MULTIBUMP SOLUTIONS AND ASYMPTOTIC EXPANSIONS FOR MESOSCOPIC ALLEN-CAHN TYPE EQUATIONS

MATTEO NOVAGA AND ENRICO VALDINOCI

Abstract. We consider a mesoscopic model for phase transitions in a periodic medium and we construct multibump solutions.

The rational perturbative case is dealt with by explicit asymptotics.

Introduction

We are concerned with the equation

$$-\Delta u + F'(u) + H(x) = 0, \qquad x \in \mathbb{R}^n, \tag{1}$$

where the smooth function F is a double-well potential.

More precisely, we assume that

- $F(t) \geqslant 0$ for any $t \in \mathbb{R}$,
- $F(t) = 0$ if and only if $t = \pm 1$, and $F''(1) = F''(-1) > 0$,
- there exist positive constants δ_0, c such that $F'(-1-s) \leqslant -c$ and $F'(1+s) \geqslant c$ for any $s \geqslant \delta_0$,
- $F(-1+s) = F(1+s)$ for any $s \in [-\delta_0, \delta_0]$.

The function $H \in L^\infty(\mathbb{R}^n)$ in (1) will be a small periodic perturbation of the operator. To this extent, we suppose that

- $\|H\|_{L^\infty(\mathbb{R}^n)}$ is suitably small,
- H is $\mathbb{Z}^n$-periodic, with zero average on $[0,1]^n$, that is

$$\begin{aligned} H(x+k) &= H(x) \qquad \forall x \in \mathbb{R}^n \text{ and } k \in \mathbb{Z}^n \\ \text{and } \int_{[0,1]^n} H(x)\,dx &= 0\,. \end{aligned} \tag{2}$$

Equation (1) is the Euler-Lagrange equation of the (formal) functional

$$\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{2} + F(u) + H(x)u \; dx. \tag{3}$$

The functional in (3) has been considered in [DLN06, NV07] as a mesoscopic model for phase transitions (see also [DY06] for the analysis of the gradient flow of (3), and [DO07] for a related problem in the random setting).

When $H = 0$, (1) is called the Ginzburg-Landau or Allen-Cahn equation, which is a popular model for superconductors and superfluids [GP58, Lan67] and for gas and solid interfaces [Row79, AC79]. Similar equations also arise in cosmology [Car95].

EV's research is partially supported by MURST Variational Methods and Nonlinear Differential Equations.

The term H may be seen as a small defect which favors locally one of the phases: condition (2) then says that such defect is "neutral" on large scales, in the sense that both the phases are equally treated.

We refer to [DLN06, DY06, NV07, DO07] for further physical motivations and geometric interpretations.

In [NV07], minimizers of (3) have been dealt with. We say that $u \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$ is a minimizer if

$$\begin{aligned}&\int_U \frac{|\nabla u|^2}{2} + F(u) + H(x)u\,dx \\ &\qquad\leqslant \int_U \frac{|\nabla(u+\psi)|^2}{2} + F(u+\psi) + H(x)(u+\psi)\,dx\end{aligned} \tag{4}$$

for any $\psi \in C^\infty_0(U)$ and any bounded domain U (minimizers of this type are often called "local", or "class A", minimizers). As usual in the calculus of variation framework, the word minimizer for (4) refers to the fact that the energy is increased by compact perturbations, even if the energy (3) in the whole of $\mathbb{R}^n$ may well be infinite.

In particular, the following result has been proved in [NV07].

Theorem 1. *There exist two $\mathbb{Z}^n$-periodic minimizers $U^\pm$ of* (3)*, with $U^+ = U^- + 2$.*
For small $\|H\|_{L^\infty(\mathbb{R}^n)}$, U^+ and U^- are uniformly close to $+1$ and -1, respectively.
Moreover, given $\omega \in S^{n-1}$, there exist minimizers $u^\pm_\omega$ of (3)*, which connects U^+ and U^- far from $\omega^\perp$.*
More explicitly, there are constants C_1, $C_2 > 0$ such that

$$|u^+_\omega(x) - U^+(x)| + |u^-_\omega(x) - U^-(x)| \leqslant C_1 e^{-C_2\langle\omega,x\rangle} \tag{5}$$

and

$$|u^+_\omega(x) - U^-(x)| + |u^-_\omega(x) - U^+(x)| \leqslant C_1 e^{C_2\langle\omega,x\rangle} \tag{6}$$

for any $x \in \mathbb{R}^n$.

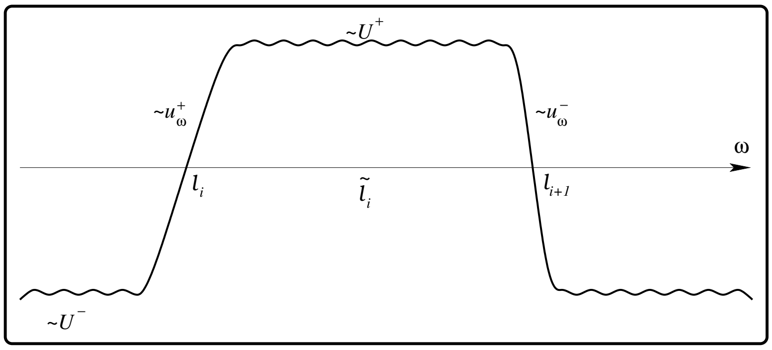
Remark 2. From the proof of [NV07, Lemma 5.1] it follows that there exists a constant $C > 0$, depending only on the potential F and the dimension n, such that

$$\|U^+ - 1\|_{L^\infty(\mathbb{R}^n)} \leqslant C\,\|H\|_{L^\infty(\mathbb{R}^n)}, \qquad \|U^- + 1\|_{L^\infty(\mathbb{R}^n)} \leqslant C\,\|H\|_{L^\infty(\mathbb{R}^n)}. \tag{7}$$

The gist of this paper is to detect multibump solutions of the mesoscopic model by gluing together pieces of $u^\pm_\omega$'s, according to the following result:

Theorem 3. *Under a suitable non-degeneracy assumption on H and $\omega \in S^{n-1}$, there exist solutions of* (1) *which connects U^+ and U^- in the direction given by ω, as many times as we want.*

Analogous layered and multibump solutions have been studied in [AJM02, RS03, RS04] and multiplicity results are also in [dlLV07]: differently from those results, the multibumps are here obtained not by perturbing the potential $F(t)$ into $Q(x)F(t)$, but by using the mesoscopic term $H(x)$.



A more formal description of Theorem 3 will be given in the subsequent Section 1.

In Section 2, we prove Theorem 3 (and, in fact, the more explicit version of it given in Theorem 6 below), while Section 3 contains comments and examples about the nondegeneracy assumption needed in Theorem 3 and an asymptotic expansion for the rational perturbative case, which we think is interesting in itself (see, in particular Theorems 13, 17 and 19 in there).

1. Formal setup and eigenvalues

First we recall an elementary property of the minimal eigenvalue:

Lemma 4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth and $\mathbb{Z}^n$-periodic function. Then,*

$$\inf_{\substack{u \in W^{1,2}(\mathbb{R}^n/\mathbb{Z}^n) \\ \|u\|_{L^2(\mathbb{R}^n/\mathbb{Z}^n)}=1}} \int_{\mathbb{R}^n/\mathbb{Z}^n} |\nabla u(x)|^2 + f(x)u^2(x)\, dx \tag{8}$$

is finite and attained at some function $v \in W^{1,2}(\mathbb{R}^n/\mathbb{Z}^n)$.

Also, $\{v = 0\} = \emptyset$ and, if $\lambda \in \mathbb{R}$ is the quantity in (8), we have that

$$-\Delta v + fv = \lambda v. \tag{9}$$

Proof. The fact that the infimum in (8) is finite and attained follows from the standard direct method in the calculus of variations. Indeed, one takes a minimizing sequence $v_n \in W^{1,2}(\mathbb{R}^n/\mathbb{Z}^n)$ of

$$E(u) := \int_{\mathbb{R}^n/\mathbb{Z}^n} |\nabla u(x)|^2 + f(x)u^2(x)\, dx$$

constrained to $\|v_n\|_{L^2(\mathbb{R}^n/\mathbb{Z}^n)} = 1$.

By comparing with a smooth compactly supported function, we may suppose that $E(v_n)$ is uniformly bounded. What is more,

$$\left| \int_{\mathbb{R}^n/\mathbb{Z}^n} f(x)v_n^2(x)\, dx \right| \leqslant \|f\|_{L^\infty(\mathbb{R}^n/\mathbb{Z}^n)}$$

and so $\|\nabla v_n\|_{L^2(\mathbb{R}^n/\mathbb{Z}^n)}$ is also uniformly bounded. This gives the compactness necessary to show that the infimum in (8) is attained, and thence finite.

Any function v attaining the minimum in (8) satisfies (9) for a Lagrange multiplier λ. By integrating (9) against v, it follows that λ agrees with the quantity in (8).

