TRAVELLING WAVES IN DISCRETE NONLINEAR SYSTEMS WITH NON-NEAREST NEIGHBOR INTERACTIONS

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Abstract. The aim of this paper is to provide a construction of travelling waves in an extended one-dimensional lattice model with non nearest neighbor interactions. These models, coming mainly from solid state physics, are known to play an important role in the mechanisms of propagation of energy. We focus on an extended version of the Klein-Gordon chains, i.e. each particle is embedded into an anharmonic potential and linearly coupled to their first and second neighbors. The technique we use is a reduction to a center manifold that goes back to [Sir05, JS05, SJ04, IK00] and relies on the reduction of this infinite-dimensional problem to a finite-dimensional one. New phenomena appear due to the extended interactions. We find propagating solutions on the lattice which were not present for models involving coupling with only the nearest neighbors.

Contents

1. Introduction

The goal of this work is to study the existence of special solutions for extended lattice models. We focus here on the existence of travelling wave solutions.

The models under consideration are discrete in space and continuous in time. Particularly, we focus on a non-static version of the so-called extended Frenkel-Kontorova model described in [CdlL08b, CdlL08a].

More precisely, we consider a chain of particles $(x_n)_{n\in\mathbb{Z}}$, each one being embedded into a nonlinear potential V, analytic in a neighborhood of 0 (and such that $V'(0)=0, V''(0)>0$) and coupled to the first and second nearest neighbors. The equations of motion of such a system write

$$\ddot{x}_n+V'(x_n)=\gamma\Big[(x_{n+1}+x_{n-1}-2x_n)+A(x_{n+2}+x_{n-2}-2x_n)\Big] \quad n\in\mathbb{Z} \tag{1}$$

where $\gamma, A>0$ are coupling constants.

In this paper, we focus on solutions of (1) satisfying

$$x_n(t)=\varphi(n-t/T) \tag{2}$$

for a fixed time $T\in\mathbb{R}^*$ and a smooth function φ.

A lot of litterature has been devoted to discrete models like (1). In the case $A=0$ in (1), MacKay and Aubry [MA94] construct breather solutions (i.e. spatially localized time-periodic solutions for low coupling γ). As far as propagating solutions are concerned, the case $A=0$ has been considered in [IK00] for travelling wave solutions. In [FW94, FP04b, FP04a, FP02, FP99] several authors use variational techniques (via concentration compactness) for the problem (1) with $A=0$ and for the so-called Fermi-Pasta-Ulam lattice, i.e. discrete models with non linear coupling (see [FPU55]). One of the authors together with Guillaume James generalized the approach initiated in [IK00] to pulsating travelling waves in the works [Sir05, JS05, SJ04] for $A=0$ in (1).

In the present work, we pursue this generalization by considering the existence of travelling waves for (1) when $A\neq 0$. Considering non-nearest neighbor interactions is reminiscent of taking into account non local effects in the propagation of energy. It is also interesting to notice that the continuous limits (at least at the formal level) depend on the sign of the coupling coefficient A. Here, we will focus on the case $A > 0$ but the computations sketched in the last section of this paper show that one can get a different behaviour when $A < 0$ is allowed. In the paper [CdlL08b], one of the author and de la Llave considered the static version of (1), i.e.

$$V'(x_n) = \gamma\Big[(x_{n+1} + x_{n-1} - 2x_n) + A(x_{n+2} + x_{n-2} - 2x_n)\Big].$$

The previous system defines a $4-$dimensional map and reduces to a $2-$dimensional one when $A = 0$. The limit $A = 0$ is in some sense highly singular and considering non-nearest neighbor interactions considerably enlarges the range of solutions one can consider.

We focus here on small amplitude solutions of (1) for several parameter regimes (A, γ, T) and describe which type of solutions bifurcate from the equilibrium 0. Besides extending the theory already developped in references [Sir05, JS05, SJ04, IK00], our purpose is to put in light which kind of phenomenon occurs when the coupling is no longer to the first neighbors but to first and second neighbors. We show that in this case the structure of the simplest bifurcations of travelling waves are already more complicated than in [IK00]. We prove that generically solitary waves (i.e. localized solutions on the lattice) do not exist. Instead, the generic (in some suitable sense) solutions which appear are superpositions of a solitary wave and exponentially small tails.

The method of construction consists in reducing the system (1)-(2) to a finite dimensional center manifold (for some parameter regimes) which allows to capture all the dynamics of bounded small amplitude solutions. Reduction theorems can be found for instance in [CHT97], [VI92] (see the Appendix of the present paper where the abstract reduction theorem we use is formulated).

We now describe the results we obtain. Searching for a travelling wave solution of (1) amounts to find a function φ such that

(3)
$$\ddot{\varphi} + T^2 V'(\varphi) = T^2\gamma\Big\{\varphi(.+1) + \varphi(.-1) - 2\varphi + A(\varphi(.+2) + \varphi(.-2) - 2\varphi)\Big\}$$

and

$$x_n(t) = \varphi(\xi) = \varphi(n - t/T).$$

Our main results are described in the following theorems.

Theorem 1.1. *(Periodic waves) There exists a non empty set* Δ_1 *of parameters* $(A, T, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ *with infinite Legesgue measure* *such that for every* $(A, T, \gamma) \in \Delta_1$*, the set of small amplitude solutions of* (3) *constitutes a one-parameter family of periodic orbits, bifurcating from* 0*.*

Theorem 1.2. *(Waves with exponentially small tails) There exists a non empty set* Δ_2 *of parameters* $(A, T, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ *with infinite Lebesgue measure such that for every* (A, T, γ) *close to* $(A_0, T_0, \gamma_0) \in \Delta_2$ *there are small amplitude travelling waves of* (3)*. These travelling waves are the superposition of a localized wave and a periodic tail with exponentially small amlitude provided that the following inequality is fullfilled*

$$s_0 = \frac{T_0^2(2aK_1 + 4a^2 + 3b)}{1 - T_0^2\gamma_0(\cos(q_0) + 4A_0\cos(2q_0))} < 0$$

where

$$K_1 = \frac{aT_0^2}{T_0^2(1 + 2\gamma_0(1 + A_0)) - 4q_0^2 - 2T_0^2\gamma_0\cos(2q_0) - 2T_0^2\gamma_0 A_0\cos(4q_0)}.$$

The parameters a *and* b *are such that* $a = \frac{-V^{(3)}(0)}{2}$ *and* $b = \frac{-V^{(4)}(0)}{6}$ *and* $q_0 > 0$ *is the imaginary part of the double non semi-simple eigenvalue of the linear operator around* 0 *associated to* (3) *for* $T = T_0$*,* $\gamma = \gamma_0$ *and* $A = A_0$*.*

Remark 1.1. *It has to be noticed that the method we use here can be adapted to systems involving couplings to third or even farther neighbors, paying the price of additional technicalities in the computations. Furthermore, our method also allows to construct pulsating travelling fronts for* (1)*, i.e. solutions satisfying*

$$x_n(t) = x_{n+p}(t + T),$$

for a given $p > 1$ *(the case* $p = 1$ *corresponding to the travelling waves studied in the present paper).*

2. Formulation of the problem

Recall the model under consideration

$$\ddot{x}_n + V'(x_n) = \gamma\Big[x_{n+1} + x_{n-1} - 2x_n + A(x_{n+2} + x_{n-2} - 2x_n)\Big] \quad n \in \mathbb{Z}. \tag{4}$$

We consider solutions of (4) such that

$$x_n(t) = \varphi(n - t/T)$$

where $T > 0$ is a parameter (one can choose $T > 0$ since the equation (4) is invariant under time reversibility). The function φ satisfies the advance-delay equation

$$\ddot{\varphi}+T^2V'(\varphi)=\gamma T^2\Big\{\varphi(\xi+1)+\varphi(\xi-1)-2\varphi(\xi)+A(\varphi(\xi+2)+\varphi(\xi-2)-2\varphi(\xi))\Big\}$$

where $\xi=n-t/T$.

In order to apply center manifold reduction, we introduce a new variable v which will play the role of an evolution variable.

We then introduce

$$U(\xi)(v)=(\varphi(\xi),\dot{\varphi}(\xi),\Phi(v,\xi))$$

where $\Phi(v,\xi)=\varphi(\xi+v)$. We denote δ_a the trace operators

$$\delta_a\Phi(t,v)=\Phi(t,a).$$

We end up with the following equation

$$\partial_\xi U=\mathcal{L}_{A,T,\gamma}U+\mathcal{N}_T(U) \tag{5}$$

where the linear operator is given by

$$\mathcal{L}_{A,T,\gamma}=\begin{pmatrix} 0 & 1 & 0\\ T^2[-1-2\gamma(1+A)] & 0 & \gamma T^2(\delta_1+\delta_{-1})+\gamma T^2A(\delta_2+\delta_{-2})\\ 0 & 0 & \partial_v\end{pmatrix} \tag{6}$$

and the nonlinear one by

$$\mathcal{N}_T(U)=\begin{pmatrix} 0\\ T^2(a\varphi^2+b\varphi^3)+h.o.t.\\ 0\end{pmatrix} \tag{7}$$

where

$$V(x)=\frac{1}{2}x^2-\frac{a}{3}x^3-\frac{b}{4}x^4+\text{h.o.t.}$$

We introduce the following Banach spaces

$$\mathbb{H}=\mathbb{R}^2\times C^0[-2,2] \tag{8}$$

and

$$\mathbb{D}=\left\{U\in\mathbb{R}^2\times C^1[-2,2]/\Phi(0)=\varphi\right\}. \tag{9}$$

The operator $\mathcal{L}_{A,T,\gamma}$ maps $\mathbb{D}$ into $\mathbb{H}$ continuously and the nonlinearity $\mathcal{N}_T:\mathbb{D}\to\mathbb{D}$ is analytic (in a neighborhood of 0) with $\mathcal{N}_T(U)=O(\|U\|^2_{\mathbb{D}})$ as $\|U\|_{\mathbb{D}}\to 0$.

