SHARP WEIGHTED HARDY-TYPE INEQUALITIES AND UNCERTAINTY PRINCIPLE INEQUALITY ON CARNOT GROUPS

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Dedicated to Jerome A. Goldstein on the occasion of his sixty-fourth birthday

Abstract. In this paper we establish sharp weighted Hardy-type inequalities on Carnot groups with homogeneous dimension $Q \geq 3$. A simple argument based on the sharp Hardy-type inequality is used to obtain uncertainty principle, Hardy-Sobolev and Rellich inequalities on Carnot groups.

1. Introduction

The classical Hardy inequality states that for $n \geq 3$

$$\int_{\mathbb{R}^n} |\nabla\phi(x)|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^2} dx, \tag{1.1}$$

where $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and the constant $(\frac{n-2}{2})^2$ is sharp. There exists a large literature dealing with the Hardy-type inequalities on the Euclidean space $\mathbb{R}^n$ and, in particular, sharp inequalities as well as their improved versions which have attracted a lot of attention because of their application to singular problems, e.g. [4], [7], [8], [25], [31]. For instance, Baras and Goldstein in their classical paper [4], showed that the following heat problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \frac{c}{|x|^2} u & \text{in} \quad \Omega \times (0,\infty), \quad 0 \in \Omega, \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \end{cases} \tag{1.2}$$

has a global solution (in the sense of distributions) if $c \leq C^*(n) = (\frac{n-2}{2})^2$ and no solution, even locally in time, if $c > C^*(n) = (\frac{n-2}{2})^2$. Thus, $C^*(n) = (\frac{n-2}{2})^2$ is the cut-off point for existence of positive solutions for the heat equation with inverse square potential $c/|x|^2$.

Recently there has been considerable interest in improving the inequality (1.1), in the sense that nonnegative terms are added in the right hand side of (1.1), and one of the important improvement has been obtained by Brezis and Vázquez [7]. They proved that for a bounded domain $\Omega \subset \mathbb{R}^n$ there holds

$$\int_\Omega |\nabla\phi(x)|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{|\phi(x)|^2}{|x|^2} dx + \mu\Big(\frac{\omega_n}{|\Omega|}\Big)^{2/n} \int_\Omega \phi^2 dx, \tag{1.3}$$

where ω_n and $|\Omega|$ denote the n-dimensional Lebesgue measure of the unit ball $B \subset \mathbb{R}^n$ and the domain Ω respectively. Here $\mu = 5.7832$ is the first eigenvalue of the Laplace operator in the two dimensional unit disk and it is optimal when Ω is a ball centered at the origin.

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A comprehensive treatment of improved Hardy inequalities with best constants, involving various kinds of distance functions in the Euclidean space $\mathbb{R}^n$ can be found in [5].

In view of these important works mentioned above it is natural to investigate Hardy-type inequalities on general Carnot groups. It is well known that the Euclidean space $\mathbb{R}^n$ with its usual abelian group structure is a trivial Carnot group. We are mainly concerned with the Hardy-type inequalities on non-trivial Carnot groups.

The first goal of this paper is to investigate the existence and the explicit determination of constants C and weight $q(x)$ on Carnot group $\mathbb{G}$ such that the Hardy-type inequality

$$\int_{\mathbb{G}} w(x)|\nabla_{\mathbb{G}}\phi(x)|^2 dx \geq C \int_{\mathbb{G}} q(x)|\phi(x)|^2 dx \tag{1.4}$$

holds for all $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Here we consider a special weight function $w(x)$ which is related to the fundamental solution of sub-Laplacian $\Delta_{\mathbb{G}}$ on Carnot group $\mathbb{G}$, and dx refers to Haar measure on $\mathbb{G}$.

It is important to emphasize that our result lead us to obtain a version of the uncertainty principle, Hardy-Sobolev and Rellich inequalities on Carnot groups.

The simplest nontrivial example of a Carnot group is given by the Heisenberg group $\mathbb{H}^n$. The following Hardy-type inequality on the Heisenberg group $\mathbb{H}^n$ was first proved by Garofalo and Lanconelli [17] :

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}\phi|^2 dzdt \geq \Big(\frac{Q-2}{2}\Big)^2 \int_{\mathbb{H}^n} \big(\frac{|z|^2}{|z|^4+t^2}\big)\phi^2 dzdt \tag{1.5}$$

where $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$, $Q = 2n+2$ and the constant $(\frac{Q-2}{2})^2$ is sharp. Here we view $\mathbb{H}^n$ as $\mathbb{C}^n \times \mathbb{R}$, and $dzdt$ refers to the usual Lebesgue measure. The L^p version of (1.5) has been found by Niu, Zhang and Wang [24]. In [10], D'Ambrosio proved a weighted version of (1.5) as well as L^p version on the Heisenberg group $\mathbb{H}^n$.

