partial_t u = \tilde{L}u + \tilde{X}(u)$ follows from the Duhamel formula

$$u(t,x) = e^{-t\tilde{L}}u_0(x) + \int_0^t e^{-(t-\tau)\tilde{L}}\tilde{X}(u(\tau,x))\,d\tau,$$

and the iteration argument from Section 4.

5.1. **Constant Coefficients.** In the case of periodic boundary conditions and if the matrix $a(x)$ is constant (i.e. independent of x), the situation is simplified because we can write down concrete formulae for the operators, and we can use classical Fourier analysis in place of some of the abstract semigroup theory. For instance, we could have avoided the theory of fractional powers of operators and instead used the definition $(\gamma+A)^\alpha u = ((\gamma+4\pi^2\xi^T a\xi)^\alpha\hat{u}(\xi))^\vee$

One can also write down the semigroup $e^{-t(\gamma+A)} = e^{-t(\gamma-\Delta)} = e^{-\gamma t}e^{t\Delta}$. The operator $e^{t\Delta}$ is convolution with the heat kernel, written as

$$\left(e^{t\Delta}u\right)(x) = \frac{1}{(4\pi t)^{d/2}}\int_0^1 \sum_{k\in\mathbb{Z}^d} e^{-|x-y+k|^2/4t}u(y)\,dy. \tag{32}$$

Combining this with equation (28) yields the comparison principle for e^{tL} immediately.

6. Application to Aubry-Mather Theory for PDEs

We can use the comparison principle from Theorem 2.1 and the results from [dlLV09] to further develop the Aubry-Mather theory for PDEs to the case of a general elliptic problem of the form (8). For this section, we restrict discussion to periodic boundary conditions. We require the potential function V and the matrix coefficient functions $a^{ij}(x)$ to be periodic over the integers. That is,

$$V(x+e,y+l) = V(x,y) \quad \forall (e,l)\in\mathbb{Z}^d\times\mathbb{Z}, \quad \forall (x,y)\in\mathbb{R}^d\times\mathbb{R}$$

$$a^{ij}(x+e) = a^{ij}(x) \quad \forall e\in\mathbb{Z}^d, \quad \forall x\in\mathbb{R}^d$$

which we write as $V:\mathbb{T}^d\times\mathbb{T}\to\mathbb{R}$ and $a^{ij}:\mathbb{T}^d\to\mathbb{R}$. An important class of functions is given by the following

Definition. A function $u:\mathbb{R}^d\to\mathbb{R}$ is said to have the *Birkhoff property* (or u is a *Birkhoff function*) if for any fixed $e\in\mathbb{Z}^d$, and $l\in\mathbb{Z}$, $u(x+e)-(u(x)+l)$ does not change sign with x. That is, $u(x+e)-(u(x)+l)$ is either ≥ 0 or ≤ 0 depending on the choices of e and l, but not x. Any such function can be seen as a surface in $\mathbb{T}^{d+1}$ without any self-crossings.

For a fixed $\omega \in \mathbb{R}^d$, we define $B_\omega = \{u : \mathbb{R}^d \to \mathbb{R} | u \text{ is Birkoff}, u(x) - \omega \cdot x \in L^\infty(\mathbb{R}^d)\}$. We note that $u \in B_\omega$ if and only if u is a Birkhoff function and if for any $e \in \mathbb{Z}^d$ and $l \in \mathbb{Z}$, either $u(x+e) - u(x) - l \leq 0$ or ≥ 0 according to whether $\omega \cdot e - l \leq 0$ or ≥ 0 (see [Mos86]). The vector ω is referred to as the *frequency* or the *rotation vector* of the function u, and is a natural generalization of the one dimensional notion of rotation number. We have the following result.