We observe that the symmetry R on $\mathbb{H}$ defined by

$$R(\varphi,\psi,\Phi(v))=(\varphi,-\psi,\Phi(-v,\xi))$$

satisfies $(\mathcal{L}_{A,T,\gamma}+\mathcal{N}_T)\circ R=-R\circ(\mathcal{L}_{A,T,\gamma}+\mathcal{N}_T)$. Therefore, if U is a solution of the evolution equation (5) then $RU(-.)$ is also a solution, i.e. the system is reversible under R. This is reminiscent of the time reversibility of the equation.

3. Spectral analysis

As described in Theorem 7.1 (see the Appendix), one of the main issue in the applicability of the reduction to a center-manifold technique is to get precise informations on the spectrum of the linear operator $\mathcal{L}_{A,T,\gamma}$.

From $\mathbb{D}$ into $\mathbb{H}$, the operator $\mathcal{L}_{A,T,\gamma}$ is closed with compact resolvent. Its spectrum is then discrete with isolated eigenvalues with finite multiplicity.

We consider the following eigenproblem

$$\mathcal{L}_{A,T,\gamma}\hat{U} = \sigma\hat{U}$$

where $\sigma \in \mathbb{C}$. This leads to the following dispersion relation

$$N(\sigma, T, A, \gamma) = \sigma^2 + T^2 - 2T^2\gamma(\cosh(\sigma) - 1) - 2AT^2\gamma(\cosh(2\sigma) - 1) = 0. \tag{10}$$

Simple eigenvalues of $\mathcal{L}_{A,T,\gamma}$ are simple roots of $N(\sigma, T, A, \gamma)$. Since we are mainly interested in the center spectrum, we will consider $\sigma = iq$ and $q > 0$ (notice that if σ is an eigenvalue so is $-\sigma$). This leads to

$$N(iq, T, A, \gamma) = -q^2 + T^2 - 2T^2\gamma(\cos(q) - 1) - 2AT^2\gamma(\cos(2q) - 1) = 0. \tag{11}$$

The eigenvalue iq is going to be a double eigenvalue if it satisfies (11) and $\frac{d}{dq}N(iq, T, A, \gamma) = 0$. Similarly, iq is going to be a triple eigenvalue if it satisfies (11), $\frac{d}{dq}N(iq, T, A, \gamma) = 0$ and $\frac{d^2}{dq^2}N(iq, T, A, \gamma) = 0$. For eigenvalues of multiplicity 4, the condition $\frac{d^3}{dq^3}N(iq, T, A, \gamma) = 0$ has to be satisfied aswell. Notice that since we are dealing with a 3-parameter problem, we do not have higher order eigenvalues.

In order to apply Theorem 7.1 it is important to have a clear picture of the spectrum of $\mathcal{L}_{A,T,\gamma}$. This is done in the following lemmas. First it is easy to check that the center spectrum is separated from the hyperbolic one.

Lemma 3.1. *For all* $(T, A, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, *there exists* p_0 *such that all eigenvalues* $\sigma = p + iq$ *for* $p \neq 0$ *satisfy* $|p| \geq p_0$.

We first consider simple eigenvalues. The object of the following lemma is to describe the spectrum of $\mathcal{L}_{A,T,\gamma}$ at low coupling A and time T, γ being fixed.

Lemma 3.2. *For every $\gamma > 0$, there exist $A_0 > 0$ and $T_0 > 0$ such that for all $(A, T) \in (0, A_0) \times (0, T_0)$, the center spectrum of $\mathcal{L}_{A,T,\gamma}$ consists of one pair of simple eigenvalues $\pm iq^*$.*

Proof. Consider $A = 0$ then the roots $q > 0$ of the dispersion relation $N(iq, T, 0, \gamma) = 0$ are the values of q so that for a given T the following holds

$$T^2 = \frac{q^2}{1 + 4\gamma \sin^2(q/2)}. \tag{12}$$

Now, define the function

$$f_\gamma(q) = \frac{q^2}{1 + 4\gamma \sin^2(q/2)}.$$

For every $\gamma > 0$, the function $f_\gamma(q)$ is stricly increasing on $(0, \pi)$ and therefore $1 - 1$ from $(0, \pi)$ into $(f_\gamma(0) = 0, f_\gamma(\pi))$. Consequently, choosing $T_0 = \sqrt{f_\gamma(\pi)}$, for all $T \in (0, T_0)$, there exists a unique q^* such that

$$q^* = f_\gamma^{-1}(T^2).$$

By applying the implicit function theorem we get the existence of a curve $A(T)$ so that $N(iq^*(A, T), T, A(T), \gamma) = 0$ for T and A small enough, γ being fixed.

□

We consider now the case of double eigenvalues. The imaginary part q has to satisfy the two following equations

$$N(iq, T, A, \gamma) = -q^2 + T^2 - 2T^2\gamma(\cos(q) - 1) - 2AT^2\gamma(\cos(2q) - 1) = 0, \tag{13}$$

$$\frac{d}{dq} N(iq, T, A, \gamma) = -2q + 2T^2\gamma \sin(q) + 4T^2 A\gamma \sin(2q) = 0. \tag{14}$$

The occurrence of double eigenvalues is important for our purpose. Indeed, we are interested in solutions of (4) connecting 0 at $\pm\infty$ (homoclinic solutions to 0). We then have to consider bifurcations in the parameter space (T, A, γ) such that the spectrum of $\mathcal{L}_{A,T,\gamma}$ consists of a pair of weakly coupled hyperbolic eigenvalues. For some values (A, T, γ), these two pairs of hyperbolic eigenvalues are going to collide on the imaginary axis, yielding a pair of double non semi-simple eigenvalues. The next lemma locates in the parameter plane such curves on which the spectrum has this structure.

Lemma 3.3. *Fix $\gamma > 0$ and consider the parametrized curve $\Gamma : \Omega_\gamma \mapsto \mathbb{R}^+ \times \mathbb{R}^+$ given by*

$$\begin{aligned} T_\gamma(q) &= \left(\tfrac{q^2-q\tan(q)}{1+4\gamma\sin^2(q/2)-\gamma\sin(q)\tan(q)}\right)^{1/2} \\ A_\gamma(q) &= \tfrac{1}{2\gamma T_\gamma(q)^2}\left(\tfrac{q-T_\gamma(q)^2\gamma\sin(q)}{\sin(2q)}\right) \end{aligned} \tag{15}$$

where Ω_γ is some subset of $\mathbb{R}^+$.

Then there exists a zero Lebesgue measure set $\tilde{\Omega}$ in $\mathbb{R}^+\times\mathbb{R}^+$ and a subset $\Omega\subset\Gamma(\Omega_\gamma)$ such that for all $(T,A)\in\Omega\backslash\tilde{\Omega}$, the center spectrum of $\mathcal{L}_{A,T,\gamma}$ consists in a pair of double non semi-simple eigenvalues and a finite number of pairs of simple eigenvalues.

Proof. We are looking for double eigenvalues of $\mathcal{L}_{A,T,\gamma}$ so we look for roots of (13) and (14). Solving $2\gamma T^2A$ from (14) leads to

$$2\gamma T^2A = \frac{q-T^2\sin(q)}{2\sin(q)\cos(q)}. \tag{16}$$

Plugging this expression back into equation (13), this gives an expression for T^2, namely

$$T^2 = \frac{q^2-q\tan(q)}{1+4\gamma\sin^2(q/2)-\gamma\sin(q)\tan(q)}.$$

The set Ω_γ is the set of $q\in\mathbb{R}^+$ such that $T^2>0$ and $A>0$. To get the expression for A we simply solve from (16).

The set $\tilde{\Omega}$ contains the values of (A,T) such that the spectrum admits triple eigenvalues and several pairs of double eigenvalues. Triple eigenvalues occur at cusp points of the curve Γ (see Figure 1). We also exclude in $\tilde{\Omega}$ the curves in the parameter plane where the simple eigenvalues collide, yielding semi-simple double eigenvalues.

This set is clearly negligible for the product measure in $\mathbb{R}^2$.

According to Lemma 3.2, there exists a region in the parameter plane where the center spectrum of $\mathcal{L}_{A,T,\gamma}$ consists of only one pair of simple eigenvalues. By continuity of the spectrum, we deduce that except on the set $\tilde{\Omega}$, the image of the curve Γ provides the set of parameters for which the spectrum just consists of a pair of double non semi-simple eigenvalues and a finite number of pairs of simple ones.

We refer the reader to Figure 1 where the complete spectrum is given in the parameter plane (T,A) for some $\gamma=\gamma_0$.