Although we prove Hardy-type inequalities on Carnot groups with an arbitrary step, we first establish sharp weighted Hardy-type inequalities on the Heisenberg group $\mathbb{H}^n$ and extend this result to the H-type groups. The main point is that the fundamental solution of the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ and H-type groups are known explicitly (see Section 3) but not for general Carnot groups. The proof of our theorem on general Carnot groups differs slightly in some steps from the Heisenberg group $\mathbb{H}^n$ and H-type group cases. The method we apply here, inspired by the work of Allegretto [1], can be applied to the Baouendi-Grushin type vector fields in that they do not arise from any Carnot group.

In a forthcoming paper, we shall present some improved Hardy-type inequalities and their applications in Carnot groups.

In order to state our theorems on Carnot group $\mathbb{G}$, we first recall the basic properties of Carnot group $\mathbb{G}$ and some well known results that will be used in the sequel.

2. CARNOT GROUP

A Carnot group (see [2], [3], [13], [15], [16], [23] and [28]) is a connected, simply connected, nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathcal{G}$ admits a stratification. That is, there exist linear subspaces $V_1, ..., V_k$ of $\mathcal{G}$ such that

$$\mathcal{G} = V_1 \oplus ... \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad \text{for} \quad i = 1, 2, ..., k-1 \quad \text{and} \quad [V_1, V_k] = 0 \tag{2.1}$$

where $[V_1, V_i]$ is the subspace of $\mathcal{G}$ generated by the elements $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. This defines a k-step Carnot group and integer $k \geq 1$ is called the step of $\mathbb{G}$.

Via the exponential map, it is possible to induce on $\mathbb{G}$ a family of automorphisms of the group, called dilations, $\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n (\lambda > 0)$ such that

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n)$$

where $1 = \alpha_1 = ... = \alpha_m < \alpha_{m+1} \leq ... \leq \alpha_n$ are integers and $m = \dim(V_1)$.

The group law can be written in the following form

$$x \cdot y = x + y + P(x, y), \quad x, y \in \mathbb{R}^n \tag{2.2}$$

where $P : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has polynomial components and $P_1 = ... = P_m = 0$. Note that the inverse x^{-1} of an element $x \in \mathbb{G}$ has the form $x^{-1} = -x = (-x_1, ..., -x_n)$.

Let $X_1, ..., X_m$ be a family of left invariant vector fields which form an orthonormal basis of $V_1 \equiv \mathbb{R}^m$ at the origin, that is, $X_1(0) = \partial_{x_1}, ..., X_m(0) = \partial_{x_m}$. The vector fields X_j have polynomial coefficients and can be assumed to be of the form

$$X_j(x) = \partial_j + \sum_{i=j+1}^{n} a_{ij}(x)\partial_i, \quad X_j(0) = \partial_j, j = 1, ..., m,$$

where each polynomial a_{ij} is homogeneous with respect to the dilations of the group, that is $a_{ij}(\delta_\lambda(x)) = \lambda^{\alpha_i - \alpha_j} a_{ij}(x)$. The horizontal gradient on Carnot group $\mathbb{G}$ is the vector valued operator

$$\nabla_{\mathbb{G}} = (X_1, ..., X_m)$$

where $X_1, ..., X_m$ are the generators of $\mathbb{G}$. The sub-Laplacian is the second-order partial differential operator on $\mathbb{G}$ given by

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{m} X_j^2.$$

The fundamental solution u for $\Delta_{\mathbb{G}}$ is defined to be a weak solution to the equation

$$-\Delta_{\mathbb{G}} u = \delta$$

where δ denotes the Dirac distribution with singularity at the neutral element 0 of $\mathbb{G}$. In [13] Folland proved that in any Carnot group $\mathbb{G}$, there exists a homogeneous norm N such that

$$u = N^{2-Q}$$

is a fundamental solution for $\Delta_{\mathbb{G}}$ (see also [6]).

