A WEIGHTED MOSER-TRUDINGER INEQUALITY AND ITS RELATION TO THE CAFFARELLI-KOHN-NIRENBERG INEQUALITIES IN TWO SPACE DIMENSIONS

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Abstract. We first prove a weighted inequality of Moser-Trudinger type depending on a parameter, in the two-dimensional Euclidean space. The inequality holds for radial functions if the parameter is larger than -1. Without symmetry assumption, it holds if and only if the parameter is in the interval $(-1,0]$.

The inequality gives us some insight on the symmetry breaking phenomenon for the extremal functions of the Hardy-Sobolev inequality, as established by Caffarelli-Kohn-Nirenberg, in two space dimensions. In fact, for suitable sets of parameters (asymptotically sharp) we prove symmetry or symmetry breaking by means of a blow-up method. In this way, the weighted Moser-Trudinger inequality appears as a limit case of the Hardy-Sobolev inequality.

1. Introduction

By Onofri's inequality on the sphere S^2, see for instance [1], we have

$$\int_{S^2} e^{2\,u-2\int_{S^2} u\,d\sigma}\,d\sigma \;\le\; e^{\|\nabla u\|^2_{L^2(S^2,d\sigma)}}\;, \tag{1}$$

for all $u \in \mathcal{E} = \{u \in L^1(S^2,d\sigma)\;:\;|\nabla u| \in L^2(S^2,d\sigma)\}$, where $d\sigma$ denotes the measure induced by Lebesgue's measure on $\mathbb{R}^3 \supset S^2$, normalized so that $\int_{S^2} d\sigma = 1$. Using the stereographic projection from S^2 onto $\mathbb{R}^2$, we see that (1) is equivalent to the following Moser-Trudinger inequality on $\mathbb{R}^2$:

$$\int_{\mathbb{R}^2} e^{v-\int_{\mathbb{R}^2} v\,d\mu}\,d\mu \;\le\; e^{\frac{1}{16\,\pi}\,\|\nabla v\|^2_{L^2(\mathbb{R}^2,dx)}}\;,$$

for all $v \in \mathcal{D} = \{v \in L^1(\mathbb{R}^2,d\mu)\;:\;|\nabla v| \in L^2(\mathbb{R}^2,dx)\}$ where $d\mu$ denotes the probability measure

$$d\mu = \frac{dx}{\pi\,(1+|x|^2)^2}\;.$$

In this paper, we first generalize the above Moser-Trudinger inequality to the family of probability measures

$$d\mu_\alpha = \frac{\alpha+1}{\pi}\,\frac{|x|^{2\alpha}\,dx}{(1+|x|^{2\,(\alpha+1)})^2}\;,$$

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for $\alpha > -1$, and investigate when the weighted inequality

$$\int_{\mathbb{R}^2} e^{v - \int_{\mathbb{R}^2} v \, d\mu_\alpha} \, d\mu_\alpha \leq e^{\frac{1}{16\,\pi\,(\alpha+1)} \, \|\nabla v\|^2_{L^2(\mathbb{R}^2,\, dx)}} \,, \tag{2}$$

holds for all v in the space

$$\mathcal{E}_\alpha = \Big\{ v \in L^1(\mathbb{R}^2, d\mu_\alpha) \;:\; |\nabla v| \in L^2(\mathbb{R}^2, dx) \Big\} \,.$$

In section 2 we prove that (2) always holds for functions in $\mathcal{E}_\alpha$ which are radially symmetric about the origin. Meanwhile, without symmetry assumption inequality (2) holds in $\mathcal{E}_\alpha$ if and only if $\alpha \in (-1, 0]$.

We use the above information to investigate possible symmetry breaking phenomena for extremal functions of the weighted Hardy-Sobolev inequality as established by Caffarelli-Kohn-Nirenberg (see [3]), in two space dimensions :

$$\begin{gathered} \left(\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^{bp}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{|x|^{2a}} \, dx \quad \forall \, u \in \mathcal{D}_{a,b} \,, \\ \text{with } a < b \leq a+1 \,, \quad p = \frac{2}{b-a} \,, \\ \mathcal{D}_{a,b} = \{ |x|^{-b} \, u \in L^p(\mathbb{R}^2, dx) \;:\; |x|^{-a} \, |\nabla u| \in L^2(\mathbb{R}^2, dx) \} \,, \end{gathered} \tag{3}$$

and an optimal constant $C_{a,b}$. Typically (3) is stated with $a < 0$ (see [3]) so that the space $\mathcal{D}_{a,b}$ can be seen as the completion of the space $C_c^\infty(\mathbb{R}^2)$ of all smooth functions on $\mathbb{R}^2$ with compact support, with respect to the norm $\|u\|^2 = \|\, |x|^{-b} \, u \,\|_p^2 + \|\, |x|^{-a} \, \nabla u \,\|_2^2$. Actually (3) holds also for $a > 0$ (see section 2), but in this case $\mathcal{D}_{a,b}$ is obtained as the completion with respect to $\| \cdot \|$ of the set $\{ u \in C_c^\infty(\mathbb{R}^2) \;:\; \mathrm{supp}(u) \subset \mathbb{R}^2 \setminus \{0\} \}$. We know that for $b = a+1$, the best constant in (3) is given by $C_{a,\, b=a+1} = a^2$ and it is never achieved (see [4, Theorem 1.1, (ii)]). On the contrary, for $a < b < a+1$, the best constant in (3) is always achieved, say at some function $u_{a,b} \in \mathcal{D}_{a,b}$ that we will call an *extremal function,* but its value is not explicitly known unless we have the additional information that $u_{a,b}$ is radially symmetric about the origin. In fact, in the class of positive radially symmetric functions, the extremals of (3) are explicitly known (see [6, 4]) and given by a multiplication by a non-zero constant and a dilation of the function

$$u_{a,b}^{\mathrm{rad}}(x) = \left(1 + |x|^{-\frac{2a\,(1+a-b)}{b-a}} \right)^{-\frac{b-a}{1+a-b}} \,. \tag{4}$$

See [4] for more details on existence and non-existence results and for a "modified inversion symmetry" property based on a generalized Kelvin transformation. Also we refer to [13, 12, 11] for further partial symmetry results about extremal functions. On the other hand, equality is achieved by non-radially symmetric extremals for a certain range of parameters (a, b) identified first in [4] and subsequently improved in [9]. In fact those results provide a rather satisfactory information about the symmetry breaking phenomenon for $u_{a,b}$ when $|a|$ is sufficiently large and also apply to any dimension $N \geq 3$, where inequality (3) reads as follows:

$$\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{2/p} \le C^N_{a,b}\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\ , \quad \forall\, u\in \mathcal{D}^N_{a,b}\ , \tag{5}$$

with $p = \frac{2\,N}{(N-2)+2\,(b-a)}$, $\mathcal{D}^N_{a,b} = \{|x|^{-b}\,u \in L^p(\mathbb{R}^N, dx) \ : \ |x|^{-a}\,|\nabla u| \in L^2(\mathbb{R}^N, dx)\}$, an optimal constant $C^N_{a,b}$, and $a, b \in \mathbb{R}$ such that $a < (N-2)/2$, $a \le b \le a+1$. Again we observe that inequality (5) makes sense also if $a > (N-2)/2$ and $a \le b \le a+1$, provided the functions are in the space $\mathcal{D}^N_{a,b}$ given by the completion with respect to $\|\cdot\|$ of the set $\{u \in C^\infty_c(\mathbb{R}^2) \,:\, \mathrm{supp}(u) \subset \mathbb{R}^2 \setminus \{0\}\}$.

For $N \ge 3$ and $0 \le a < (N-2)/2$, the extremal $u_{a,b}$ of (5) (which again exists for every $a < b < a+1$) is always radially symmetric (see [6], and for a survey on previous results see [4]). On the other hand, when $a < 0$, this is ensured only in some special cases described in [12, 11]. Also see [13, Theorem 4.8] for an earlier but slightly less general result.

In this paper, we focus on the less investigated bidimensional case $N = 2$, and besides symmetry breaking phenomena, we explore the possibility of ensuring radial symmetry (which cannot be studied as in [13, 12, 11]) for the extremal $u_{a,b}$ according to an admissible range of parameters (a, b) (see in particular [13, Remark 4.9]).

To this purpose we check in section 2.2 that (3) (or more generally, (5)) holds for all $a \ne 0$ (or $a \ne (N-2)/2$ if $N \ge 3$) and not only for $a < 0$ (or $a < (N-2)/2$) as it is usually found in literature. In this way we can analyze radial symmetry of the extremal $u_{a,b}$ of (3), in the range $a \ne 0$ and for all $b \in (a, a+1)$. We find that if $N = 2$, $a \ne 0$, $b \in (a, h(a))$, with

$$h(a) = a + \frac{|a|}{\sqrt{1+a^2}}\ ,$$

no extremal $u_{a,b}$ for (3) is radially symmetric. This result is inspired by [9], and it is even stated without proof for $a < 0$ in [12, 11]. Since as $|a| \to +\infty$,

$$0 < a+1-h(a) \to 0\ ,$$

it is reasonable to look for radially symmetric extremals when $|a|$ is small. Indeed, we will show that, if $a \to 0_+$, then $h'_+(0) = 2$ (or if $a \to 0_-$, then $h'_-(0) = 0$) gives the "sharp" slope of the ratio b/a that signs the transition between radial symmetry and symmetry breaking. That is, we identify two regions in the set of parameters a and b relative to which $u_{a,b}$ is radially symmetric, or not. The precise statement of our result is as follows (also see Figure 1 below).

Theorem 1. *Let* $a \ne 0$ *and* $N = 2$.

- (i) *If* $a < b < h(a) = a + \frac{|a|}{\sqrt{1+a^2}}$, *then* (3) *admits only* non radially symmetric *extremals.*
- (ii) *For every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $|a| \in (0, \delta)$, $b \in (a, a+1)$ *and either* $b/a > 2 + \varepsilon$ *if* $a > 0$, *or* $b/a < -\varepsilon$ *if* $a < 0$, *then the extremals of* (3) *are* radially symmetric, *and given by a multiplication by a non-zero constant and a dilation of the function* $u^{\mathrm{rad}}_{a,b}$ *defined in* (4).

As a consequence of (i), we can contrast (ii) with the following statement:

(i') *For every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $|a| \in (0, \delta)$*,* $b \in (a, a+1)$ *and either* $b/a < 2 - \varepsilon$ *if* $a > 0$*, or* $b/a > \varepsilon$ *if* $a < 0$*, then any extremal of* (3) is not radially symmetric.

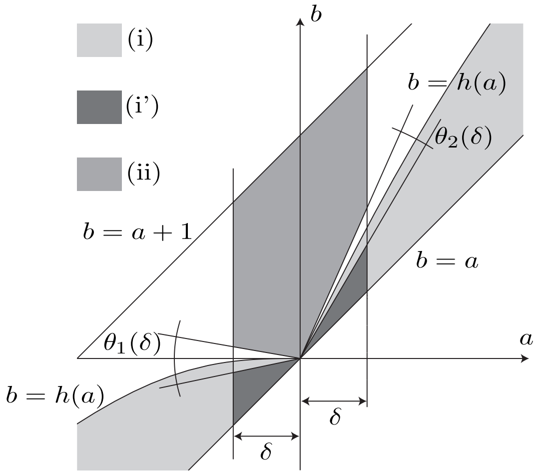

Figure 1. *Radial symmetry occurs in Region* (ii). *Optimal functions are not radially symmetric in Region* (i), *and in particular in Region* (i'). *The angles* $\theta_1(\delta)$ *and* $\theta_2(\delta)$ *are such that* $\lim_{\delta \to 0_+} \theta_k(\delta) = 0$ *for* $k = 1, 2$.

