On threshold eigenvalues and resonances for the linearized NLS equation [1]

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Abstract

We prove the instability of threshold resonances and eigenvalues of the linearized NLS operator. We compute the asymptotic approximations of the eigenvalues appearing from the endpoint singularities in terms of the perturbations applied to the original NLS equation. Our method involves such techniques as the Birman-Schwinger principle and the Feshbach map.

1. Introduction

In this work we consider the nonlinear Schrödinger (NLS) equation in three dimensions,

$$i\psi_t = -\Delta\psi + U_0(x)\psi + F_0(|\psi|^2)\psi, \tag{1}$$

where $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$, $\psi \in \mathbb{C}$. The problem in other dimensions can be treated similarly. For certain functions $U_0(x)$ and $F_0(|\psi|^2)$, the NLS equation (1) possesses solitary wave solutions of the form,

$$\psi = \phi_0(x)e^{i\omega t}, \qquad \omega > 0, \tag{2}$$

where $\phi_0(x)$ is an exponentially decreasing solution of the elliptic problem,

$$-\Delta\phi_0 + \omega\phi_0 + U_0(x)\phi_0 + F_0({\phi_0}^2)\phi_0 = 0, \tag{3}$$

such that $\phi_0 : \mathbb{R}^3 \to \mathbb{R}$, $\phi_0 \in \mathbb{C}^\infty$. The existence of such solutions was proven for a large class of nonlinearities (see e.g. [A 02], [BL 83I], [BL 83II], [BLP 81], [M 93], [Str 77]).

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These solutions are called the ground states if they are positive in the whole space $\mathbb{R}^3$ and the excited states if they are sign-indefinite. Linearization of the NLS equation near the solution (2) leads to the spectral problem for the operator $\mathcal{L}_0 := \sigma_1 \mathrm{H}_0$ considered on $L^2(\mathbb{R}^3, \mathbb{C}^2)$:

$$\mathcal{L}_0 \mathbf{u} = z\mathbf{u}, \tag{4}$$

where $\mathbf{u} = (u, w)^T, \sigma_1$ is the standard Pauli matrix and the energy operator H_0 is the second variation of the Hamiltonian functional for the NLS equation (1):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{H}_0 = \begin{pmatrix} L_{+,\ 0} & 0 \\ 0 & L_{-,\ 0} \end{pmatrix} \tag{5}$$

where $L_{\pm,\ 0} = -\Delta + \omega + V_{\pm,\ 0}$ is the pair of Schrödinger operators with the potentials $V_{\pm,\ 0}$ given by

$$V_{+,\ 0}(x) = U_0(x) + F_0(\phi_0^2) + 2F_0'(\phi_0^2)\phi_0^2, \qquad V_{-,\ 0} = U_0(x) + F_0(\phi_0^2) \tag{6}$$

$V_{\pm,\ 0} \in C^\infty$ and decay exponentially. The operator $\mathcal{L}_0$ is not self-adjoint since σ_1 and H_0 do not commute in general. The spectral analysis of the operator $\mathcal{L}_0$ plays the crucial role in proving the asymptotic stability of the solitary waves of the NLS equation (see e.g. [C 03], [BS 03], [P 04], [GS 05]). Spectral properties of the linearized NLS operator were studied recently in (see e.g. [CPV 05], [CP 05], [V], [VP 06], [CGNT 07]). Its spectrum $\sigma(\mathcal{L}_0)$ is symmetric with respect to the real and imaginary axes. The essential spectrum consists of two branches $(-\infty, -\omega]$ and $[\omega, \infty)$ such that the points $\pm\omega$ are called the *thresholds*, while the eigenvalues could be located anywhere in the complex plane. For the operator $\mathcal{L}_0$ its point spectrum is denoted as $\sigma_p(\mathcal{L}_0)$. The notations $\langle \mathbf{f}, \mathbf{g} \rangle$ are used for the inner product of $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$, (f, g) for $f, g \in L^2(\mathbb{R}^3)$, $\|f\|_2$ for the norm of $f \in L^2(\mathbb{R}^3)$, $\langle \mathbf{u}, \mathbf{v} \rangle_N$ for the scalar product of $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ and $|\mathbf{u}|$ for the length of $\mathbf{u} \in \mathbb{C}^3$.

A solution $\mathbf{u}_0$ of the spectral problem (4) at $z = \omega$ such that $(1+|x|^2)^{-\frac{s}{2}}\mathbf{u}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^2),\ s > \frac{1}{2}$ and $\nabla \mathbf{u}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^6)$ but $\mathbf{u}_0 \notin L^2(\mathbb{R}^3, \mathbb{C}^2)$ is called the *threshold resonance*. In the paper we extend the results of the work [CP 05] on the bifurcations from a zero resonance or a simple eigenvalue at $z = \omega$ in three dimensions by estimating these bifurcations in terms of the perturbation applied to the original NLS equation.The main result of the work is the proof of the structural instability of the singularities of both kinds at the end of the essential spectrum $z = \omega$ under generic perturbations and the computation of the asymptotics of the eigenvalues which can appear in the neighborhood of the point $z = \omega$ when such perturbations are applied. We omit the studies near $z = -\omega$ due to the symmetry of the problem.

We consider two types of perturbations to the equation (1). The first one is when the potential in the NLS is perturbed

$$U(x) = U_0(x) + \varepsilon U_p(x), \ \varepsilon > 0 \tag{7}$$

In the work all the unperturbed terms are denoted with the 0 index, the perturbed ones without any indices and the perturbations applied with the p index.

In the second case the perturbation is applied to the nonlinearity term:

$$F(|\psi|^2) = F_0(|\psi|^2) + \varepsilon F_p(|\psi|^2), \ \varepsilon > 0 \tag{8}$$

The ε is a small positive parameter and the frequency ω of oscillations of the standing wave solution is assumed to be fixed and $\phi(x)$ is the perturbed solitary wave solution.

Thus the energy operator will be

$$L_+ = -\Delta + \omega + U_0(x) + \varepsilon U_p(x) + 2F_0'(\phi^2)\phi^2 + F_0(\phi^2) \tag{9}$$

$$L_- = -\Delta + \omega + U_0(x) + \varepsilon U_p(x) + F_0(\phi^2) \tag{10}$$

when perturbation is applied to the potential term in the NLS equation and

$$L_+ = -\Delta + \omega + U_0(x) + 2(F_0'(\phi^2) + \varepsilon F_p'(\phi^2))\phi^2 + F_0(\phi^2) + \varepsilon F_p(\phi^2) \tag{11}$$

$$L_- = -\Delta + \omega + U_0(x) + F_0(\phi^2) + \varepsilon F_p(\phi^2) \tag{12}$$

when perturbation is applied to the nonlinearity term.

Clearly the kernel of the linearized operator $\sigma_1 \mathrm{H}_0$ contains the element $\boldsymbol{\phi}_0 = (0, \ \phi_0)^T$. The zero modes $(\nabla\phi_0, \ 0)^T$ disappear in the presence of the potential $U_0(x)$. We make the following assumption on the kernel (cf [CPV 05]).

Assumption 1.1 $\mathrm{Ker}(\sigma_1 \mathrm{H}_0) = \{\boldsymbol{\phi}_0\}$

We can write the energy operator as $\mathrm{H} = \mathrm{H}_0 + \varepsilon V_p + O(\varepsilon^2)$ in the $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ space. The explicit form of V_p is given in the Proposition 2.1. According to our assumption the operator $L_{+,\ 0}^{-1}$ has a bounded inverse, $L_{+,\ 0}^{-1} : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$ exists and $\|L_{+,\ 0}^{-1}\|_{L^2 \to H^2} < \infty$. For the non-invertible cases we refer to [FW 86, Str 77]. As for the potential and the nonlinear term involved in the NLS equation (1) and the perturbations applied to them we assume the following.

Assumption 1.2 *The potentials $U_0(x) \in C^\infty,\ U_p(x) \in C^\infty$ and are exponentially decreasing, the nonlineaer terms $F_0 \in C^\infty,\ F_p \in C^\infty$ and $F_0(0) = 0,\ F_p(0) = 0$.*

Theorem 1.3 *Let the operator $\mathcal{L}_0$ have a threshold resonance and no eigenvalues at the end of its essential spectrum ω, Assumptions 1.1 and 1.2 be satisfied and, NLS equation be perturbed either as (7) or (8) and the quantity $\langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle$ does not vanish. Let V_p be defined in Proposition 2.1. If $\langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle < 0$ then $\mathcal{L}$ has a simple eigenvalue $z(\varepsilon)$, on the real axis, such that*

$$z(\varepsilon) = \omega - 4\varepsilon^2 \langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle^2 + o(\varepsilon^2)$$

If $\langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle > 0$, then $\mathcal{L}$ has no eigenvalue in the neighborhood of $z = \omega$. In both cases, the zero resonance at $z = \omega$ disappears at $\varepsilon \neq 0$.