We now set

$$N(x) := \begin{cases} u^{\frac{1}{2-Q}} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases} \tag{2.3}$$

We recall that a homogeneous norm on $\mathbb{G}$ is a continuous function $N : \mathbb{G} \longrightarrow [0, \infty)$ smooth away from the origin which satisfies the conditions : $N(\delta_\lambda(x)) = \lambda N(x)$, $N(x^{-1}) = N(x)$ and $N(x) = 0$ iff $x = 0$.

The curve $\gamma : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{G}$ is called horizontal if its tangents lie in V_1, i.e, $\gamma'(t) \in$ *span*$\{X_1, ..., X_m\}$ for all t. Then, the Carnot-Caréthedory distance $d_{CC}(x, y)$ between two points $x, y \in \mathbb{G}$ is defined to be the infimum of all horizontal lengths $\int_a^b \langle \gamma'(t), \gamma'(t) \rangle^{1/2} dt$

over all horizontal curves $\gamma : [a,b] \longrightarrow \mathbb{G}$ such that $\gamma(a) = x$ and $\gamma(y) = b$. Notice that d_{CC} is a homogeneous norm and satisfies the invariance property

$$d_{CC}(z \cdot x, z \cdot y) = d_{CC}(x,y), \quad \forall\, x,y,z \in \mathbb{G},$$

and is homogeneous of degree one with respect to the dilation δ_λ, i.e.

$$d_{CC}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_{CC}(x,y), \quad \forall\, x,y,z \in \mathbb{G}, \forall\, \lambda > 0.$$

The Carnot-Carethédory balls are defined by $B(x,R) = \{y \in \mathbb{G} | d_{CC}(x,y) < R\}$. By left-translation and dilation, it is easy to see that the Haar measure of $B(x,R)$ is proportional by R^Q. More precisely

$$|B(x,R)| = R^Q |B(x,1)| = R^Q |B(0,1)|$$

where

$$Q = \sum_{j=1}^{k} j(\mathrm{dim} V_j)$$

is the homogeneous dimension of $\mathbb{G}$.

It is well known that Sobolev inequalities are important in the study of partial differential equations, especially in the study of those arising from geometry and physics. The following Sobolev inequality holds on $\mathbb{G}$

$$\Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}} \phi(x)|^2 dx\Big)^{1/2} \geq C\Big(\int_{\mathbb{G}} |\phi(x)|^{\frac{2Q}{Q-2}} dx\Big)^{\frac{Q-2}{2Q}} \tag{2.4}$$

(see [13], [30]). It is a more difficult problem to determine the sharp constant C in (2.4) on general Carnot groups. The only results that have so far been proven are in the case of Heisenberg group $\mathbb{H}^n$ by Jerison and Lee [21] and Iwasawa-type groups (a particular sub-class of H-type groups) by Garofalo and Vassilev [18]. We should mention that the sharp constants in [21] and [18] lead us to obtain explicit constant in Corollary 4.2.

3. Hardy-type inequalities on Carnot groups of step 2

Among Carnot groups of step two, the Heisenberg group and Heisenberg type (H-type) groups are of particular significance. These groups appear naturally in analysis, geometry, representation theory and mathematical physics. In this section, we first prove Hardy-type inequalities on the Heisenberg group and we extend this result to the H-type group.

Heisenberg group. The Heisenberg group $\mathbb{H}^n$ is an example of a noncommutative Carnot group. Denoting points in $\mathbb{H}^n$ by (z,t) with $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $t \in \mathbb{R}$ we have the group law given as

$$(z,t) \circ (z',t') = (z + z', t + t' + 2\sum_{j=1}^{n} Im(z_j \bar{z}'_j))$$

With the notation $z_j = x_j + iy_j$, the horizontal space V_1 is spanned by the basis

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}.$$

The one dimensional center V_2 is spanned by the vector field $T = \frac{\partial}{\partial t}$. We have the commutator relations $[X_j, Y_j] = -4T$, and all other brackets of $\{X_1, Y_1, ..., X_n, Y_n\}$ are zero. The sub-elliptic gradient is the $2n$ dimensional vector field given by

$$\nabla_{\mathbb{H}^n} = (X_1, ..., X_n, Y_1, ..., Y_n)$$

and the Kohn Laplacian on $\mathbb{H}^n$ is the operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

A homogeneous norm on $\mathbb{H}^n$ is given by

$$\rho = |(z,t)| = (|z|^4 + t^2)^{1/4}$$

and the homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n + 2$.