We will first prove (i') as a consequence of the weighted Moser-Trudinger inequality (2). We emphasize that such an approach makes no use of the linearized problem around the radial solution (4) and could be helpful in other contexts. To prove the more complete result stated in (i), we use the Emden-Fowler transformation in order to formulate (3) (or more generally (5)) as the Sobolev inequality on the cylinder $\mathbb{R} \times S^1$ (or more generally $\mathbb{R} \times S^{N-1}$). In this way we can analyze the linearized elliptic problem around the solution corresponding to (4) and see in which case it yields to a "local" minimizer. We shall obtain precise informations about the linearized problem in section 3. This will lead us directly to the proof of (i), and will be used also to handle part (ii) of Theorem 1 via a blow-up analysis.

In concluding we wish to bring the reader's attention to a weighted Moser-Trudinger inequality on the cylinder $\mathbb{R} \times S^1$ (see Proposition 23 in section 5). We believe that it helps to illustrate the nature of the symmetry breaking phenomenon analyzed here.

2. A weighted Moser-Trudinger inequality and its connection with the weighted Hardy-Sobolev inequality

Consider the measure μ_α and the Banach space $\mathcal{E}_\alpha$, $\alpha > -1$, defined in section 1. Here and from now on, $\|v\|_2$ means $\|v\|_{L^2(\mathbb{R}^2, dx)}$.

2.1. **A weighted Moser-Trudinger inequality on $\mathbb{R}^2$.**

Proposition 2. *Let $\alpha > -1$. For all $v \in \mathcal{E}_\alpha$, there holds*

$$\int_{\mathbb{R}^2} e^{v - \int_{\mathbb{R}^2} v \, d\mu_\alpha} \, d\mu_\alpha \;\le\; e^{\frac{1}{16\,\pi\,(\alpha+1)}\left(\|\nabla v\|_2^2 + \alpha\,(\alpha+2)\,\|\,\frac{1}{r}\,\partial_\theta v\,\|_2^2\right)} \,. \tag{6}$$

Proof. We use polar coordinates in $\mathbb{R}^2 \approx \mathbb{C}$. For $x \in \mathbb{R}^2$, we let $x = r\,e^{i\theta}$, $r \ge 0$, $\theta \in [0, 2\pi)$. We also consider cylindrical coordinates in $\mathbb{R}^3$, so that for $(y, z) \in \mathbb{R}^2 \times \mathbb{R}$, we let $y = \rho\,e^{i\theta}$, $\rho \ge 0$, $\theta \in [0, 2\pi)$ and $z \in \mathbb{R}$. In this way, we can write $\mathbb{R}^3 \supset S^2 = \{(\rho\,e^{i\theta}, z) \,:\, \rho^2 + z^2 = 1 \text{ and } \theta \in [0, 2\pi)\}$. We recall that the inverse Σ_0 of the usual stereographic projection from S^2 onto $\mathbb{R}^2$ is defined by

$$\Sigma_0\big(r\,e^{i\theta}\big) = (\rho\,e^{i\theta}, z) = \left(\frac{2\,r\,e^{i\theta}}{1 + r^2}\,,\ \frac{r^2 - 1}{1 + r^2}\right) .$$

If u is defined on S^2, then $v = u \circ \Sigma_0$ is defined on $\mathbb{R}^2$ and for any continuous real function f on $\mathbb{R}$, we have

$$\pi \int_{S^2} f(u)\,d\sigma = \int_{\mathbb{R}^2} \frac{f(v)}{(1 + |x|^2)^2}\,dx \quad \text{and} \quad 4\pi \int_{S^2} |\nabla u|^2\,d\sigma = \int_{\mathbb{R}^2} |\nabla v|^2\,dx$$

whenever $f(u)$ and $|\nabla u|^2$ belong to $L^1(S^2)$.

In order to prove the proposition, we are going to use the inverse of a *dilated stereographic projection* given for all $\alpha > -1$ by the function $\Sigma_\alpha : \mathbb{R}^2 \to S^2$ such that

$$\Sigma_\alpha\big(r\,e^{i\theta}\big) = \left(\frac{2\,r^{\alpha+1}\,e^{i\theta}}{1 + r^{2(\alpha+1)}}\,,\ \frac{r^{2(\alpha+1)} - 1}{1 + r^{2(\alpha+1)}}\right) .$$

Note that for any $r \ge 0$, $\theta \in [0, 2\pi)$, $\Sigma_\alpha(r\,e^{i\theta}) = \Sigma_0(r^{1+\alpha}\,e^{i\theta})$ and, for any $\rho \ge 0$, $\theta \in [0, 2\pi)$ and $z \in [-1, 1]$,

$$\Sigma_\alpha^{-1}\big((\rho\,e^{i\theta}, z)\big) = \big(\tfrac{\rho}{1-z}\big)^{1/(\alpha+1)} e^{i\theta} \,.$$

Now, if f is a continuous real function on $\mathbb{R}$, $f(u)$, $|\nabla u|^2 \in L^1(S^2)$ and $v = u \circ \Sigma_\alpha$, then an elementary computation (see the Appendix) shows that

$$\begin{aligned} &\int_{S^2} f(u)\,d\sigma = \int_{\mathbb{R}^2} f(v)\,d\mu_\alpha \,, \\ &4\pi \int_{S^2} |\nabla u|^2\,d\sigma = \frac{1}{\alpha+1} \int_{\mathbb{R}^2} \left(|\nabla v|^2 + \alpha\,(\alpha+2)\left|\frac{1}{r}\,\partial_\theta v\right|^2\right) dx \,. \end{aligned}$$

The result follows from Onofri's inequality (1). □

Notice that we will recover Onofri's inequality as a consequence of Proposition 7 and the symmetry result of Theorem 1, (ii). See Remark 8 for details.

Corollary 3. *If $\alpha \in (-1, 0]$, then (2) holds true for any $v \in \mathcal{E}_\alpha$.*

Proof. It is an immediate consequence of Proposition 2 since for $\alpha \in (-1, 0]$, we have $\alpha\,(\alpha + 2) \le 0$. □

This result is optimal. While (2) remains valid for all $\alpha > -1$ among radially symmetric functions (about the origin), it fails in $\mathcal{E}_\alpha$ for $\alpha > 0$:

Proposition 4. *If $\alpha > 0$, then inequality* (2) *fails to hold in $\mathcal{E}_\alpha$.*

Proof. Let us exhibit a counter-example to (2), which is valid for all $\alpha > 0$. For any $\varepsilon \in (0,1)$, let us consider the function $v_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$2\,v_\varepsilon = \begin{cases} \log\left(\frac{\varepsilon}{(\varepsilon+\pi\,|x-\bar{x}|^2)^2}\right) & \text{if} \quad |x-\bar{x}| \le 1 \\ \log\left(\frac{\varepsilon}{(\varepsilon+\pi)^2}\right) & \text{if} \quad |x-\bar{x}| > 1 \end{cases}$$

where $\bar{x}$ denotes the point $(1,0)$. For this function we can calculate the various terms of (2).

First we compute the l.h.s., and see that

$$\mu_\alpha(e^{2v_\varepsilon}) = \int_{\mathbb{R}^2} e^{2v_\varepsilon}\;d\mu_\alpha = I_{\alpha,\varepsilon} + A_\alpha\,\frac{\varepsilon}{(\varepsilon+\pi)^2}$$

where

$$I_{\alpha,\varepsilon} = \frac{1}{\varepsilon}\int_{|x-\bar{x}|<1} \frac{1}{\left(1+\pi\,|\frac{x-\bar{x}}{\sqrt{\varepsilon}}|^2\right)^2}\;d\mu_\alpha$$

and $A_\alpha = \int_{|x-\bar{x}|>1} d\mu_\alpha$ is finite for all $\alpha > -1$. Now, by the change of variables $x = \bar{x} + \sqrt{\varepsilon}\,y$ and dominated convergence, we find

$$\lim_{\varepsilon\to 0}\int_{|y|<1} \frac{|\bar{x}+\sqrt{\varepsilon}\,y|^{2\alpha}}{\left(1+|\bar{x}+\sqrt{\varepsilon}\,y|^{2(\alpha+1)}\right)^2(1+\pi\,|y|^2)^2}\;dy \;=\; \frac{1}{4}\int_{\mathbb{R}^2}\frac{dy}{(1+\pi\,|y|^2)^2}\;,$$

So, for the function v_ε, the l.h.s. of (2) satisfies

$$\lim_{\varepsilon\to 0}\mu_\alpha(e^{2v_\varepsilon}) = \lim_{\varepsilon\to 0} I_{\alpha,\varepsilon} = \frac{\alpha+1}{4\pi}\;.$$

Next we compute the r.h.s. of (2), that is $\frac{1}{4\pi\,(\alpha+1)}\,\|\nabla v_\varepsilon\|_2^2 + 2\,\mu_\alpha(v_\varepsilon)$ and see that

$$\|\nabla v_\varepsilon\|_2^2 = 4\,\pi\,\log\left(\frac{\varepsilon+\pi}{\varepsilon}\right) - \frac{4\,\pi^2}{(\varepsilon+\pi)}$$

and

$$2\,\mu_\alpha(v_\varepsilon) = J_{\alpha,\varepsilon} + A_\alpha\,\log\frac{\varepsilon}{(\varepsilon+\pi)^2}\;,$$

where

$$J_{\alpha,\varepsilon} = \int_{|x-\bar{x}|<1}\log\left(\frac{\varepsilon}{(\varepsilon+\pi\,|x-\bar{x}|^2)^2}\right)\,d\mu_\alpha\;.$$

Using $A_\alpha = 1 - \int_{|x-\bar{x}|<1} d\mu_\alpha$, we get

$$2\,\mu_\alpha(v_\varepsilon) = \log\frac{\varepsilon}{(\varepsilon+\pi)^2} + B_{\alpha,\varepsilon}\;,\quad B_{\alpha,\varepsilon} = \int_{|x-\bar{x}|<1}\log\left(\frac{\varepsilon+\pi}{\varepsilon+\pi\,|x-\bar{x}|^2}\right)^2 d\mu_\alpha\;,$$

$$\lim_{\varepsilon\to 0} B_{\alpha,\varepsilon} = \int_{|x-\bar{x}|<1}\log\left(\frac{1}{|x-\bar{x}|^4}\right)\,d\mu_\alpha\;.$$

Hence

$$\frac{1}{4\pi\,(\alpha+1)}\,\|\nabla v_\varepsilon\|_2^2 + 2\,\mu_\alpha(v_\varepsilon) = \frac{\alpha}{1+\alpha}\log\varepsilon + O(1) \quad \text{as} \quad \varepsilon \to 0\;,$$

and comparing with the estimate above, we violate (2) for $\varepsilon > 0$ small enough. □

2.2. The weighted Hardy-Sobolev inequality.

The range in which inequalities (3) and (5) are usually considered can be extended as follows.