□

It is important to notice that the bifurcation curve described in the previous lemma involves branches of bifurcations of double eigenvalues which were not present in the works [IK00, Sir05, JS05]. These new branches are the ones starting from $A=0$ and they are present for each

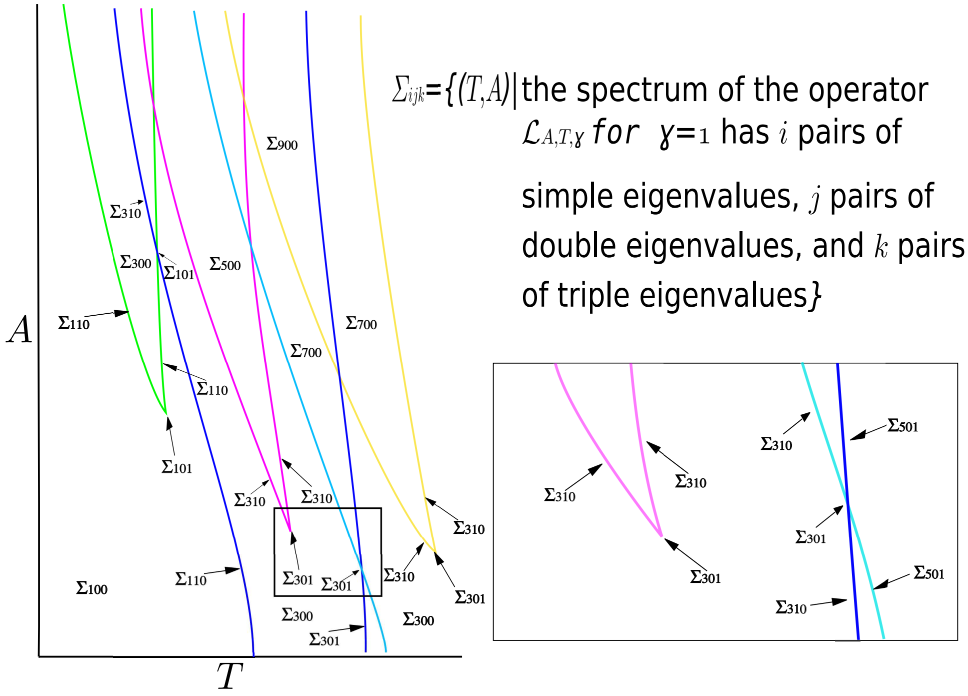

FIGURE 1. Curves parametrized by (15) for $\gamma = 1$ on the parameter space (T, A). The different colors represent different branches of the curves.

$\gamma > 0$ (see figures 2). More precisely, by just analyzing the equation $A_\gamma(q) = 0$, one can see that they occur as soon as

$$\gamma T^2 > 1.$$

This is a new feature of coupling to second nearest neighbors and not just to first nearest neighbors. As a consequence, the bifurcation picture is more involved and new solutions appear. Indeed, in the work [IK00] (i.e. $A = 0$ in our context), one can see that for every $T_0 > 0$, there exists $\gamma_0 > 0$ such that for all $T < T_0$ and $\gamma < \gamma_0$, the center spectrum of the linear operator consists of *one* pair of simple eigenvalues.

In our context, we have the following lemma, whose proof follows from a continuity argument as a consequence of the two previous lemmata. We refer the reader to Figure 3 also.

Lemma 3.4. *Fix* $\gamma > 0$ *and* A *small enough. Then there exists an increasing sequence of times* $\{T_n\}_{n\geq 0}$ *such that*

- $T_0 > \sqrt{\frac{1}{\gamma}}$

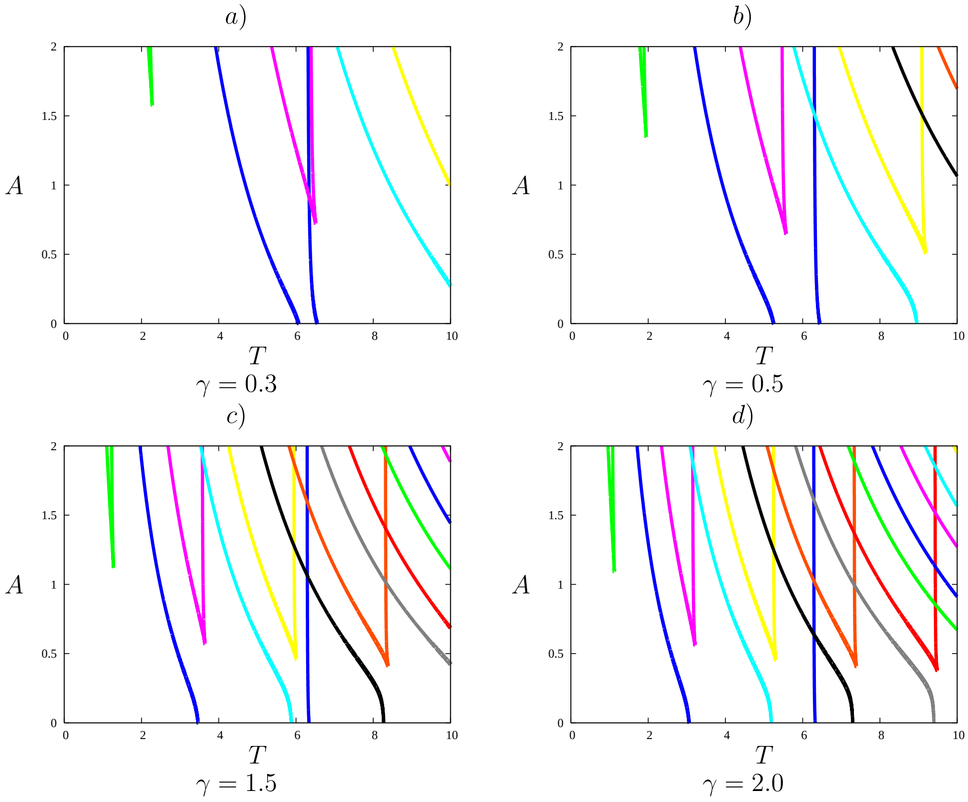

FIGURE 2. Curves parametrized by (15) for several values of γ on the parameter space (T, A).

- *There exists a sequence* $\{\delta_n > 0\}_{n\geq 0}$ *such that for every* $T \in (T_n, T_n + \delta_n)$ *the center spectrum consists of* $2n+3$ *pairs of simple eigenvalues.*

In order to study bifurcations of travelling waves near 0, we restrict ourselves to the set Δ of A, T and γ in the parameter space such that the center spectrum Σ_c of $\mathcal{L}_{A,T,\gamma}$ has the following structure

$$\Sigma_c = \{\pm iq_0, \pm iq_1, ..., \pm iq_p\}$$

where $\pm iq_0$ is a pair of double non semi-simple eigenvalues and $\pm iq_j$ $(j = 1, ..., p)$ are pairs of simple eigenvalues. In this case, the linear space generated by Σ_c is $2p+4$-dimensional.

As previously mentioned, the reason why we consider this type of bifurcations is that the pair of double non semi-simple eigenvalues gives rise to homoclinic connections for the normal form of the evolution

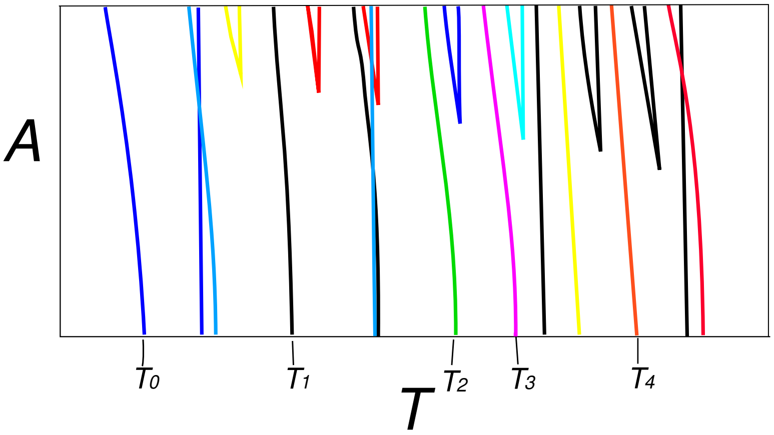

FIGURE 3. Curves parametrized by (15) for $\gamma = 1$ on the parameter space (T, A) for low coupling A. The sequence of times $T_0, T_1, \dots$ illustrates the statement of lemma 3.4.

equation (5) on the center manifold. The additional pairs of simple eigenvalues will lead to oscillatory tails.

One can check the following properties.

Lemma 3.5. *Fix $(A, \gamma, T) \in \Delta$ and let V_0, V_j for $j = 1, \dots, p$ be the eigenvectors for the associated to iq_0 and iq_j respectively. Denote by $\hat{V}_0$ the generalized eigenvector associated to iq_0. The eigenvectors can be chosen in the following way:*

$$V_0 = \begin{pmatrix} 1 \\ iq_0 \\ e^{iq_0 v} \end{pmatrix}, \quad \hat{V}_0 = \begin{pmatrix} 0 \\ 1 \\ ve^{iq_0 v} \end{pmatrix}, \quad \text{and} \quad V_j = \begin{pmatrix} 1 \\ iq_j \\ e^{iq_j v} \end{pmatrix}$$

for $j = 1, \dots, p$. Moreover, these eigenvectors satisfy

$$RV_0 = \bar{V}_0, \quad RV_j = \bar{V}_j, \quad R\hat{V}_0 = -\bar{\hat{V}}_0.$$

4. PROBLEM ON THE HYPERBOLIC SUBSPACE AND REDUCTION TO A CENTER MANIFOLD

In this section we compute the spectral projection on the hyperbolic subspace (invariant subspace under $\mathcal{L}_{A,T,\gamma}$ corresponding to the hyperbolic spectral part) and prove a regularity result for the associated inhomogeneous linearized equation. This result is a crucial assumption for applying center manifold reduction theory (see Theorem 7.1 in the Appendix). Our proof closely follows the method given in [IK00]. For sake of completeness, we however give the proof in the present case, making sure that the additional coupling term in A is not an obstruction to the solvability.