A remarkable analogy between Kohn Laplacian and the classical Laplace operator has been obtained by Folland [12]. He found that the fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at zero is given by

$$\Psi(z,t) = \frac{c_Q}{\rho(z,t)^{Q-2}} \quad \text{where} \quad c_Q = \frac{2^{(Q-2)/2}\Gamma((Q-2)/4)^2}{\pi^{Q/2}}.$$

We now prove the following theorem on the Heisenberg group $\mathbb{H}^n$. In the various integral inequalities below (Section 3 and Section 4), we allow the values of the integrals on the left-hand sides to be $+\infty$. Before we proceed, we should emphasize that the constant $C(Q,\alpha) = (\frac{Q+\alpha-2}{2})^2$ obtained in Section 3 and Section 4 is sharp in the sense that if it is replaced by an grater number the inequality fails.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$. Then we have :*

$$\int_{\mathbb{H}^n} \rho^\alpha |\nabla_{\mathbb{H}^n}\phi|^2 dzdt \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{H}^n} \rho^\alpha \frac{|z|^2}{\rho^4}\phi^2 dzdt$$

where $\rho = (|z|^4 + l^2)^{1/4}$ is the homogeneous norm on $\mathbb{H}^n$. Moreover, the constant $(\frac{Q+\alpha-2}{2})^2$ is sharp provided $Q + \alpha - 2 > 0$.

Proof. Let $\phi = \rho^\beta \psi$ where $\beta \in \mathbb{R} \setminus \{0\}$ and $\psi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$. A direct calculation shows that

$$\rho^\alpha |\nabla_{\mathbb{H}^n}\phi|^2 = \beta^2 \rho^{\alpha+2\beta-2} |\nabla_{\mathbb{H}^n}\rho|^2 \psi^2 + 2\beta\rho^{\alpha+2\beta-1}\psi \nabla_{\mathbb{H}^n}\rho \cdot \nabla_{\mathbb{H}^n}\psi + \rho^{\alpha+2\beta}|\nabla_{\mathbb{H}}\psi|^2. \tag{3.1}$$

It is easy to see that

$$|\nabla_{\mathbb{H}^n}\rho|^2 = \frac{|z|^2}{\rho^2}$$

and integrating (3.1) over $\mathbb{H}^n$, we get

$$\begin{aligned} \int_{\mathbb{H}^n} \rho^\alpha |\nabla_{\mathbb{H}^n}\phi|^2 dzdt = \int_{\mathbb{H}^n} \beta^2 \rho^{\alpha+2\beta-4} |z|^2 \psi^2 dzdt + \int_{\mathbb{H}^n} 2\beta\rho^{\alpha+2\beta-1}\psi \nabla_{\mathbb{H}^n}\rho \cdot \nabla_{\mathbb{H}^n}\psi dzdt \\ + \int_{\mathbb{H}^n} \rho^{\alpha+2\beta} |\nabla_{\mathbb{H}^n}\psi|^2 dzdt \end{aligned} \tag{3.2}$$

Applying integration by parts to the middle integral on the right-hand side of (3.2), we obtain

$$\begin{aligned}\int_{\mathbb{H}^n}\rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi|^2dzdt=&\int_{\mathbb{H}^n}\beta^2\rho^{\alpha+2\beta-4}|z|^2\psi^2dzdt-\frac{\beta}{\alpha+2\beta}\int_{\mathbb{H}^n}\Delta_{\mathbb{H}^n}(\rho^{\alpha+2\beta})\psi^2dzdt\\&+\int_{\mathbb{H}^n}\rho^{\alpha+2\beta}|\nabla_{\mathbb{H}^n}\psi|^2dzdt.\end{aligned}\tag{3.3}$$

One can show that

$$\Delta_{\mathbb{H}^n}(\rho^{\alpha+2\beta})=|z|^2\rho^{\alpha+2\beta-4}(\alpha+2\beta)(\alpha+2\beta+Q-2).\tag{3.4}$$