To formulate our next result we introduce the following auxiliary matrices:

$$(\hat{V}_p)_{i,j} := \langle \mathbf{u}_0^i, V_p \mathbf{u}_0^j \rangle, \quad 1 \le i, j \le N,$$

where $\{\mathbf{u}_0^i\}_{i=1}^N$ are the eigenvectors corresponding to the N-fold degenerate endpoint eigenvalue of the problem (4) and V_p is given in Proposition 2.1. We assume that the eigenvalues of the hermitian matrix $\hat{V}_p$ are simple since the multiple ones are not generic and do not vanish, i.e.

$$\hat{V}_p \mathbf{v}_p^i = \lambda_p^i \mathbf{v}_p^i, \quad \langle \mathbf{v}_p^i, \mathbf{v}_p^j \rangle_N = \delta_{i,j}, \quad \lambda_p^i \neq \lambda_p^j, \quad i \neq j, \quad i, j = 1, ..., N$$

Let

$$\hat{B}_{i,j} := \langle \mathbf{u}_0^i, \sigma_1 \mathbf{u}_0^j \rangle, \quad 1 \le i, j \le N$$

and assume that $\langle \mathbf{v}_p^i, \hat{B} \mathbf{v}_p^i \rangle \neq 0, \quad i = 1, ..., N$. The number of eigenvalues λ_p^i which are positive (negative) such that $\langle \mathbf{v}_p^i, \hat{B} \mathbf{v}_p^i \rangle$ is positive (negative) is being denoted as $N_{B^\pm}^\pm$. Thus

$$N = N_{B^+}^+ + N_{B^-}^- + N_{B^+}^- + N_{B^-}^+$$

Let

$$\hat{S}_{k,i} := \int_{\mathbb{R}^3} x_k s^i(x) \mathrm{d}x, \quad 1 \leq k \leq 3, \quad 1 \leq i \leq N$$

where $s^i(x) = V_{+,\,0}(x)u_0^i(x) + V_{-,\,0}(x)w_0^i(x)$.

Theorem 1.4 *Let the operator $\mathcal{L}_0$ have the N-fold degenerate eigenvalue at the end of its essential spectrum ω, the Assumptions 1.1 and 1.2 be satisfied, the NLS equation be perturbed either as (7) or (8), the quantities λ_p^i, $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle$, $\hat{S}\mathbf{v}_p^i$, $i = 1, ..., N$ do not vanish. Let V_p be defined in the Proposition 2.1. Then the N-fold degenerate eigenvalue disappears as $\varepsilon \neq 0$. Moreover, we have the following three cases for the spectrum of the perturbed operator $\mathcal{L}$.*

(i) When $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle < 0$ and $\lambda_p^i > 0$ or $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle > 0$ and $\lambda_p^i < 0$ the eigenvalues move in the negative direction of the real axis, such that

$$z_i(\varepsilon) = \omega + \varepsilon \frac{\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle} + O(\varepsilon^{\frac{3}{2}}), \quad i = 1, ..., N_{B^-}^+ + N_{B^+}^-$$

(ii) When $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle < 0$ and $\lambda_p^i < 0$ the eigenvalues move in the complex plane, such that they become complex conjugate pairs and their asymptotics are given by

$$z_i(\varepsilon) = \omega + \varepsilon \frac{\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle} \pm i \frac{\varepsilon^{\frac{3}{2}}}{24\pi} \frac{|\hat{S}\mathbf{v}_p^i|^2 \lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle^2} \sqrt{\frac{\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle}} + O(\varepsilon^2), \quad i = 1, ..., N_{B^-}^-$$

(iii) When $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle > 0$ and $\lambda_p^i > 0$ the $N_{B^+}^+$ eigenvalues will disappear from the neighborhood of the endpoint $z = \omega$ of the essential spectrum of the linearized NLS operator.

Remark 1.5 *A straightforward computation yields that the results of Theorem 1.3 and Theorem 1.4 when $N = 1$ are analogous to the results on the bifurcations of a zero resonance and a simple eigenvalue from the edge of the essential spectrum of the linearized NLS operator obtained by Pelinovsky and Cuccagna (see Propositions 3.4 and 4.5 of [CP 05]) with the perturbation potential computed in terms of the perturbation applied to the original NLS equation.*

Remark 1.6 *When we deal with a soliton, which remains to be the ground state under perturbation for ε small enough, only the bifurcations of the (i) and the (iii) kind in Theorem 1.4 are possible, since the eigenvalues of the linearized operator are restricted to the real and imaginary axes in this case.*

Bifurcations of resonances at the endpoints were studied in one dimension using the method of Evans functions (see [PKA 98], [KS 02], [KS 04]), numerically in one and two dimensions in ([CGNT 07]). The primary goal of the study of such bifurcations is to show the structural instability of such resonances and eigenvalues, that they disappear when generic perturbations are applied to the NLS equation. Nonexistence of resonances and eigenvalues at the thresholds $\pm\omega$ is one of the key assumptions in the works on the asymptotic stability of the NLS solitons (see [GS 05]). The assumption of nonexistence of eigenvalues embedded in the essential spectrum plays the crucial role in proving the dispersive estimates for the problem ([ES 06]), the existence of stable manifolds for an orbitally unstable NLS ([S]). Another question arising here is whether the endpoint bifurcations lead to the appearance of the eigenvalues lying in the upper half-plane which leads to instabilitary of solitary waves. Theorems 1.3 and 1.4 are the statements about the fate of the threshold resonance and the N-fold degenerate eigenvalue when perturbation is applied to the NLS equation, which is the generalization of the work of ([CP 05]), proven via techniques different than the resolvent expansions and the mappings between the weighted Sobolev spaces.

In the next section we investigate how the perturbations to the equation (1) entail modifications of the operators $L_{\pm,\ 0}$ and thus determine the V_p.

2. Corrections to the energy operator for the perturbed NLS

We have the following statement.

Proposition 2.1 *Let the Assumptions 1.1 and 1.2 be satisfied. Then the application of perturbation to the NLS equation (1) translates into perturbation of the energy operator, such that in the linear order, in the* $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ *space*

$$V_{+,p} = U_p(x) - 2\phi_0\{3F_0'(\phi_0^2) + 2F_0''(\phi_0^2)\phi_0^2\}(L_{+,\ 0})^{-1}(U_p\phi_0)$$

$$V_{-,p} = U_p(x) - 2\phi_0 F_0'(\phi_0^2)(L_{+,\ 0})^{-1}(U_p\phi_0)$$

in the first case and

$$V_{+,p} = -2\phi_0\{3F_0'(\phi_0^2) + 2\phi_0^2F_0''(\phi_0^2)\}(L_{+,\ 0})^{-1}(F_p(\phi_0^2)\phi_0) + F_p(\phi_0^2) + 2F_p'(\phi_0^2)\phi_0^2$$

$$V_{-,p} = -2\phi_0 F_0'(\phi_0^2)(L_{+,\ 0})^{-1}(F_p(\phi_0^2)\phi_0) + F_p(\phi_0^2)$$

in the second case.

To evaluate the corrections to the standing solitary wave solution of the NLS and thus to the energy operator of the problem we use the Implicit Function Theorem technique. The perturbed solitary wave solution can be expressed as

$$\phi(x) = \phi_0(x) + \phi_p(x)$$

Let us first consider the case when the potential is being perturbed in the original NLS equation, such that the nonlinear elliptic problem for the standing wave solution is

$$-\Delta\phi + \omega\phi + [U_0(x) + \varepsilon U_p(x)]\phi + F_0(\phi^2)\phi = 0$$

A straightforward computation using (3) leads to the equation for the correction term to the solitary wave solution

$$\phi_p = L_{+,\ 0}^{-1}[-\varepsilon\mathcal{K}_1(\phi_0) - \mathcal{K}_2(\phi_0, \phi_p)] \tag{13}$$

with $\mathcal{K}_1(\phi_0) = U_p\phi_0$ and $\mathcal{K}_2(\phi_0, \phi_p) = F_0(\phi^2)\phi + \varepsilon U_p\phi_p - 2F_0'(\phi_0^2)\phi_0^2\phi_p - F_0(\phi_0^2)\phi_p - F_0(\phi_0^2)\phi_0$.