Theorem 6.1. *Let $V \in C^2(\mathbb{T}^d \times \mathbb{T}, \mathbb{R})$, and let A be a self-adjoint, uniformly elliptic operator given by $Au = -div(a(x)\nabla u)$ with coefficients $a^{ij} \in C^\infty(\mathbb{T}^d, \mathbb{R})$. Then for any $\omega \in \mathbb{R}^d$, there exists a solution $u \in B_\omega$ to equation (8). That is $Au + V_2(x, u) = 0$, $u(x) - \omega \cdot x \in L^\infty(\mathbb{R}^d)$, and u is Birkoff.*

The method of proof from [dlLV09] is to first show the result holds for rational frequencies (i.e. for any $\omega_N \in \frac{1}{N}\mathbb{Z}^d$ with $N \in \mathbb{N}$). Then we obtain solutions for arbitrary $\omega \in \mathbb{R}^d$ as limits of solutions with rational frequencies. Passing to the limit requires an oscillation lemma of De Giorgi-Moser type, as well as classical C^ϵ elliptic estimates.

For minimal solutions, the oscillation lemma was given by Moser in Theorem 2.2 of [Mos86]. This gives a bound on the supremum of the solution u_N by the norm of its associated frequency $\omega_N \in \frac{1}{N}\mathbb{Z}^d$, and independent of N. We provide the arguments for the oscillation lemma in Section 6.2.

To have $\omega_N \to \omega$, we must have $N \to \infty$, and the size of the fundamental domain, $N\mathbb{T}^d$, becomes unbounded. If the bound on u_N depended on the size of the domain, then the C^ϵ-estimates on ∇u_N would degenerate. Instead, because the bounds on the u_N are in terms of ω_N, the C^ϵ-estimates on ∇u_N are uniform in N. Hence we can conclude the convergence of u_N to a continuous function u with associated frequency ω.

Note that the regularity assumption on V is weaker than in Theorem 2.1 because we can choose a smooth initial condition (i.e. $u_0(x) = \omega \cdot x$).

6.1. Rational Frequencies.

Lemma 6.2. *If $\omega \in \frac{1}{N}\mathbb{Z}^d$ and $u(x,t)$ solves (3) on $N\mathbb{T}^d$ with initial condition $u_0(x) = \omega \cdot x$, then there exists a sequence $t_n \to \infty$ as $n \to \infty$ such that $u(x, t_n) \to u^*_\omega$ in L^2 and u^*_ω solves (8).*

Proof. Suppose $u(x,t)$ solves (3) on $[0, N]^d$ with periodic boundary conditions and intial condition $u_o = \omega \cdot x$. In Section 1.2 we showed that $DS_N(u)\eta = \langle (\gamma + A)^{1-\beta} u - (\gamma + A)^{-\beta} (\gamma u - V_2(x, u)), \eta\rangle_\beta$. This together with the assumption that $u(x,t)$ solves (3) allows us to show that $S_N(u(x,t))$ is decreasing in t. More precisely,

$$\begin{aligned}
\frac{d}{dt} S_N(u(t)) &= DS_N(u(t))\partial_t u \\
&= \langle (\gamma + A)^{1-\beta} u - (\gamma + A)^{-\beta} (\gamma u - V_2(x, u)), \partial_t u\rangle_{H^\beta(N\mathbb{T}^d)} \\
&= -\|(\gamma + A)^{1-\beta} u - (\gamma + A)^{-\beta} (\gamma u - V_2(x, u))\|^2_{H^\beta(N\mathbb{T}^d)} \leq 0.
\end{aligned}$$

Recalling that $\Lambda_1 \leq a(x) \leq \Lambda_2$, we have $S_N(u(t)) \leq S_N(u_0) \leq \frac{N^d}{2}\Lambda_2|\omega|^2 + N^d\|V\|_{L^\infty}$ for all $t > 0$. We can conclude that

$$\begin{aligned}
\frac{\Lambda_1}{2}\int_{[0,N]^d} |\nabla u(x,t)|^2 dx &\leq \int_{[0,N]^d} \frac{1}{2} a(x)\nabla u(x,t) \cdot \nabla u(x,t) dx \\
&\leq \frac{N^d}{2}\Lambda_2|\omega|^2 + 2N^d\|V\|_{L^\infty([0,N]^d)}.
\end{aligned} \tag{33}$$

Thus, $\|\nabla u(t)\|_{L^2}$ is bounded uniformly in t.