Lemma 5. *If $N = 2$, then inequality (3) holds for any $a \neq 0$ and b such that $a < b \leq a+1$. If $N \geq 3$, then inequality (5) holds for any $a \neq (N-2)/2$ and b such that $a \leq b \leq a+1$.*

Proof. We use Kelvin's transformation and deal with the case $N = 2$. If $u \in \mathcal{D}_{a,b}$, then $v(x) = u\left(x/|x|^2\right)$ is such that $|x|^a\,|\nabla v| \in L^2(\mathbb{R}^2, dx)$. Hence, for $a > 0$, $b \in (a, a+1]$, define $a' = -a$, $b' = b - 2a \in (-a, -a+1]$ and apply (3) to the pair (a', b') with $p = 2/(b' - a')$ to obtain

$$\int_{\mathbb{R}^2} \left(\frac{|v|^p}{|x|^{b'p}}\,dx\right)^{2/p} \leq C_{a',b'} \int_{\mathbb{R}^2} \frac{|\nabla v|^2}{|x|^{2a'}}\,dx \quad \text{in } \mathcal{D}_{a',b'}\ .$$

Now, we make the change of variables $y = x/|x|^2$ and get

$$\int_{\mathbb{R}^2} \left(\frac{|u|^p}{|y|^{4-b'p}}\,dy\right)^{2/p} \leq C_{a',b'} \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{|y|^{-2a'}}\,dy \quad \text{in } \mathcal{D}_{a,b}\ .$$

Thus we arrive at the desired conclusion with $C_{a,b} = C_{a',b'}$, since

$$4 - b'p = bp\ , \quad -2a' = 2a \quad \text{and} \quad p = 2/(b' - a') = 2/(b - a)\ .$$

Similarly in dimension $N \geq 3$, argue as above with $a = N - 2 - a'$, $b\,p = 2N - b'\,p$ and $p = 2N/(N - 2 - 2(b' - a')) = 2N/(N - 2 - 2(b - a))$. □

Surprisingly, the case $a > 0$ if $N = 2$, or $a > (N-2)/2$ if $N \geq 3$, has apparently never been considered. According to our argument, it requires to define with care the space $\mathcal{D}_{a,b}$. Indeed if a function $u \in C^\infty_c(\mathbb{R}^N) \cap \mathcal{D}_{a,b}$ for $a > (N-2)/2$, $N \geq 2$, then u must satisfy $u(0) = 0$. Although optimal functions for inequality (5), $a > (N-2)/2$, $N \geq 2$, have not been studied, it has been noted in [4, Theorem 1.4] that whenever $u > 0$ satisfies the corresponding Euler-Lagrange equations, then, up to a scaling, it satisfies the "modified inversion symmetry" property, that is, there exists $\tau > 0$ such that

$$u(x) = \left|\frac{x}{\tau}\right|^{-(N-2-2a)} u\left(\tau^2\,\frac{x}{|x|^2}\right) \quad \forall\, x \in \mathbb{R}^N\ .$$

The transformation $u \mapsto |x|^{-(N-2-2a)}\ u(x/|x|^2)$ is sometimes called the generalized Kelvin transformation, see *e.g.* [6]. The modified inversion symmetry formula can be shown for an optimal function u using the fact that v given in terms of u as in the proof of Lemma 5 is also an optimal function for inequality (5), with parameters a', b'.

2.3. The Moser-Trudinger inequality as a limit case of the weighted Hardy-Sobolev inequality on $\mathbb{R}^2$.

We now relate inequalities (2) and (3). In this section, we will only consider the case $a < 0$. The case $a > 0$ follows by Lemma 5.

For $N = 2$, $\alpha > -1$, $\varepsilon \in (0,1)$, let us make the following special choice of parameters:

$$a = -\frac{\varepsilon}{1-\varepsilon}\,(\alpha + 1)\ , \quad b = a + \varepsilon \quad \text{and} \quad p = \frac{2}{\varepsilon}\ .$$

Let $u_\varepsilon = u^{\mathrm{rad}}_{a,b}$ be given in (4), that is

$$u_\varepsilon(x) = \Big(1 + |x|^{2(\alpha+1)}\Big)^{-\frac{\varepsilon}{1-\varepsilon}} \;.$$

We consider the functions

$$f_\varepsilon = \left[\frac{u_\varepsilon}{|x|^{a+\varepsilon}}\right]^{2/\varepsilon} \;, \quad g_\varepsilon = \left[\frac{|\nabla u_\varepsilon|}{|x|^a}\right]^2 ,$$

and the integrals

$$\kappa_\varepsilon = \int_{\mathbb{R}^2} f_\varepsilon \; dx \quad \text{and} \quad \lambda_\varepsilon = \int_{\mathbb{R}^2} g_\varepsilon \; dx \;.$$

Straightforward computations show that

$$\kappa_\varepsilon = \int_{\mathbb{R}^2} \frac{|x|^{2\alpha}}{\big(1 + |x|^{2(1+\alpha)}\big)^2} \, \frac{u_\varepsilon^2}{|x|^{2a}} \; dx = \frac{\pi}{\alpha+1} \int_0^\infty \frac{s^{\frac{\varepsilon}{1-\varepsilon}}}{(1+s)^{\frac{2}{1-\varepsilon}}} \; ds \;,$$

$$\lambda_\varepsilon = 4a^2 \int_{\mathbb{R}^2} \frac{|x|^{2(2\alpha+1-a)}}{\big(1+|x|^{2(1+\alpha)}\big)^{\frac{2}{1-\varepsilon}}} \; dx \;.$$

Notice that we can use Euler's Gamma function $\Gamma(x) = \int_0^\infty s^{x-1} \, e^{-s} \, ds$, and on the basis of the well known identity:

$$2 \int_0^\infty s^{2a-1} (1+s^2)^{-b} \; ds = \frac{\Gamma(a)\,\Gamma(b-a)}{\Gamma(b)} \;,$$

deduce for λ_ε the following expression:

$$\lambda_\varepsilon = 4\pi \, |a| \, \frac{\Gamma\big(\frac{2-\varepsilon}{1-\varepsilon}\big)\,\Gamma\big(\frac{1}{1-\varepsilon}\big)}{\Gamma\big(\frac{2}{1-\varepsilon}\big)} \;.$$

Lemma 6. *Let $\alpha_0 > -1$, $v \in C^\infty_c(\mathbb{R}^2)$, $w_\varepsilon = (1 + \varepsilon\, v)\, u_\varepsilon$. With the above notations, we have*

$$\frac{1}{\kappa_\varepsilon} \int_{\mathbb{R}^2} \frac{|w_\varepsilon|^p}{|x|^{bp}} \; dx = \int_{\mathbb{R}^2} |1 + \varepsilon\, v|^{\frac{2}{\varepsilon}} \; \frac{f_\varepsilon \; dx}{\int_{\mathbb{R}^2} f_\varepsilon \; dx}$$

and, as $\varepsilon \to 0$, uniformly with respect to $\alpha \geq \alpha_0$,

$$\int_{\mathbb{R}^2} \frac{|\nabla w_\varepsilon|^2}{|x|^{2a}} \; dx = \lambda_\varepsilon + \varepsilon^2 \left[\tfrac{8(1+\alpha)^2}{(1-\varepsilon)^2} \int_{\mathbb{R}^2} \frac{u_\varepsilon^{2/\varepsilon} \; v}{|x|^{2(a-\alpha)}} \; dx + \int_{\mathbb{R}^2} |\nabla v|^2 \, \frac{u_\varepsilon^2}{|x|^{2a}} \; dx + O(a^2 \varepsilon) \right] .$$

Proof. By definition of g_ε, we can write

$$\int_{\mathbb{R}^2} \frac{|\nabla w_\varepsilon|^2}{|x|^{2a}} \; dx = \lambda_\varepsilon + 2\,\varepsilon \underbrace{\int_{\mathbb{R}^2} \nabla u_\varepsilon \cdot \nabla(u_\varepsilon \, v) \; \frac{dx}{|x|^{2a}}}_{\text{(I)}} + \varepsilon^2 \underbrace{\int_{\mathbb{R}^2} |\nabla(u_\varepsilon \, v)|^2 \; \frac{dx}{|x|^{2a}}}_{\text{(II)}} \;.$$

A simple algebraic computation shows that

$$-\nabla \cdot \left(\frac{\nabla u_\varepsilon}{|x|^{2a}}\right) = \frac{4\,a^2}{\varepsilon} \, u_\varepsilon^{\frac{2}{\varepsilon}-1} |x|^{2(\alpha-a)} \;. \tag{7}$$

Using (7) and an integration by parts, we obtain

$$\text{(I)} = \frac{4\,a^2}{\varepsilon} \int_{\mathbb{R}^2} |x|^{2(\alpha-a)} \, u_\varepsilon^{2/\varepsilon} \, v \; dx \;.$$

As for (II), we expand $|\nabla(u_\varepsilon\, v)|^2$ and write

$$(\mathrm{II}) = \int_{\mathbb{R}^2} \left[v^2\, |\nabla u_\varepsilon|^2 + u_\varepsilon\, \nabla(v^2) \cdot \nabla u_\varepsilon + u_\varepsilon^2\, |\nabla v|^2 \right] \frac{dx}{|x|^{2a}}$$

where the first two terms can be evaluated as above using (7) and an integration by parts. Hence,

$$\int_{\mathbb{R}^2} \left(v^2\, |\nabla u_\varepsilon|^2 + u_\varepsilon \nabla(v^2) \cdot \nabla u_\varepsilon \right) \frac{dx}{|x|^{2a}} = \frac{4\,a^2}{\varepsilon} \int_{\mathbb{R}^2} |x|^{2(\alpha-a)}\, u_\varepsilon^{2/\varepsilon}\, v^2\, dx\ .$$

To complete the proof we just remark that the function $|x|^{2(\alpha-a)} u_\varepsilon^{2/\varepsilon}$ is uniformly bounded for $\alpha \ge \alpha_0 > -1$. □

For a given $\alpha > -1$, we now investigate the limit as $\varepsilon \to 0$. We prove that inequality (2) is a limiting case of inequality (3), whenever (3) admits a radially symmetric extremal for any ε small enough. In such a case, we can write (3) as follows:

$$\frac{1}{\kappa_\varepsilon} \int_{\mathbb{R}^2} \frac{|w|^p}{|x|^{bp}}\, dx \le \left(\frac{1}{\lambda_\varepsilon} \int_{\mathbb{R}^2} \frac{|\nabla w|^2}{|x|^{2a}}\, dx \right)^{1/\varepsilon}\ . \tag{8}$$

Thus, if we take $w = w_\varepsilon = (1+\varepsilon\, v)\, u_\varepsilon$, then we have:

$$\frac{1}{\kappa_\varepsilon} \int_{\mathbb{R}^2} \frac{|w_\varepsilon|^p}{|x|^{bp}}\, dx \le \left(1 + \tfrac{\varepsilon^2}{\lambda_\varepsilon} \left[\tfrac{8(1+\alpha)^2}{(1-\varepsilon)^2} \int_{\mathbb{R}^2} \tfrac{u_\varepsilon^{2/\varepsilon}\, v}{|x|^{2(a-\alpha)}}\, dx + \int_{\mathbb{R}^2} \tfrac{|\nabla v|^2\, u_\varepsilon^2}{|x|^{2a}}\, dx \right] \right)^{1/\varepsilon} + O(a^2 \varepsilon^2)\,.$$

In particular, observe that

$$\frac{|x|^{-bp}\, f_\varepsilon\, dx}{\int_{\mathbb{R}^2} f_\varepsilon\, dx} \sim \frac{\alpha+1}{\pi}\, |x|^{2\alpha}\, u_\varepsilon^{2/\varepsilon}\, dx \sim d\mu_\alpha(x) \quad \text{as} \quad \varepsilon \to 0_+\ .$$

Proposition 7. *Let us fix $\alpha > -1$ and suppose that there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0 such that the radial extremal function u_{ε_n} is also extremal for (3) with $(a, b, p) = (a_n, b_n, p_n)$ specified a follows,*

$$p_n = \frac{2}{\varepsilon_n}\ , \quad a_n = -\frac{\varepsilon_n}{1-\varepsilon_n}\, (\alpha+1)\ , \quad b_n = a_n + \varepsilon_n\ .$$

Then the weighted Moser-Trudinger inequality (2) holds true on $\mathcal{E}_\alpha$.