We call P_0, P_1, ..., P_p respectively the spectral projection on the 4-dimensional invariant subspace associated to $\pm iq_0$ and on the p 2-dimensional subspaces corresponding to $\pm iq_j$, for $j = 1, ..., p$. We also define $P = \sum_{j=0}^{p} P_j$ (spectral projection on the $2p + 4$-dimensional central subspace) and use the notations $\mathbb{D}_h = (\mathbb{I} - P)\mathbb{D}$, $\mathbb{H}_h = (\mathbb{I} - P)\mathbb{H}$, $\mathbb{D}_c = P\mathbb{D}$, $U_h = (\mathbb{I} - P)U$. The affine linearized system on $\mathbb{H}_h$ reads

$$\frac{dU_h}{d\xi} = \mathcal{L}_{A,T,\gamma} U_h + F_h(\xi) \tag{17}$$

where $F(\xi) = (0, f(\xi), 0)^T$ lies in the range of the nonlinear operator (7). We shall note $U_h = (\varphi^h, \dot{\varphi}^h, \Phi^h(v))^T$.

Our aim is to check the regularity property of equation (17) (see [VI92], property (ii) p.127 or see the appendix). This property can be stated as follows. We introduce the following Banach space, for a given Banach space Z and $\alpha \in \mathbb{R}^+$:

$$E_j^\alpha(Z) = \left\{ f \in C^j(\mathbb{R}, Z) \; \|f\|_j = \max_{0 \le k \le j} \sup_{t \in \mathbb{R}} e^{-\alpha|t|} |D^k f(t)| < \infty \right\}. \tag{18}$$

We have to prove that system (17) admits a unique solution U_h in $E_0^\alpha(\mathbb{D}_h) \bigcap E_1^\alpha(\mathbb{H}_h)$ for $0 \le \alpha < \alpha_0$ (for some $\alpha_0 > 0$), the operator $K_h : E_0^\alpha(\mathbb{R}) \to E_0^\alpha(\mathbb{D}_h)$, $f \mapsto U_h$ being bounded.

We do not know *a priori* if the operator $\mathcal{L}_{A,T,\gamma}$ is sectorial, providing explicit estimates on U_h. Following the method in [IK00, SJ04], we compute by hand the solution U_h and provide necessary estimates.

4.1. **Computation of the spectral projection on the hyperbolic subspace.**

The spectral projection on the central subspace is defined by the Dunford integral

$$P = \frac{1}{2i\pi} \int_C (\sigma \mathbb{I} - \mathcal{L}_{A,T,\gamma})^{-1} dC, \tag{19}$$

where C is a regular curve surrounding $\pm iq_0, \pm iq_1, ..., \pm iq_p$. The spectral projection on the hyperbolic subspace is $P_h = \mathbb{I} - P$.
We shall use the following result for computing P_h.

Lemma 4.1. *Let $h(z) = \frac{f(z)}{g(z)}$ be a function of $z \in \mathbb{C}$. Assume the function $f(z)$ is entire and the function $g(z)$ admits a double pole at $z = z_0$. Then the residue of h at $z = z_0$ is given by*

$$Res(h, z_0) = \frac{2f'(z_0)g''(z_0) - \frac{2}{3} f(z_0) g'''(z_0)}{g''(z_0)^2}. \tag{20}$$

In the following lemma, we compute the spectral projection on the hyperbolic subspace of a vector F lying in the range of the nonlinear operator (7).

Lemma 4.2. *Let* $F \in \mathbb{D}$ *be a vector of the type* $F = (0, f, 0)^T$. *Then the projection of* F *on the hyperbolic subspace reads*

$$F_h = (0, k_1 f, k_2(v) f)^T, \tag{21}$$

where $k_1 \in \mathbb{R}$ *and* $k_2 \in C^\infty([-2, 2])$ *depend on* A, γ *and* T.

Proof. We first compute the resolvent of $\mathcal{L}_{A,T,\gamma}$. One has to solve $(\sigma \mathbb{I} - \mathcal{L}_{A,T,\gamma})U = F$ for $U = (u_1, u_2, u_3)^T$, which yields the system

$$\sigma u_1 = u_2 \tag{22}$$

$$\beta_1 u_1 - \gamma T^2[(u_3(1) + u_3(-1)) + A(u_3(2) + u_3(-2))] = f \tag{23}$$

$$u_3 = u_1 e^{\sigma v} \tag{24}$$

for $\beta_1 = T^2(1 + 2\gamma(1 + A)) + \sigma^2$. We have then that

$$u_1 = \frac{f}{N(\sigma, T, A, \gamma)}.$$

Now we compute the spectral projection P_j. Since $\sigma_j = iq_j$, for $j = 1, ..., p$, are simple roots of the dispersion relation, one has

$$Res(u_1, iq_j) = \frac{f}{-2q_j + 2\gamma T^2 \sin(q_j) + 4A\gamma T^2 \sin(2q_j)},$$

Denoting $(P_1 F)_i$ the ith component of $P_j F$, we get consequently

$$(P_j F)_1 = Res(u_1, iq_j) + Res(u_1, -iq_j) = 0,$$

In the same spirit

$$(P_j F)_2 = \frac{-2q_j f}{-2q_j + 2\gamma T^2 \sin(q_j) + 4A\gamma T^2 \sin(2q_j)},$$

$$(P_j F)_3 = \frac{2\cos(q_j v) f}{-2q_j + 2\gamma T^2 \sin(q_j) + 4A\gamma T^2 \sin(2q_j)},$$

which completes the computation of the projections $P_j F$. For computing the spectral projection P_0 associated to the double eigenvalues $\pm iq_0$, we use formula (20). These computations lead to equation (21). □

4.2. **Resolution of the affine equation for bounded functions of t.** We first solve (17) in the spaces E_j^α with $\alpha = 0$, i.e. we consider bounded functions of t (note that $E_j^0(\mathbb{H}) = C_b^j(\mathbb{H})$). Fixing $\alpha = 0$ will allow us to take the Fourier transform in time of the system in the space of tempered distributions $S'(\mathbb{R})$.

From (17), we directly deduce

$$\begin{aligned}\Phi^h(t,v) &= \varphi^h(t+v) + \int_0^v k_2(s) f(t+v-s)ds \\ &= \varphi^h(t+v) + \int_t^{t+v} k_2(t+v-s) f(s)ds,\end{aligned} \tag{25}$$

(this expression comes from the two last equations of the affine linear system and from conditions $\Phi(0,t) = \varphi(t)$).
From the previous equations and the fact that k_2 and its derivatives are bounded functions of v, we deduce that

$$\|\Phi^h\|_{E_0^0(C^1[-2,2])} \leq \|\varphi^h\|_{E_1^0} + C\|f\|_{E_0^0}. \tag{26}$$

We now have to estimate φ^h and $\dot\varphi^h$. Taking the Fourier transform in time of the system (17) in the tempered distributional space $S'(\mathbb{R})$, we have

$$(ik - \mathcal{L}_{A,T,\gamma})\hat{U}_h = \hat{F}_h. \tag{27}$$

We deduce

$$\hat{\dot\varphi}^h = ik\hat\varphi^h,$$

$$\hat{\varphi}^h = e^{ikv}\hat\varphi^h + \hat f \int_0^v e^{ik(v-s)} k(s)ds.$$

For $\hat\varphi^h$, we have

$$N(ik, T, A, \gamma)\hat\varphi^h = \hat f. \tag{28}$$

Then from (28) we define the function $\hat G$ as

$$\hat\varphi^h = \hat G \hat f.$$

The operator $(ik - \mathcal{L}_{A,T,\gamma}^h)^{-1}$ ($\mathcal{L}_{A,T,\gamma}^h$ denoting the restriction of $\mathcal{L}_{A,T,\gamma}$ to the hyperbolic space) is analytic in a strip around the real axis. We deduce that $\hat G$ is an analytic function in the same strip. Moreover, $\hat G$ behaves like $O(\frac{1}{k^2})$ as $k \to \pm\infty$ due to the fact that $N(ik, A, T, \gamma) = O(k^2)$ as $k \to \pm\infty$. The fact that the dispersion relation behaves like k^2 is important in our context. Here, even if the coupling term is more complicated, involving the second nearest neighbors, the perturbation is still $O(1)$ in terms of k.