We establish the two facts needed to prove the existence of the perturbed solitary wave solution in the appropriate Sobolev space and to evaluate its asymptotics.

Fact 1 *If* $\phi_p \in H^2(\mathbb{R}^3)$ *then* $L_{+,\ 0}^{-1}[\varepsilon\mathcal{K}_1(\phi_0) + \mathcal{K}_2(\phi_0, \phi_p)] \in H^2(\mathbb{R}^3)$.

Fact 2 *If* $\varepsilon,\ \|\phi_p^{(1)}\|_{H^2},\ \|\phi_p^{(1)}\|_{H^2}$ *are small enough, then*

$$\|L_{+,\ 0}^{-1}[\mathcal{K}_2(\phi_0, \phi_p^{(1)})] - L_{+,\ 0}^{-1}[\mathcal{K}_2(\phi_0, \phi_p^{(2)})]\|_{H^2} \leq \frac{1}{2}\|\phi_p^{(1)} - \phi_p^{(2)}\|_{H^2}$$

Proof of the Fact 1. Since the potential perturbation $|U_p(x)| \leq C,\ x \in \mathbb{R}^3$, we have $\|\mathcal{K}_1(\phi_0)\|_2 \leq C\|\phi_0\|_2 < \infty$. Here and further with a slight abuse of notation C stands for a finite, positive constant. Thus $L_{+,\ 0}^{-1}[\varepsilon\mathcal{K}_1(\phi_0)] \in H^2(\mathbb{R}^3)$.

Clearly $|\varepsilon U_p\phi_p| \leq \varepsilon C|\phi_p|$. Let $\eta(\phi) := F_0(\phi^2)\phi$. For this smooth function $\eta(\phi) \in C^\infty(\mathbb{R})$ we have

$$\eta(\phi) - \eta(\phi_0) - \eta'(\phi_0)(\phi - \phi_0) = \int_{\phi_0}^{\phi}\Big(\int_{\phi_0}^{s}\eta''(q)dq\Big)ds$$

and $\phi_p \in L^\infty(\mathbb{R}^3)$ by means of the Sobolev embedding theorem. Therefore,

$$|F_0(\phi^2)\phi - 2F_0'(\phi_0^2)\phi_0^2\phi_p - F_0(\phi_0^2)\phi_p - F_0(\phi_0^2)\phi_0| \le C|\phi_p|^2, \tag{14}$$

which yields the upper bound

$$|\mathcal{K}_2(\phi_0, \phi_p)| \le \varepsilon C|\phi_p| + C|\phi_p|^2$$

Since $\phi_p \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we have $\phi_p \in L^4(\mathbb{R}^3)$. Thus $\mathcal{K}_2(\phi_0, \phi_p) \in L^2(\mathbb{R}^3)$ and, therefore $L_{+,\ 0}^{-1}[\mathcal{K}_2(\phi_0, \phi_p)] \in H^2(\mathbb{R}^3)$.

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Proof of the Fact 2. Let $\phi_1 := \phi_0 + \phi_p^{(1)}$ and $\phi_2 := \phi_0 + \phi_p^{(2)}$. The norms $\|\phi_p^{(1,2)}\|_\infty$ are finite and sufficiently small by means of the Sobolev embedding theorem. A trivial computation yields

$$\mathcal{K}_2(\phi_0, \phi_p^{(1)}) - \mathcal{K}_2(\phi_0, \phi_p^{(2)}) = F_0(\phi_1^2)\phi_1 - F_0(\phi_2^2)\phi_2 + \varepsilon U_p(\phi_p^{(1)} - \phi_p^{(2)}) -$$
$$-2F_0'(\phi_0^2)\phi_0^2(\phi_p^{(1)} - \phi_p^{(2)}) - F_0(\phi_0^2)(\phi_p^{(1)} - \phi_p^{(2)})$$

Due to the bound $|U_p(x)| \le C,\ x \in \mathbb{R}^3$ we have $\|\varepsilon U_p(\phi_p^{(1)} - \phi_p^{(2)}\|_2 \le \varepsilon C\|\phi_p^{(1)} - \phi_p^{(2)}\|_2$. To estimate the remaining terms in the right side of the identity above we use the formula

$$\int_{\phi_2}^{\phi_1} \Big(\int_{\phi_0}^{s} \eta''(q)dq\Big)ds = \eta(\phi_1) - \eta(\phi_2) - \eta'(\phi_0)(\phi_1 - \phi_2),$$

which implies

$$\|F_0(\phi_1^2)\phi_1 - F_0(\phi_2^2)\phi_2 - 2F_0'(\phi_0^2)\phi_0^2(\phi_p^{(1)} - \phi_p^{(2)}) - F_0(\phi_0^2)(\phi_p^{(1)} - \phi_p^{(2)})\|_2 \le \tag{15}$$
$$\le C(\|\phi_p^{(1)}\|_\infty + \|\phi_p^{(2)}\|_\infty)\|\phi_p^{(1)} - \phi_p^{(2)}\|_2,$$

and therefore

$$\|\mathcal{K}_2(\phi_0, \phi_p^{(1)}) - \mathcal{K}_2(\phi_0, \phi_p^{(2)}\|_2 \le C(\varepsilon + \|\phi_p^{(1)}\|_\infty + |\phi_p^{(2)}\|_\infty)\|\phi_p^{(1)} - \phi_p^{(2)}\|_2$$

Hence we arrive at

$$\|L_{+,\ 0}^{-1}[\mathcal{K}_2(\phi_0, \phi_p^{(1)})] - L_{+,\ 0}^{-1}[\mathcal{K}_2(\phi_0, \phi_p^{(2)})]\|_{H^2} \le$$
$$\le \|L_{+,\ 0}^{-1}\|_{L^2 \to H^2} C(\varepsilon + \|\phi_p^{(1)}\|_\infty + \|\phi_p^{(1)}\|_\infty)\|\phi_p^{(1)} - \phi_p^{(2)}\|_{H^2},$$

which completes the proof of the Fact 2 due to the smallness of ε and the norms $\|\phi_p^{(1,2)}\|_\infty$.

$\blacksquare$

Now we turn to the studies of the situation when the nonlinear term is being perturbed in the NLS equation. Hence the standing wave solution will satisfy the elliptic equation

$$-\Delta\phi + \omega\phi + U_0(x)\phi + [F_0(\phi^2) + \varepsilon F_p(\phi^2)]\phi = 0$$

We easily obtain the equation for the correction term to the solitary wave solution via (3).

$$\phi_p = L_{+,\,0}^{-1}[-\varepsilon\mathcal{K}_3(\phi_0) - \mathcal{K}_4(\phi_0, \phi_p)], \tag{16}$$

where $\mathcal{K}_3(\phi_0) = F_p(\phi_0^2)\phi_0$ and $\mathcal{K}_4(\phi_0, \phi_p) = F_0(\phi^2)\phi + \varepsilon F_p(\phi^2)\phi - F_0(\phi_0^2)\phi_0 - F_0(\phi_0^2)\phi_p - F_0'(\phi_0^2)2\phi_0^2\phi_p - \varepsilon F_p(\phi_0^2)\phi_0$. Analogously to the first case when the potential term was being perturbed we prove the two crucial facts.

Fact 3 *If* $\phi_p \in H^2(\mathbb{R}^3)$ *then* $L_{+,\,0}^{-1}[\varepsilon\mathcal{K}_3(\phi_0) + \mathcal{K}_4(\phi_0, \phi_p)] \in H^2(\mathbb{R}^3)$.