Proof. As $n \to \infty$, we have

$$\lambda_{\varepsilon_n} = 4\pi\, |a_n| + o(\varepsilon_n)\ , \quad \kappa_{\varepsilon_n} = \frac{\pi}{\alpha+1} + o(1)\ .$$

Using Lebesgue's theorem of dominated convergence repeatedly and Lemma 6, for any $v \in C_c^\infty(\mathbb{R}^2)$ and $w_{\varepsilon_n} = (1+\varepsilon_n\, v)\, u_{\varepsilon_n}$, we have

$$\frac{1}{\kappa_{\varepsilon_n}} \int_{\mathbb{R}^2} \frac{|w_{\varepsilon_n}|^{p_n}}{|x|^{b_n p_n}}\, dx = \int_{\mathbb{R}^2} |1+\varepsilon_n\, v|^{\frac{2}{\varepsilon_n}}\, \frac{f_{\varepsilon_n}\, dx}{\int_{\mathbb{R}^2} f_{\varepsilon_n}\, dx} \to \int_{\mathbb{R}^2} e^{2v}\, d\mu_\alpha\ ,$$

$$\frac{1}{\lambda_{\varepsilon_n}} \int_{\mathbb{R}^2} \frac{|\nabla w_{\varepsilon_n}|^2}{|x|^{2a_n}}\, dx = 1 + \varepsilon_n \left(\int_{\mathbb{R}^2} 2v\, d\mu_\alpha + \frac{1}{4\,(1+\alpha)\,\pi}\, \|\nabla v\|_2^2 \right) + O(\varepsilon_n^2)$$

as $n \to +\infty$. The proposition follows by applying inequality (3) with $(a, b, p) = (a_n, b_n, p_n)$. By density we can finally choose v in the larger space $\mathcal{E}_\alpha$. □

Remark 8. *Incidentally let us note that if we temporarily admit the result* (ii) *of Theorem 1, then we find a sequence of optimal functions as required by Proposition 7. In particular, for $\alpha = 0$, this gives a proof of the Moser-Trudinger inequality on $\mathbb{R}^2$ as a consequence of inequality* (3)*. Using the inverse Σ_0 of the stereographic projection, this also proves Onofri's inequality* (1) *on S^2.*

Let us now consider another asymptotic regime in which $\alpha \to \infty$.

Proposition 9. *If $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$ are two sequences of positive real numbers such that as $n \to +\infty$,*

$$\lim_{n\to+\infty} \varepsilon_n = 0\ , \quad \lim_{n\to+\infty} \alpha_n = +\infty \quad \textit{and} \quad a_n = -\frac{\varepsilon_n}{1-\varepsilon_n}\,(1+\alpha_n) \underset{n\to+\infty}{\longrightarrow} 0_-\ ,$$

then for n large enough, the radially symmetric extremal u_{ε_n} cannot be a global extremal for inequality (3)*.*

Proof. We argue by contradiction and assume that (8) holds with respect to the given choice of parameters. By definition of λ_{ε_n}, κ_{ε_n}, and Lebesgue's theorem of dominated convergence, we know that

$$\lim_{n\to+\infty} \frac{\lambda_{\varepsilon_n}}{|a_n|} = 4\pi \quad \text{and} \quad \lim_{n\to+\infty} (\alpha_n+1)\,\kappa_{\varepsilon_n} = \pi\ .$$

If $v \in C^\infty_c(\mathbb{R}^2)$, then by a direct computation, we find:

$$\begin{aligned}
&(\alpha_n+1)\int_{\mathbb{R}^2} \frac{|u_{\varepsilon_n}(1+\varepsilon_n\,v)|^{p_n}}{|x|^{b_n p_n}}\,dx\\
&= (\alpha_n+1)\int_0^{2\pi}\!\!\int_0^{+\infty} r^{2\frac{\alpha_n+\varepsilon_n}{1-\varepsilon_n}+1}\,\frac{\big(1+\varepsilon_n\,v(r\cos\theta, r\sin\theta)\big)^{2/\varepsilon_n}}{\big(1+r^{2(\alpha_n+1)}\big)^{\frac{2}{1-\varepsilon_n}}}\,dr\,d\theta\\
&= \int_0^{2\pi}\!\!\int_0^{+\infty} \frac{t^{\frac{1+\varepsilon_n}{1-\varepsilon_n}}}{(1+t^2)^{\frac{2}{1-\varepsilon_n}}}\,\big(1+\varepsilon_n\,v(\,t^{\frac{1}{1+\alpha_n}}\cos\theta,\ t^{\frac{1}{1+\alpha_n}}\sin\theta)\big)^{2/\varepsilon_n}\,dt\,d\theta\,.
\end{aligned}$$

We pass to the limit as $n \to +\infty$ and obtain:

$$\begin{aligned}
\lim_{n\to+\infty} \frac{1}{\kappa_{\varepsilon_n}}\int_{\mathbb{R}^2} \frac{|u_{\varepsilon_n}(1+\varepsilon_n\,v)|^{p_n}}{|x|^{b_n p_n}}\,dx &= \frac{1}{\pi}\int_0^{2\pi} e^{2\,v(\cos\theta,\,\sin\theta)}\,d\theta \int_0^{+\infty} \frac{t\,dt}{(1+t^2)^2}\\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{2\,v(\cos\theta,\,\sin\theta)}\,d\theta\ .
\end{aligned}$$

Analogously,

$$\begin{aligned}
&(\alpha_n+1)\int_{\mathbb{R}^2} \frac{u_{\varepsilon_n}^{2/\varepsilon_n}}{|x|^{2(a_n-\alpha_n)}}\,v\,dx\\
&= (\alpha_n+1)\int_{\mathbb{R}^2} |x|^{2\frac{\alpha_n+\varepsilon_n}{1-\varepsilon_n}}\,\frac{v(x)}{\big(1+|x|^{2(1+\alpha_n)}\big)^{\frac{2}{1-\varepsilon_n}}}\,dx\\
&= \int_0^{2\pi}\!\!\int_0^{+\infty} \frac{t^{\frac{1+\varepsilon_n}{1-\varepsilon_n}}}{(1+t^2)^{\frac{2}{1-\varepsilon_n}}}\,v(\,t^{\frac{1}{\alpha_n+1}}\cos\theta,\ t^{\frac{1}{\alpha_n+1}}\sin\theta)\,dt\,d\theta\,.
\end{aligned}$$

By Lemma 6, we see that

$$\frac{1}{\lambda_{\varepsilon_n}} \int_{\mathbb{R}^2} \frac{|\,\nabla[u_{\varepsilon_n}(1+\varepsilon_n\,v)]\,|^2}{|x|^{2a_n}}\,dx \;=\; 1+\frac{\varepsilon_n^2}{\lambda_{\varepsilon_n}}\,\frac{8\,(\alpha_n+1)^2}{(1-\varepsilon_n)^2}\int_{\mathbb{R}^2}\frac{u_{\varepsilon_n}^{2/\varepsilon_n}}{|x|^{2(a_n-\alpha_n)}}\,v\,dx + O\left(\frac{\varepsilon_n}{1+\alpha_n}\right)+O\left(\frac{\varepsilon_n\,a_n^2}{\lambda_{\varepsilon_n}}\right)\,,$$

and so

$$\lim_{n\to+\infty}\left(\frac{1}{\lambda_{\varepsilon_n}}\int_{\mathbb{R}^2}\frac{|\,\nabla[u_{\varepsilon_n}(1+\varepsilon_n\,v)]\,|^2}{|x|^{2a_n}}\,dx\right)^{1/\varepsilon_n} = e^{\frac{2}{\pi}\int_0^{2\pi}v(\cos\theta,\,\sin\theta)\,d\theta\int_0^{+\infty}\frac{t\,dt}{(1+t^2)^2}} = e^{\frac{1}{\pi}\int_0^{2\pi}v(\cos\theta,\,\sin\theta)\,d\theta}\,.$$

Hence the validity of (8) would imply that for all $v\in C_c^\infty(\mathbb{R}^2)$, there holds:

$$\frac{1}{2\pi}\int_0^{2\pi}e^{2\,v(\cos\theta,\,\sin\theta)}\,d\theta\le e^{\frac{1}{\pi}\int_0^{2\pi}v(\cos\theta,\,\sin\theta)\,d\theta}\,.$$

But this is clearly impossible, since such an inequality is violated for instance by the function $v(x)=v(x_1,x_2)=x_1^2\,\eta(x)$, with η a standard cut-off function such that $\eta(x)=1$ if $|x|\le 1$, $\eta(x)=0$ if $|x|\ge 2$. □

3. SYMMETRY BREAKING

This section is devoted to the proof of Theorem 1, (i). We start by establishing Property (i'), which is weaker, but it follows as an easy consequence of the results of section 2.

3.1. **Proof of Property (i').** By Lemma 5 and Kelvin transformation, we can reduce the proof to the case $a<0$. Let us argue by contradiction and assume that there exists $\varepsilon_0\in(0,1)$, $a_n\to 0_-$ and b_n such that $\varepsilon_0<\frac{b_n}{a_n}<1$ and u_{a_n,b_n} is radially symmetric. Set $\varepsilon_n=b_n-a_n>0$ and define α_n such that $\alpha_n+1=-a_n\,(1-\varepsilon_n)/\varepsilon_n$. Notice that $\varepsilon_n\to 0_+$ while $\alpha_n+1=a_n-a_n/(b_n-a_n)=a_n-(b_n/a_n-1)^{-1}>a_n+(1-\varepsilon_0)^{-1}$. Hence, $\liminf_{n\to+\infty}\alpha_n\ge\alpha_0=\varepsilon_0/(1-\varepsilon_0)>0$. But this is impossible since it contradicts Proposition 9 in case $\liminf_{n\to+\infty}\alpha_n=+\infty$, or Propositions 4 and 7 if $\limsup_{n\to+\infty}\alpha_n<+\infty$; and we conclude the proof of (i').

3.2. **Proof of (i) of Theorem 1.** It is well known (see [4]) that by means of the following Emden-Fowler transformations:

$$t=\log|x|\;,\quad \theta=\frac{x}{|x|}\in S^{N-1}\,,\quad w(t,\theta)=|x|^{\frac{N-2-2a}{2}}\,v(x)\;, \tag{9}$$

inequality (5) for u is equivalent to the Sobolev inequality for w on $\mathbb{R}\times S^{N-1}$. Namely,

$$\|w\|^2_{L^p(\mathbb{R}\times S^{N-1})}\le C^N_{a,b}\left[\|\nabla w\|^2_{L^2(\mathbb{R}\times S^{N-1})}+\tfrac14(N-2-2a)^2\|w\|^2_{L^2(\mathbb{R}\times S^{N-1})}\right],$$

for $w\in H^1(\mathbb{R}\times S^{N-1})$, with $p=2\,N/[(N-2)+2\,(b-a)]$ and the same optimal constant $C^N_{a,b}$ as in (5). This inequality is consistent with the statement of Lemma 5, as it makes sense for any $a\neq(N-2)/2$, independently of the sign of $N-2-2a$.

For $N=2$, the inequality holds for functions $w=w(t,\theta)$ defined over the two-dimensional cylinder $\mathcal{C}=\mathbb{R}\times S^1\approx(\mathbb{R}/2\pi\mathbb{Z})$, *i.e.*, such that $w(t,\cdot)$ is 2π-periodic for a.e. $t\in\mathbb{R}$. The inequality then takes the form

$$\|w\|^2_{L^p(\mathcal{C})}\le C_{a,a+2/p}\left(\|\nabla w\|^2_{L^2(\mathcal{C})}+a^2\,\|w\|^2_{L^2(\mathcal{C})}\right)\quad\forall\,w\in H^1(\mathcal{C}) \tag{10}$$

for all $a\neq 0$ and $p>2$. Here $C_{a,b}$ is the optimal constant in (3) which enters in (10) with $b=a+2/p$.