Since $N(iq_j, A, T, \gamma) = 0$, $N'(iq_0, A, T, \gamma) = 0$ and $N'(iq_j, A, T, \gamma)$ and $N''(iq_0, A, T, \gamma)$ do not vanish, equation (28) yields

$$\hat{\varphi}^h = \hat{G}\hat{f} + b_0^+\delta'_{iq_0} + b_0^-\delta'_{-iq_0} + \sum_{j=1}^{p}\left(a_j^+\delta_{iq_j} + a_j^-\delta_{-iq_j}\right) \tag{29}$$

Furthermore, $k \to (1+|k|^2)^{1/2}\hat{G}$ belongs to $L^2(\mathbb{R})$. Therefore, using the inverse Fourier Transform and lemma 3 p.448 of [IK00], there exists a function $G \in H^1_\delta(\mathbb{R})$ (i.e $e^{\delta|t|}G \in H^1(\mathbb{R})$, $\delta > 0$ small enough) such that $\hat{G}$ is the unique Fourier transform of G. We have the following estimates

$$\begin{aligned} \|\frac{dG}{dt} * f\|_{C^0_b} &= \sup_{t\in\mathbb{R}} |\int_{\mathbb{R}} \frac{dG}{dt}(t-s)f(s)ds| \\ &\le C(\delta)\|f\|_{C^0_b}\|G\|_{H^1_\delta(\mathbb{R})}. \end{aligned} \tag{30}$$

Now we make the solution of (17) explicit. We set $\tilde{U}_h = (\tilde{u}^h_1, \tilde{u}^h_2, , \tilde{u}^h_3)^T$ and

$$\begin{aligned} \tilde{u}^h_1 &= G * f, \\ \tilde{u}^h_2 &= \frac{d\tilde{u}^h_1}{dt}, \\ \tilde{u}^h_3(t,v) = \tilde{u}^h_1(t+v) &+ \int_0^v k_2(s)f(t+v-s) + k_8(s)\,ds, \end{aligned}$$

By construction, $\tilde{u}^h$ satisfies (17) and $P\hat{\tilde{U}}_h = 0$ (hence $P\tilde{U}_h = 0$) for $f \in E^\alpha_0(\mathbb{R})$ with $\alpha < 0$ ($\hat{f}_i$ are analytic functions in a strip around the real axis). Since the computations are formally the same for $\alpha = 0$, we have $P\hat{\tilde{U}}_h = 0$ for $\alpha = 0$, hence $P\tilde{U}_h = 0$ for $\alpha = 0$.
Moreover, we have the estimate

$$\|\tilde{U}_h\|_{C^0_b(\mathbb{D}_h)\bigcap C^1_b(\mathbb{H}_h)} \le C\|f\|_{C^0_b(\mathbb{R})} \tag{31}$$

due to estimates (26), (30) (with analogous estimates on H_2). For $\alpha = 0$, we obtain u^h by adding to $\tilde{u}^h$ the inverse Fourier transforms of Dirac measures, i.e

$$\begin{aligned} u^h_1 \;\; = \;\; & \tilde{u}^h_1 + \sum_{j=1}^{p}(a_j^+e^{iq_1t} + a_j^-e^{-iq_1t}) \\ & +(a_0^+ + itb_0^+)e^{iq_0t} + (a_0^- - itb_0^-)e^{-iq_0t}, \end{aligned} \tag{32}$$

$$u_2^h = \tilde{u}_2^h + \sum_{j=1}^{p}(c_j^+ e^{iq_jt} + c_j^- e^{-iq_jt}) \\ +(c_0^+ + itd_0^+)e^{iq_0t} + (c_0^- - itd_0^-)e^{-iq_0t}. \tag{33}$$

Since $P\tilde{U}_h = 0$, we have $PU_h = 0$ if and only if

$$a_j^\pm = c_j^\pm = b_0^\pm = a_0^\pm = d_0^\pm = 0 \quad \text{for } j = 1, ..., p. \tag{34}$$

It follows that $U_h = \tilde{U}_h$. Finally, we have proved the following

Lemma 4.3. *Assume $F = (0, f, 0)^T$ and $f \in C_b^0(\mathbb{R})$. Then the affine linear system (17) has a unique bounded solution $U_h \in C_b^0(\mathbb{D}_h) \bigcap C_b^1(\mathbb{H}_h)$ and the operator $K_h : C_b^0(\mathbb{R}) \to C_b^0(\mathbb{D}_h)$, $f \mapsto U_h$ is bounded.*

4.3. **Affine equation in exponentially weighted spaces.** The problem now is to extend lemma 4.3 to the case $f \in E_0^\alpha(\mathbb{R})$, with $\alpha > 0$ sufficiently close to 0. This has been done in [IK00] by constructing a suitable distribution space, but the following lemma gives an alternative proof (see [Mie87]).

Lemma 4.4. *Consider Banach spaces $\mathbb{D}$,$\mathbb{Y}$ and $\mathbb{X}$ such that: $\mathbb{D} \hookrightarrow \mathbb{Y} \hookrightarrow \mathbb{X}$.*
Let L be a closed linear operator in $\mathbb{X}$, of domain $\mathbb{D}$, such that the equation

$$\frac{dU}{dt} = LU + f \tag{35}$$

admits for any fixed $f \in C_b^0(\mathbb{Y})$ a unique solution

$$U = Kf$$

in $C_b^0(\mathbb{D}) \bigcap C_b^1(\mathbb{X})$, with in addition $K \in \mathcal{L}(C_b^0(\mathbb{Y}), C_b^0(\mathbb{D}))$. Then there exists $\alpha_0 > 0$ such that if $0 \le \alpha < \alpha_0$, for all $f \in E_0^\alpha(\mathbb{Y})$ the system (35) admits a unique solution in $E_0^\alpha(\mathbb{D}) \bigcap E_1^\alpha(\mathbb{X})$ with

$$\|U\|_{E_0^\alpha(\mathbb{D})} \le C(\alpha)\|f\|_{E_0^\alpha(\mathbb{Y})}. \tag{36}$$

Applying this result to our problem yields the following result.

Proposition 4.5. *There exists $\alpha_0 > 0$ such that for all $F = (0, f, 0)^T$ with $f \in E_0^\alpha(\mathbb{R})$ and $\alpha \in [0, \alpha_0]$, the affine linear system (17) has a unique solution $U_h \in E_0^\alpha(\mathbb{D}_h) \bigcap E_1^\alpha(\mathbb{H}_h)$. Moreover, the operator $K_h : E_0^\alpha(\mathbb{R}) \to E_0^\alpha(\mathbb{D}_h)$, $f \mapsto U_h$ is bounded (uniformly in $\alpha \in [0, \alpha_0]$).*

Remark 4.6. *It has to be remarked that the previous argument is also valid if the system involves more general linear coupling terms (to the third or farther neighbors). Indeed, the perturbation in the dispersion relation still involves $O(1)$ terms in k and as a consequence*

$N(ik, A, T, \gamma)$ is still $O(k^2)$. Hence the regularity argument with the kernel G still holds.

However, for the case of pulsating travelling waves, as it was pointed out in [JS05, Sir05]*, one has to be more careful. Indeed, one can use for instance the alternative argument developped in* [Sir05] *to overcome the fact that the dispersion relation does not have the suitable behaviour.*

5. Normal form computations

This section is devoted to the computation of the normal form after reduction to a center manifold. Normal form theory (see for instance [IA98]) ensures that there exists a change of variables $U_c = \tilde{U}_c + \tilde{P}_{\gamma,T}(\tilde{U}_c)$ (here $U_c = PU$ an $\tilde{P}$ is close to the identity). The simplifies the reduced equation on the center manifold. In this section, we provide this normal form at order 3.

5.1. Normal forms.

The linear operator $\mathcal{L}_{A,T,\gamma}$ restricted to the $(2p+4)$-dimensional subspace $\mathbb{D}_c$ has the following structure in the basis $(V_0, \hat{V}_0, V_1, ..., V_p, \bar{V}_0, \bar{\hat{V}}_0, \bar{V}_1, ..., \bar{V}_p)$ of eigenvectors in Lemma 3.5

$$\mathcal{L}_c = \begin{pmatrix} J_1 & & & \\ & \Lambda & & \\ & & J_2 & \\ & & & \Lambda' \end{pmatrix}$$

where

$$J_1 = \begin{pmatrix} iq_0 & 1 \\ 0 & iq_0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} -iq_0 & 1 \\ 0 & -iq_0 \end{pmatrix},$$
$$\Lambda = \begin{pmatrix} iq_1 & & \\ & \ddots & \\ & & iq_p \end{pmatrix}, \quad \Lambda' = \begin{pmatrix} -iq_1 & & \\ & \ddots & \\ & & -iq_p \end{pmatrix}. \tag{37}$$

To compute the normal form, we exclude points of Δ which are close to points where $sq_0 + \sum_{j=1}^{p} r_j q_j = 0$ for $s, r_j \in \mathbb{Z}$ and $0 < |s| + \sum_{j=1}^{p} |r_j| \leq 4$ which correspond to strong resonances. We denote this new set by Δ_0. The normal form computation is very similar to [IK00], to which we refer for details.

In what follows we set $\tilde{U}_c = \mathcal{A}V_0 + \mathcal{B}\hat{V}_0 + \sum_{j=1}^{p} \mathcal{C}_j V_j + \bar{\mathcal{A}}\bar{V}_0 + \bar{\mathcal{B}}\bar{\hat{V}}_0 + \sum_{j=1}^{p} \bar{\mathcal{C}}_j \bar{V}_j$. The normal form of the evolution equation (5) restricted to the center space up to order 3 is given in the following lemma.