Moreover, the function $|v|$ also attains the minimum in (8) and so it satisfies (9) in the weak distributional sense. Thus, by the strong maximum principle, $|v| > 0$. ☐

We now consider the linearization of (1) around a function $u \in L^\infty(\mathbb{R}^n)$:

$$-\Delta v + F''(u)v = \lambda v, \qquad \lambda \in \mathbb{R} \tag{10}$$

and we investigate the properties of its eigenvalues.
Notice that, by Theorem 1 and Remark 2, we have

$$F''(U^+) = F''(U^-)\,. \tag{11}$$

Then, the following is a plain consequence of Lemma 4 and (11):

Proposition 5. *Let λ_1 be the minimal eigenvalue of the operator $-\Delta + F''(U^\pm)$ in $L^2(\mathbb{R}^n/\mathbb{Z}^n)$. Then $\lambda_1 > 0$ and there exists a $\mathbb{Z}^n$-periodic function $w > 0$ satisfying*

$$-\Delta w + F''(U^\pm)w = 0.$$

We now state the non-degeneracy condition needed in our paper.
For this, we introduce the following equivalence relation on $\mathbb{R}^n$. Given $\omega \in S^{N-1}$ and x, $y \in \mathbb{R}^n$, we say that $x \sim_\omega y$ if and only if $\langle \omega, x-y\rangle = 0$ and $x - y \in \mathbb{Z}^n$.
The quotient space $\mathbb{R}^n/\sim_\omega$ will be denoted by $\mathbb{R}^n_\omega$.
Let $\omega \in S^{n-1}$ be such that

(**A**) the minimal eigenvalues λ^+_ω and λ^-_ω of $-\Delta + F''(u^\pm_\omega)$ in $L^2(\mathbb{R}^n_\omega)$ are strictly positive and belong to the discrete spectrum of the operator.

Note that condition (**A**) is an assumption on both ω and H, since u^+_ω and u^-_ω depend on H (recall Theorem 1).
An equivalent formulation of condition (**A**) is that

$$\lambda^\pm_\omega := \inf_{\|u\|_{L^2(\mathbb{R}^n_\omega)}=1} \int_{\mathbb{R}^n_\omega} |\nabla u|^2 + F''(u^\pm_\omega)u^2\, dx \tag{12}$$

are strictly positive and attained at some eigenfunction $v^\pm_\omega$.

Note that, even when (**A**) fails, the quantity in (12) is non-negative, due to the minimizing properties of $u^\pm_\omega$ (recall (4) and Theorem 1).
We reckon that assumption (**A**) is satisfied for a generic function H. Such condition is analogous to the stability condition assumed in [DY06], and a formal computation is performed in Section 4.2 of [DY06] to justify such assumption. Related asymptotic expansion of eigenvalues are also in [Mar06, Bor05].
Here, in Section 3, we will make rigorous expansions, interesting in themselves, to make condition (**A**) more explicit in the rational perturbative case.
We are now in the position of giving a formal statement of Theorem 3, which is the main result of the paper.

Theorem 6. *Let H and $\omega \in \mathbb{R}^n$ be such that assumption (**A**) holds. Then, there exist solutions of (1) which connects U^+ and U^- in the direction given by ω, as many times as we want. More precisely, there exists a constant $C > 0$ such that for any $\epsilon > 0$ there exists $K > 0$ with the following property.*
Let $N \in \mathbb{Z} \cup \{-\infty\}$ and $M \in \mathbb{Z} \cup \{+\infty\}$ with $N < M$.
Let $\ell_i \in \mathbb{R}$, with $i \in \mathbb{Z} \cap [N, M]$, with $\ell_{j+1} - \ell_j \geqslant K$ for any $j \in \mathbb{Z} \cap [N, M-1]$.
Let $\ell_{N-1} := -\infty$ if $N > -\infty$ and $\ell_{M+1} := +\infty$ if $M < +\infty$ and set $\tilde{\ell}_i := (\ell_i + \ell_{i+1})/2$.
Then, there exists a solution u of (1) such that $u(x)$ has distance less than $C\epsilon$ from, alternately, $u^+_\omega(x - \ell_i\omega)$ and $u^-_\omega(x - \ell_i\omega)$, for any $x \in \mathbb{R}^n$ such that $\langle \omega, x\rangle \in (\tilde{\ell}_{i-1} + 1, \tilde{\ell}_i - 1)$.

Moreover, $u(x)$ has distance less than $C\epsilon$ from, alternately, U^+ and U^- for any $x \in \mathbb{R}^n$ such that $\langle \omega, x\rangle \in (\tilde{\ell}_i - 2, \tilde{\ell}_i + 2)$.

In Theorem 6 above, we made use of the obvious notation

$$[-\infty, a] := (-\infty, a] \cup \{-\infty\}, \qquad [a, +\infty] := [a, +\infty) \cup \{+\infty\}$$
$$\text{and} \qquad [-\infty, +\infty] := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}.$$

The content of Theorem 6 is visualized in the figure on page 3. Namely, the multibump solution we construct has one and only one excursion from (the vicinity of) one phase to (the vicinity of) the other one in a large interval around ℓ_i, while each of these transitions is suitable glued with the opposite one near $\tilde{\ell}_i$.

2. Proof of Theorem 6

First, let us suppose that $N \neq -\infty$ and $M \neq +\infty$. Up to relabelling ℓ_i, we may suppose that $N = 0$, so the points ℓ_i are just

$$\ell_0, \ell_1, \dots, \ell_M.$$

Let $\phi \in C^\infty(\mathbb{R}, [0, 1])$ be such that $\phi(t) = 1$ for any $t \geqslant 1$ and $\phi(t) = 0$ for any $t \leqslant -1$.
For any $i \in \mathbb{Z} \cap [0, M-1]$, let $\phi_i(x) := \phi(\langle \omega, x\rangle - \tilde{\ell}_i)$.
For any $i \in \mathbb{Z} \cap [0, M]$ let also u_i be alternately $u^+_\omega(x - \ell_i\omega)$ and $u^-_\omega(x - \ell_i\omega)$, as prescribed by Theorem 6.
We also set v_i to be either v^+_ω, if $u_i = u^+_\omega$, or v^-_ω, if $u_i = u^-_\omega$, where $v^\pm_\omega$ is given by condition (**A**), according to (12). The eigenvalue $\lambda^\pm_\omega$ corresponding to v_i will be denoted by λ_i.
Analogously, we set z_i (resp., $\hat{z}_i$) to be either U^+ (resp., U^-), if $u_i = u^+_\omega$, or U^- (resp., U^+), if $u_i = u^-_\omega$, where $U^\pm$ is given in Theorem 1.
Note that, by Theorem 1, there exists a suitable $L > 0$ in such a way that

$$\begin{aligned} |u_i(x) - z_i(x)| &\leqslant \frac{\lambda_1}{2\|F'''\|_{L^\infty([-2,2])}} \quad \text{as long as } \langle \omega, x\rangle \geqslant L, \text{ and} \\ |u_i(x) - \hat{z}_i(x)| &\leqslant \frac{\lambda_1}{2\|F'''\|_{L^\infty([-2,2])}} \quad \text{as long as } \langle \omega, x\rangle \leqslant -L. \end{aligned} \tag{13}$$

Recalling (11), we also define

$$\Phi(x) := F''(z_i(x)) = F''(\hat{z}_i(x)). \tag{14}$$

Note that

$$\begin{aligned} &|\Phi(x) - F''(u_i(x))| \\ &\quad \leqslant \|F'''\|_{L^\infty([-2,2])} \min\{|u_i(x) - z_i(x)|,\ |u_i(x) - \hat{z}_i(x)|\} \quad \text{for any } x \in \mathbb{R}^n. \end{aligned} \tag{15}$$

Given $C > 0$, to take suitably large in the sequel, we define

$$\tilde{u}^\pm_i := u_i \pm \epsilon(w + Cv_i),$$

where w is the one given by Proposition 5, and

$$\begin{aligned}\beta^{\pm}(x) \;&:=\; \left(\prod_{j=0}^{M-1}(1-\phi_j(x))\right)\tilde u_0^{\pm}(x)\\ &+\; \sum_{i=1}^{M-1}\left(\left(\prod_{j=i}^{M-1}(1-\phi_j(x))\right)\phi_{i-1}(x)\tilde u_i^{\pm}(x)\right)\\ &+\; \phi_{M-1}(x)\tilde u_M^{\pm}(x).\end{aligned}$$

Note that, if $k\in\mathbb{Z}\cap[1,M-1]$ and $\langle\omega,x\rangle\in[\tilde\ell_{k-1}+1,\tilde\ell_k-1]$, we have that $\phi_j(x)=0$ for any $j\geqslant k$ and $\phi_j(x)=1$ for any $j\leqslant k-1$, thence

$$\beta^{\pm}=\tilde u_k^{\pm}\ \text{ if }\ \langle\omega,x\rangle\in[\tilde\ell_{k-1}+1,\tilde\ell_k-1]. \tag{16}$$