Fact 4 *If* $\varepsilon, \|\phi_p^{(1)}\|_{H^2}, \|\phi_p^{(2)}\|_{H^2}$ *are small enough, then*

$$\|L_{+,\,0}^{-1}[\mathcal{K}_4(\phi_0, \phi_p^{(1)})] - L_{+,\,0}^{-1}[\mathcal{K}_4(\phi_0, \phi_p^{(2)})]\|_{H^2} \le \frac{1}{2}\|\phi_p^{(1)} - \phi_p^{(2)}\|_{H^2}$$

Proof of the Fact 3. Since $|F_p(\phi_0^2)| \le C,\ x \in \mathbb{R}^3$, we obtain $\|\mathcal{K}_3(\phi_0)\|_2 \le C\|\phi_0\|_2 < \infty$. Hence $L_{+,\,0}^{-1}[\varepsilon\mathcal{K}_3(\phi_0)] \in H^2(\mathbb{R}^3)$. Let $\eta_p(\phi) := F_p(\phi^2)\phi$. For this smooth function $\eta_p \in C^\infty$ we have $\eta_p(\phi) - \eta_p(\phi_0) = \displaystyle\int_{\phi_0}^{\phi} \eta_p'(s)ds$. By means of the Sobolev embedding theorem $\phi_p \in L^\infty$. Thus $|\varepsilon F_p(\phi^2)\phi - \varepsilon F_p(\phi_0^2)\phi_0| \le \varepsilon C|\phi_p|$, which along with the estimate (14) yields

$$|\mathcal{K}_4(\phi_0, \phi_p)| \le \varepsilon C|\phi_p| + C|\phi_p|^2$$

Since the correction to the solitary wave solution $\phi_p \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we have $\phi_p \in L^4(\mathbb{R}^3)$. Therefore, $\mathcal{K}_4(\phi_0, \phi_p) \in L^2(\mathbb{R}^3)$ and $L_{+,\,0}^{-1}[\mathcal{K}_4(\phi_0, \phi_p)] \in H^2(\mathbb{R}^3)$.

$\blacksquare$

Proof of the Fact 4. By means of the Sobolev embedding theorem the norms $\|\phi_p^{(1,2)}\|_\infty$ are sufficiently small. An elementary computation yields

$$\mathcal{K}_4(\phi_0, \phi_p^{(1)}) - \mathcal{K}_4(\phi_0, \phi_p^{(2)}) = F_0(\phi_1^2)\phi_1 - F_0(\phi_2^2)\phi_2 + \varepsilon F_p(\phi_1^2)\phi_1 - \varepsilon F_p(\phi_2^2)\phi_2 -$$

$$-F_0(\phi_0^2)(\phi_p^{(1)}-\phi_p^{(2)})-F_0'(\phi_0^2)2\phi_0^2(\phi_p^{(1)}-\phi_p^{(2)})$$

A trivial estimate $|\eta_p(\phi_1)-\eta_p(\phi_2)|=|\int_{\phi_2}^{\phi_1}\eta_p'(s)ds|\le C|\phi_p^{(1)}-\phi_p^{(2)}|$, where the function η_p is defined above implies the inequality $\varepsilon\|F_p(\phi_1^2)\phi_1-F_p(\phi_2^2)\phi_2\|_2\le C\varepsilon\|\phi_p^{(1)}-\phi_p^{(2)}\|_2$, which along with the bound (15) gives us

$$\|\mathcal{K}_4(\phi_0,\phi_p^{(1)})-\mathcal{K}_4(\phi_0,\phi_p^{(2)})\|_2\le C(\varepsilon+\|\phi_p^{(1)}\|_\infty+\|\phi_p^{(2)}\|_\infty)\|\phi_p^{(1)}-\phi_p^{(2)}\|_2$$

Thus we arrive at the estimate

$$\|L_{+,\ 0}^{-1}[\mathcal{K}_4(\phi_0,\phi_p^{(1)})]-L_{+,\ 0}^{-1}[\mathcal{K}_4(\phi_0,\phi_p^{(2)})\|_{H^2}\le$$

$$\le\|L_{+,\ 0}^{-1}\|_{L^2\to H^2}C(\varepsilon+\|\phi_p^{(1)}\|_\infty+\|\phi_p^{(2)}\|_\infty)\|\phi_p^{(1)}-\phi_p^{(2)}\|_{H^2},$$

which completes the Proof of the Fact 4 due to the smallness of ε and the norms $\|\phi_p^{(1,2)}\|_\infty$.

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Having established the Facts 1-4 enables us to prove the Proposition 2.1.

Proof of the Proposition 2.1. Part I. When the potential term in the NLS equation is being perturbed, by means of the Facts 1 and 2 along with the contaction lemma we obtain that the solution ϕ_p to the equation (13) exists and is unique and has the asymptotics

$$\phi_p(x)=\phi(x)-\phi_0(x)=-\varepsilon L_{+,\ 0}^{-1}(U_p\phi_0)+O(\varepsilon^2)$$

in the $H^2(\mathbb{R}^3)\subset L^\infty(\mathbb{R}^3)$ space. A straightforward computation using the identity above gives

$$F_0(\phi^2)=F_0(\phi_0^2)-2\varepsilon F_0'(\phi_0^2)\phi_0(L_{+,\ 0})^{-1}(U_p\phi_0)+O(\varepsilon^2),$$

$$F_0'(\phi^2)=F_0'(\phi_0^2)-2\varepsilon F_0''(\phi_0^2)\phi_0(L_{+,\ 0})^{-1}(U_p\phi_0)+O(\varepsilon^2)$$

Substituting these expressions into formulas (9) and (10) we obtain

$$L_+=L_{+,\ 0}+\varepsilon U_p(x)-2\varepsilon\phi_0\{3F_0'(\phi_0^2)+2F_0''(\phi_0^2)\phi_0^2\}(L_{+,\ 0})^{-1}(U_p\phi_0)+O(\varepsilon^2)$$

$$L_-=L_{-,\ 0}+\varepsilon U_p(x)-2\varepsilon\phi_0F_0'(\phi_0^2)(L_{+,\ 0})^{-1}(U_p\phi_0)+O(\varepsilon^2)$$

in $H^2(R^3)\subset L^\infty(\mathbb{R}^3)$, which completes the proof in the first case.

Part II. When the perturbation is being applied to the nonlinear term in the NLS equation, via the Facts 3, 4 and the contraction lemma we arrive at the existence of the unique solution to the equation (16) having the asymptotics

$$\phi_p(x) = \phi(x) - \phi_0(x) = -\varepsilon(L_{+,\ 0})^{-1}(F_p(\phi_0^2)\phi_0) + O(\varepsilon^2)$$

in the $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ space. Hence

$$F_0(\phi^2) = F_0(\phi_0^2) - 2\varepsilon\phi_0 F_0'(\phi_0^2)(L_{+,\ 0})^{-1}(F_p(\phi_0^2)\phi_0) + O(\varepsilon^2)$$

$$F_0'(\phi^2) = F_0'(\phi_0^2) - 2\varepsilon\phi_0 F_0''(\phi_0^2)(L_{+,\ 0})^{-1}(F_p(\phi_0^2)\phi_0) + O(\varepsilon^2)$$

$$\varepsilon F_p(\phi^2) = \varepsilon F_p(\phi_0^2) + O(\varepsilon^2), \qquad \varepsilon F_p'(\phi^2)\phi^2 = \varepsilon F_p'(\phi_0^2)\phi_0^2 + O(\varepsilon^2)$$

A trivial calculation using (11) and (12) gives

$$L_+ = L_{+,\ 0} - 2\varepsilon\phi_0\{3F_0'(\phi_0^2) + 2\phi_0^2 F_0''(\phi_0^2)\}(L_{+,\ 0})^{-1}(F_p(\phi_0^2)\phi_0) + \varepsilon F_p(\phi_0^2) + 2\varepsilon F_p'(\phi_0^2)\phi_0^2 + O(\varepsilon^2)$$

$$L_- = L_{-,\ 0} - 2\varepsilon\phi_0 F_0'(\phi_0^2)(L_{+,\ 0})^{-1}(F_p(\phi_0^2)\phi_0) + \varepsilon F_p(\phi_0^2) + O(\varepsilon^2)$$

in $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$, which completes the proof of the Proposition 2.1.

■

Having proved the Proposition 2.1 we turn to the spectral analysis of the linearized NLS operator in the presence of threshold eigenvalues and resonances.