Lemma 5.1. *The normal form of the reduced equation at order 3 reads*
(38)

$$\begin{aligned}\frac{d\mathcal{A}}{d\xi} &= iq_0\mathcal{A} + \mathcal{B} + i\mathcal{A}\mathcal{P} + O((|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p} |\mathcal{C}_j|)^4),\\ \frac{d\mathcal{B}}{d\xi} &= iq_0\mathcal{B} + i\mathcal{B}\mathcal{P} + \mathcal{A}\mathcal{S} + O((|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p} |\mathcal{C}_j|)^4),\\ \frac{d\mathcal{C}_j}{d\xi} &= iq_j\mathcal{C}_j + i\mathcal{C}_j\mathcal{Q}_j + O((|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p} |\mathcal{C}_j|)^4) \quad \textit{for } j = 1, ..., p.\end{aligned}$$

where $\mathcal{P}, \mathcal{S}, \mathcal{Q}_k$ are polynomials of the variables $u_0 = \mathcal{A}\bar{\mathcal{A}}, v_0 = i(\mathcal{A}\bar{\mathcal{B}} - \bar{\mathcal{A}}\mathcal{B}), u_j = \mathcal{C}_j\bar{\mathcal{C}}_j$, with real coefficients depending smoothly on the parameters (T, A, γ) in the neigborhood of Δ. Furthermore, we have

$$\begin{aligned}\mathcal{P}(u_0, v_0, u_1, ..., u_p) &= p_0'(A, T, \gamma) + \sum_{j=0}^{p} p_j u_k + p_{p+1} v_0\\ \mathcal{S}(u_0, v_0, u_1, ..., u_p) &= s_0'(A, T, \gamma) + \sum_{j=0}^{p} s_j u_k + s_{p+1} v_0 \qquad (39)\\ \mathcal{Q}_k(u_0, v_0, u_1, ..., u_p) &= \tilde{q}_{0,k}'(A, T, \gamma) + \sum_{l=0}^{p} \tilde{q}_{l,k} u_l + \tilde{q}_{p+1,k} v_0\end{aligned}$$

where $p_0', s_0', \tilde{q}_{0j}'$ vanish on Δ_0.

The truncated normal form (obtained by neglecting terms of orders 4 and higher) is integrable with the following first integrals

$$\begin{gathered} A\bar{B} - \bar{A}B,\\ |B|^2 - \int_0^{|A|^2} \mathcal{S}(x, |C|^2, |D|^2, i(A\bar{B} - \bar{A}B))dx, \qquad (40)\\ |C_j|^2, \text{for } j = 1, ..., p.\end{gathered}$$

The existence of homoclinic orbits is linked to the sign on the coefficient s_0 in the polynomial $\mathcal{S}$. We consider values of the parameters A and T so that $s_0'(A, T) > 0$. The following section is devoted to the computation of s_0.

5.2. **Computation of the coefficient s_0.** First we expand the evolution equation (5) as follows
(41)

$$\frac{dU}{d\xi} = \mathcal{L}^{(0)}U + (A - A_0)\mathcal{L}^{(1)}U + (T - T_0)\mathcal{L}^{(2)}U + (\gamma - \gamma_0)\mathcal{L}^{(3)}U + M_2(U, U) + M_3(U, U, U) + ...$$

where $\mathcal{L}^{(0)}$ is the linear operator $\mathcal{L}_{A,T,\gamma}$ for $(A_0, T_0, \gamma_0) \in \Delta_0$,

$$\mathcal{L}^{(1)} = \gamma T^2 \begin{pmatrix} 0 \\ -2\varphi + \Phi(2) + \Phi(-2) \\ 0 \end{pmatrix},$$

$$\mathcal{L}^{(2)} = 2T \begin{pmatrix} 0 \\ (-1 - 2\gamma(1+A))\varphi + \gamma(\Phi(1) + \Phi(-1)) + \gamma A(\Phi(2) + \Phi(-2)) \\ 0 \end{pmatrix},$$

$$\mathcal{L}^{(3)} = T^2 \begin{pmatrix} 0 \\ -2(1+A)\varphi + (\Phi(1) + \Phi(-1)) + A(\Phi(2) + \Phi(-2)) \\ 0 \end{pmatrix},$$

$$M_2(U,U) = aT^2 \begin{pmatrix} 0 \\ \varphi^2 \\ 0 \end{pmatrix},$$

and

$$M_3(U,U,U) = bT^2 \begin{pmatrix} 0 \\ \varphi^3 \\ 0 \end{pmatrix},$$

We now expand the solution U around (A_0, T_0, γ_0) in the following way

$$\begin{gathered} U = \mathcal{A}V_0 + \mathcal{B}\hat{V}_0 + \sum_{j=1}^{p} \mathcal{C}_j V_j + \bar{\mathcal{A}}\bar{V}_0 + \bar{\mathcal{B}}\bar{\hat{V}}_0 + \sum_{j=1}^{p} \bar{\mathcal{C}}_j \bar{V}_j + \\ \sum (A - A_0)^m (T - T_0)^n (\gamma - \gamma_0)^l \mathcal{A}^{r_0} \mathcal{B}^{\hat{r_0}} C_1^{r_1} C_2^{r_2} \ldots C_p^{r_p} \\ \bar{\mathcal{A}}^{s_0} \bar{\mathcal{B}}^{\hat{s_0}} \bar{C}_1^{s_1} \bar{C}_2^{s_2} \ldots \bar{C}_p^{s_p} \phi^{(m,n,l)}_{r_0 \hat{r_0} r_1 \ldots r_p s_0 \hat{s_0} s_1 \ldots s_p} \end{gathered} \tag{42}$$

Then we use the use the expression of the normal form and the expansion (41) to find the terms of order $\mathcal{A}^2, |\mathcal{A}|^2, \mathcal{A}|\mathcal{A}|^2$ by identification. We get the expressions (we omit the index $(m,n,l) = (0,0,0)$ in the notations)

(43)

$$\begin{aligned} (2iq_0 - \mathcal{L}_{A_0,T_0,\gamma_0})\phi_{20\ldots000\ldots0} &= M_2(V_0, V_0), \\ -\mathcal{L}_{A_0,T_0,\gamma_0}\phi_{10\ldots010\ldots0} &= 2M_2(V_0, \bar{V}_0), \\ s_0\hat{V}_0 + ip_0V_0 + (iq_0 - \mathcal{L}_{A_0,T_0,\gamma_0})\phi_{20\ldots010\ldots0} &= 2M_2(\bar{V}_0, \phi_{20\ldots000\ldots0}) \\ &\quad +2M_2(V_0, \phi_{10\ldots010\ldots0}) \\ &\quad +3M_3(V_0, V_0, \bar{V}_0). \end{aligned}$$

This leads to

(44)

$$
\begin{aligned}
\phi_{20\ldots000\ldots0} &= K_1 \begin{pmatrix} 1 \\ 2iq_0 \\ e^{2iq_0 v} \end{pmatrix} \\
\phi_{10\ldots010\ldots0} &= 2a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
\phi_{20\ldots010\ldots0} &= \varphi \begin{pmatrix} 1 \\ iq_0 \\ e^{iq_0 v} \end{pmatrix} + ip_0 \begin{pmatrix} 0 \\ 1 \\ ve^{iq_0 v} \end{pmatrix} + \frac{s_0}{2} \begin{pmatrix} 0 \\ 0 \\ v^2 e^{iq_0 v} \end{pmatrix}
\end{aligned}
$$

with p_0 and φ still unknown and

(45)

$$
K_1 = \frac{aT_0^2}{T_0^2(1 + 2\gamma(1 + A_0)) - 4q_0^2 - 2T_0^2\gamma\cos(2q_0) - 2T_0^2\gamma A_0 \cos(4q_0)}
$$

(46) $$[1 - \gamma_0 T_0^2 \cos(q_0) - 4A_0\gamma_0 T_0^2 \cos(2q_0)]s_0 = T_0^2(2aK_1 + 4a^2 + 3b)$$

The other coefficients can be computed in a similar way.

6. Small amplitude solutions of the evolution equation (5)

This section is devoted to the study of the small amplitude solutions of (1) after reduction to a center manifold. This relies mainly on the study of the normal form given below

(47)

$$
\begin{aligned}
\frac{d\mathcal{A}}{d\xi} &= iq_0\mathcal{A} + \mathcal{B} + i\mathcal{A}\mathcal{P} + O((|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p} |\mathcal{C}_j|)^4), \\
\frac{d\mathcal{B}}{d\xi} &= iq_0\mathcal{B} + i\mathcal{B}\mathcal{P} + \mathcal{A}\mathcal{S} + O((|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p} |\mathcal{C}_j|)^4), \\
\frac{d\mathcal{C}_j}{d\xi} &= iq_j\mathcal{C}_j + i\mathcal{C}_j\mathcal{Q}_j + O((|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p} |\mathcal{C}_j|)^4) \quad \text{for } j = 1, ..., p.
\end{aligned}
$$

Depending on the parameter regime, we study several types of solutions for equation (5): periodic, quasi-periodic and homoclinic solutions.

6.1. **Periodic solutions of the evolution equation** (5)**.** Consider Δ_1 as the set of parameters for which the center spectrum consists of one pair of simple eigenvalues $\pm iq^*$. By the spectral theory in Section 3, this set is non empty. The theory developped in Section 4 applies (it is a particular case when the total projection P is just the projection P_* onto the 2-dimensional invariant subspace associated to $\pm iq^*$). Therefore, one gets the following reduction theorem since the analysis in Section 4 shows that the assumptions of Theorem 7.1 in the Appendix are satisfied.