Substituting (3.4) into (3.3) gives the following

$$\begin{aligned}\int_{\mathbb{H}^n}\rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi|^2dzdt&=(\beta^2-\beta(\alpha+2\beta+Q-2))\int_{\mathbb{H}^n}\rho^{\alpha+2\beta-4}|z|^2\psi^2dzdt+\int_{\mathbb{H}^n}\rho^{\alpha+2\beta}|\nabla_{\mathbb{H}^n}\psi|^2dzdt\\&\geq(-\beta^2-\beta(\alpha+Q-2))\int_{\mathbb{H}^n}\rho^{\alpha+2\beta-4}|z|^2\psi^2dzdt.\end{aligned}$$

Note that the function $\beta\longrightarrow-\beta^2-\beta(\alpha+Q-2)$ attains the maximum for $\beta=\frac{2-\alpha-Q}{2}$, and this maximum is equal to $(\frac{Q+\alpha-2}{2})^2$. Therefore we have the following inequality

$$\int_{\mathbb{H}^n}\rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi|^2dzdt\geq\Big(\frac{Q+\alpha-2}{2}\Big)^2\int_{\mathbb{H}^n}\rho^{\alpha}\frac{|z|^2}{\rho^4}\phi^2dzdt.\tag{3.5}$$

It only remains to show that the constant $(\frac{Q+\alpha-2}{2})^2$ is sharp. Let $\phi_\epsilon(z,t)$ be the family of functions defined by

$$\phi_\epsilon(z,t)=\begin{cases}1 & \text{if}\quad \rho\in[0,1],\\ \rho^{-(\frac{Q+\alpha-2}{2}+\epsilon)} & \text{if}\quad \rho>1,\end{cases}\tag{3.6}$$

where $\epsilon>0$ and $\rho=|(z,t)|=(|z|^4+t^2)^{1/4}$. It follows that

$$|\rho^{\alpha}\nabla_{\mathbb{H}^n}\phi_\epsilon(z,t)|^2=\Big(\frac{Q+\alpha-2}{2}+\epsilon\Big)^2|z|^2\rho^{-(Q+2+2\epsilon)}.$$

In the sequel we indicate $B_1=\{(z,t):\rho\leq 1\}$ ρ-ball centered at the origin in $\mathbb{H}^n$ with radius 1.

By direct computation we get

$$\begin{aligned}\int_{\mathbb{H}^n}\rho^{\alpha}\frac{|z|^2}{\rho^4}\phi_\epsilon^2dzdt&=\int_{B_1}\rho^{\alpha-4}|z|^2\phi_\epsilon^2dzdt+\int_{\mathbb{H}^n\backslash B_1}\rho^{\alpha-4}|z|^2\phi_\epsilon^2dzdt\\&=\int_{B_1}\rho^{\alpha-4}|z|^2dzdt+\int_{\mathbb{H}^n\backslash B_1}|z|^2\rho^{-(Q+2+2\epsilon)}dzdt\\&=\int_{B_1}\rho^{\alpha-4}|z|^2dzdt+(\frac{Q+\alpha-2}{2}+\epsilon)^{-2}\int_{\mathbb{H}^n}\rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi_\epsilon|^2dzdt.\end{aligned}\tag{3.7}$$

Since $Q+\alpha-2>0$ then the first integral on the right hand side of (3.7) is integrable and we conclude by $\epsilon\longrightarrow 0$.

□

The following theorem shows that the weight function ρ^{α} has a significant effect on the sharp constant $(\frac{Q+\alpha-2}{2})^2$ whereas the new weight function $|\nabla_{\mathbb{H}^n}\rho|^{\gamma}$ has no effect.

Theorem 3.2. *Let* $\alpha, \gamma \in \mathbb{R}$ *and* $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$. *Then we have :*

$$\int_{\mathbb{H}^n} \rho^\alpha |\nabla_{\mathbb{H}^n}\rho|^\gamma |\nabla_{\mathbb{H}^n}\phi|^2 dz dt \geq (\frac{Q+\alpha-2}{2})^2 \int_{\mathbb{H}^n} \rho^{\alpha-2} |\nabla_{\mathbb{H}^n}\rho|^{\gamma+2} \phi^2 dz dt$$

where $\rho = (|z|^4 + l^2)^{1/4}$ *is the homogeneous norm on* $\mathbb{H}^n$. *Moreover, the constant* $(\frac{Q+\alpha-2}{2})^2$ *is sharp provided* $Q+\alpha-2>0$.