3. The endpoint solutions, the Birman-Schwinger principle and the Feschbach map for the linearized NLS operator

The spectral problem for the operator $\mathcal{L}$ can be written in the equivalent form via the transformation $\boldsymbol{\psi} = (u + w,\ u - w)^T$:

$$\sigma_3 \mathcal{H} \boldsymbol{\psi} = z\boldsymbol{\psi}, \tag{17}$$

where σ_3 is the Pauli matrix and $\mathcal{H}$ is the new energy operator:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} -\Delta + \omega + f(x) & g(x) \\ g(x) & -\Delta + \omega + f(x) \end{pmatrix}, \tag{18}$$

such that $V_{\pm}(x) = f(x) \pm g(x)$. The solution of the spectral problem above $\boldsymbol{\psi}_0 = (\psi_{0,\ 1}, \psi_{0,\ 2})^T$ at the end of the essential spectrum $z = \omega$ satisfies the system

$$\begin{cases} \Delta\psi_{0,\ 1} = f_0\psi_{0,\ 1} + g_0\psi_{0,\ 2} \\ (\Delta - 2\omega)\psi_{0,\ 2} = g_0\psi_{0,\ 1} + f_0\psi_{0,\ 2} \end{cases} \tag{19}$$

We consider the solutions of the system (19), such that $(1 + |x|^2)^{-\frac{s}{2}}\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ with $s > \frac{1}{2}$ or $\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^2)$. By inverting the Laplacian the first component of the solution of this system can be written as

$$\psi_{0,\ 1} = -\frac{C_0}{4\pi|x|} + \chi, \tag{20}$$

where

$$C_0 = \int_{\mathbb{R}^3} (f_0(x)\psi_{0,\ 1}(x) + g_0(x)\psi_{0,\ 2}(x))\mathrm{d}x$$

and $\chi(x) = \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^3} \Big(\frac{1}{|x|} - \frac{1}{|x-y|}\Big)(f_0(y)\psi_{0,1}(y) + g_0(y)\psi_{0,2}(y))\mathrm{d}y$. Clearly $\psi_{0,\ 2} \in L^2(\mathbb{R}^3)$. Let us show that χ is square integrable as well.

Fact 5 *For the second term in the right side of the identity (20) we have* $\chi(x) \in L^2(\mathbb{R}^3)$

Proof. Let us introduce the auxiliary function

$$s(x) := f_0(x)\psi_{0,1}(x) + g_0(x)\psi_{0,2}(x)\ ,$$

which is square integrable since $\psi_{0,1}$ and $\psi_{0,2}$ belong to weighted L^2 spaces and f_0 and g_0 decay exponentially. Hence $\chi(x) = \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^3} \Big(\frac{1}{|x|} - \frac{1}{|x-y|}\Big)s(y)dy$ and therefore

$$\widehat{\chi}(p) = \frac{\widehat{s}(0) - \widehat{s}(p)}{p^2}$$

Here and further down the hat symbol stands for the Fourier transform. For proving our statement we use the decomposition

$$\widehat{\chi}(p) = \widehat{\chi}_1(p) + \widehat{\chi}_2(p)$$

with $\widehat{\chi}_1(p) := \dfrac{\widehat{s}(0) - \widehat{s}(p)}{p^2}\chi_{\{|p|\le 1\}}$ and $\widehat{\chi}_2(p) := \dfrac{\widehat{s}(0) - \widehat{s}(p)}{p^2}\chi_{\{|p|>1\}}$, where $\chi_{\{|p|\le 1\}}$ stands for the characteristic function of the unit ball in the Fourier space centered at the origin and $\chi_{\{|p|>1\}}$ for the characteristic function of its complement. The functions $s(x)$ and $|x|s(x)$ have finite norms in $L^1(\mathbb{R}^3)$, which can be easily shown by means of the Schwarz inequality using that $f_0(x)$ and $g_0(x)$ decay exponentially and $\psi_{0,1}(x)$ and $\psi_{0,2}(x)$ belong to the weighted L^2 space. Thus we have

$$|\widehat{\chi}_2(p)| \leq \Big(\frac{2}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|s(x)|dx\Big)\frac{\chi_{\{|p|>1\}}}{p^2} \in L^2(\mathbb{R}^3)$$

In order to show the square integrability of the remaining term we will make use of the following representation

$$\widehat{s}(p) = \widehat{s}(0) + \int_0^{|p|}\frac{\partial}{\partial|\eta|}\widehat{s}(|\eta|,\Omega)d|\eta| \ ,$$

where Ω stands for the angle variables on the sphere. This implies

$$\widehat{\chi}_1(p) = -\frac{\int_0^{|p|}\frac{\partial}{\partial|\eta|}\widehat{s}(|\eta|,\Omega)d|\eta|}{p^2}\chi_{\{|p|\leq 1\}}$$

Clearly $|\nabla_p\widehat{s}(p)| \leq \dfrac{1}{(2\pi)^{\frac{3}{2}}}\displaystyle\int_{\mathbb{R}^3}|s(x)||x|dx$, which yields

$$|\widehat{\chi}_1(p)| \leq \Big(\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|s(x)||x|dx\Big)\frac{1}{|p|}\chi_{\{|p|\leq 1\}} \in L^2(\mathbb{R}^3)$$

■

Hence there are two distinct types of solutions for the spectral problem (17) at the end of the essential spectrum (see e.g. [CP 05]).

I. Threshold resonance

This situation occurs when $C_0 \neq 0$. Therefore the solution $\boldsymbol{\psi}_0$ of the system (19) such that $(1+|x|^2)^{-\frac{s}{2}}\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3,\mathbb{C}^2)$ with $s > \frac{1}{2}$ decays as $\frac{1}{|x|}$. From the representation (20) of the first component of the solution $\psi_{0,\ 1}$ it follows that the threshold resonance is always simple. Indeed, if there were two of them: $\boldsymbol{\psi}_0^1$ and $\boldsymbol{\psi}_0^2$, we could assume them having the same constant C_0 in the formula (20), which can be achieved by multiplying the equations of the system (19) by the appropriate constant. Then their difference $\boldsymbol{\psi}_0^1 - \boldsymbol{\psi}_0^2 \in L^2(\mathbb{R}^3,\mathbb{C}^2)$ will be a solution of the system (19). But the coexistence of a zero resonance and an eigenvalue at the threshold is beyond the scope of our work, which is assumed in the Theorem 1.3. Therefore, $\boldsymbol{\psi}_0^1$ and $\boldsymbol{\psi}_0^2$ coincide.

We normalize $\boldsymbol{\psi}_0$ such that $C_0 = \sqrt{4\pi}$ and in the original variables for our spectral problem the solution is $\mathbf{u}_0 = (u_0, w_0)^T$.

II. Eigenvalue

This is the case when $C_0 = 0$ and, therefore $\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^2)$.

The eigenvalue at the end of the essential spectrum now is assumed to be N-fold degenerate, such that the corresponding eigenvectors are $\boldsymbol{\psi}_0^i = (\psi^i_{0,\ 1}, \psi^i_{0,\ 2})^T, \quad i = 1, ..., N$, for which we have

$$\int_{\mathbb{R}^3} s^i(x)\mathrm{d}x = 0, \ \ i = 1, ..., N \tag{21}$$

For the spectral problem (17) according to the Birman-Schwinger principle (see e.g. [S 79], [LSW 76]) each eigenvalue z is correspondent to the eigenvalue -1 of the same multiplicity of the problem

$$K_{\alpha,\ \varepsilon}\mathbf{w} = -\mathbf{w} \ , \tag{22}$$

where the Birman-Schwinger operator is defined as

$$K_{\alpha,\ \varepsilon} := |Q_0|^{\frac{1}{2}} R_\alpha Q |Q_0|^{-\frac{1}{2}} \tag{23}$$

on $L^2(\mathbb{R}^3, \mathbb{C}^2)$ with the operator-valued terms

$$R_\alpha = \begin{pmatrix} G_\alpha & 0 \\ 0 & G_{\sqrt{2\omega-\alpha^2}} \end{pmatrix}, \ G_\alpha = (-\Delta + \alpha^2)^{-1} \text{ and } G_{\sqrt{2\omega-\alpha^2}} = (-\Delta + 2\omega - \alpha^2)^{-1},$$

the family of potentials

$$Q = Q_0 + \varepsilon Q_p = \begin{pmatrix} f_0 & g_0 \\ g_0 & f_0 \end{pmatrix} + \varepsilon \begin{pmatrix} f_p & g_p \\ g_p & f_p \end{pmatrix}$$

and the solution of the spectral problem (22)

$$\mathbf{w} = |Q_0|^{\frac{1}{2}} \boldsymbol{\psi}$$

The spectral parameter α is defined relatively to the spectrum of the problem (17) via

$$z = \omega - \alpha^2,$$

such that α belonging to the first sheet of the Riemann surface $\{\alpha : \mathrm{Re}\alpha > 0\}$ corresponds to an eigenvalue near ω and $\sigma_p(\mathcal{L})$ is empty in the neighborhood of ω when α belongs to the second sheet of the Riemann surface $\{\alpha : \mathrm{Re}\alpha < 0\}$. For the matrix valued potentials here and further we use the notations in the sense of the spectral calculus