For any $a\neq 0$ and $p>2$, inequality (10) is attained at an extremal function $w_{a,p}\in H^1(\mathcal{C})$ which satisfies

$$\left\{\begin{array}{l}-(w_{tt}+w_{\theta\theta})+a^2\,w=w^{p-1}\quad\text{in}\quad\mathbb{R}\times[-\pi,\pi]\;,\\[2ex] w>0\;,\quad w(t,\cdot)\quad\text{is }2\pi\text{-periodic}\quad\forall\,t\in\mathbb{R}\;,\end{array}\right. \tag{11}$$

and such that

$$\left(C_{a,a+2/p}\right)^{-1}=\|w_{a,p}\|^{p-2}_{L^p(\mathcal{C})}=\inf_{w\in H^1(\mathcal{C})\setminus\{0\}}\mathcal{F}(w)\;,$$

where the functional

$$\mathcal{F}(w)=\frac{\|\nabla w\|^2_{L^2(\mathcal{C})}+a^2\,\|w\|^2_{L^2(\mathcal{C})}}{\|w\|^2_{L^p(\mathcal{C})}}$$

is well defined on $H^1(\mathcal{C})\setminus\{0\}$. Moreover, according to [4], we can further assume that

$$\left\{\begin{array}{l}w_{a,p}(t,\theta)=w_{a,p}(-t,\theta)\quad\forall\,t\in\mathbb{R}\;,\quad\theta\in[-\pi,\pi)\;,\\ \frac{\partial w_{a,p}}{\partial t}(t,\theta)<0\quad\forall\,t>0\;,\quad\forall\,\theta\in[-\pi,\pi)\,,\\ \max_{\mathbb{R}\times[-\pi,\pi)}w_{a,p}=w_{a,p}(0,0)\;.\end{array}\right. \tag{12}$$

This symmetry result is easy to establish for a minimizer, but the monotonicity requires more elaborate tools like the sliding method and we refer to [4] for more details. For a solution of (11) which does not depend on θ, the conditions in (12) allow to determine its value at 0 simply by multiplying the ODE by w_t and integrating from 0 to ∞. In fact, in this way, one deduces the relation: $a^2\,w^2(0)/2=w^p(0)/p$, which uniquely determines $w(0)>0$. In turn this yields to the following unique θ-independent solution for (11) and (12):

$$w^*_{a,p}(t)=\left(\tfrac{a^2\,p}{2}\right)^{1/(p-2)}\left[\cosh\left(\tfrac{p-2}{2}\,a\,t\right)\right]^{-2/(p-2)}\;,$$

as a consequence of the classification result in [4]. Such a solution is an extremal for (10) on the set of functions which are independent of the θ-variable, and

$$\|w^*_{a,p}\|^{p-2}_{L^p(\mathbb{R})}=\inf_{f\in H^1(\mathbb{R})\setminus\{0\}}\mathcal{F}^*(f)\quad\text{with}\quad\mathcal{F}^*(f)=\frac{\|f'\|^2_{L^2(\mathbb{R})}+a^2\,\|f\|^2_{L^2(\mathbb{R})}}{\|f\|^2_{L^p(\mathbb{R})}}\;.$$

For simplicity, we will also write $\mathcal{F}(f)=(\pi)^{1-2/p}\,\mathcal{F}^*(f)$ for all functions f which are independent of θ. As a useful consequence of the above considerations, we have the following result.

Lemma 10. *Let $p > 2$. For any $a \neq 0$,*

$$\big(C_{a,a+2/p}\big)^{-\frac{p}{p-2}} = \|w_{a,p}\|^p_{L^p(\mathcal{C})} \leq \|w^*_{a,p}\|^p_{L^p(\mathcal{C})} = 4\,\pi\,(2\,a)^{\frac{p}{p-2}}\,(a\,p)^{\frac{2}{p-2}}\,c_p$$

where c_p is an increasing function of p such that

$$\begin{aligned} c_p &\to 0 &\quad \text{as} \quad & p \to 2_+\ , \\ c_p &\to \tfrac{1}{2} &\quad \text{as} \quad & p \to +\infty\ . \end{aligned} \tag{13}$$

As a consequence, if $a = a(p)$ is such that $\lim_{p\to\infty} a(p)\,p = 2\,(\alpha+1)$, then

$$\lim_{p\to\infty} p \int_{\mathcal{C}} |w^*_{a(p),p}|^p\,dx = 8\,(\alpha+1)\ . \tag{14}$$

Proof. Observe that

$$\|w_{a,p}\|^p_{L^p(\mathcal{C})} = \big(C_{a,a+2/p}\big)^{-\frac{p}{p-2}} = (\mathcal{F}(w_{a,p}))^{\frac{p}{p-2}} \leq \big(\mathcal{F}(w^*_{a,p})\big)^{\frac{p}{p-2}} = \|w^*_{a,p}\|^p_{L^p(\mathcal{C})}\,.$$

On the other hand,

$$\begin{aligned} \|w^*_{a,p}\|^p_{L^p(\mathcal{C})} &= 2\pi\,\big(\tfrac{a^2\,p}{2}\big)^{\frac{p}{p-2}} \int_{-\infty}^{\infty} \Big[\cosh\Big(\tfrac{a\,(p-2)}{2}\,t\Big)\Big]^{-\frac{2p}{p-2}}\,dt \\ &= 4\pi\,\big(\tfrac{a^2\,p}{2}\big)^{\frac{p}{p-2}} \int_0^{\infty} \frac{2^{\frac{2p}{p-2}}\,e^{-a\,p\,t}}{\big(1+e^{-a\,(p-2)\,t}\big)^{\frac{2p}{p-2}}}\,dt \\ &= 4\pi\,\big(\tfrac{a^2\,p}{2}\big)^{\frac{p}{p-2}}\,\frac{2^{\frac{2p}{p-2}}}{a\,p} \int_0^1 \frac{ds}{\big(1+s^{(p-2)/p}\big)^{\frac{2p}{p-2}}}\,. \end{aligned}$$

Hence by setting:

$$c_p = \int_0^1 \frac{ds}{\big(1+s^{(p-2)/p}\big)^{\frac{2p}{p-2}}}\ ,$$

we easily check (13) and the fact that c_p is monotonically increasing in p. The limiting behavior of c_p stated in (13) is a direct consequence of Lebesgue's dominated convergence theorem. ☐

We can now reformulate Theorem 1 in the cylinder $\mathcal{C}$, in terms of w, as follows.

Theorem 11. *Let $a \neq 0$ and $p > 2$.*

- (i) *If $|a|\,p > 2\sqrt{1+a^2}$, then $\mathcal{F}(w_{a,p}) < \mathcal{F}(w^*_{a,p})$.*
- (ii) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $0 < |a| < \delta$ and $|a|\,p < 2-\varepsilon$, then $\mathcal{F}(w_{a,p}) = \mathcal{F}(w^*_{a,p})$.*

Part (ii) of Theorem 11 will be proved in the next section. Concerning part (i), we define the quadratic form

$$Q(\psi) = \|\nabla\psi\|^2_{L^2(\mathcal{C})} + a^2\,\|\psi\|^2_{L^2(\mathcal{C})} - (p-1)\int_{\mathcal{C}} |w^*_{a,p}|^{p-2}\,|\psi|^2\,dx$$

on $H^1(\mathcal{C})$. In fact, property (i) is a consequence of the following result, inspired by [4, 9] (at least for the case $a < 0$):

Proposition 12. *Let* $a \neq 0$ *and* $p > 2$. *Then*

$$\inf_{\substack{\psi \in H^1(\mathcal{C}) \\ \int_{-\pi}^{\pi} \psi(t,\theta)\, d\theta = 0,\ t \in \mathbb{R}\ a.e.}} \frac{Q(\psi)}{\|\psi\|^2_{L^2(\mathcal{C})}} = a^2 + 1 - \left(\frac{a\,p}{2}\right)^2$$

is achieved by

$$\psi(t,\theta) = \big(\cosh((\alpha+1)\,t)\big)^{-p/(p-2)}\,, \quad \textit{with} \quad \alpha = (p-2)\,a/2 - 1\,.$$

In particular, if $|a|\,p > 2\sqrt{1+a^2}$, *then* $w^*_{a,p}$ *is a critical point for* $\mathcal{F}$ *of saddle-type.*

Proof. Since $w^*_{a,p}$ is a local minimum for $\mathcal{F}$ when restricted to the set of functions independent of θ, to search for negative directions of the Hessian of $\mathcal{F}$ around $w^*_{a,p}$, we have to analyze the quadratic form $Q(\psi)$ on the space of functions $\psi \in H^1(\mathcal{C})$ such that $\int_{-\pi}^{\pi} \psi(t,\theta)\, d\theta = 0$ for a.e. $t \in \mathbb{R}$. To this purpose, we use the Fourier expansion of ψ,

$$\psi(t,\theta) = \sum_{k\neq 0} f_k(t)\, \frac{e^{ik\theta}}{\sqrt{2\pi}}\,, \quad f_{-k}(t) = \overline{f_k}(t)\,,$$

$$Q(\psi) = 2\sum_{k=1}^{+\infty} \Big(\|f_k'\|^2_{L^2(\mathbb{R})} + (a^2+k^2)\,\|f_k\|^2_{L^2(\mathbb{R})} - (p-1)\int_{\mathbb{R}} |w^*_{a,p}|^{p-2}\,|f_k|^2\, dt\Big)\,.$$

Hence we obtain a negative direction for Q if and only if

$$\mu^1_{a,p} = \inf_{f\in H^1(\mathbb{R})\setminus\{0\}} \frac{\|f'\|^2_{L^2(\mathbb{R})} + (a^2+1)\|f\|^2_{L^2(\mathbb{R})} - (p-1)\int_{\mathbb{R}} |w^*_{a,p}|^{p-2}\,|f|^2\, dt}{\|f\|^2_{L^2(\mathbb{R})}} < 0\,.$$

Setting $1+\alpha = (p-2)\,a/2$ and $\beta = a^2\,p\,(p-1)/2 = 2\,(1+\alpha)^2\,p\,(p-1)/(p-2)^2 > 0$, the question is reduced to the eigenvalue problem

$$-f'' - \frac{\beta\, f}{\big(\cosh((\alpha+1)\,t)\big)^2} = \lambda\, f\,.$$

in $H^1(\mathbb{R})$. The eigenfunction $f_1(t) = \big(\cosh((\alpha+1)\,t)\big)^{-p/(p-2)}$ corresponds to the first eigenvalue $\lambda_1 = -(a\,p/2)^2$. See [10, 9] for a discussion of the above eigenvalue problem. Hence $\mu^1_{a,p} = 1 + a^2 - (a\,p/2)^2$, and the proof is completed. □

4. A SYMMETRY RESULT

The section is devoted to the proof of part (ii) of Theorem 11.

Without loss of generality, by Lemma 5, we can restrict our analysis to the case $a > 0$.