Theorem 6.1. *Fix* $(A_0, T_0, \gamma_0) \in \Delta_1$ *and* $k \geq 1$. *There exists a neighborhood* $\mathcal{U}\times\mathcal{V}$ *of* $(0, A_0, T_0, \gamma_0)$ *in* $\mathbb{D}\times\mathbb{R}^3$ *and a map* $\psi \in C_b^k(\mathbb{D}_c\times\mathbb{R}^3, \mathbb{D}_h)$ *such that the following properties hold for all* $(A, T, \gamma) \in \mathcal{V}$ *(with* $\psi(0, A, T, \gamma) = 0, D\psi(0, A_0, T_0, \gamma_0) = 0$*).*

- *If* $U : \mathbb{R} \to \mathbb{D}$ *solves (5) and* $U(t) \in \mathcal{U}$ $\forall t \in \mathbb{R}$ *then* $U_h(t) = \psi(U_c(t), A, T, \gamma)$ *for all* $t \in \mathbb{R}$ *and* U_c *is a solution of*

$$\frac{dU_c}{dt} = \mathcal{L}_{A_0,T_0,\gamma_0}U_c + P_*\mathcal{N}_T(U_c + \psi(U_c, A, T, \gamma)). \tag{48}$$

- *If* $U_c : \mathbb{R} \to \mathbb{D}_c$ *is a solution of (48) with* $U_c \in \mathcal{U}_c = P_*\mathcal{U}$ $\forall t \in \mathbb{R}$, *then* $U = U_c + \psi(U_c, A, T, \gamma)$ *is a solution of (5).*
- *The map* $\psi(., A, T, \gamma)$ *commutes with* R. *Moreover, the reduced system (48) is reversible under* R.

The normal form (47) then reduces in this case to

$$\frac{d\mathcal{C}_*}{d\xi} = iq_*\mathcal{C}_* + i\mathcal{C}_*\mathcal{Q}_* + O(|\mathcal{C}_*|^4)$$

together with the conjugate equation.

We now apply the center Devaney-Lyapunov theorem to get the following result

Theorem 6.2. *For every* $(A, T, \gamma) \in \Delta_1$ *and for* U *near the origin in* $\mathbb{D}$, *the evolution equation* (5) *reduces to a two-dimensional reversible smooth vector field.*

Moreover, the set of solutions near 0 *of* (5) *constitutes a one-parameter family of periodic orbits, bifurcating from* 0.

As a corollary of the previous theorem, one gets Theorem 1.1.

6.2. **Quasi-periodic solutions of** (5)**.** We are now interested in quasi-periodic motions for equation (5). Let Δ_2 be the set of parameters for which the center spectrum consists of a finite number of pairs of simple eigenvalues $\pm iq_k$ for $k = 1, ..., p$ with $p > 1$. Once again, Section 3 shows that this set is non empty. We have the following reduction theorem where $P = \sum_{k=1}^{p} P_k$ is the total projection with the same notation as in Section 4.

Theorem 6.3. *Fix* $(A_0, T_0, \gamma_0) \in \Delta_2$ *and* $k \geq 1$. *There exists a neighborhood* $\mathcal{U} \times \mathcal{V}$ *of* $(0, A_0, T_0, \gamma_0)$ *in* $\mathbb{D} \times \mathbb{R}^3$ *and a map* $\psi \in C_b^k(\mathbb{D}_c \times \mathbb{R}^3, \mathbb{D}_h)$ *such that the following properties hold for all* $(A, T, \gamma) \in \mathcal{V}$ *(with* $\psi(0, A, T, \gamma) = 0, D\psi(0, A_0, T_0, \gamma_0) = 0$*).*

- *If* $U : \mathbb{R} \to \mathbb{D}$ *solves (5) and* $U(t) \in \mathcal{U}$ $\forall t \in \mathbb{R}$ *then* $U_h(t) = \psi(U_c(t), A, T, \gamma)$ *for all* $t \in \mathbb{R}$ *and* U_c *is a solution of*

$$\frac{dU_c}{dt} = \mathcal{L}_{A_0, T_0, \gamma_0} U_c + P\mathcal{N}_T(U_c + \psi(U_c, A, T, \gamma)). \tag{49}$$

- *If* $U_c : \mathbb{R} \to \mathbb{D}_c$ *is a solution of (49) with* $U_c \in \mathcal{U}_c = P\mathcal{U}$ $\forall t \in \mathbb{R}$, *then* $U = U_c + \psi(U_c, A, T, \gamma)$ *is a solution of (5).*
- *The map* $\psi(., A, T, \gamma)$ *commutes with* R. *Moreover, the reduced system (49) is reversible under* R.

In this case, the normal form (47) writes

$$\frac{d\mathcal{C}_j}{d\xi} \;=\; iq_j\mathcal{C}_j + i\mathcal{C}_j\mathcal{Q}_j + O((\sum_{k=1}^{p} |\mathcal{C}_k|)^4) \quad \text{for } j = 1, ..., p \tag{50}$$

together with the conjugate equations.

Considering the standard (almost-complex) symplectic structure J in $\mathbb{R}^{2p}$ (up to some complexification), it is then easy to see that the previous system (50) inheritates an Hamiltonian structure $H = H_0 + \mathcal{R}$, where H_0 is integrable (in the Arnold-Liouville sense) and $\mathcal{R}$ stands for higher order terms.

In order to apply a KAM theorem to ensure the existence of invariant $p-$*dimensional* tori (we refer to [dlL01] for an account on KAM theorems), one has to check several assumptions. The phase space is foliated by invariant tori for the unperturbed system (i.e. getting rid of high order terms in (50), $\mathcal{R} = 0$), which are given by

$$q_1\mathbb{S}^1 \times \cdots \times q_p\mathbb{S}^1 \times \{a\}^p,$$

provided that $(q_1, ..., q_p)$ is Diophantine in $\mathbb{R}^p$, $a \in \mathbb{R}$ and where $q_j\mathbb{S}^1$ stands for the torus $\mathbb{R}/(q_j\mathbb{Z})$. To be able to apply the standard KAM theorem in our context, one has to check: first, some regularity and smallness assumptions on the underlying Hamiltonian H and second, the twist condition. Our Hamiltonian H is not analytic since it comes from the reduction to a center manifold, which are not analytic manifolds. However, it is a well-known fact in KAM theory that one can overcome this difficulty by introducing suitable smoothing operators (see for instance [dlL01]). The smallness assumption amounts to control the remainder $\mathcal{R}$ in a suitable norm. This can be done by shrinking a little bit the amplitude of our solutions. The most intricate assumption to check is the twist condition. In our context, according to the form of the Hamiltonian H this amounts to check the invertibility of a matrix involving the coefficients $\tilde{q}_{l,j}$ of the normal form (47).

We are not going to state a result since it involves lengthy computations but one can say that generically, there exists an open set of parameters such that system (5) admits invariant tori of dimension p.

6.3. **Homoclinic solutions of** (5)**.** We now concentrate on homoclinic solutions (possibly to 0) for system (5). In this case, these solutions for the truncated normal form for $C_j = 0$ $(j = 1, ..., p)$ are given by, provided $s_0(A_0, T_0, \gamma_0) < 0$ and $s_0'(A, T, \gamma) > 0$

$$\mathcal{A}(\xi) = r_0(\xi)e^{i(q_0\xi+\psi(\xi)+\theta)}, \quad \mathcal{B}(\xi) = r_1(\xi)e^{i(q_0\xi+\psi(\xi)+\theta)}, \tag{51}$$

where

$$\begin{aligned} r_0(\xi) &= \sqrt{\tfrac{2s_0}{-s_0'}}\operatorname{sech}(\xi\sqrt{s_0}) \\ r_1(\xi) &= \frac{dr_0(\xi)}{d\xi} \\ \psi(\xi) &= p_0\xi - \frac{2p_0'\sqrt{s_0}}{s_0'}\tanh(\xi\sqrt{s_0'}) \end{aligned} \tag{52}$$

This orbits are reversible under R if one chooses θ equal to 0 or π. We first state the reduction theorem. Let Δ_0 be the set of parameters introduced in Section 3. We have

Theorem 6.4. *Fix* $(A_0, T_0, \gamma_0) \in \Delta_0$ *and* $k \geq 1$. *There exists a neighborhood* $\mathcal{U}\times\mathcal{V}$ *of* $(0, A_0, T_0, \gamma_0)$ *in* $\mathbb{D}\times\mathbb{R}^3$ *and a map* $\psi \in C_b^k(\mathbb{D}_c\times\mathbb{R}^3, \mathbb{D}_h)$ *such that the following properties hold for all* $(A, T, \gamma) \in \mathcal{V}$ *(with* $\psi(0, A, T, \gamma) = 0, D\psi(0, A_0, T_0, \gamma_0) = 0$*).*

- *If* $U : \mathbb{R} \to \mathbb{D}$ *solves (5) and* $U(t) \in \mathcal{U}$ $\forall t \in \mathbb{R}$ *then* $U_h(t) = \psi(U_c(t), A, T, \gamma)$ *for all* $t \in \mathbb{R}$ *and* U_c *is a solution of*

$$\frac{dU_c}{dt} = \mathcal{L}_{A_0,T_0,\gamma_0}U_c + P\mathcal{N}_T(U_c + \psi(U_c, A, T, \gamma)). \tag{53}$$

- *If* $U_c : \mathbb{R} \to \mathbb{D}_c$ *is a solution of (53) with* $U_c \in \mathcal{U}_c = P\mathcal{U}$ $\forall t \in \mathbb{R}$, *then* $U = U_c + \psi(U_c, A, T, \gamma)$ *is a solution of (5).*
- *The map* $\psi(., A, T, \gamma)$ *commutes with* R. *Moreover, the reduced system (53) is reversible under* R.