Also,

$$\beta^{\pm}(x)=\tilde u_0^{\pm}(x) \tag{17}$$

if $\langle\omega,x\rangle\leqslant\tilde\ell_0-1$ and

$$\beta^{\pm}(x)=\tilde u_M^{\pm}(x) \tag{18}$$

if $\langle\omega,x\rangle\geqslant\tilde\ell_M+1$.
Also, if $k\in\mathbb{Z}\cap[0,M-1]$, we have that $\phi_j(x)=0$ if $k<j\leqslant M-1$ and $\phi_j(x)=1$ if $0\leqslant j<k$, when $\langle\omega,x\rangle\in(\tilde\ell_k-2,\tilde\ell_k+2)$. Accordingly,

$$\beta^{\pm}=(1-\phi_k)u_k^{\pm}+\phi_k u_{k+1}^{\pm}\qquad\text{if }\langle\omega,x\rangle\in(\tilde\ell_k-2,\tilde\ell_k+2). \tag{19}$$

Recalling Theorem 1, we have that both u_k^+ and u_{k+1}^+ (resp., u_k^- and u_{k+1}^-) are uniformly close (hence C^2-close, by elliptic estimates) to z_k (resp., $\hat z_k$) in $\{\langle\omega,x\rangle\in(\tilde\ell_k-2,\tilde\ell_k+2)\}$, as long as $\ell_{k+1}-\ell_k$ is large enough.
Therefore, by (19),

$$\|\beta^{\pm}-u_k^{\pm}\|_{C^2(\{\langle\omega,x\rangle\in(\tilde\ell_k-2,\tilde\ell_k+2)\})}\ \text{is as small as we like,} \tag{20}$$

as long as $\ell_{k+1}-\ell_k$ is large enough.
As a consequence of (16), (17), (18) and (20), we have that for any $x\in\mathbb{R}^n$ there exists i in such a way that

$$\sum_{|j|=0}^{2}|D^j(\beta^{\pm}-u_i^{\pm})(x)|\ \text{is as small as we like,} \tag{21}$$

provided that the ℓ_i's are conveniently far apart.
We now claim that there exists $c>0$ such that

$$-\|F'''\|_{L^\infty([-2,2])}\min\{|u_i-z_i|,\,|u_i-\hat z_i|\}w+\lambda_1 w+C\lambda_i v_i\geqslant c, \tag{22}$$

as long as C is chosen suitably large (recall that w and λ_1 are the ones given by Proposition 5).

To prove (22), we distinguish two cases. If $|\langle\omega, x\rangle| \geqslant L$, we use (13) to get

$$\begin{aligned} & -\|F'''\|_{L^\infty([-2,2])}\min\{|u_i - z_i|,\, |u_i - \hat{z}_i|\}w + \lambda_1 w + C\lambda_i v_i \\ & \geqslant \frac{\lambda_1}{2} w + C\lambda_i v_i \\ & \geqslant \frac{\lambda_1}{2}\inf_{\mathbb{R}^n/\mathbb{Z}^n} w. \end{aligned} \tag{23}$$

If, on the other hand, $|\langle\omega, x\rangle| \leqslant L$, we have

$$\begin{aligned} & -\|F'''\|_{L^\infty([-2,2])}\min\{|u_i - z_i|,\, |u_i - \hat{z}_i|\}w + \lambda_1 w + C\lambda_i v_i \\ & \geqslant -5\|F'''\|_{L^\infty([-2,2])} + C\lambda_i \inf_{|\langle\omega,x\rangle|\leqslant L} v_i. \end{aligned} \tag{24}$$

Then, (22) follows from (23) and (24) if C is conveniently large.
Furthermore, recalling the setting of (14), we see that

$$\begin{aligned} & -\Delta\tilde{u}_i^\pm + F'(\tilde{u}_i^\pm) + H(x) \\ & \quad = -\Delta u_i + F'(u_i \pm \epsilon(w + Cv_i)) + H(x) \pm \epsilon(-\Delta w - C\Delta v_i) \\ & \quad = F'\big(u_i \pm \epsilon(w + Cv_i)\big) - F'(u_i) \pm \epsilon(\lambda_1 w + C\lambda_i v_i - \Phi w - CF''(u_i)v_i) \\ & \quad = \pm\epsilon\Big(\big(F''(u_i) - \Phi\big)w + \lambda_1 w + C\lambda_i v_i\Big) + O(\epsilon^2). \end{aligned}$$

As a consequence of the latter estimate, (15) and (22), we deduce that

$$\begin{aligned} & -\Delta\tilde{u}_i^+ + F'(\tilde{u}_i^+) + H(x) \geqslant c\epsilon/2 \qquad \text{and} \\ & -\Delta\tilde{u}_i^- + F'(\tilde{u}_i^-) + H(x) \leqslant -c\epsilon/2. \end{aligned} \tag{25}$$

By (21) and (25), we gather that

$$\begin{aligned} & -\Delta\beta^+ + F'(\beta^+) + H(x) \geqslant c\epsilon/4 \qquad \text{and} \\ & -\Delta\beta^- + F'(\beta^-) + H(x) \leqslant -c\epsilon/4, \end{aligned} \tag{26}$$

as long as ℓ_{i+1} and ℓ_i are all distanced enough (possibly in dependence of ϵ).
Let $\eta := (\beta^+ + \beta^-)/2$. Then, η is smooth and $\beta^- < \eta < \beta^+$. Thus, for any $R > 0$, we let u_R be a solution of

$$-\Delta u_R + F'(u_R) + H(x) = 0$$

in the open ball B_R, with $u = \eta$ on ∂B_R.
Note that the existence of such u_R is warranted, for instance, by direct minimization and that $\beta^- \leqslant u_R \leqslant \beta^+$ by Comparison Principle and (26).
Also, by elliptic regularity theory, u_R converges, up to subsequences, to some u, which is a solution of (1) and which is trapped between β^- and β^+.
Such u is the desired multibump solution, thanks to (16), (17) (18), (20) and Remark 2, thus proving Theorem 6 when both N and M are finite.
The case in which N or/and M become infinite is then obtained by taking limits, due to elliptic estimates. This ends the proof of Theorem 6.

Remark 7. From the above proof it also follows that when $\lambda_\omega^+ > 0$ (but possibly $\lambda_\omega^- = 0$), then there are homoclinic type connections between $u_\omega^+(x - \ell_0)$ and $u_\omega^-(x - \ell_1)$, for $\ell_1 - \ell_0$ suitably large.
Analogously, when $\lambda_\omega^- > 0$ (but possibly $\lambda_\omega^+ = 0$), then there are homoclinic type connections between $u_\omega^-(x - \ell_0)$ and $u_\omega^+(x - \ell_1)$.

That is, if we control only one eigenvalue in **(A)**, we are still able to construct onebump solutions.

3. On the validity of the non-degeneracy assumption

We consider now the case in which ω is rational, i.e., $\omega \in \mathbb{Q}^n$. Notice that in this case $\mathbb{R}^n_\omega$ is the topological product of $\mathbb{R}$ and a $(n-1)$-dimensional torus.
We also suppose that

$$H_\epsilon = \epsilon h, \tag{27}$$

and we show that, even if assumption **(A)** is violated for $\epsilon = 0$, it does hold, for somewhat generic h's, if $\epsilon \neq 0$ (see for instance Theorem 19 below).

Lemma 8. *Let $u^\pm_\epsilon = u^\pm_\omega$ be the function given by Theorem 1 when $H = \epsilon h$ is as in* (27)*. Then, there exists a sequence $\epsilon_n \to 0$ and a smooth function $\gamma^\pm$ which is a minimal solution of*

$$-\Delta\gamma^\pm + F'(\gamma^\pm) = 0, \qquad \text{for any } x \in \mathbb{R}^n \tag{28}$$

satisfying

$$\gamma^\pm(x) = \gamma^\pm_o(\langle\omega, x\rangle) \qquad \text{for any } x \in \mathbb{R}^n \tag{29}$$

for suitable $\gamma^\pm_o : \mathbb{R} \to \mathbb{R}$, with

$$\lim_{t\to+\infty} \gamma^\pm_o(t) = \pm 1, \qquad \text{and} \qquad \lim_{t\to-\infty} \gamma^\pm_o(t) = \mp 1, \tag{30}$$

for which

$$\lim_{\epsilon\to 0} u^\pm_\epsilon = \gamma^\pm, \tag{31}$$

uniformly on $\mathbb{R}^n_\omega$.