Proof. The proof is similar to the proof of Theorem 3.1. We only need to note that

$$\nabla_{\mathbb{H}^n}\rho \cdot \nabla_{\mathbb{H}^n}(|\nabla_{\mathbb{H}^n}\rho|^\gamma) = 0.$$

□

Heisenberg-type group. Another important model of Carnot groups are the H-type (Heisenberg type) groups which were introduced by Kaplan [22] as direct generalizations of the Heisenberg group $\mathbb{H}^n$. An H-type group is a Carnot group with a two-step Lie algebra $\mathcal{G} = V_1 \oplus V_2$ and an inner product $\langle , \rangle$ in $\mathcal{G}$ such that the linear map

$$J : V_2 \longrightarrow \mathrm{End} V_1,$$

defined by the condition

$$\langle J_z(u), v\rangle = \langle z, [u,v]\rangle, \quad u, v \in V_1, z \in V_2$$

satisfies

$$J_z^2 = -||z||^2 \mathbf{Id}$$

for all $z \in V_2$, where $||z||^2 = \langle z, z\rangle$.

Sub-Laplacian is defined in terms of a fixed basis $X_1, ..., X_m$ for V_1:

$$\Delta_{\mathbb{G}} = \sum_{i=1}^m X_i^2. \tag{3.8}$$

The exponential mapping of a simply connected Lie group is an analytic diffeomorphism. One can then define analytic mappings $v : \mathbb{G} \longrightarrow V_1$ and $z : \mathbb{G} \longrightarrow V_2$ by

$$x = \exp(v(x) + z(x))$$

for every $x \in \mathbb{G}$. In [22] Kaplan proved that there exists a constant $c > 0$ such that the function

$$\Phi(x) = c\Big(|v(x)|^4 + 16|z(x)|^2\Big)^{\frac{2-Q}{4}}$$

is a fundamental solution for the operator $-\Delta_{\mathbb{G}}$. We note that

$$K(x) = \Big(|v(x)|^4 + 16|z(x)|^2\Big)^{\frac{1}{4}} \tag{3.9}$$

defines a homogeneous norm and $Q = m + 2k$ is the homogeneous dimension of $\mathbb{G}$ where $m = \mathrm{dim}V_1$ and $k = \mathrm{dim}V_2$. This result generalized Folland's fundamental solution for the Heisenberg group $\mathbb{H}^n$ [12]. It is remarkable that the homogeneous norm $K(x)$ is involved also in the expression of the fundamental solution of the following $p-$sub-Laplace operator

$$\mathcal{L}_p u = \sum_{i=1}^m X_i(|Xu|^{p-2} X_i u), \quad 1 < p < \infty. \tag{3.10}$$

More precisely, Capogna, Danielli and Garofalo [9] proved that for every $1 < p < \infty$ there exists $c_p > 0$ such that the function

$$\Gamma_p(x) = \begin{cases} c_p K^{(p-Q)/(p-1)} & \text{when} \quad p \neq Q, \\ -c_p \log K & \text{when} \quad p = Q, \end{cases} \tag{3.11}$$

is a fundamental solution for the operator $-\mathcal{L}_p$ (see also [19] for the case $p = Q$).

We cite, without proof of the following, useful formulas which can be found in [9]. Let u be a radial function, i.e., $u(x) = f(K(x))$ where $f \in C(\mathbb{R})$ then

$$|\nabla_{\mathbb{G}} u|^2 = \frac{|v|^2}{K^2} |f'(K)|^2.$$

Moreover if $u(x) = f(K(x))$ and $f \in C^2(\mathbb{R})$ then

$$\begin{aligned} \Delta_{\mathbb{G}} u &= |\nabla_{\mathbb{G}} K(x)|^2 \Big[f''(K) + \frac{Q-1}{K} f'(K) \Big] \\ &= \frac{v^2}{K^2} \Big[f''(K) + \frac{Q-1}{K} f'(K) \Big] \end{aligned} \tag{3.12}$$

at every point $x \in \mathbb{G} \setminus \{0\}$ where $f'(K(x)) \neq 0$.