Because $S_N(u)$ is bounded below and $\frac{d}{dt}S_N(u(t_n)) \leq 0$, there is a sequence $t_n \to \infty$ such that $\frac{d}{dt}S_N(u(t_n)) \to 0$ as $n \to \infty$. The periodicity of V ensures that for any sequence of integers, $\{k_n\}$, we have $S_N(u(t_n)+k_n) = S_N(u(t_n))$. By selecting $k_n = -\lfloor N^{-d}\int_{[0,N]^d} u(x,t_n)dx\rfloor$, we can replace the sequence $u(t_n)$ by $u(t_n)+k_n$ without affecting S_N. Therefore we assume, without loss of generality, that for each n, the average of $u(x,t_n)$ lies in the range $[0,1]$. The gradients $\nabla u(x,t_n)$ are uniformly bounded in L^2, so by Poincaré's inequality we have that $\{u(t_n)\}$ is precompact, and there is a subsequence $u(t_{n_k})$ that converges weakly in H^1 and strongly in L^2. Denote the limit by u^*_ω. V_2 is Lipschitz, so $\|V_2(\cdot,u(t_{n_k})) - V_2(\cdot,u^*_\omega)\|_{L^2} \to 0$ as $k\to\infty$. A is an L^2-closed operator, hence $Au^*_\omega + V_2(x,u^*_\omega) = 0$. $\square$

Lemma 6.3. *If $u(x,t)$ solves (3) with initial condition $u(x,0) = u_0(x) \in B_\omega$, then $u(t)\in B_\omega$ for all $t>0$.*

Proof. As before, we denote the solution u with initial condition u_0 of (3) as $u(x,t) = \Phi_t u_0(x)$. We can conclude from Theorem 2.1 that if $u_0 \le v_0$ then $\Phi_t u_0(x) \le \Phi_t v_0(x)$. For convenience we let $\mathcal{C}_k$ and $\mathcal{R}_l$ denote the family of operators

$$\mathcal{C}_k u(x) = u(x+k) \quad \text{and} \quad \mathcal{R}_l u(x) = u(x)+l$$

for each $k\in\mathbb{Z}^d$ and $l\in\mathbb{Z}$.

Let $l\in\mathbb{Z}$ and define $Y_l(x,t) = \mathcal{R}_l\Phi_t u_0(x) = u(x,t)+l$. Recall theabbreviated notation

$$Lu = -(\gamma+A)^{1-\beta}u \quad \text{and} \quad X(u) = (\gamma+A)^{-\beta}(\gamma u - V_2(x,u)),$$

which allows us to write (3) as $\partial_t u = Lu + X(u)$, or $\partial_t\Phi_t u_0 = L\Phi_t u_0 + X(\Phi_t u_0)$.

Now, V_2 is periodic over the integers, so $V_2(x,u) = V_2(x,\mathcal{R}_l u)$. Also, $(\gamma+A)^\alpha c = \gamma^\alpha c$ for any constant $c\in\mathbb{R}$. Hence

$$\begin{aligned} X(Y_l) = X(\mathcal{R}_l\Phi_t u_0) &= X(u(x,t)+l) = (\gamma+A)^{-\beta}(\gamma u + \gamma l - V_2(x,u+l)) \\ &= \gamma^{1-\beta}l + (\gamma+A)^{-\beta}(\gamma u - V_2(x,u)) = \gamma^{1-\beta}l + X(\Phi_t u_0). \end{aligned}$$

Similarly,

$$\begin{aligned} LY_l = L(\mathcal{R}_l\Phi_t u_0) &= L(u(x,t)+l) = -(\gamma+A)^{1-\beta}(u(x,t)+l) \\ &= -\gamma^{1-\beta}l - (\gamma+A)^{1-\beta}u(x,t) = -\gamma^{1-\beta}l + L(\Phi_t u_0). \end{aligned}$$

Therefore $L\Phi_t u_0 + X(\Phi_t u_0) = LY_l + X(Y_l)$ and we have $\partial_t Y_l = \partial_t\Phi_t u_0 = L\Phi_t u_0 + X(\Phi_t u_0) = LY_l + X(Y_l)$. So Y_l solves (3) with initial condition $Y_l(x,0) = u_0(x)+l$. But $\partial_t(\Phi_t\mathcal{R}_l u_0) = L(\Phi_t\mathcal{R}_l u_0)+X(\Phi_t\mathcal{R}_l u_0)$ with the same initial condition $\Phi_0\mathcal{R}_l u_0 = u_0+l$. Thus,

$$\Phi_t\mathcal{R}_l u_0 = \mathcal{R}_l\Phi_t u_0 \tag{34}$$

by the uniqueness of solutions to (3) as shown in Proposition 3.6.