Proof. By elliptic regularity estimates and the Ascoli-Arzelà Theorem, $u^\pm_\epsilon$ converges locally uniformly, up to subsequence, to some $\gamma^\pm$. Since $u^\pm_\epsilon$ is a solution of (1) with H as in (27), passing to the limit we get (28). More precisely, since $u^\pm_\epsilon$ minimizes the energy (4) under compact perturbations with H as in (27), passing to the limit we conclude that $\gamma^\pm$ minimizes the energy under compact perturbations with $H = 0$.
In fact, the limit in (31) is uniform, not only locally uniform, in $\mathbb{R}^n_\omega$. Indeed, suppose, by contradiction, that there exists an infinitesimal sequence ϵ_m and $x_m \in \mathbb{R}^n_\omega$ such that

$$|u^+_{\epsilon_m}(x_m) - \gamma^+(x_m)| \geqslant a, \tag{32}$$

for some $a > 0$. From (5), (6) and (7),

$$|u^+_{\epsilon_m}(x) - \gamma^+(x)| \leqslant C\big(e^{-C|\langle\omega,x\rangle|} + \epsilon_m\big)$$

and so $|\langle\omega, x_m\rangle| \leqslant \bar{C}$, for a suitable $\bar{C} > 0$, due to (32). Then, by the locally uniform convergence,

$$|u^+_{\epsilon_m}(x_m) - \gamma^+(x_m)| \leqslant \|u^+_{\epsilon_m} - \gamma^+\|_{L^\infty(\{|\langle\omega,x\rangle|\leqslant\bar{C}\})} \leqslant a/2$$

for large m, in contradiction with (32)
This proves the limit in (31) to be uniform in $\mathbb{R}^n_\omega$.
Accordingly, the limits of $\gamma^\pm$ for $\langle\omega, x\rangle \to \pm\infty$ are uniformly attained, because so are the ones of $u^\pm_\epsilon$, in the light of (5), (6) and (7).
Then, the results in the literature on the De Giorgi-Gibbons conjecture (see, e.g., [Far03]) imply the one-dimensional symmetry claimed in (29). □

From now on, we will fix the sequence ϵ_n, which for simplicity we will still call ϵ, and the limit functions $\gamma^\pm$ given by Lemma 8.

Lemma 9. *The functions $(\gamma_o^\pm)'$ are strictly positive on the whole of $\mathbb{R}$.*

Proof. First of all

$$\text{solutions of } \ddot{x}(t) + F'(x(t)) = 0 \text{ are even with respect to critical points.} \tag{33}$$

Indeed, if $\dot{x}(\tau) = 0$ for some $\tau \in \mathbb{R}$, we set $X_\pm(t) := x(\pm t + \tau)$, so that $X_+(0) = X_-(0) = x(\tau)$ and $\dot{X}_+(0) = \dot{X}_-(0) = 0$. The uniqueness of ODE solutions then implies that $X_+(t) = X_-(t)$, that is $x(t + \tau) = x(-t + \tau)$, proving (33).
Note that $\gamma_o^\pm$ cannot have one and only one critical point, because of (30) and (33). Consequently, if the claim of Lemma 9 were false, $\gamma_o^\pm$ would have at least two critical points, and so, by (33), they would be periodic. This is in contradiction with (30) and proves the desired result. □

In what follows, when no confusion is possible, the subindex of $\gamma_o^\pm$ will be dropped and $\gamma^\pm$ will be identified with $\gamma_o^\pm$ without further comments. In particular, we will denote by $(\gamma^\pm)'$ the derivative of $\gamma^\pm$ in the direction given by ω, i.e. $(\gamma^\pm)' = \langle \nabla\gamma^\pm, \omega\rangle = (\gamma_o^\pm)'(\langle\omega, x\rangle)$.
We now introduce the Schrödinger operator

$$T^\pm = -\Delta + F''(\gamma^\pm(x)).$$

We observe that, as a consequence of (28),

$$\begin{aligned} T^\pm\big((\gamma^\pm)''\big) &= T^\pm\big(F'(\gamma^\pm)\big) - \Delta\big(F'(\gamma^\pm)\big) + F''(\gamma^\pm)F'(\gamma^\pm) \\ &= -F'''(\gamma^\pm)\left((\gamma^\pm)'\right)^2 - F''(\gamma^\pm)\,(\gamma^\pm)'' + F''(\gamma^\pm)F'(\gamma^\pm) \\ &= -F'''(\gamma^\pm)\left((\gamma^\pm)'\right)^2. \end{aligned} \tag{34}$$

Lemma 10. *The spectrum of $T^\pm$ is composed of an essential spectrum, corresponding to the unbounded interval $[F''(1), +\infty)$, and of a discrete spectrum, given by a finite number of eigenvalues $0 = \lambda_0^\pm < \cdots < \lambda_N^\pm < F''(1)$, with finite multiplicities.*
Moreover, the eigenspace corresponding to $\lambda_0^\pm = 0$ is spanned by the eigenfunction $(\gamma^\pm)' \in L^2(\mathbb{R}^n_\omega)$.

Proof. The first assertion follows from [Kat95, Theorem 5.7 in Chapter V.5.3].
The fact that $\lambda_0^\pm$ has multiplicity one follows from the minimality property of $(\gamma^\pm)'$ and the strong maximum principle, applied to the equation $T^\pm v = 0$ (indeed, the argument in [Eva98, page 340] may be repeated verbatim here). □

We now define

$$\Im := \left((\gamma^\pm)'\right)^\perp = \left\{\psi \in L^2(\mathbb{R}^n_\omega) \text{ s.t. } \int_{\mathbb{R}^n_\omega} \psi\,(\gamma^\pm)'\,dx = 0\right\}.$$

Lemma 11. *For any $g_0 \in \Im$ there exists a unique $g_1 \in \Im$ such that $T^\pm g_1 = g_0$.*

Proof. Notice that $T^\pm$ is self-adjoint and its domain is dense in $L^2(\mathbb{R}^n_\omega)$, thence it is a closed operator, and its image is the orthogonal to the kernel (see, e.g., Section II.6 in [Bre83]). Since the kernel of $T^\pm$ is spanned by $(\gamma^\pm)'$, due to Lemma 10, we get that given any $g_0 \in \Im$ there exists $\tilde{g}_1 \in L^2(\mathbb{R}^n_\omega)$ such that $T^\pm\tilde{g}_1 = g_0$.

We now set

$$g_1 := \tilde{g}_1 - \frac{\int_{\mathbb{R}^n_\omega} \tilde{g}_1 (\gamma^\pm)' \, dx}{\|(\gamma^\pm)'\|^2_{L^2(\mathbb{R}^n_\omega)}} (\gamma^\pm)'.$$

Such g_1 lies in $\Im$ and $T^\pm g_1 = T^\pm \tilde{g}_1 = g_0$.
Moreover, if $T^\pm g_2 = g_0$, with $g_2 \in \Im$, we have that $T^\pm(g_1 - g_2) = 0$ and so, by Lemma 10, $g_1 - g_2 = C(\gamma^\pm)'$, for some $C \in \mathbb{R}$. Therefore,

$$C\|(\gamma^\pm)'\|^2_{L^2(\mathbb{R}^n_\omega)} = \int_{\mathbb{R}^n_\omega} (g_1 - g_2)(\gamma^\pm)' \, dx = 0,$$

so $C = 0$ and $g_1 = g_2$. □

In the light of Lemma 11, given $g_0 \in \Im$, we define $(T^\pm)^{-1} g_0$ to be the unique element g_1 in $\Im$ for which $T^\pm g_1 = g_0$.
Since $T^\pm$ is self-adjoint, we have that

$$\int_{\mathbb{R}^n_\omega} \Big((T^\pm)^{-1} f \Big) g \, dx = \int_{\mathbb{R}^n_\omega} f \Big((T^\pm)^{-1} g \Big) \, dx, \tag{35}$$

for any f, $g \in \Im$.
Given $x \in \mathbb{R}^n_\omega$, we let

$$\Omega_x := \{ y \in \mathbb{R}^n_\omega : \ \langle \omega, x - y \rangle = 0 \}.$$

Note that Ω_x is an $(n-1)$-dimensional torus.