4.1. **Pohozaev's identity.**

Lemma 13. *If* $w \in H^1(\mathcal{C})$ *satisfies* (11), *then for all* $t \in \mathbb{R}$, $w = w(t,\theta)$ *satisfies the identity*

$$\int_{-\pi}^{\pi} \Big(\frac{\partial w}{\partial \theta}\Big)^2\, d\theta = \int_{-\pi}^{\pi} \Big(\frac{\partial w}{\partial t}\Big)^2\, d\theta - a^2 \int_{-\pi}^{\pi} w^2\, d\theta + \frac{2}{p}\int_{-\pi}^{\pi} w^p\, d\theta\,.$$

Proof. Multiply the equation in (11) by $\frac{\partial w}{\partial t}$ and integrate over $[-\pi,\pi]$ to obtain:

$$\int_{-\pi}^{\pi}\left(-\frac{\partial^2 w}{\partial t^2}\,\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial \theta^2}\,\frac{\partial w}{\partial t}+a^2\,\frac{\partial w}{\partial t}\,w\right)d\theta=\int_{-\pi}^{\pi}w^{p-1}\,\frac{\partial w}{\partial t}\,d\theta\;,$$

that is

$$\int_{-\pi}^{\pi}\left\{-\tfrac{\partial}{\partial\theta}\left(\tfrac{\partial w}{\partial\theta}\tfrac{\partial w}{\partial t}\right)+\tfrac{1}{2}\,\tfrac{d}{dt}\left[\left(\tfrac{\partial w}{\partial\theta}\right)^2-\left(\tfrac{\partial w}{\partial t}\right)^2+a^2\,w^2\right]\right\}d\theta=\frac{1}{p}\int_{-\pi}^{\pi}\frac{d\,(w^p)}{dt}\,d\theta\;.$$

Since $\int_{-\pi}^{\pi}\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\theta}\frac{\partial w}{\partial t}\right)d\theta=0$, we get

$$\frac{d}{dt}\int_{-\pi}^{\pi}\left[\left(\frac{\partial w}{\partial t}\right)^2-\left(\frac{\partial w}{\partial\theta}\right)^2-a^2\,w^2+\frac{2}{p}\,w^p\right]d\theta=0$$

for all $t\in\mathbb{R}$. Hence as a function of t, the above integral must be a constant. Since it is also integrable over $\mathbb{R}$, then it must vanish identically. □

4.2. **Proof of Theorem 11, (ii).** We argue by contradiction and suppose that there exists $\varepsilon_0\in(0,1)$ and, for all $n\in\mathbb{N}$, $a_n>0$, $p_n>2$, such that:

$$\lim_{n\to+\infty}a_n=0\;,\quad a_n\,p_n<2-\varepsilon_0\quad\text{and}\quad\mathcal{F}(w_{a_n,p_n})<\mathcal{F}(w^*_{a_n,p_n})\;. \tag{15}$$

For simplicity, set

$$w_n=w_{a_n,p_n}\quad\text{and}\quad w_n^*=w^*_{a_n,p_n}\;,$$

and recall that we can assume

$$w_n(t,\theta)=w_n(-t,\theta)\;,\quad\frac{\partial w_n}{\partial t}(t,\theta)<0\quad\forall\,t>0\quad\text{and}\quad w_n(0,0)=\max_{\mathcal{C}}w_n\;.$$

Notice in particular that $\frac{\partial w_n}{\partial t}(0,\theta)=0$ for any $\theta\in[-\pi,\pi]$. If we apply Lemma 13 to $w=w_n$ and $t=0$, we obtain

$$\frac{p_n^2\,a_n^2}{2}\int_{-\pi}^{\pi}w_n^2(0,\theta)\,d\theta\le p_n\int_{-\pi}^{\pi}w_n^{p_n}(0,\theta)\,d\theta\le p_n\,\|w_n\|_{L^\infty(\mathcal{C})}^{p_n-2}\int_{-\pi}^{\pi}w_n^2(0,\theta)\,d\theta\;,$$

and deduce that

$$p_n\,\|w_n\|_{L^\infty(\mathcal{C})}^{p_n-2}\ge\frac{1}{2}\,p_n^2\,a_n^2\;.$$

Lemma 14.

$$\liminf_{n\to+\infty}p_n\,\|w_n\|_{L^\infty(\mathcal{C})}^{p_n-2}\ge 1\;.$$

Proof. We can write $w_n(t,\theta)=\varphi_n(t)+\psi_n(t,\theta)$ with

$$\varphi_n(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}w_n(t,\theta)\,d\theta\;,\quad\int_{-\pi}^{\pi}\psi_n(t,\theta)\,d\theta=0\quad\text{a.e. }t\in\mathbb{R}\quad\text{and}\quad\psi_n\neq 0\;.$$

Multiplying (11) by ψ_n and using the fact that $\int_{-\pi}^{\pi} \psi_n(t,\theta)\,d\theta = 0$ for any $t \in \mathbb{R}$, we find

$$\begin{aligned}
&\left\| \frac{\partial \psi_n}{\partial t} \right\|^2_{L^2(\mathcal{C})} + \left\| \frac{\partial \psi_n}{\partial \theta} \right\|^2_{L^2(\mathcal{C})} + a_n^2 \, \|\psi_n\|^2_{L^2} \\
&= \int_{\mathcal{C}} w_n^{p_n-1}\, \psi_n \; dt\, d\theta \\
&= \int_{\mathcal{C}} w_n^{p_n-1}\, \psi_n \; dt\, d\theta - \int_{\mathcal{C}} \varphi_n^{p_n-1}\, \psi_n \; dt\, d\theta \\
&= (p_n - 1) \int_0^1 \left\{ \int_{\mathcal{C}} \left| s\, \varphi_n + (1-s)\, w_n \right|^{p_n-2} |\psi_n|^2 \; dt\, d\theta \right\} ds \\
&\le (p_n - 1)\, \|w_n\|^{p_n-2}_{L^\infty(\mathcal{C})} \int_{\mathcal{C}} |\psi_n|^2 \; dt\, d\theta \; .
\end{aligned}$$

By Poincaré's inequality, we know that $\|\psi_n\|^2_{L^2(\mathcal{C})} \le \left\| \frac{\partial \psi_n}{\partial \theta} \right\|^2_{L^2(\mathcal{C})}$, and this proves the claim. $\square$

Next we introduce the new parameters:

$$\varepsilon_n = \frac{2}{p_n} \quad \text{and} \quad \alpha_n = -1 + (1 - \varepsilon_n)\, \frac{a_n}{\varepsilon_n} = -1 + \frac{1}{2}\, (1 - \varepsilon_n)\, a_n\, p_n \; .$$

Lemma 15. *Up to a subsequence we have:*

$$\lim_{n \to +\infty} \alpha_n = \alpha \in [-1, 0) \; ,$$

and $\lim_{n \to +\infty} p_n = +\infty$, *or equivalently,*

$$\lim_{n \to +\infty} \varepsilon_n = 0 \; .$$

Proof. From the condition: $a_n\, p_n < 2 - \varepsilon_0$, we deduce that $\alpha_n + 1 \le (1 - \varepsilon_n)\, (1 - \varepsilon_0/2)$. Thus, along a subsequence, we can assume that α_n converges to some $\alpha \in [-1, 0)$ and $\lim_{n \to +\infty} p_n \in [2, \infty]$.

To rule out the possibility that $\lim_{n \to +\infty} p_n = \bar{p} \in [2, \infty)$, notice that if this would be the case, then by Lemma 10,

$$\lim_{n \to +\infty} \|w_n\|_{L^{p_n}(\mathcal{C})} = 0 \; .$$

By applying local elliptic estimates in a neighborhood of the origin $(0,0)$ then we would deduce that $\lim_{n \to +\infty} \|w_n\|_{L^\infty(\mathcal{C})} = \lim_{n \to +\infty} w_n(0,0) = 0$, in contradiction with Lemma 14. $\square$

Corollary 16.

$$\liminf_{n \to +\infty} w_n(0,0) \ge 1 \; .$$

Proof. If by contradiction we assume that $\liminf_{n \to +\infty} w_n(0,0) < 1$, then $\liminf_{n \to +\infty} p_n\, \|w_n\|^{p_n-2}_{L^\infty(\mathcal{C})} = 0$, and again this is impossible by Lemma 14. $\square$

Lemma 17.

$$\limsup_{n \to +\infty} p_n\, \|w_n\|^{p_n-2}_{L^\infty(\mathcal{C})} < +\infty \; .$$

Proof. Argue by contradiction, and assume that, along a subsequence, $\delta_n = \left(p_n \, \|w_n\|_{L^\infty(\mathcal{C})}^{p_n-2}\right)^{-1/2}$ converges to 0 as $n \to +\infty$. We consider the function

$$W_n(t,\theta) = p_n \left(\frac{w_n(\delta_n\, t,\, \delta_n\, \theta)}{w_n(0,0)} - 1 \right)$$

defined in $\mathcal{C}_n = \mathbb{R} \times [-\pi/\delta_n,\, \pi/\delta_n]$, which satisfies

$$\begin{cases} -\Delta W_n = \left(1 + \frac{W_n}{p_n}\right)^{p_n-1} - \, a_n^2\, p_n\, \delta_n^2 \, \left(1 + \frac{W_n}{p_n}\right) \quad \text{in} \quad \mathcal{C}_n \ , \\ \\ W_n \leq 0 = W_n(0,0) \ . \end{cases} \tag{16}$$

Furthermore, by Lemma 10, we find

$$\int_{\mathcal{C}_n} \left(1 + \frac{W_n}{p_n}\right)^{p_n} dx = \frac{p_n}{w_n(0,0)^2} \int_{\mathcal{C}} w_n^{p_n}\, dx \leq \frac{1}{w_n(0,0)^2}\, p_n \int_{\mathcal{C}} |w_n^*|^{p_n}\, dx \ .$$

Recalling that $\liminf_{n\to+\infty} w_n(0,0) \geq 1$ and $\lim_{n\to+\infty} a_n\, p_n = 2\,(1+\alpha)$ by (14), we can pass to the limit above and by virtue of (13)-(14), conclude:

$$\lim_{n\to+\infty} \|1 + W_n/p_n\|_{L^{p_n}(\mathcal{C}_n)}^{p_n} \leq 8\pi\,(1+\alpha) \ .$$

Since the right hand side in (16) is uniformly bounded in $L^\infty_{\rm loc}(\mathbb{R}^2)$, we can use Harnack's inequality (see for instance [2, 14] in similar cases) to deduce that W_n is uniformly bounded in $L^\infty_{\rm loc}$. Hence, by elliptic regularity theory, W_n is uniformly bounded in $C^{2,\alpha}_{\rm loc}$. So we can find a subsequence along which W_n converges pointwise (uniformly on every compact set in $\mathbb{R}^2$) to a function W which satisfies

$$-\Delta W = e^W \quad \text{in} \quad \mathbb{R}^2 \, . \tag{17}$$

Furthermore, by Fatou's Lemma,

$$\int_{\mathbb{R}^2} e^W\, dx \leq \lim_{n\to+\infty} \int_{\mathcal{C}_n} \left(1 + \frac{W_n}{p_n}\right)^{p_n} dx \leq 8\pi\,(1+\alpha) < 8\pi \ ,$$

as $\alpha \in [-1,0)$. But this is impossible, since according to [5], every solution W of (17) with $e^W \in L^1(\mathbb{R}^2)$, must satisfy $\int_{\mathbb{R}^2} e^W\, dx = 8\pi$ (also see [7, 8]). □

Corollary 18. *For a subsequence of* $\|w_n\|_{L^\infty(\mathcal{C})} = w_n(0,0)$ *(denoted the same way) we have:*

$$\lim_{n\to+\infty} w_n(0,0) = 1 \ ,$$
$$\lim_{n\to+\infty} \left[w_n(0,0)\right]^{p_n} = 0 \ ,$$
$$\lim_{n\to+\infty} p_n \left[w_n(0,0)\right]^{p_n-2} = \mu \in [1,+\infty) \ .$$

Proof. The existence of a limit $\mu \geq 1$ is just a consequence of Lemmata 14 and 17. Furthermore by Lemma 15, $p_n = 2/\varepsilon_n \to +\infty$ as $n \to +\infty$, which proves that $[w_n(0,0)]^{p_n}$ converges to 0. Finally, according to Corollary 16, $\liminf_{n\to+\infty} w_n(0,0) \geq 1$ and if this limit were not 1, we would get a contradiction to the existence of μ. □