Another important fact that $K(x)$ satisfies the so-called ∞-sub-Laplace equation :

$$\Delta_{\mathbb{G},\infty} K = \frac{1}{2} \langle \nabla_{\mathbb{G}} (|\nabla_{\mathbb{G}} K|^2), \nabla_{\mathbb{G}} K \rangle = 0$$

at every point $x \in \mathbb{G} \setminus \{0\}$. (See [22] and [9] for further information on H-type groups)

We now have the following theorem on the H-type group :

Theorem 3.3. *Let $\mathbb{G}$ be an H-type group with homogeneous dimension $Q = m + 2k$ and let $\alpha, \gamma \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Then the following inequality is valid :*

$$\int_{\mathbb{G}} K^\alpha |\nabla_{\mathbb{G}} K|^\gamma |\nabla_{\mathbb{G}} \phi|^2 dx \geq (\frac{Q + \alpha - 2}{2})^2 \int_{\mathbb{G}} K^{\alpha-2} |\nabla_{\mathbb{G}} K|^{\gamma+2} \phi^2 dx \tag{3.13}$$

where $K(x) = (|v(x)|^4 + 16|z(x)|^2)^{1/4}$. Moreover, the constant $(\frac{Q+\alpha-2}{2})^2$ is sharp provided $Q + \alpha - 2 > 0$.

Proof. The proof is identical to the Heisenberg group case. □

4. Hardy-type inequalities on Carnot groups of arbitrary step

In this section, we consider Carnot group $\mathbb{G}$ of any step k with the homogeneous norm $N = u^{1/(2-Q)}$ associated to Folland's solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$ [13]. We have the following theorem:

Theorem 4.1. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and let $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $\alpha \in \mathbb{R}$, $Q + \alpha - 2 > 0$. Then the following inequality is valid*

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} \phi|^2 dx \geq \Big(\frac{Q + \alpha - 2}{2}\Big)^2 \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^2} \phi^2 dx. \tag{4.1}$$

Furthermore, the constant $C(Q, \alpha) = (\frac{Q+\alpha-2}{2})^2$ is sharp.

Proof. Let $\phi = N^{\beta}\psi$ where $\psi \in C_0^{\infty}(\mathbb{G} \setminus \{0\})$ and $\beta \in \mathbb{R} \setminus \{0\}$. A direct calculation shows that

$$|\nabla_{\mathbb{G}}(N^{\beta}\psi)|^2 = \beta^2 N^{2\beta-2}|\nabla_{\mathbb{G}}N|^2\psi^2 + 2\beta N^{2\beta-1}\psi\nabla_{\mathbb{G}}N \cdot \nabla_{\mathbb{G}}\psi + N^{2\beta}|\nabla_{\mathbb{G}}\psi|^2. \tag{4.2}$$

Multiplying both sides of (4.2) by the N^{α} and applying integration by parts over $\mathbb{G}$ gives

$$\begin{aligned}\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^2 dx &= \beta^2 \int_{\mathbb{G}} N^{\alpha+2\beta-2}|\nabla_{\mathbb{G}}N|^2\psi^2 dx - \frac{\beta}{\alpha+2\beta}\int_{\mathbb{G}} \Delta_{\mathbb{G}}(N^{\alpha+2\beta})\psi^2 dx\\ &\quad + \int_{\mathbb{G}} N^{\alpha+2\beta}|\nabla_{\mathbb{G}}\psi|^2 dx\\ &\geq \beta^2 \int_{\mathbb{G}} N^{\alpha+2\beta-2}|\nabla_{\mathbb{G}}N|^2\psi^2 dx - \frac{\beta}{\alpha+2\beta}\int_{\mathbb{G}} \Delta_{\mathbb{G}}(N^{\alpha+2\beta})\psi^2 dx.\end{aligned} \tag{4.3}$$

A straightforward calculation shows that

$$-\frac{\beta}{\alpha+2\beta}\Delta_{\mathbb{G}}(N^{\alpha+2\beta}) = -\beta(\alpha+2\beta+Q-2)N^{\alpha+2\beta-2}|\nabla_{\mathbb{G}}N|^2 - \frac{\beta}{2-Q}N^{\alpha+2\beta+Q-2}\Delta_{\mathbb{G}}u. \tag{4.4}$$

Substituting (4.4) into (4.3) and using the fact that $\psi^2 = N^{-2\beta}\phi^2$, we get the following :

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^2 dx \geq (-\beta^2 - \beta(\alpha+Q-2))\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx - \frac{\beta}{2-Q}\int_{\mathbb{G}}(\Delta_{\mathbb{G}}u)N^{\alpha+Q-2}\phi^2 dx.$$