Lemma 12. *Let $x_0^\pm \in \mathbb{R}^n_\omega$ be such that $\gamma^\pm(x_0^\pm) = 0$. Then, there exists an infinitesimal sequence $M^\pm_\epsilon$ for which*

$$\int_{\Omega_{x_0^\pm}} u^\pm_\epsilon(x) \ dx = \gamma^\pm(x_0^\pm + M^\pm_\epsilon \omega) |\Omega_{x_0^\pm}|.$$

Proof. Let

$$m^\pm_\epsilon := \frac{1}{|\Omega_{x_0^\pm}|} \int_{\Omega_{x_0^\pm}} u^\pm_\epsilon(x) \ dx.$$

Thanks to (31) we get $m^\pm_\epsilon \to 0$, as $\epsilon \to 0$. By Lemma 9, we know that γ_o is invertible. Thus, the thesis follows by letting $M^\pm_\epsilon := (\gamma^\pm_o)^{-1}(m^\pm_\epsilon) - \langle \omega, x_0^\pm \rangle$. □

We will now consider the translated etheroclinic

$$\gamma^\pm_\epsilon(x) := \gamma^\pm(x + M^\pm_\epsilon \omega),$$

for which there holds

$$\int_{\Omega_{x_0^\pm}} \gamma^\pm_\epsilon \ dx = \int_{\Omega_{x_0^\pm}} u^\pm_\epsilon \ dx. \tag{36}$$

We are in the position of improving the asymptotics of Lemma 8:

Theorem 13. *For all $\epsilon > 0$, there exist smooth functions $\phi^\pm \in L^\infty(\mathbb{R}^n_\omega)$ such that*

$$u^\pm_\epsilon(x) = \gamma^\pm_\epsilon(x) + \epsilon \phi^\pm(x) + o(\epsilon). \tag{37}$$

Moreover, $\phi^\pm$ are solutions of

$$-\Delta \phi^\pm + F''(\gamma^\pm) \phi^\pm + h = 0. \tag{38}$$

Proof. We introduce the cylindrical slab

$$\mathcal{B}_R := \{x \in \mathbb{R}^n_\omega \text{ such that } |\langle \omega, x\rangle| \leqslant R\}.$$

Let

$$\phi^\pm_\epsilon := \frac{u^\pm_\epsilon - \gamma^\pm_\epsilon}{\epsilon} \tag{39}$$

and

$$c^\pm_\epsilon := \int_0^1 F''(\gamma^\pm_\epsilon + \tau(u^\pm_\epsilon - \gamma^\pm_\epsilon))\, d\tau\,.$$

Note that $c^\pm_\epsilon$ is a smooth function, which is uniformly bounded in ϵ and close to $F''(\gamma^\pm)$ for small ϵ, by (31), and that

$$L^\pm_\epsilon \phi^\pm_\epsilon + h = 0\,, \tag{40}$$

where we defined the operator

$$L^\pm_\epsilon := -\Delta + c^\pm_\epsilon.$$

We claim that, for any $R \geqslant 1$ there exists $C_R > 0$, independent of ϵ, such that

$$\|\phi^\pm_\epsilon\|_{L^2(\mathcal{B}_R)} \leqslant C_R. \tag{41}$$

For this, we denote by $U^\pm_\epsilon = U^\pm$ the $\mathbb{Z}^n$-periodic minimizers of Theorem 1 and we consider the functions

$$\psi^\pm_\epsilon := \frac{U^\pm_\epsilon \mp 1}{\epsilon},$$

which solve the equation

$$-\Delta\psi^\pm_\epsilon + d^\pm_\epsilon \psi^\pm_\epsilon + h = 0,$$

where

$$d^\pm_\epsilon := \int_0^1 F''(\pm 1 + \tau(U^\pm_\epsilon \mp 1))\, d\tau\,.$$

Recall that, from Remark 2,

$$\|\psi^\pm_\epsilon\|_{L^\infty(\mathbb{R}^n_\omega)} \leqslant C\|h\|_{L^\infty(\mathbb{R}^n_\omega)}, \tag{42}$$

where the constant C does not depend on ϵ.
We now let

$$\eta^\pm_\epsilon := \phi^\pm_\epsilon - \psi^\pm_\epsilon.$$

From Theorem 1, we have that the functions $\eta^\pm_\epsilon$ lie in $W^{2,2}(\mathbb{R}^n_\omega)$, and solve

$$L^\pm_\epsilon \eta^\pm_\epsilon = \left(d^\pm_\epsilon - c^\pm_\epsilon\right)\psi^\pm_\epsilon. \tag{43}$$

Notice that, since $u^\pm_\epsilon$ converge exponentially to $U^\pm_\epsilon$ *independently* of ϵ, we have

$$\|d^\pm_\epsilon - c^\pm_\epsilon\|_{L^2(\mathbb{R}^n_\omega)} \leqslant C, \tag{44}$$

for some constant $C > 0$ independent of ϵ.
Let now $\mu^\pm_\epsilon$ be the minimal eigenvalue of the operator $L^\pm_\epsilon$ on $L^2(\mathbb{R}^n_\omega)$, and $w^\pm_\epsilon > 0$ the corresponding eigenvector, which we may take with $L^2(\mathbb{R}^n_\omega)$-norm equal to 1. Notice that, as $\epsilon \to 0$, we have that $\mu^\pm_\epsilon$ is simple, $\mu^\pm_\epsilon \to 0$ and $w^\pm_\epsilon \to \pm(\gamma^\pm)'/\|(\gamma^\pm)'\|_{L^2(\mathbb{R}^n_\omega)}$, uniformly on compact subsets of $\mathbb{R}^n_\omega$, due to Lemma 10, the continuity properties of the eigenvalues [Kat95, Chapter IV.3.5], and the regularity estimates for $w^\pm_\epsilon$ [GT83, Theorem 8.13].
In particular, by Lemma 9 there exists $c > 0$ such that

$$w^\pm_\epsilon(y) \geqslant c \qquad \text{for all } y \in \Omega_{x_0^\pm}. \tag{45}$$

Let us now split $\eta_\epsilon^\pm = \tilde\eta_\epsilon^\pm + \alpha_\epsilon^\pm w_\epsilon^\pm$, where $\alpha_\epsilon^\pm = \langle \eta_\epsilon^\pm, w_\epsilon^\pm\rangle_{L^2(\mathbb{R}^n_\omega)}$. Recalling (31), we see that $L_\epsilon^\pm$ is a perturbation of $T^\pm$ and so, by Lemma 10 and [Kat95, page 208, Theorem 3.1], we see that $[-\sigma_o, \sigma_o]$ does not meet the spectrum of $L_\epsilon^\pm$ except that in $\mu_\epsilon^\pm$, for some suitably small $\sigma_o > 0$, independent of ϵ.
As a consequence, we get

$$\int_{\mathbb{R}^n_\omega} (L_\epsilon^\pm \eta_\epsilon^\pm)\tilde\eta_\epsilon^\pm \, dx = \int_{\mathbb{R}^n_\omega} (L_\epsilon^\pm \tilde\eta_\epsilon^\pm)\tilde\eta_\epsilon^\pm \, dx \geqslant \sigma_o \|\tilde\eta_\epsilon^\pm\|^2_{L^2(\mathbb{R}^n_\omega)}$$

and so, recalling (43), (44) and (42), we get

$$\begin{aligned} \|\tilde\eta_\epsilon^\pm\|_{L^2(\mathbb{R}^n_\omega)} &\leqslant \frac{C}{\|\tilde\eta_\epsilon^\pm\|_{L^2(\mathbb{R}^n_\omega)}} \int_{\mathbb{R}^n_\omega} (L_\epsilon^\pm \eta_\epsilon^\pm)\tilde\eta_\epsilon^\pm \, dx \\ &\leqslant C\|d_\epsilon^\pm - c_\epsilon^\pm\|_{L^2(\mathbb{R}^n_\omega)} \|\psi_\epsilon^\pm\|_{L^\infty(\mathbb{R}^n_\omega)} \\ &\leqslant C\|h\|_{L^\infty(\mathbb{R}^n_\omega)}. \end{aligned} \tag{46}$$

Since, by (43),

$$\begin{aligned} \alpha_\epsilon^\pm L_\epsilon^\pm w_\epsilon^\pm &= \langle \eta_\epsilon^\pm, w_\epsilon^\pm\rangle_{L^2(\mathbb{R}^n_\omega)} \, \mu_\epsilon^\pm w_\epsilon^\pm \\ &= \langle \eta_\epsilon^\pm, L_\epsilon^\pm w_\epsilon^\pm\rangle_{L^2(\mathbb{R}^n_\omega)} \, w_\epsilon^\pm \\ &= \langle L_\epsilon^\pm \eta_\epsilon^\pm, w_\epsilon^\pm\rangle_{L^2(\mathbb{R}^n_\omega)} \, w_\epsilon^\pm \\ &= \langle (d_\epsilon^\pm - c_\epsilon^\pm)\psi_\epsilon^\pm, w_\epsilon^\pm\rangle_{L^2(\mathbb{R}^n_\omega)} \, w_\epsilon^\pm, \end{aligned}$$

we see that $\tilde\eta_\epsilon^\pm$ solves the equation

$$L_\epsilon^\pm \tilde\eta_\epsilon^\pm = \left(d_\epsilon^\pm - c_\epsilon^\pm\right)\psi_\epsilon^\pm - \langle \left(d_\epsilon^\pm - c_\epsilon^\pm\right)\psi_\epsilon^\pm, w_\epsilon^\pm\rangle_{L^2(\mathbb{R}^n_\omega)} \, w_\epsilon^\pm \, .$$

Therefore, recalling (42), (44) and (46), elliptic regularity [GT83, Theorem 8.12] yields

$$\begin{aligned} \|\tilde\eta_\epsilon^\pm\|_{W^{2,2}(\mathbb{R}^n_\omega)} &\leqslant C\left(\|\tilde\eta_\epsilon^\pm\|_{L^2(\mathbb{R}^n_\omega)} + \|\left(d_\epsilon^\pm - c_\epsilon^\pm\right)\psi_\epsilon^\pm\|_{L^2(\mathbb{R}^n_\omega)}\right) \\ &\leqslant C\|h\|_{L^\infty(\mathbb{R}^n_\omega)}. \end{aligned} \tag{47}$$