Define the function

$$V_n(t,\theta) = p_n \left(\frac{w_n(t,\theta)}{w_n(0,0)} - 1 \right) \quad \forall\, (t,\theta) \in \mathcal{C}\ .$$

It satisfies:

$$-\Delta V_n = p_n \left(w_n(0,0)\right)^{p_n-2} \left(1 + \frac{V_n}{p_n}\right)^{p_n-1} - a_n^2\, p_n \left(1 + \frac{V_n}{p_n}\right) \quad \text{in} \quad \mathcal{C}\,,$$
$$V_n \le 0 = V_n(0,0)\ , \qquad V_n(t,\cdot) \quad \text{is } 2\pi\text{-periodic}\ .$$

We also observe that

$$p_n \left(w_n(0,0)\right)^{p_n} \int_{\mathcal{C}} \left(1 + \frac{V_n}{p_n}\right)^{p_n} dx = p_n \int_{\mathcal{C}} |w_n|^{p_n}\, dx \le p_n \int_{\mathcal{C}} |w_n^*|^{p_n}\, dx$$

and by (14), $\lim_{n\to\infty} p_n \int_{\mathcal{C}} |w_n^*|^{p_n}\, dx = 8\,\pi\,(\alpha+1)$. In particular, by Corollary 18, we obtain

$$\lim_{n\to+\infty} p_n \left(w_n(0,0)\right)^{p_n-2} \int_{\mathcal{C}} \left(1 + \frac{V_n}{p_n}\right)^{p_n} dx \ \le\ 8\pi\,(1+\alpha)\ .$$

Lemma 19. *Up to a subsequence, V_n converges to a function V pointwise and C^2-uniformly on any compact set in $\mathbb{R}\times[-\pi,\pi]$. Furthermore V satisfies:*

$$(18) \qquad \begin{cases} -\Delta V = \mu\, e^V \quad \textit{in} \quad \mathcal{C}\ , \\[1ex] \max_{\mathcal{C}} V \le 0 = V(0,0)\ , \quad V(t,\cdot) \quad \textit{is } 2\pi\textit{-periodic} \quad \forall\, t \in \mathbb{R}\ , \\[1ex] \mu \int_{\mathcal{C}} e^V\, dx \ \le\ 8\pi\,(1+\alpha)\ , \end{cases}$$

$$V(t,\theta) = V(-t,\theta)\ , \quad \frac{\partial V}{\partial t}(t,\theta) < 0 \quad \forall\, t > 0\ , \quad \forall\, \theta \in [-\pi,\pi]\ ,$$

and

$$(19) \quad \int_{-\pi}^{\pi} \left(\frac{\partial V}{\partial \theta}\right)^2 d\theta = \int_{-\pi}^{\pi} \left(\frac{\partial V}{\partial t}\right)^2 d\theta - 8\pi\,(1+\alpha)^2 + 2\mu \int_{-\pi}^{\pi} e^V\, d\theta \quad \forall\, t \in \mathbb{R}\ .$$

Proof. Since $-\Delta V_n$ is uniformly bounded in $L^\infty_{\rm loc}(\mathbb{R}^2)$, by Harnack's inequality, we see that V_n is uniformly bounded in $L^\infty_{\rm loc}$. Hence, by elliptic regularity theory, V_n is uniformly bounded in $C^{2,\alpha}_{\rm loc}$. Therefore, up to a subsequence, V_n converges pointwise, and uniformly on every compact set in $\mathcal{C}$, to a function V which satisfies (18) with $0 \le 1+\alpha < 1$, and also inherits the symmetric properties of V_n. To obtain (19) observe first that the result of Lemma 13 can be rewritten as follows,

$$\int_{-\pi}^{\pi} \left(\frac{\partial V_n}{\partial \theta}\right)^2 d\theta = \int_{-\pi}^{\pi} \left(\frac{\partial V_n}{\partial t}\right)^2 d\theta - \frac{a_n^2\, p_n^2}{w_n^2(0,0)} \int_{-\pi}^{\pi} |w_n|^2\, d\theta + \frac{2\, p_n}{w_n^2(0,0)} \int_{-\pi}^{\pi} |w_n|^{p_n}\, d\theta\,,$$

for any $t \in \mathbb{R}$, and that w_n converges uniformly to 1 on any compact set in $\mathbb{R}\times[-\pi,\pi]$. Hence by means of Lemma 15 and Corollary 18, we can pass to the limit in the above identity and deduce (19). □

Lemma 20. *The following estimates hold:*

$$\lim_{n\to+\infty} p_n \left(\|w_n\|^{p_n}_{L^{p_n}(\mathcal{C})} - \|w^*_n\|^{p_n}_{L^{p_n}(\mathcal{C})} \right) = 0 \,,$$

$$\int_{\mathcal{C}} e^V \, dx = \lim_{n\to+\infty} \int_{\mathcal{C}_n} \left(1 + \frac{V_n}{p_n}\right)^{p_n} dx = \frac{4\pi}{\alpha+1} \,.$$

Moreover,

$$\mu = 2\,(\alpha+1)^2 \,,$$

and V takes the form

$$V(t) = -2 \log \left[\cosh((\alpha+1)\, t) \right] \,. \tag{20}$$

Proof. In order to identify the given solution of (18), we consider the function φ expressed in polar coordinates as follows:

$$\varphi(r,\theta) = V\,(-\log r \,,\, \theta) - 2 \log r + \log \mu \quad \forall\, r > 0 \,, \quad \forall\, \theta \in [-\pi, \pi] \,.$$

By straightforward calculations we see that φ satisfies:

$$-\Delta\varphi = -\frac{1}{r^2}\, (V_{tt} + V_{\theta\theta})\, (-\log r \,,\, \theta) = e^{\varphi} \quad \text{in } \mathbb{R}^2 \backslash \{0\} \,,$$

$$\int_{\mathbb{R}^2} e^{\varphi}\, dx \;\leq\; 8\pi\,(1+\alpha) \,,$$

and

$$\varphi\left(r^{-1} \,,\, \theta\right) = \varphi(r,\theta) + 4\, \log r \quad \forall\, r > 0 \,, \quad \forall\, \theta \in [-\pi, \pi] \,. \tag{21}$$

A classification result of Chou and Wan (see [7, Theorem 3, 1.] and [8]) concerning solutions of Liouville equations on the punctured disk allows us to conclude that (in complex notations):

$$\varphi(z) = \log \left[\frac{8\,|f'(z)|^2}{\left(1 + |f(z)|^2\right)^2} \right] \,,$$

with f locally univalent in $\mathbb{C} \backslash \{0\}$, possibly multivalued and,

(i) either $f(z) = z^{\gamma}\, g(z)$,
(ii) or $f(z) = \phi(\sqrt{z})$ and $\phi(z)\, \phi(-z) = 1$,

where g and ϕ are holomorphic in $\mathbb{C} \backslash \{0\}$. Since the case (ii) implies that ϕ must admit an essential singularity either at the origin or at infinity, this can be excluded in account of the integrability condition of e^{φ}.

On the other hand, in case (i), if we take into account the fact that $f' \neq 0$ for any $z \neq 0$, and the integrability of e^{φ}, we can allow only the choice:

$$f(z) = a\left(z^{\beta+1} - b\right) \,,$$

with $\beta \in \mathbb{R}$, a, $b \in \mathbb{C}$ and $b \neq 0$ only if $\beta + 1 \in \mathbb{N}$ (as otherwise φ would be multivalued). For the corresponding solution φ we find:

$$\varphi(z) = \log \left[\frac{8\, \lambda\, (\beta+1)^2\, |z|^{2\beta}}{(1 + \lambda\, |z^{\beta+1} - b|^2)^2} \right] \,, \quad \text{with} \quad \lambda = |a|^2 \,.$$

The symmetry property (21) implies that

$$\varphi\left(\frac{z}{|z|^2}\right) = \varphi(z) + 4 \log |z| \,,$$

and so, necessarily $b = 0$ and $\lambda = 1$. Hence,

$$\varphi(z) = \varphi(r) = \log\left[\frac{8\,(\beta+1)^2\, r^{2\beta}}{\left(1+r^{2(\beta+1)}\right)^2}\right] .$$

By direct calculation, we get

$$\int_{\mathbb{R}^2} e^{\varphi}\, dx = 8\pi\,(1+\beta) \;\leq\; 8\pi\,(1+\alpha) \,.$$

In other words, $-1 < \beta \leq \alpha < 0$. As a consequence, we find that $V = V(t)$ is given by

$$V(t) = \varphi(e^{-t}) - 2t - \log\mu = \log\left[\frac{2\,(\beta+1)^2}{\mu\left(\cosh((\beta+1)\,t)\right)^2}\right] ,$$

with $-1 < \beta \leq \alpha < 0$. The condition $V(0) = 0$ implies $\mu = 2\,(\beta+1)^2$. On the other hand, from (19) we also have:

$$\left(\frac{\partial V}{\partial t}\right)^2 = 4\,(1+\alpha)^2 - \frac{4\,(\beta+1)^2}{\left(\cosh((\beta+1)\,t)\right)^2} ,$$

that gives:

$$4\,(\beta+1)^2 \frac{\left(\sinh((\beta+1)\,t)\right)^2}{\left(\cosh((\beta+1)\,t)\right)^2} = 4\,(1+\alpha)^2 - \frac{4\,(\beta+1)^2}{\left(\cosh((\beta+1)\,t)\right)^2} ,$$

and we get $\beta = \alpha$. Therefore (20) is established and necessarily

$$\lim_{n\to\infty} p_n\left(w_n(0,0)\right)^{p_n} \int_{\mathcal{C}} \left(1+\frac{V_n}{p_n}\right)^{p_n} dx = 2(\alpha+1)^2 \int_{\mathcal{C}} e^V\, dx = 8\,\pi\,(\alpha+1) \,.$$

Thus, by recalling (14), we complete the proof. □

Define

$$r_n = \sup_{\mathcal{C}} \left| \left(\frac{w_n}{w_n(0,0)}\right)^{p_n-2} - e^V \right| .$$

Lemma 21. *With the above notations,* $\lim_{n\to+\infty} r_n = 0$.