$$Q_0 = signQ_0|Q_0|, \quad Q_0^{\frac{1}{2}} = signQ_0|Q_0|^{\frac{1}{2}}$$

The Birman-Schwinger operator $K_{\alpha,\ \varepsilon}$ is compact due to the exponential decay of the potentials and of the kernel of the operator R_α in three dimension. The resolvent operator can be expanded as

$$R_\alpha = R_0 + \alpha R_1 + \alpha^2 R_2 + \alpha^3 R_3 + O(\alpha^4),$$

where the kernels of the operators are given by

$$R_0(x,y) = \begin{pmatrix} \frac{1}{4\pi|x-y|} & 0 \\ 0 & \frac{e^{-\sqrt{2\omega}|x-y|}}{4\pi|x-y|} \end{pmatrix}, \quad R_1(x,y) = \begin{pmatrix} -\frac{1}{4\pi} & 0 \\ 0 & 0 \end{pmatrix},$$

$$R_2(x,y) = \begin{pmatrix} \frac{|x-y|}{8\pi} & 0 \\ 0 & \frac{e^{-\sqrt{2\omega}|x-y|}}{8\pi\sqrt{2\omega}} \end{pmatrix}, \quad R_3(x,y) = \begin{pmatrix} -\frac{|x-y|^2}{24\pi} & 0 \\ 0 & 0 \end{pmatrix}$$

and $O(\alpha^4)$ here and below stands for the terms of the order four and higher and analytic in α for α small enough. The expansion above yields the convergent expansion for $K_{\alpha,\ \varepsilon}$ in the operator norm ($L^2 \to L^2$) due to the exponential decay of the potentials. First we consider the case of a multiple eigenvalue at $z = \omega$ with corresponding eigenfunctions $\boldsymbol{\psi}_0^i,\ i = 1, ..., N$. By letting $\alpha, \varepsilon \to 0$ we obtain the Birman-Schwinger problem at the edge of the essential spectrum of the linearized NLS operator $K_{0,\ 0}\mathbf{w}_0^i = -\mathbf{w}_0^i,\ i = 1, ..., N$ with $\mathbf{w}_0^i = |Q_0|^{\frac{1}{2}}\boldsymbol{\psi}_0^i \in L^2$ and $K_{0,\ 0} = |Q_0|^{\frac{1}{2}}R_0Q_0^{\frac{1}{2}}$. This is equivalent to

$$\boldsymbol{\psi}_0^i = -R_0Q_0\boldsymbol{\psi}_0^i,\ i = 1, ..., N \tag{24}$$

Let $\mathbf{q}_0^i := (q_{0,\ 1}^i, q_{0,\ 2}^i)^T = Q_0\boldsymbol{\psi}_0^i,\ i = 1, ..., N$. A trivial calculation using (21) yields

$$R_1\mathbf{q}_0^i = 0, \quad \int_{\mathbb{R}^3} q_{0,\ 1}^i(x)\mathrm{d}x = \int_{\mathbb{R}^3} s^i(x)\mathrm{d}x = 0, \quad i = 1, ..., N \tag{25}$$

The spectral problem adjoint to the unperturbed Birman-Schwinger principle is

$$K_{0,\ 0}^*\mathbf{w}_0^{i\ *} = -\mathbf{w}_0^{i\ *},\ i = 1, ..., N$$

with

$$K^*_{0,\ 0} = Q_0^{\frac{1}{2}} R_0 |Q_0|^{\frac{1}{2}}$$

and the solutions of the adjoint problem are related to the original ones as

$$\mathbf{w}_0^{i\ *} = s_0^i sign Q_0 \mathbf{w}_0^i,\ i = 1, ..., N,$$

where $s_0^i = sign\langle sign Q_0 \mathbf{w}_0^i, \mathbf{w}_0^i\rangle$. Let us denote the eigenspace of the operator $K_{0,\ 0}$ correspondent to the eigenvalue -1 as X_0.We choose the basis of eigenvectors in the subspace X_0 orthonormal with respect to the eigenvectors of the adjoint operator via the Gram-Schmidt procedure, such that

$$\langle \mathbf{w}_0^{i\ *}, \mathbf{w}_0^j\rangle = \delta_{i,\ j}, \quad i,\ j = 1, ..., N \tag{26}$$

We define the projection operator onto the subspace X_0.

Definition 3.1 *For an arbitrary function* $\mathbf{f} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$

$$P_0 \mathbf{f} := \sum_{i=1}^{N} \langle \mathbf{w}_0^{i\ *}, \mathbf{f}\rangle \mathbf{w}_0^i$$

The operator defined above satisfies $P_0^2 = P_0$ *and its adjoint is given by*

$$P_0^* \mathbf{g} = \sum_{i=1}^{N} \langle \mathbf{w}_0^i, \mathbf{g}\rangle \mathbf{w}_0^{i\ *}$$

for any $\mathbf{g} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$.

This enables us to restrict the problem to $Ran P_0$, the range of the projection P_0, which is the space of N dimensions.

Definition 3.2 *The Feshbach map (see p.207 of [GS 03], also [BFS 98]) for the Birman-Schwinger operator* $K_{\alpha,\ \varepsilon}$ *is the operator* $F : Ran P_0 \to Ran P_0$, *such that*

$$F := P_0(K_{\alpha,\ \varepsilon} - K_{\alpha,\ \varepsilon}\bar{P}_0(\bar{K}_{\alpha,\ \varepsilon} + 1)^{-1}\bar{P}_0 K_{\alpha,\ \varepsilon})P_0\ ,$$

where $\bar{P}_0 = 1 - P_0$ *and* $\bar{A} = \bar{P}_0 A \bar{P}_0$ *for any operator* $A \neq P_0$.

Note that since -1 is an isolated eigenvalue of the Hilbert-Schmidt operator $K_{\alpha,\ \varepsilon}$, the spectrum of $\bar{K}_{\alpha,\ \varepsilon} + 1$ is empty in the neighborhood of the origin and therefore, the operator

$(\bar{K}_{\alpha,\ \varepsilon}+1)^{-1}$ is well defined. We denote the first term in the definition of the Feschbach map as

$$U := P_0 K_{\alpha,\ \varepsilon} P_0$$

and the second one as

$$D := P_0 K_{\alpha,\ \varepsilon} \bar{P}_0 (\bar{K}_{\alpha,\ \varepsilon}+1)^{-1} \bar{P}_0 K_{\alpha,\ \varepsilon} P_0$$

Since the Feshbach operator has the isolated eigenvalue -1 of multiplicity N as well as the Birman-Schwinger operator (see [GS 03], [BFS 98]) we have the spectral problem

$$F\mathbf{v} = -\mathbf{v},$$

where $\mathbf{v} = \sum_{k=1}^{N} c_k \mathbf{w}_0^k$. Thus

$$det(F+I) = 0 \tag{27}$$

Hence the implicit dependence of α on ε is given by this equation and the goal is to estimate the matrix elements of the Feshbach operator. Due to the orthogonality relations (26) they are given by

$$F_{i,j} = \langle \mathbf{w}_0^{i\ *}, F\mathbf{w}_0^j \rangle, \quad 1 \le i,j \le N$$

Let us first evaluate the matrix elements of the operator U. We have the following lemma.

Lemma 3.3 *For all $i,j = 1,...,N$ we have*

$$U_{i,j} = -\delta_{i,j} - 2\varepsilon s_0^i (\hat{V}_p)_{i,j} - 2\alpha^2 s_0^i \hat{B}_{i,j} + \alpha^2 \varepsilon s_0^i \langle \mathbf{q}_0^i, R_2 Q_p \boldsymbol{\psi}_0^j \rangle +$$
$$+ \frac{\alpha^3 s_0^i}{12\pi} (\hat{S}^* \hat{S})_{i,j} + \alpha^3 \varepsilon s_0^i \langle \mathbf{q}_0^i, R_3 Q_p \boldsymbol{\psi}_0^j \rangle + O(\alpha^4),$$

where $O(\alpha^4)$ is analytic in α for α small enough.