We let $\overline{\eta}_\epsilon^\pm : \mathbb{R} \to \mathbb{R}$ be the average of $\tilde\eta_\epsilon^\pm$ on sections of $\mathbb{R}^n_\omega$ orthogonal to ω, i.e.

$$\overline{\eta}_\epsilon^\pm(t) := \frac{1}{|\Omega_{t\omega}|}\int_{\Omega_{t\omega}} \tilde\eta_\epsilon^\pm \, dx.$$

From (47) and the one-dimensional Sobolev Embedding Theorem [Bre83, Theorem IX.12], we get

$$\|\overline{\eta}_\epsilon^\pm\|_{L^\infty(\mathbb{R})} \leqslant C\|\overline{\eta}_\epsilon^\pm\|_{W^{2,2}(\mathbb{R})} \leqslant C\|\tilde\eta_\epsilon^\pm\|_{W^{2,2}(\mathbb{R})} \leqslant C\|h\|_{L^\infty(\mathbb{R}^n_\omega)}. \tag{48}$$

In order to obtain (41), it remains to bound the coefficient $\alpha_\epsilon^\pm$. Recalling (36) and (42), we have

$$\frac{1}{|\Omega_{x_0^\pm}|}\left|\int_{\Omega_{x_0^\pm}} \eta_\epsilon^\pm \, dx\right| \leqslant C\|h\|_{L^\infty(\mathbb{R}^n_\omega)}.$$

Therefore, by (45) and (48),

$$\begin{aligned} c\,|\alpha_\epsilon^\pm| &\leqslant \frac{|\alpha_\epsilon^\pm|}{|\Omega_{x_0^\pm}|}\left|\int_{\Omega_{x_0^\pm}} w_\epsilon^\pm \, dx\right| \\ &= \frac{1}{|\Omega_{x_0^\pm}|}\left|\int_{\Omega_{x_0^\pm}} \left(\eta_\epsilon^\pm - \tilde{\eta}_\epsilon^\pm\right)\, dx\right| \leqslant C\|h\|_{L^\infty(\mathbb{R}^n_\omega)}. \end{aligned}$$

This estimate, together with (42) and (46), gives (41).

It follows from (40), (41) and standard elliptic estimates (see, e.g., [Eva98, Section 6.3.1]) that $\phi_\epsilon^\pm$ converges, up to subsequence, to some $\phi^\pm \in L^\infty(\mathbb{R}^n_\omega)$, uniformly on compact subsets of $\mathbb{R}^n_\omega$. Hence, (37) is a consequence of (39).

Passing to the limit in (40) and recalling Lemma 8, we finally obtain (38). ☐

Proposition 14. *Let*

$$\lambda_\epsilon^\pm := \inf_{\|u\|_{L^2(\mathbb{R}^n_\omega)}=1} \int_{\mathbb{R}^n_\omega} |\nabla u|^2 + F''(u_\epsilon^\pm)u^2 \, dx.$$

Then, $\lambda_\epsilon^\pm$ belongs to the discrete spectrum of the operator and

$$\lambda_\epsilon^\pm = \frac{\epsilon}{\|(\gamma^\pm)'\|^2_{L^2(\mathbb{R}^n_\omega)}} \int_{\mathbb{R}^n_\omega} F'''(\gamma^\pm)\Big((\gamma^\pm)'\Big)^2 \phi^\pm \, dx + o(\epsilon). \tag{49}$$

Proof. Since, by (37),

$$\begin{aligned} &\int_{\mathbb{R}^n_\omega} |\nabla u|^2 + F''(u_\epsilon^\pm)u^2 dx \\ &\qquad \leqslant \int_{\mathbb{R}^n_\omega} |\nabla u|^2 + F''(\gamma_\epsilon^\pm)u^2 dx + \epsilon\|F'''\|_{L^\infty([-2,2])}\|\phi^\pm\|_{L^\infty(\mathbb{R}^n_\omega)} \int_{\mathbb{R}^n_\omega} u^2 dx, \end{aligned}$$

we get

$$\lambda_\epsilon^\pm \leqslant C\epsilon. \tag{50}$$

Since $\lambda_\epsilon^\pm$ is small, according to Lemma 10 and the continuity properties of the spectrum (see [Kat95, Chapter IV]), it does not lie in the essential spectrum of $-\Delta + F''(u_\epsilon^\pm)$, hence it belongs to the discrete spectrum.

Let now $w_\epsilon^\pm$ be the eigenvector corresponding to $\lambda_\epsilon^\pm$ such that

$$\|w_\epsilon^\pm\|_{L^2(\mathbb{R}^n_\omega)} = 1, \tag{51}$$

i.e. there holds

$$\lambda_\epsilon^\pm = \int_{\mathbb{R}^n_\omega} |\nabla w_\epsilon^\pm|^2 + F''(u_\epsilon^\pm)(w_\epsilon^\pm)^2 \, dx. \tag{52}$$

Then, by (37),

$$\lambda_\epsilon^\pm = \int_{\mathbb{R}^n_\omega} |\nabla w_\epsilon^\pm|^2 + F''(\gamma_\epsilon^\pm)(w_\epsilon^\pm)^2 + \epsilon F'''(\gamma_\epsilon^\pm)(w_\epsilon^\pm)^2\phi^\pm \, dx + o(\epsilon). \tag{53}$$

In particular, $\|\nabla w_\epsilon^\pm\|_{L^2(\mathbb{R}^n_\omega)}$ is uniformly bounded, thence we may suppose that

(54) $w_\epsilon^\pm$ converges to some $w^\pm$ weakly in $W^{1,2}(\mathbb{R}^n_\omega)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^n_\omega)$.

Recall that from Lemma 10 and the spectral theorem we have

$$\int_{\mathbb{R}^n_\omega} |\nabla u|^2 + F''(\gamma^\pm_\epsilon) u^2 \, dx \geqslant \lambda_1^\pm \int_{\mathbb{R}^n_\omega} \hat{u}^2 \, dx, \tag{55}$$

where $\lambda_1^\pm > 0$ (here we set $\lambda_1^\pm = F''(1)$ if 0 is the only discrete eigenvalue),

$$\kappa := 1/\|(\gamma^\pm)'\|_{L^2(\mathbb{R}^n)} \text{ and } \hat{u} := u - \kappa^2 \langle u, (\gamma^\pm_\epsilon)'\rangle_{L^2(\mathbb{R}^n_\omega)} (\gamma^\pm_\epsilon)'.$$

Since

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n_\omega} |\nabla w^\pm_\epsilon|^2 + F''(\gamma^\pm_\epsilon)(w^\pm_\epsilon)^2 \, dx = 0,$$

due to (50) and (53), it follows from (55) that

$$\|\widehat{w}^\pm_\epsilon\|^2_{L^2(\mathbb{R}^n_\omega)} \leqslant C\,\epsilon. \tag{56}$$

As a consequence, recalling also (51) and (54), we conclude that

$$w^\pm_\epsilon \text{ converges to } w^\pm := \kappa(\gamma^\pm)' \text{ in } L^2(\mathbb{R}^n_\omega), \text{ as } \epsilon \to 0. \tag{57}$$

Moreover, since $\widehat{w}^\pm_\epsilon$ solves the equation

$$T^\pm_\epsilon \widehat{w}^\pm_\epsilon := -\Delta \widehat{w}^\pm_\epsilon + F''(\gamma^\pm_\epsilon)\widehat{w}^\pm_\epsilon = \lambda^\pm_\epsilon \widehat{w}^\pm_\epsilon + \big(F''(\gamma^\pm_\epsilon) - F''(u^\pm_\epsilon)\big)\, w^\pm_\epsilon,$$

by elliptic regularity [GT83, Corollary 8.7] and recalling Theorem 13, (50) and (56) we get

$$\begin{aligned}
\|\widehat{w}^\pm_\epsilon\|^2_{W^{1,2}(\mathbb{R}^n_\omega)} &\leqslant C\left(\|\lambda^\pm_\epsilon \widehat{w}^\pm_\epsilon\|^2_{L^2(\mathbb{R}^n_\omega)} + \|\big(F''(\gamma^\pm_\epsilon) - F''(u^\pm_\epsilon)\big)\, w^\pm_\epsilon\|^2_{L^2(\mathbb{R}^n_\omega)}\right) \\
&\leqslant C\,\|u^\pm_\epsilon - \gamma^\pm_\epsilon\|^2_{L^2(\mathbb{R}^n_\omega)} + o(\epsilon) = o(\epsilon).
\end{aligned}$$

In particular, it follows that

$$\int_{\mathbb{R}^n_\omega} T^\pm_\epsilon w^\pm_\epsilon \, w^\pm_\epsilon \, dx = \int_{\mathbb{R}^n_\omega} T^\pm_\epsilon \widehat{w}^\pm_\epsilon \, \widehat{w}^\pm_\epsilon \, dx = \int_{\mathbb{R}^n_\omega} |\nabla w^\pm_\epsilon|^2 + F''(\gamma^\pm_\epsilon)(w^\pm_\epsilon)^2 \, dx = o(\epsilon). \tag{58}$$