Proof. Fix $\varepsilon > 0$ and choose $R_\varepsilon > 0$ sufficiently large so that

$$e^{V(R_\varepsilon)} = \frac{1}{\left(\cosh((\alpha+1)\,R_\varepsilon)\right)^2} \;<\; \frac{\varepsilon}{4} \,.$$

Furthermore, $(w_n(t,\theta)/w_n(0,0))^{p_n-2} = (1+V_n/p_n)^{p_n-2}$ converges to e^V uniformly on any compact set in $\mathbb{R}\times[-\pi,\pi]$, and so we can find $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$\sup_{|t|\leq R_\varepsilon\,,\,|\theta|\leq\pi} \left| \left(\frac{w_n(t,\theta)}{w_n(0,0)}\right)^{p_n-2} - e^V \right| \;<\; \frac{\varepsilon}{4} \,.$$

Thus, recalling that $(w_n(t,\theta)/w_n(0,0))^{p_n-2}$ and e^V are even in t and monotone decreasing in $t > 0$ by Lemma 19, for $n \geq n_\varepsilon$ we find the estimate

$$r_n \leq \underbrace{\sup_{|t|\leq R_\varepsilon\,,\,|\theta|\leq\pi} \left| \left(\frac{w_n(t,\theta)}{w_n(0,0)}\right)^{p_n-2} - e^V \right|}_{<\varepsilon/4} + \underbrace{\sup_{|t|\geq R_\varepsilon} \left(\frac{w_n(t,\theta)}{w_n(0,0)}\right)^{p_n-2}}_{e^{V(R_\varepsilon)}+\varepsilon/4<\varepsilon/2} + \underbrace{\sup_{|t|\geq R_\varepsilon} e^V}_{\varepsilon/4} ,$$

which proves the result. □

Lemma 22. *For n large enough, we have $w_n = w_n^*$.*

Proof. Let $\chi_n = \partial w_n / \partial\theta$. Clearly $\int_{-\pi}^{\pi} \chi_n(t,\theta)\, d\theta = 0$, and since $w_n \in H^1(\mathcal{C})$, then $\chi_n \in L^2(\mathcal{C})$. Moreover, χ_n satisfies

$$-\Delta\chi_n + a_n^2\, \chi_n = (p_n - 1)\,\big(w_n(t,\theta)\big)^{p_n-2}\, \chi_n$$

(in the sense of distributions), where

$$\begin{aligned} \Big|(p_n-1)\,\big(w_n(t_n,\theta)\big)^{p_n-2}\chi_n\Big| \;&\le\; (p_n-1)\,(w_n(0,0))^{p_n-2}\Big(\frac{w_n(t,\theta)}{w_n(0,0)}\Big)^{p_n-2}|\chi_n| \\ &\le\; (p_n-1)\,(w_n(0,0))^{p_n-2}\,|\chi_n| \in L^2(\mathcal{C})\ . \end{aligned}$$

In other words, $-\Delta\chi_n + a_n^2\, \chi_n \in L^2(\mathcal{C})$, and hence $\chi_n \in H^1(\mathcal{C})$ satisfies:

$$\|\nabla\chi_n\|_{L^2}^2 + a_n^2\, \|\chi_n\|_{L^2}^2 = (p_n-1)\,\int_{\mathcal{C}}\Big(\frac{w_n(t,\theta)}{w_n(0,0)}\Big)^{p_n-2}\chi_n^2\, dx\ .$$

By Proposition 12, we know that if $\psi \in H^1(\mathcal{C})$ and $\int_{-\pi}^{\pi}\psi(t,\theta)\, d\theta = 0$ a.e. $t \in \mathbb{R}$, then

$$\|\nabla\psi\|_2^2 - \beta_n \int_{\mathcal{C}} \frac{|\psi(t,\theta)|^2}{\big(\cosh((\alpha_n+1)\, t)\big)^2}\, dt\, d\theta \ge \Big[1 - \Big(\frac{a_n\, p_n}{2}\Big)^2\Big]\, \|\psi\|_{L^2(\mathcal{C})}^2$$

with $\beta_n = a_n^2\, p_n\, (p_n-1)/2$. Passing to the limit as $n \to +\infty$, we get

$$\|\nabla\psi\|_2^2 - 2\,(\alpha+1)^2 \int_{\mathcal{C}} \frac{|\psi(t,\theta)|^2}{\big(\cosh((\alpha+1)\, t)\big)^2}\, dt\, d\theta \ge \Big[1 - (\alpha+1)^2\Big]\, \|\psi\|_{L^2(\mathcal{C})}^2\ .$$

Consequently, for $\psi = \chi_n$, we obtain

$$\begin{aligned} 0 =\;& \|\nabla\chi_n\|_2^2 + a_n^2\, \|\chi_n\|_{L^2}^2 - (p_n-1)\, \textstyle\int_{\mathcal{C}} \big(w_n(t,\theta)\big)^{p_n-2}\chi_n^2\, dx \\ =\;& \|\nabla\chi_n\|_{L^2}^2 - 2\,(\alpha+1)^2 \textstyle\int_{\mathcal{C}} \frac{\chi_n^2}{(\cosh((\alpha+1)\, t))^2}\, dx + a_n^2\, \|\chi_n\|_{L^2(\mathcal{C})}^2 \\ & +(p_n-1)\,\big(w_n(0,0)\big)^{p_n-2} \int_{\mathcal{C}} \Big[\tfrac{1}{(\cosh((\alpha+1)\, t))^2} - \Big(\tfrac{w_n(t,\theta)}{w_n(0,0)}\Big)^{p_n-2}\Big]\chi_n^2\, dx \\ & +\big[2\,(\alpha+1)^2 - (p_n-1)\,(w_n(0,0))^{p_n-2}\big] \int_{\mathcal{C}} \frac{\chi_n^2}{(\cosh((\alpha+1)\, t))^2}\, dx \\ \ge\;& \big[1 + a_n^2 - (\alpha+1)^2 - (p_n-1)\,\big(w_n(0,0)\big)^{p_n-2}\, r_n\big]\, \|\chi_n\|_{L^2(\mathcal{C})}^2 \\ & +\big[2\,(\alpha+1)^2 - (p_n-1)\,(w_n(0,0))^{p_n-2}\big] \int_{\mathcal{C}} \frac{\chi_n^2}{(\cosh((\alpha+1)\, t))^2}\, dx \end{aligned}$$

with $r_n = \sup_{\mathcal{C}} \big|\big(w_n(t,\theta)/w_n(0,0)\big)^{p_n-2} - e^V\big|$. Recall that by Lemma 20,

$$\lim_{n\to+\infty} (p_n-1)(w_n(0,0))^{p_n-2} = \mu = 2\,(\alpha+1)^2\ ,$$

and by Lemma 21, $\lim_{n\to+\infty} r_n = 0$. Since $a_n \to 0$ as $n \to +\infty$ and $(1+\alpha)^2 < 1$, we readily get a contradiction for large n, unless $\chi_n \equiv 0$. This means that w_n is independent of the variable θ, and so $w_n = w_n^*$. □

5. Concluding remarks

It is interesting to note that, via the Emden-Fowler transformation (9), for any $\alpha > -1$, inequality (2) can be stated on the space

$$\mathfrak{E}_\alpha = \Big\{ w = w(t,\theta) \in L^1(\mathcal{C}, d\nu_\alpha) \ : \ |\nabla w| \in L^2(\mathcal{C}, dx) \Big\}$$

where

$$d\nu_\alpha := \frac{\alpha+1}{2}\,\frac{dt}{\big[\cosh\big((\alpha+1)\,t\big)\big]^2}\ .$$

Proposition 23. *If $\alpha > -1$, then*

$$\int_{\mathcal{C}} e^{w - \int_{\mathcal{C}} w\, d\nu_\alpha}\, d\nu_\alpha \ \le\ e^{\frac{1}{16\,\pi\,(\alpha+1)}\left(\|\nabla w\|^2_{L^2(\mathcal{C})} + \alpha\,(\alpha+2)\,\|\,\partial_\theta w\,\|^2_{L^2(\mathcal{C})}\right)} \quad \forall\ w \in \mathfrak{E}_\alpha\ .$$

As in Section 2.1, when $\alpha \le 0$, there holds

$$\int_{\mathcal{C}} e^{w - \int_{\mathcal{C}} w\, d\nu_\alpha}\, d\nu_\alpha \ \le\ e^{\frac{1}{16\,\pi\,(\alpha+1)}\,\|\nabla w\|^2_{L^2(\mathcal{C})}} \quad \forall\ w \in \mathfrak{E}_\alpha\ ,$$

with extremals obtained from (20) up to translations, scalings and addition of constants.

However, when $\alpha > 0$, while the latter inequality is always valid for functions depending only on the variable $t \in \mathbb{R}$, in general it fails to hold in $\mathfrak{E}_\alpha$.

The above inequality is one of the three equivalent versions of the weighted Moser-Trudinger inequalities that we prove in this paper: on the sphere S^2, on the euclidean space $\mathbb{R}^2$ and on the cylinder $\mathcal{C}$. The symmetry breaking phenomenon is easily understood in this case, as clearly, the corresponding extremals are symmetric if and only if $\alpha \in (-1, 0]$.

On the contrary, the symmetry breaking phenomenon in Caffarelli-Kohn-Nirenberg inequality is a more subtle issue, since it is less evident how the weights conspire against symmetry. Our key observation is that weighted Moser-Trudinger inequalities appear as limits of Caffarelli-Kohn-Nirenberg inequalities in an appropriate blow-up limit. In this asymptotics, the case $b < h(a)$ yields to $\alpha > 0$, while the case $b > h(a)$ leads to $\alpha \in (-1, 0)$.

Appendix. The dilated stereographic projection

We use spherical coordinates $(\phi, \theta) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi)$ on $S^2 \subset \mathbb{R}^3$ and radial coordinates $(r, \theta) \in [0, \infty) \times [0, 2\pi)$ on $\mathbb{R}^2$. By definition of the dilated stereographic projection, we have

$$\cos\phi = \frac{2\, r^{\alpha+1}}{1 + r^{2(\alpha+1)}} \quad \text{and} \quad \sin\phi = \frac{r^{2(\alpha+1)} - 1}{1 + r^{2(\alpha+1)}}\ ,$$

from which we deduce

$$\cos\phi\,\frac{d\phi}{dr} = \frac{4\,(\alpha+1)\, r^{2\alpha+1}}{(1 + r^{2(\alpha+1)})^2}\ .$$

The normalized measure of the sphere S^2 is given by

$$d\sigma = \frac{1}{2}\,\cos\phi\,\frac{d\theta}{2\pi}$$

and a simple change of variables shows that, if $u(\phi,\theta)=v(r,\theta)$, then

$$\int_{S^2} f(u)\,d\sigma=\int_{\mathbb{R}^2} f(v)\,\frac{\cos\phi}{2}\,\frac{d\phi}{dr}\,dr\,\frac{d\theta}{2\pi}=\int_{\mathbb{R}^2} f(v)\,d\mu_\alpha$$

where $d\mu_\alpha=\frac{\alpha+1}{\pi}\,\frac{r^{2\alpha}}{(1+r^{2(\alpha+1)})^2}\,r\,dr\,d\theta$. Using spherical and radial coordinates respectively on S^2 and $\mathbb{R}^2$, the expressions of the gradients are given respectively as follows

$$|\nabla u|^2=|\partial_\phi u|^2+\frac{1}{\cos^2\phi}\,|\partial_\theta u|^2 \quad\text{and}\quad |\nabla v|^2=|\partial_r v|^2+\frac{1}{r^2}\,|\partial_\theta r|^2\ .$$

Knowing that $\partial_\phi u=\partial_r v\left(\frac{d\phi}{dr}\right)^{-1}$, we get

$$\int_{S^2}|\partial_\phi u|^2\,d\sigma=\int_{\mathbb{R}^2}|\partial_r v|^2\,\frac{\cos\phi}{2}\left(\frac{d\phi}{dr}\right)^{-1}dr\,\frac{d\theta}{2\pi}=\frac{1}{4\pi\,(\alpha+1)}\int_{\mathbb{R}^2}|\partial_r v|^2\,r\,dr\,d\theta\ .$$

While using that $\partial_\theta u=\partial_\theta v$, we get

$$\int_{S^2}\frac{1}{\cos^2\phi}\,|\partial_\theta u|^2\,d\sigma=\int_{\mathbb{R}^2}|\partial_\theta v|^2\,\frac{1}{2\cos\phi}\,\frac{d\phi}{dr}\,dr\,\frac{d\theta}{2\pi}=\frac{\alpha+1}{4\pi}\int_{\mathbb{R}^2}\frac{|\partial_\theta v|^2}{r^2}\,r\,dr\,d\theta\ .$$

Thus, observing that $(\alpha+1)^2-1=\alpha\,(\alpha+2)$, we conclude

$$\int_{S^2}|\nabla u|^2\,d\sigma=\frac{1}{4\pi\,(\alpha+1)}\left[\int_{\mathbb{R}^2}|\nabla v|^2\,dx+\alpha\,(\alpha+2)\int_{\mathbb{R}^2}\frac{|\partial_\theta v|^2}{r^2}\,r\,dr\,d\theta\right]\ .$$

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