Proof. Step I. For the first term of the Feshbach operator we have the expansion

$$U = \sum_{s=0}^{3} \alpha^s U_s + O(\alpha^4)\ ,$$

where $U_s = P_0 K_{s,\ \varepsilon} P_0$ and $K_{s,\ \varepsilon} = |Q_0|^{\frac{1}{2}} R_s Q |Q_0|^{-\frac{1}{2}},\ s = 0,...,3$, the term $O(\alpha^4)$ is analytic in α for α small enough and the expansion converges in the operator norm ($L^2 \to L^2$) due to the exponential decay of the potentials. Let us evaluate the matrix elements for each of the operators U_s.

Step II. Using the expression for the projection operator from Lemma 3.1, the relations (26), (24), the variable change for the eigenfunctions of the spectral problem (17) and for the elements of the energy operator we obtain

$$(U_0)_{i,j} = \langle \mathbf{w}_0^{i\,*}, P_0 K_{0,\,\varepsilon} P_0 \mathbf{w}_0^j \rangle = \langle \mathbf{w}_0^{i\,*}, K_{0,\,0} \mathbf{w}_0^j \rangle +$$

$$+\, \varepsilon \langle \mathbf{w}_0^{i\,*}, |Q_0|^{\frac{1}{2}} R_0 Q_p |Q_0|^{-\frac{1}{2}} \mathbf{w}_0^j \rangle = -\delta_{i,\,j} - 2\varepsilon s_0^i (\hat{V}_p)_{i,j}, \quad 1 \le i, j \le N$$

Step III. For any operator A such that the quadratic form below is defined, using (25) we have

$$\langle \mathbf{w}_0^{i\,*}, P_0 K_{1,\,\varepsilon} A \mathbf{w}_0^j \rangle = s_0^i \langle R_1 \mathbf{q}_0^i, Q |Q_0|^{-\frac{1}{2}} A \mathbf{w}_0^j \rangle = 0,\ 1 \le i, j \le N$$

Thus choosing $A = P_0$ yields

$$(U_1)_{i,j} = \langle \mathbf{w}_0^{i\,*}, P_0 K_{1,\,\varepsilon} P_0 \mathbf{w}_0^j \rangle = 0,\ 1 \le i, j \le N$$

Step IV. For the quadratic term in the expansion of the operator U we use the definitions of the terms involved to derive

$$(U_2)_{i,j} = \langle \mathbf{w}_0^{i\,*}, P_0 K_{2,\,\varepsilon} P_0 \mathbf{w}_0^j \rangle = s_0^i \{ \langle \mathbf{q}_0^i, R_2 \mathbf{q}_0^j \rangle + \varepsilon \langle \mathbf{q}_0^i, R_2 Q_p \boldsymbol{\psi}_0^j \rangle \},\ 1 \le i, j \le N$$

Thus it remains to estimate

$$\langle \mathbf{q}_0^i, R_2 \mathbf{q}_0^j \rangle = \frac{1}{8\pi} \{ (q_{0,\,1}^i, |x| * q_{0,\,1}^j) + (q_{0,\,2}^i, \frac{1}{\sqrt{2\omega}} e^{-\sqrt{2\omega}|x|} * q_{0,\,2}^j) \},$$

where $*$ stands for the convolution. We write the first term in the right side of the equality above as

$$\frac{1}{8\pi}(q_{0,\,1}^i, |x| * q_{0,\,1}^j) = -\frac{1}{8\pi} \frac{\partial}{\partial t}|_{t=0^+} (q_{0,\,1}^i, e^{-t|x|} * q_{0,\,1}^j)$$

Having the formula for the heat kernel of the root of the Laplacian handy (see e.g. [LL 97]) we easily obtain

$$\widehat{e^{-t|x|}}(p) = \frac{(2\pi)^{\frac{3}{2}}}{\pi^2} \frac{t}{[t^2 + p^2]^2}, \quad p \in \mathbb{R}^3, \quad t > 0$$

Thus

$$\frac{1}{8\pi}(q_{0,\,1}^i, |x| * q_{0,\,1}^j) = -(\widehat{q_{0,\,1}^i}(p), \frac{1}{p^4} \widehat{q_{0,\,1}^j}(p)) = -((-\Delta)^{-1} q_{0,\,1}^i, (-\Delta)^{-1} q_{0,\,1}^j)$$

For the quadratic form remaining to estimate via the above mentioned heat kernel with $t = \sqrt{2\omega}$ we derive

$$\frac{1}{8\pi}(q^i_{0,\,2}, \frac{1}{\sqrt{2\omega}}e^{-\sqrt{2\omega}|x|} * q^j_{0,\,2}) = (\widehat{q^i_{0,\,2}}(p), \frac{1}{(2\omega+p^2)^2}\widehat{q^j_{0,\,2}}(p)) =$$

$$= ((-\Delta+2\omega)^{-1}q^i_{0,\,2}, (-\Delta+2\omega)^{-1}q^j_{0,\,2})$$

Therefore

$$\langle \mathbf{q}^i_0, R_2\mathbf{q}^j_0\rangle = -((-\Delta)^{-1}q^i_{0,\,1}, (-\Delta)^{-1}q^j_{0,\,1}) + ((-\Delta+2\omega)^{-1}q^i_{0,\,2}, (-\Delta+2\omega)^{-1}q^j_{0,\,2})$$

By means of the Birman-Schwinger principle at the edge of the essential spectrum of the linearized NLS operator (24) the right side of the equality above equals to $-\langle \boldsymbol{\psi}^i_0, \sigma_3\boldsymbol{\psi}^j_0\rangle$, which is identical to $-2\langle \mathbf{u}^i_0, \sigma_1\mathbf{u}^j_0\rangle$ via the change of variables (17). Hence

$$(U_2)_{i,j} = -2s^i_0\hat{B}_{i,j} + s^i_0\varepsilon\langle \mathbf{q}^i_0, R_2Q_p\boldsymbol{\psi}^j_0\rangle,\ 1 \le i,j \le N$$

Step V. For the cubic term in the expansion of the operator U using the definitions of the operators and the vector-functions involved in it along with the identity (25) we easily obtain

$$(U_3)_{i,j} = \langle \mathbf{w}^{i\,*}_0, P_0K_{3,\,\varepsilon}P_0\mathbf{w}^j_0\rangle = s^i_0\langle \mathbf{q}^i_0, R_3\mathbf{q}^j_0\rangle + \varepsilon s^i_0\langle \mathbf{q}^i_0, R_3Q_p\boldsymbol{\psi}^j_0\rangle =$$

$$= \frac{s^i_0}{12\pi}(\hat{S}^*\hat{S})_{i,j} + \varepsilon s^i_0\langle \mathbf{q}^i_0, R_3Q_p\boldsymbol{\psi}^j_0\rangle,\ 1 \le i,j \le N$$

Collecting all the estimates derived in the *Steps* $I - V$ we complete the proof of the lemma.

$\blacksquare$

Lemma 3.4 *For all* $i, j = 1, ..., N$ *we have*

$$D_{i,j} = \varepsilon^2(D_0)_{i,j} + \alpha\varepsilon^2(D_1)_{i,j} + \alpha^2\varepsilon(D_2)_{i,j} + \alpha^3\varepsilon(D_3)_{i,j} + O(\alpha^4),$$

where here and further with a slight abuse of notations $(D_k)_{i,j},\ k = 0, ..., 3$ *stand for the terms analytic in* ε, *such that their exact values are unimportant and* $O(\alpha^4)$ *is analytic in* α *for* α *small enough.*

Proof. Let us expand each of the three operator valued factors involved in the second term D of the Feshbach map in terms of the parameter α up to the third order inclusively. Thus the first one can be written as

$$P_0 K_{\alpha,\ \varepsilon} \bar{P}_0 = \sum_{s=0}^{3} \alpha^s P_0 K_{s,\ \varepsilon} \bar{P}_0 + O(\alpha^4)$$

and the third one analogously as

$$\bar{P}_0 K_{\alpha,\ \varepsilon} P_0 = \sum_{s=0}^{3} \alpha^s \bar{P}_0 K_{s,\ \varepsilon} P_0 + O(\alpha^4)$$

with the $K_{s,\ \varepsilon}$ operators given in the Lemma 3.3. To expand the middle term we use $\bar{K}_{\alpha,\ \varepsilon} = \sum_{s=0}^{3} \alpha^s \bar{K}_{s,\varepsilon} + O(\alpha^4)$ and therefore,

$$(\bar{K}_{\alpha,\ \varepsilon} + 1)^{-1} = (1 + \sum_{s=1}^{3} \alpha^s M_{s,\ \varepsilon} + O(\alpha^4))^{-1} M_{0,\ \varepsilon}\ ,$$

where the operators

$$M_{s,\ \varepsilon} = \begin{cases} (1 + \bar{K}_{0,\ \varepsilon})^{-1} \bar{K}_{s,\ \varepsilon},\ \ s \geq 1 \\ (1 + \bar{K}_{0,\ \varepsilon})^{-1},\ \ s = 0 \end{cases}$$