Accordingly, exploiting (53), (57) and (58), we get

$$\begin{aligned}
\lambda^\pm_\epsilon + o(\epsilon) &= \int_{\mathbb{R}^n_\omega} |\nabla w^\pm_\epsilon|^2 + F''(\gamma^\pm_\epsilon)(w^\pm_\epsilon)^2 + \epsilon F'''(\gamma^\pm_\epsilon)(w^\pm_\epsilon)^2 \phi^\pm \, dx \\
&= \int_{\mathbb{R}^n_\omega} T^\pm_\epsilon w^\pm_\epsilon \, w^\pm_\epsilon \, dx + \epsilon \int_{\mathbb{R}^n_\omega} F'''(\gamma^\pm_\epsilon)(w^\pm_\epsilon)^2 \phi^\pm \, dx \\
&= \epsilon\,\kappa^2 \int_{\mathbb{R}^n_\omega} F'''(\gamma^\pm)\big((\gamma^\pm)'\big)^2 \phi^\pm \, dx + o(\epsilon).
\end{aligned}$$

This proves (49). ☐

Lemma 15. *We have that*

$$\int_{\mathbb{R}^n_\omega} h(\gamma^\pm)' \, dx = 0. \tag{59}$$

Proof. From Theorem 13,

$$\begin{aligned}
-\int_{\mathbb{R}^n_\omega} h(\gamma^\pm)' \, dx &= \int_{\mathbb{R}^n_\omega} -\Delta\phi^\pm (\gamma^\pm)' + F''(\gamma^\pm)\phi^\pm(\gamma^\pm)' \, dx \\
&= \int_{\mathbb{R}^n_\omega} \big(-(\gamma^\pm)''' + F''(\gamma^\pm)(\gamma^\pm)'\big)\, \phi^\pm \, dx = 0\,,
\end{aligned}$$

as desired. ☐

Notice that condition (59) identifies $\gamma^{\pm}$, which is determined up to a translation along ω, in dependence of the function h.

Lemma 16. *Let $f \in \Im$, and assume that f decays exponentially, possibly with its derivatives, in the directions given by $\pm\omega$. Then, $v^{\pm} := (T^{\pm})^{-1} f \in \Im$ enjoys the same decay properties of f, and*

$$\int_{\mathbb{R}^n_\omega} f\phi^{\pm}\, dx = -\int_{\mathbb{R}^n_\omega} v^{\pm} h\, dx\,. \tag{60}$$

Proof. We first observe that, thanks to Lemma 11, there exists a unique $v^{\pm} \in \Im$ such that $T^{\pm} v^{\pm} = f$. The decay properties of $v^{\pm}$ then follow from the decay properties of f by elliptic regularity [GT83, Theorem 8.13]. In particular, $v^{\pm} \in L^1(\mathbb{R}^n_\omega)$ so that the right-hand side of (60) makes sense.
Since, by (38), $T^{\pm}\phi^{\pm} = -h$ and $T^{\pm}$ is self-adjoint on $L^2(\mathbb{R}^n_\omega)$, (60) can now be easily obtained by approximating $\phi^{\pm}$ with functions $\phi^{\pm}_R := \phi^{\pm}\rho_R$, where ρ_R are suitable cut-off functions with support in $\mathcal{B}_R$. ☐

Theorem 17. *Suppose that F is even. Then,*

$$\lambda^{\pm}_\epsilon = \frac{\epsilon}{\|(\gamma^{\pm})'\|^2_{L^2(\mathbb{R}^n_\omega)}} \int_{\mathbb{R}^n_\omega} h(x)\,(\gamma^{\pm})''(x)\, dx + o(\epsilon). \tag{61}$$

Proof. Since F is even, we have that $\gamma^{\pm}_o(\cdot + \langle \omega, x^{\pm}_0\rangle)$ is odd, and so

$$\int_{\mathbb{R}^n_\omega} F'''(\gamma^{\pm})\left((\gamma^{\pm})'\right)^2 (\gamma^{\pm})'\, dx = 0, \tag{62}$$

so that we can apply Lemma 16 with $f = F'''(\gamma^{\pm})\left((\gamma^{\pm})'\right)^2$.
Then, from (60) we get

$$\int_{\mathbb{R}^n_\omega} F'''(\gamma^{\pm})\left((\gamma^{\pm})'\right)^2 \phi^{\pm}\, dx = -\int_{\mathbb{R}^n_\omega} (T^{\pm})^{-1}\left(F'''(\gamma^{\pm})\left((\gamma^{\pm})'\right)^2\right) h\, dx.$$

Hence, by (49) we have

$$\begin{aligned} \lambda^{\pm}_\epsilon &= \frac{\epsilon}{\|(\gamma^{\pm})'\|^2_{L^2(\mathbb{R}^n_\omega)}} \int_{\mathbb{R}^n_\omega} F'''(\gamma^{\pm})\left((\gamma^{\pm})'\right)^2 \phi^{\pm}\, dx + o(\epsilon) \\ &= -\frac{\epsilon}{\|(\gamma^{\pm})'\|^2_{L^2(\mathbb{R}^n_\omega)}} \int_{\mathbb{R}^n_\omega} (T^{\pm})^{-1}\left(F'''(\gamma^{\pm})\left((\gamma^{\pm})'\right)^2\right) h\; dx + o(\epsilon). \end{aligned}$$

The desired claim then follows from (34). ☐

We are now in the position to give explicit conditions that imply (**A**) in the rational perturbative setting, when the potential F is even.
For this, we also recall that

$$\lambda_\epsilon \geqslant 0, \tag{63}$$

due to the minimality of $u^{\pm}_\epsilon$.

Proposition 18. *Let F be an even function, and suppose that h satisfies*

$$\int_{\mathbb{R}^n_\omega} h(x)\,(\gamma^{\pm})''(x)\, dx \neq 0. \tag{64}$$

*Then, condition (**A**) is fulfilled by $H = \epsilon h$, for ϵ small enough.*

Proof. By Theorem 17 and (64), we have that $\lambda_\epsilon^\pm \neq 0$, for ϵ small enough. In fact, from (63), we necessarily have that $\lambda_\epsilon^\pm > 0$, for ϵ small enough.
Thus, $\lambda_\epsilon^\pm$ is strictly positive, and lies in the discrete spectrum of the operator by Proposition 14. □

We now better clarify (64). Note that γ^+ and γ^- are determined by h itself, in the sense that h selects the translation of $\gamma_o^\pm$ from which $u_\epsilon^\pm$ bifurcates. This selection occours due to (59) and to the minimality of $u_\epsilon^\pm$.
We introduce the notation

$$f_t(x) := f(x + \omega t)$$

for a given function f and $t \in \mathbb{R}$.
We observe that, if F is even, the two etheroclinic orbits γ_o^+ and γ_o^- are the same up to sign-change and translation, that is we can write $\gamma^+ = \gamma_{\theta^+}$ and $\gamma^- = -\gamma_{\theta^-}$ for a suitable etheroclinic γ and suitable $\theta^\pm \in \mathbb{R}$.
We consider the function

$$\mathbb{R} \ni t \longmapsto \mathcal{F}(t) := \int_{\mathbb{R}^n_\omega} h(x)\gamma_t'(x)\,dx. \tag{65}$$

The function $\mathcal{F}$ is periodic since h is periodic and ω is rational. Also, condition (59) says that

$$\mathcal{F}(\theta^+) = 0 = \mathcal{F}(\theta^-). \tag{66}$$

In this spirit, we now prove that condition (**A**) is assured if these zeroes are non-degenerate:

Theorem 19. *Let F be even and suppose that*

$$\{\mathcal{F} = 0\} \cap \{\mathcal{F}' = 0\} = \emptyset. \tag{67}$$

Then, condition (**A**) *holds true for $H = \epsilon h$, and ϵ small enough.*

Proof. By (66) and (67),

$$0 \neq \mathcal{F}'(\theta^\pm) = \int_{\mathbb{R}^n_\omega} h\gamma''_{\theta^\pm}\,dx = \pm\int_{\mathbb{R}^n_\omega} h\,(\gamma^\pm)''\,dx \tag{68}$$

thence (64) is fulfilled. Recalling Proposition 18, we obtain the desired result. □

Remark 20. The proof of Theorem 19 also characterizes θ^+ and θ^- according to the way $\mathcal{F}$ cuts the abscissa. Indeed, from (61), (63), (66) and (68) we obtain

$$\theta^+ \in \{\mathcal{F} = 0\} \cap \{\mathcal{F}' > 0\} \qquad \text{and} \qquad \theta^- \in \{\mathcal{F} = 0\} \cap \{\mathcal{F}' < 0\}. \tag{69}$$

Remark 21. It would be suggestive to define the function

$$\mathbb{R} \ni t \longmapsto \mathcal{E}(t) := \int_{\mathbb{R}^n_\omega} h(x)\gamma_t(x)\,dx \tag{70}$$

and to use critical points of $\mathcal{E}$ instead of zeroes of $\mathcal{F}$ in Theorem 19.
Analogously, it would be nice to write (69) by charachterizing $\theta^\pm$ in terms of the minimality or maximality attained by $\mathcal{E}$.
Notice that these are only *formal* statements, since the integral in (70) does not converge in general.