Now define $Z_k(x,t) = \mathcal{C}_k\Phi_t u_0 = u(x+k,t)$. The periodicity of V_2 ensures that $\mathcal{C}_k V_2(x,u(x,t)) = V_2(x+k,u(x+k,t)) = V_2(x,u(x+k,t)$ so that $X(\mathcal{C}_k u(x,t)) = \mathcal{C}_k X(u(x,t))$. Clearly $L(\mathcal{C}_k u(x,t)) = \mathcal{C}_k Lu(x,t)$, so

$$\begin{aligned} \partial_t Z_k = \partial_t u(x+k,t) &= \mathcal{C}_k Lu(x,t) + \mathcal{C}_k X(u(x,t)) \\ &= L\mathcal{C}_k u(x,t) + X(\mathcal{C}_k u(x,t)) = LZ_k + X(Z_k). \end{aligned}$$

So Z_k solves (3) with initial condition $Z_l(x,0) = u_0(x+k)$. But $\partial_t(\Phi_t\mathcal{C}_k u_0) = L(\Phi_t\mathcal{C}_k u_0) + X(\Phi_t\mathcal{C}_k u_0)$ with the same initial condition $\Phi_0\mathcal{C}_k u_0 = u_0(x+k)$. Thus,

$$\Phi_t\mathcal{C}_k u_0 = \mathcal{C}_k\Phi_t u_0 \tag{35}$$

by the uniqueness of solutions to (3) as shown in Proposition 3.6.

Now suppose $u_0\in\mathcal{B}_\omega$ so that $\mathcal{C}_k\mathcal{R}_l u_0 \le 0$ or ≥ 0 according to whether $\omega\cdot k+l\le 0$ or $\omega\cdot k+l\ge 0$.Then the comparison principle from Theorem 2.1 yields $\Phi_t\mathcal{C}_k\mathcal{R}_l u_0 \le 0$ or ≥ 0 according to whether $\omega\cdot k+l\le 0$ or $\omega\cdot k+l\ge 0$. But equations (34) and

(35) together give $\Phi_t\mathcal{C}_k\mathcal{R}_l = \mathcal{C}_k\mathcal{R}_l\Phi_t$, so $\mathcal{C}_k\mathcal{R}_l\Phi_t u_0 \le 0$ or ≥ 0 according to whether $\omega \cdot k + l \le 0$ or $\omega \cdot k + l \ge 0$, hence $\Phi_t u_0 \in \mathcal{B}_\omega$. □

The set $\mathcal{B}_\omega$ is closed under L^2-limits, so $u^*_\omega \in \mathcal{B}_\omega$. This establishes Theorem 6.1 in the case $\omega \in \frac{1}{N}\mathbb{Z}^d$.

6.2. **Irrational Frequencies.** For the case of irrational frequency, let $\omega \in \mathbb{R}^d \setminus \mathbb{Q}^d$ and let (ω_n) be a sequence such that $\omega_n \in \frac{1}{n}\mathbb{Z}^d$ for each n and $\omega_n \to \omega$. Let $\mathrm{div}(a(x)\nabla u_n) = V_2(x, u_n)$, where $u_n \in \omega_n \cdot x + H^1(N\mathbb{T}^d)$, $\lambda < a(x) < \Lambda$, and let B_R denote a ball of radius R centered at some point in the domain Ω such that the concentric ball B_{4R} is also in the domain Ω. Then Theorem 8.22 of [GT01] gives for each n that

$$\mathrm{osc}_{B_R}(u_n) \le (1 - C^{-1})\mathrm{osc}_{B_{4R}}(u_n) + k(R)$$

where $k = \lambda^{-1}R^{2(1-d/q)}\|V_2(x, u_n)\|_{L^{q/2}(\Omega)}$ for $q > d$, which is bounded by the volume of Ω times $\|V_2\|_{L^\infty}$, and $C = C(d, \Lambda/\lambda, q) > 1$ (pages 200-201 of [GT01]). Following Moser's methods from the proof of Theorem 2.2 of [Mos86], we can obtain a local result on the cube $Q = \{x \in \mathbb{R}^d : |x_j| \le 1/2\}$, and then consider translations.