Collecting the terms for α small enough we easily obtain

$$(\bar{K}_{\alpha,\ \varepsilon} + 1)^{-1} = M_{0,\ \varepsilon} - \alpha M_{1,\ \varepsilon} M_{0,\ \varepsilon} + \alpha^2 (M_{1,\ \varepsilon}^2 - M_{2,\ \varepsilon}) M_{0,\ \varepsilon} +$$

$$+ \alpha^3 (M_{1,\ \varepsilon} M_{2,\ \varepsilon} + M_{2,\ \varepsilon} M_{1,\ \varepsilon} - M_{3,\ \varepsilon} - M_{1,\ \varepsilon}^3) M_{0,\ \varepsilon} + O(\alpha^4)$$

We multiply out the expansions derived above for each of the three factors contributing to the operator D and estimate the matrix elements $D_{i,j} = \langle \mathbf{w}_0^{i\ *}, D\mathbf{w}_0^j \rangle,\ i,j = 1,...,N$. After long but straightforward computations using the *Step III* of the Lemma 3.3, the basic properties of the projection operators $(I - P_0)\mathbf{w}_0^i = 0$ and $(I - P_0^*)\mathbf{w}_0^{i\ *} = 0,\ i = 1,...,N$ along with the identities (24) and (25) we arrive at the statement of the lemma.

■

Armed with the Lemmas 3.3 and 3.4 we prove the Theorem 1.4.

Proof of Theorem 1.4. By means of the lemmas mentioned above the equation (27) is equivalent to

$$det W(\varepsilon, \alpha) = 0,$$

where

$$W_{i,j}(\varepsilon,\alpha)=\varepsilon(\hat{V}_p)_{i,j}+\alpha^2\hat{B}_{i,j}-\frac{\alpha^3}{24\pi}(\hat{S}^*\hat{S})_{i,j}+\varepsilon^2(D_0)_{i,j}+$$
$$+\alpha\varepsilon^2(D_1)_{i,j}+\alpha^2\varepsilon(D_2)_{i,j}+\alpha^3\varepsilon(D_3)_{i,j}+O(\alpha^4),\ i,j=1,...,N$$

and $O(\alpha^4)$ is analytic in α for α small enough. The terms proportional to $\alpha^2\varepsilon$ and $\alpha^3\varepsilon$ from $U_{i,j}$ were absorbed by $(D_2)_{i,j}$ and $(D_3)_{i,j}$. By the standard argument of the perturbation theory (see e.g. the Chapter 15 of [HS 96]) we obtain

$$\varepsilon\lambda_p^i+\alpha_i^2\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle-\frac{\alpha_i^3}{24\pi}|\hat{S}\mathbf{v}_p^i|^2+O(\alpha_i^4)=0,\quad i=1,...,N$$

A straightforward computation yields the expression for the spectral parameter

$$\alpha_i=-\frac{\varepsilon}{48\pi}\frac{|\hat{S}\mathbf{v}_p^i|^2\lambda_p^i}{\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle^2}\pm\sqrt{\varepsilon}\sqrt{\frac{-\lambda_p^i}{\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle}}+O(\varepsilon^{\frac{3}{2}}),\quad i=1,...,N$$

Thus we have the following possibilities.

(i) $\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle<0$ and $\lambda_p^i>0$ or $\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle>0$ and $\lambda_p^i<0$. The situation when the spectral parameter has a positive real part corresponds to the eigenvalues bifurcating along the real line having the asymptotics

$$z_i=\omega+\varepsilon\frac{\lambda_p^i}{\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle}+O(\varepsilon^{\frac{3}{2}}),\quad i=1,...,N_{B-}^{+}+N_{B+}^{-}$$

When the spectral parameter moves to the second sheet of the Riemann surface, the eigenvalues near the edge of the essential spectrum of the linearized NLS operator are not being produced.

(ii) $\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle<0$ and $\lambda_p^i<0$. Hence $\mathrm{Re}\alpha_i>0$ and the eigenvalues bifurcate in the complex plane becoming the complex conjugate pairs having the asymptotics

$$z_i=\omega+\varepsilon\frac{\lambda_p^i}{\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle}\pm i\frac{\varepsilon^{\frac{3}{2}}}{24\pi}\frac{|\hat{S}\mathbf{v}_p^i|^2\lambda_p^i}{\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle^2}\sqrt{\frac{\lambda_p^i}{\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle}}+O(\varepsilon^2),\quad i=1,...,N_{B-}^{-}$$

(iii) $\langle\mathbf{v}_p^i,\hat{B}\mathbf{v}_p^i\rangle>0$ and $\lambda_p^i>0$. In this case $\mathrm{Re}\alpha_i<0$ and therefore N_{B+}^{+} eigenvalues will disappear from the neighborhood of the endpoint of the essential spectrum of the linearized NLS operator under perturbation.

■

Proof of Theorem 1.3 Though the threshold resonance solution by its definition $(1+|x|^2)^{-\frac{s}{2}}\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3,\mathbb{C}^2)$ with $s>\frac{1}{2}$, $\boldsymbol{\psi}_0 \notin L^2(\mathbb{R}^3,\mathbb{C}^2)$, satisfying

$$\boldsymbol{\psi}_0 = -R_0Q_0\boldsymbol{\psi}_0$$

analogously to (24), the weighted one $\mathbf{w}_0 = |Q_0|^{\frac{1}{2}}\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3,\mathbb{C}^2)$ and therefore it is the eigenfunction of the unperturbed Birman-Schwinger principle corresponding to the eigenvalue -1. The projection P_0 onto this eigenspace is defined analogously to Definition 3.1 with $N=1$. As distinct from the eigenvalue case, for the resonance solution $C_0=\sqrt{4\pi}$ (see the beginning of the Part 3) and we easily obtain

$$R_1Q_0\boldsymbol{\psi}_0 = (-\frac{1}{\sqrt{4\pi}},0)^T$$

Using the identities above along with the property of the projection operator $(I-P_0)\mathbf{w}_0=0$ we compute the expansions of the first and the second term of the determinant of the Feshbach map analogously to the Lemmas 3.3 and 3.4. Note that the terms of the form which were studied in the Step III of the Lemma 3.3 will not be vanishing in the threshold resonance case. We derive

$$U_{1,1} = \langle \mathbf{w}_0^*, U\mathbf{w}_0\rangle = -1 - 2\varepsilon s_0\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle - \alpha s_0 - \frac{\alpha s_0\varepsilon}{\sqrt{4\pi}}\int_{\mathbb{R}^3} s_p(x)dx + O(\alpha^2)$$

and

$$D_{1,1} = \varepsilon^2(D_0)_{1,1} + \alpha\varepsilon(D_1)_{1,1} + O(\alpha^2),$$

where $s_p(x) = V_{+,p}(x)u_0(x) + V_{-,p}(x)w_0(x)$ and $s_0 = sign\langle signQ_0\mathbf{w}_0, \mathbf{w}_0\rangle$. Then we arrive at

$$W(\varepsilon,\alpha) = \varepsilon\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle + \frac{\alpha}{2} + \varepsilon^2(D_0)_{1,1} + \alpha\varepsilon(D_1)_{1,1} + O(\alpha^2) = 0$$

The term proportional to $\alpha\varepsilon$ from $U_{1,1}$ was absorbed by $(D_1)_{1,1}$. Hence in the case of the threshold resonance $\varepsilon = O(\alpha)$ or explicitly

$$\alpha = -2\varepsilon\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle + O(\varepsilon^2)$$

Therefore, when $\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle > 0$ we have $\mathrm{Re}\alpha < 0$ and $\sigma_p(\mathcal{L})$ does not contain eigenvalues in the neighborhood of $z=\omega$. In the situation when $\langle \mathbf{u}_0, V_p\mathbf{u}_0\rangle < 0$ and thus $\mathrm{Re}\alpha > 0$ the assymptotics of the eigenvalues bifurcating from the edge of the essential spectrum are given by

$$z(\varepsilon) = \omega - 4\varepsilon^2 \langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle^2 + o(\varepsilon^2)$$

■

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