Theorem 19 easily gives concrete examples of h's for which Theorem 6 applies:

Corollary 22. *Let $\kappa > 0$, $\bar{F}$ be an even double-well potential and $F = \kappa \bar{F}$. Given $\omega \in S^{n-1}$, we let*

$$h_\omega(t) := \int_{\Omega_{t\omega}} h(z)\, dz\,, \qquad \forall t \in \mathbb{R}.$$

Suppose that $h \in C^1(\mathbb{R}^n/\mathbb{Z}^n)$ and that

$$\{h_\omega = 0\} \cap \{h'_\omega = 0\} = \emptyset. \tag{71}$$

Then, there exists $\delta > 0$ such that condition $(\mathbf{A})$ holds true for $H = \epsilon h$, provided that $\epsilon \in (0,\delta)$ and $\kappa \geqslant 1/\delta$.

Proof. If $\bar{\gamma}$ is the etheroclinic of $\bar{F}$, then the etheroclinic of F is

$$\gamma(x) := \bar{\gamma}\left(x + (\sqrt{\kappa} - 1)\langle \omega, x\rangle \omega\right)\,.$$

Accordingly, from (65) we get

$$\mathcal{F}(t) = \int_{\mathbb{R}^n_\omega} h\left(y + \left(\frac{1}{\sqrt{\kappa}} - 1\right)\langle \omega, y\rangle \omega - t\omega\right) \bar{\gamma}'(y)\, dy\,, \tag{72}$$

and therefore

$$\mathcal{F}'(t) = -\int_{\mathbb{R}^n_\omega} \partial_\omega h\left(y + \left(\frac{1}{\sqrt{\kappa}} - 1\right)\langle \omega, y\rangle \omega - t\omega\right) \bar{\gamma}'(y)\, dy \tag{73}$$

We now claim

$$\text{that (67) holds if } \kappa \text{ is large enough.} \tag{74}$$

The proof of (74) is by contradiction: if not, by (72) and (73), there would exist a diverging sequnce κ_j and points $t_j \in \mathbb{R}$ for which

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n_\omega} h\left(y + \left(\frac{1}{\sqrt{\kappa_j}} - 1\right)\langle \omega, y\rangle \omega - t_j\omega\right) \bar{\gamma}'(y)\, dy \\ &= \int_{\mathbb{R}^n_\omega} \partial_\omega h\left(y + \left(\frac{1}{\sqrt{\kappa_j}} - 1\right)\langle \omega, y\rangle \omega - t_j\omega\right) \bar{\gamma}'(y)\, dy. \end{aligned} \tag{75}$$

Since $\mathcal{F}$ is periodic, say of period $\mathcal{T}$, we may suppose that $t_j \in [0,\mathcal{T})$. Hence, there exists $t_\star \in [0,\mathcal{T}]$ and a subsequence for which

$$\lim_{\ell \to +\infty} t_{j_\ell} = t_\star.$$

Therefore, by (75) and the Dominated Convergence Theorem,

$$\begin{aligned} 0 &= \frac{1}{|\Omega_{-t_\star\omega}|}\int_{\Omega_{-t_\star\omega}} h(z)\, dz \int_{\mathbb{R}^n_\omega} \bar{\gamma}'(y)\, dy = \int_{\mathbb{R}^n_\omega} h(y - \langle \omega, y\rangle\omega - t_\star\omega)\bar{\gamma}'(y)\, dy \\ &= \int_{\mathbb{R}^n_\omega} \partial_\omega h(y - \langle \omega, y\rangle\omega - t_\star\omega)\bar{\gamma}'(y)\, dy = \frac{1}{|\Omega_{-t_\star\omega}|}\int_{\Omega_{-t_\star\omega}} \partial_\omega h(z)\, dz \int_{\mathbb{R}^n_\omega} \bar{\gamma}'(y)\, dy, \end{aligned}$$

that is

$$-t_\star \in \{h_\omega = 0\} \cap \{h'_\omega = 0\}.$$

This is in contradiction with (71) and thus proves (74).
Then, the desired claim follows from Theorem 19. □

As an example, we observe that if, say $\omega = (1, 0, \dots, 0)$, the function

$$h(x) = \sin(2\pi x_1)$$

satisfies the assumption of Corollary 22 and so it gives rise to the multibump solutions of Theorem 6.

More generally, when $\omega = p/q$, with $0 \neq p \in \mathbb{Z}^n$, $0 \neq q \in \mathbb{N}$, a concrete example is given by

$$h(x) = \sin(2\pi p \cdot x).$$

Also, the function

$$h(x) = \sum_{i=1}^{N} \sin(2\pi x_i)$$

provides an example for any coordinate direction $\omega =(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$,$\dots$,$(0, 0, \dots, 1)$.

References

[AC79] S. Allen and J. Cahn. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metallurgica*, 27:1084–1095, 1979.

[AJM02] Francesca Alessio, Louis Jeanjean, and Piero Montecchiari. Existence of infinitely many stationary layered solutions in $\mathbb{R}^2$ for a class of periodic Allen-Cahn equations. *Comm. Partial Differential Equations*, 27(7-8):1537–1574, 2002.

[Bor05] D. I. Borisov. On the spectrum of a Schrödinger operator perturbed by a fast oscillating potential. *Preprint,* `http://arxiv.org/abs/math-ph/0505068v1`, 2005.

[Bre83] Haïm Brezis. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[Car95] Gilles Carbou. Unicité et minimalité des solutions d'une équation de Ginzburg-Landau. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 12(3):305–318, 1995.

[dlLV07] Rafael de la Llave and Enrico Valdinoci. Multiplicity results for interfaces of Ginzburg-Landau-Allen-Cahn equations in periodic media. *Adv. Math. (to appear)*, 2007.

[DLN06] N. Dirr, M. Lucia, and M. Novaga. Γ-convergence of the Allen-Cahn energy with an oscillating forcing term. *Interfaces Free Bound.*, 8(1):47–78, 2006.

[DO07] Nicolas Dirr and Enza Orlandi. Sharp-interface limit of a Ginzburg-Landau functional with a random external field. *Preprint,* `http://www.mat.uniroma3.it/users/orlandi/pubb.html`, 2007.

[DY06] N. Dirr and N. K. Yip. Pinning and de-pinning phenomena in front propagation in heterogeneous media. *Interfaces Free Bound.*, 8(1):79–109, 2006.

[Eva98] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 1998.

[Far03] Alberto Farina. Rigidity and one-dimensional symmetry for semilinear elliptic equations in the whole of $\mathbb{R}^N$ and in half spaces. *Adv. Math. Sci. Appl.*, 13(1):65–82, 2003.

[GP58] V. L. Ginzburg and L. P. Pitaevskiĭ. On the theory of superfluidity. *Soviet Physics. JETP*, 34 (7):858–861 (1240–1245 Ž. Eksper. Teoret. Fiz.), 1958.

[GT83] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, second edition, 1983.

[Kat95] Tosio Kato. *Perturbation Theory for Linear Operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[Lan67] L. D. Landau. *Collected papers of L. D. Landau.* Edited and with an introduction by D. ter Haar. Second printing. Gordon and Breach Science Publishers, New York, 1967.

[Mar06] Magali Marx. On the eigenvalues for slowly varying perturbations of a periodic Schrödinger operator. *Asymptot. Anal.*, 48(4):295–357, 2006.

[NV07] Matteo Novaga and Enrico Valdinoci. The geometry of mesoscopic phase transition interfaces. *Discrete Contin. Dyn. Syst.*, 19(4):777–798, 2007.

[Row79] J. S. Rowlinson. Translation of J. D. van der Waals' "The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density". *J. Statist. Phys.*, 20(2):197–244, 1979.

[RS03] Paul H. Rabinowitz and Ed Stredulinsky. Mixed states for an Allen-Cahn type equation. *Comm. Pure Appl. Math.*, 56(8):1078–1134, 2003. Dedicated to the memory of Jürgen K. Moser.

[RS04] Paul H. Rabinowitz and Ed Stredulinsky. Mixed states for an Allen-Cahn type equation. II. *Calc. Var. Partial Differential Equations*, 21(2):157–207, 2004.

Matteo Novaga, Dipartimento di Matematica, Università di Pisa, Via Buonarroti, 2, I-56127 Pisa (Italy)
E-mail address: `novaga@dm.unipi.it`

Enrico Valdinoci, Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, 1, I-00133 Roma (Italy)
E-mail address: `valdinoci@mat.uniroma2.it`