On the center manifold, the normal form is given by Lemma 5.1. The question is now to study the persistence of the solution given by equations (51). Several regimes of parameters have to be considered.

We choose $(A,T,\gamma) \approx (A_0,T_0,\gamma_0)$ ($(A_0,T_0,\gamma_0) \in \Delta_0$), in such a way that the linearized operator $\mathcal{L}_{A,T,\gamma}$ has four symmetric eigenvalues close to $\pm iq_0$ and having non zero real parts ($s_0'(A,T,\gamma) > 0$). We will assume that (A,T,γ) is such that Δ_0 contains only one pair of eigenvalues (see Section 3) and that $s_0(A,T,\gamma) < 0$. Roughly speaking, (A,T,γ) is close to the *first* tongue in the parameter plane (see Figure 1). We are now in position to apply the results by Lombardi (see [Lom00], [IL05]): homoclinic solutions to 0 do not persist for the full normal form (47). Instead, solutions homoclinic to exponentially small periodic orbits whose size is of order $O(e^{-C/d^{1/2}})$ where d is the distance from (A,T,γ) to the bifurcation surface.

This allows to get the following theorem

Theorem 6.5. *Let* (A,T,γ) *be close to* $(A_0,T_0,\gamma_0) \in \Delta_0$ *(except of some exceptional points corresponding for instance to strong resonances) such that the center spectrum consists of one pair of double non semi-simple eigenvalues and one pair of simple eigenvalues.*

Then the evolution equation (5) *is reducible to a 6 dimensional vector-field for* U *close to the origin in* $\mathbb{D}$ *and there are travelling waves which are the superposition of a localized wave with an exponentially small periodic tail provided that the coefficient depending only on constants of the problem satisfies* $s_0(A_0,T_0,\gamma_0) < 0$.

The previous theorem implies Theorem 1.2. In the previous situation where the center spectrum consists of one pair of double non semi-simple and only one pair of simple eigenvalues, one can give a complete result thanks to Lombardi's results. However, in our case (see Section 3), even at low coupling A there are some regions in the parameter plane for which the center spectrum consists of one pair of double non semi-simple eigenvalue and *several* pairs of simple eigenvalues. In this context, a complete theory is not yet available. The structure of the possible oscillatory tails is more involved in this case.

As in [JS05, Sir05], one can conjecture the persistence of homoclinic connections to higher dimensional tori.

7. Formal Multiscale expansions

This section is devoted to a multi-scale expansion yielding a continuous approximation of the lattice equation (1). This type of analysis has been done by Remoissenet (from a formal point of view) in [Rem86] and justified in [GM04] in the case $A = 0$.

In this section, we use the approach of Remoissenet to get our continuous approximation for $A > 0$. However, it could be justified rigourously using the methods developped in [GM04]. The whole point consists in searching for a solution of (1) in the form of modulated plane waves. We write a solution of (1) as

$$x_n(t) = F_{1n}(t)e^{i\theta_n} + c.c. + \varepsilon\left[F_{0n}(t) + F_{2n}(t)e^{2i\theta_n} + c.c.\right] \tag{54}$$

with $\theta_n = q_0(n - t/T_0)$. We use a continuum approximation for the functions F_{jn} (replacing $F_{jn}(t)$ by $F_j(t,x)$). Making use of a multi-scale expansion as in [Rem86], we end up with the following nonlinear Schrödinger equation (NLS) given by

$$iF_{1s} + PF_{1ZZ} + Q|F_1|^2F_1 = 0, \tag{55}$$

where s, Z are slow variables (s stands for a time variable and Z for a spatial variable). The previous equation (55) is obtained by plugging (54) into (1) and identifying up to order ε^2 and the coefficients P and Q are given by the formulas

$$P = \frac{\gamma_0 T_0}{q_0}\left(\cos(q_0) + 4A\cos(2q_0) - \frac{\gamma_0 T_0^2}{q_0^2}(\sin(q_0) + 2A\sin(2q_0))^2\right) \tag{56}$$

and

$$Q = \frac{T_0}{2q_0}\left(4a^2 - \frac{2a^2}{3 + 16\gamma_0(\sin^4(q_0/2) + A\sin^4(q_0))} + 3b\right). \tag{57}$$

Now, using the dispersion relations (13) and (14), we notice that

$$\begin{aligned} &T_0^2(3 + 16\gamma_0(\sin^4(q_0/2) + A\sin^4(q_0))) \\ &\quad= T_0^2(-1 - 2\gamma_0(1 + A_0)) + 4q_0^2 + 2\gamma_0 T_0^2\cos(2q_0) + 2A\gamma_0 T_0^2\cos(4q_0) \end{aligned} \tag{58}$$

and

$$\begin{aligned} &\gamma_0 T_0^2(\cos(q_0) + 4A_0\cos(2q_0) - \frac{\gamma_0 T_0^2}{q_0^2}(\sin(q_0) + 2A\sin(2q_0))^2) \\ &\qquad\qquad = (\gamma_0 T_0^2(\cos(q_0) + 4A_0\cos(q_0)) - 1). \end{aligned} \tag{59}$$

So, we can express P and Q in the following way

$$P = \frac{1}{q_0 T_0}\left(\gamma_0 T_0^2(\cos(q_0) + 4A_0\cos(q_0)) - 1\right) \tag{60}$$

and

$$Q = \frac{T_0}{2q_0}\left(4a^2 + 2aK_1 + 3b\right) \tag{61}$$

where K_1 is given by the expression (45).

The existence of solitons in NLS for the specific wave number $\frac{q_0}{2}$ relies on the condition $PQ > 0$. Using the expressions (60) and (61), one can see that this condition is equivalent to have $s_0 < 0$ where s_0 is given by (46).

It has to be noticed and this is an important feature of the model, that the previous computations work for $A > 0$. The case $A < 0$ has also a physical meaning. It corresponds to anti-ferromagnetic interactions. In this case, an easy computation shows that the continuous approximation is no longer NLS but an elliptic equation involving a 4th derivative operator. We believe that the case $A < 0$ deserves more investigation and we postpone it to future work.

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Appendix: reduction to a center manifold theorem

We recall here the theorem we use to reduce to a center manifold. We state it as it is written in [VI92].

Let X, Y and Z be three Banach spaces with X continuously embedded in Y and Y continuously embedded in Z. Let $A \in \mathcal{L}(X, Z)$ and $g \in C^k(X, Y)$ for some $k \geq 1$. We consider the following differential equation

$$\dot{x} = Ax + g(x). \tag{62}$$

Before stating the theorem, we introduce some definitions and notations. Let E and F be two Banach spaces, $V \subset E$ an open subset, $k \in \mathbb{N}$ and $\eta \geq 0$. Then we define

$$C_b^k = \left\{ w \in C^k(V, F) \mid |w|_{j,V} = \sup_{x \in V} \|D^j w(x)\| < \infty,\, 0 \leq j \leq k \right\}$$

and

$$BC^\eta(\mathbb{R}, E) = \left\{ w \in C^0(\mathbb{R}, E) \mid \|w\|_\eta = \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|w(x)\|_E < \infty \right\}.$$

Theorem 7.1. *Assume that*

- *The function $g \in C^k(X, Y)$ for some $k \geq 1$ and $g(0) = 0$, $Dg(0) = 0$.*

- *There exists a continuous projection* $\pi_c \in \mathcal{L}(Z, X)$ *onto a finite-dimensional subspace* $Z_c = X_c \subset X$ *such that for all* $x \in X$
$$A\pi_c x = \pi_c A x$$
and such that if we set
$$Z_h = (I - \pi_c)Z, \;\; X_h = (I - \pi_c)X, \;\; Y_h = (I - \pi_c)Y,$$
$$A_c = A|_{X_c} \in \mathcal{L}(X_c), \;\; A_h = A|_{X_h} \in \mathcal{L}(X_h, Z_h),$$
then the following holds:
  1. *The spectrum of* A_c *is on the imaginary axis*
  2. *There exists some* $\beta > 0$ *such that for each* $\eta \in [0, \beta)$ *and for each* $f \in BC^\eta(\mathbb{R}, Y_h)$ *the affine problem*
$$\dot{x}_h = A_h x_h + f(t)$$
has a unique solution $x_h = K_h f$ *where*
$$K_h \in \mathcal{L}(BC^\eta(\mathbb{R}, Y_h), BC^\eta(\mathbb{R}, X_h))$$
for each $\eta \in [0, \beta)$ *and*
$$\|K_h\|_\eta \le \gamma(\eta)$$
for all $\eta \in [0, \beta)$ *and some continuous curve* γ*.*

Then there exists a neighborhood Ω *of the origin in* X *and a mapping* $\psi \in C_b^k(X_c, X_h)$ *with* $\psi(0) = 0$ *and* $D\psi(0) = 0$ *such that*
$$\mathcal{M} = \{x_c + \psi(x_c) \,|\, x_c \in X_c\}$$
is a local center manifold for (62)*.*

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