Using a variant of the above inequality we can say

$$\mathrm{osc}_Q(u_n) \le \theta \mathrm{osc}_{4Q}(u_n) + c_1$$

where $\theta \in (0, 1)$ depends on λ, Λ, d, since we can fix $q = d + 1$, and c_1 depends on λ, d, and $\|V_2(x, u_n)\|_{L^{q/2}(\Omega)}$. Here we will take $\Omega = 5Q = \{|x_j| \le 5/2\}$ so that $\|V_2(x, u_n)\|_{L^{q/2}(\Omega)} \le 5^d\|(V_2)^{(d+1)/2}\|_{L^\infty}$, independent of n.

Since $u_n(x + k) - u_n(x) - l$ has the same sign as $\omega_n \cdot k - l$, for any $x \in \mathbb{R}^d$ and any $k \in \mathbb{Z}^d$, $l \in \mathbb{Z}$, we can conclude $|u_n(x + k) - u_n(x) - \omega_n \cdot k| \le 1$ for any such x and k. For an arbitrary $y \in \mathbb{R}^d$ select $k \in \mathbb{Z}^d$ such that $y - k \in Q$, then we have

$$\begin{aligned} |u_n(x + y) - u_n(x) - y \cdot \omega_n| &\le |u_n(x + y) - u_n(x + k)| + \\ &\quad + |u_n(x + k) - u_n(x) - \omega_n \cdot k| + |\omega_n \cdot k - \omega_n \cdot y| \\ &\le \mathrm{osc}_{x+k+Q}(u_n) + 1 + |\omega_n| \end{aligned}$$

Then an argument parallel to the one on page 240 of [Mos86] shows that $\mathrm{osc}_{4Q}(u_n) \le \mathrm{osc}_Q(u_n) + 2\sum_{j=1}^d (1 + 2|(\omega_n)_j|)$ so that with the result from [GT01] we have $\mathrm{osc}_Q(u_n) \le c_2\sqrt{1 + |\omega_n|^2}$. This estimate holds in translated cubes $x + Q$, so if we assume that $|\omega_n| \le c|\omega|$, then the $|u_n(x + y) - u_n(x) - \omega_n \cdot y|$ are bounded in $\mathbb{R}^d$ by $c_3\sqrt{1 + |\omega|^2}$, independently of n.

This bound is crucial, because it allows the application of Theorem 5.2, page 277 of [LU68], which gives $u_n \in \omega_n \cdot x + C^{1,\epsilon}$ (Assuming $V \in C^{2,\epsilon}$), and bounds

$$|\nabla u_n| \le \gamma_1$$

with γ_1 depending on the ellipticity constants and ω_n (in fact, monotone in ω_n), so bounded for our sequence of (ω_n). So, convergence of $\omega_n \to \omega$ and the Arzelà-Ascoli Theorem imply the existence of a subsequence $u_{n_k} \to u^*_\omega$ in C^0_{loc}. Then, $\mathrm{div}(a(x)\nabla)$ is closed under C^0_{loc} limits, and we have

$$Au^*_\omega + V_2(x, u^*_\omega) = -\mathrm{div}(a(x)\nabla u^*_\omega) + V_2(x, u^*_\omega) = 0.$$

This completes the proof of Theorem 6.1. □

Remark. If we replace the operator A from Theorem 6.1 with A^α for $\alpha > 0$, then the same method of proof for Theorem 6.1 will allow us to find solutions of $A^\alpha u + V_2(x, u) = 0$, with $u \in \mathcal{B}_\omega$ provided that ω is rational. However, to find solutions for irrational ω, we require a oscillation lemma for the critical points with rational frequencies. With this, one may be able to find solutions with irrational frequencies using methods from [CSM05]. The work of [CS09] establishes a Harnack inequality in the autonomous case, and can perhaps be extended to our case as well.

For this method to work, one would need both the oscillation lemma of De Giorgi-Moser type as well as C^ϵ estimates for the fractional power of an elliptic operator.

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