Since u is the fundamental solution of sub-Laplacian $\Delta_{\mathbb{G}}$ on Carnot group $\mathbb{G}$, we get

$$-\int_{\mathbb{G}}(\Delta_{\mathbb{G}}u)N^{\alpha+Q-2}\phi^2 dx = N^{\alpha+Q-2}(0)\phi^2(0) = 0.$$

We now obtain

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^2 dx \geq (-\beta^2 - \beta(\alpha+Q-2))\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx.$$

Choosing

$$\beta = \frac{2-Q-\alpha}{2}$$

gives the following sharp inequality

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^2 dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx. \tag{4.5}$$

□

Remark 4.1. In the abelian case, when $\mathbb{G} = \mathbb{R}^n$ with the ordinary dilations, one has $\mathcal{G} = V_1 = \mathbb{R}^n$ so that $Q = n$. Now it is clear that the inequality (4.1) with the homogeneous norm $N(x) = |x|$ and $\alpha = 0$ recovers the Hardy inequality (1.1).

Uncertainty Principle Inequality. The classical uncertainty principle was developed in the context of quantum mechanics by Heisenberg [20]. It says that the position and momentum of a particle cannot be determined exactly at the same time but only with an "uncertainty". The harmonic analysis version of uncertainty principle states that a function on the real line and its Fourier transform can not be simultaneously well localized. It has been widely studied in quantum mechanics and signal analysis. There are various forms of the uncertainty principle. For an overview we refer to Folland's and Sitaram's paper [14].

The uncertainty principle on the Euclidean space $\mathbb{R}^n$ can be stated in the following way:

$$\Big(\int_{\mathbb{R}^n} |x|^2|f(x)|^2dx\Big)\Big(\int_{\mathbb{R}^n} |\nabla f(x)|^2dx\Big) \geq \frac{n^2}{4}\Big(\int_{\mathbb{R}^n} |f(x)|^2dx\Big)^2 \tag{4.6}$$

for all $f \in L^2(\mathbb{R}^n)$. An analogue of the above inequality (4.6) for the Heisenberg group $\mathbb{H}^n$ was established by Garofalo and Lanconelli [17]. We should mention that Thangavelu [29], and Sitaram, Sundari and Thangavelu [27] have also obtained related but inequivalent analogues of Heisenberg's inequality for functions on the Heisenberg group $\mathbb{H}^n$.

We now give an analogue of (4.6) on general Carnot groups which is an immediate consequence of the Hardy-type inequality (4.1) with $\alpha = 0$ and the Cauchy-Schwarz inequality.

Corollary 4.1. *(Uncertainty inequality). Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$. Then for every $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$*

$$\Big(\int_{\mathbb{G}} N^2|\nabla_{\mathbb{G}}N|^2\phi^2dx\Big)\Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^2dx\Big) \geq \Big(\frac{Q-2}{2}\Big)^2\Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}}N|^2\phi^2dx\Big)^2. \tag{4.7}$$

The following Corollary is an extension of Hardy and Sobolev inequalities on Carnot groups. The proof based on the Hardy-type inequality (4.1) with $\alpha = 0$, Hölder inequality and Sobolev inequality (2.4).

Corollary 4.2. *(Hardy-Sobolev) Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and $0 \leq s \leq 2$. Then for every $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$ then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^2dx \geq C\Big(\int_{\mathbb{G}} \Big(\frac{|\nabla_{\mathbb{G}}N|}{N}\Big)^s|\phi|^{2(\frac{Q-s}{Q-2})}dx\Big)^{\frac{Q-2}{Q-s}}.$$

Another important consequence of the Hardy-type inequality (4.1) is the weighted Rellich-type inequality. Let us recall the Rellich inequality.

Rellich inequality. The classical Rellich inequality [26] states that for $n \geq 5$

$$\int_{\mathbb{R}^n} |\Delta\phi(x)|^2dx \geq \frac{n^2(n-4)^2}{16}\int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^4}dx \tag{4.8}$$

for all $\phi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ and the constant $\frac{n^2(n-4)^2}{16}$ is sharp. The Rellich inequality is the first generalization of Hardy's inequality to higher-order derivatives. It has been extended to the Heisenberg group by Niu, Zhang and Wang [24] and D'Ambrosio [10]. In this paper we give an analog of Rellich inequality for general Carnot groups.

Corollary 4.3. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and let $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $\alpha \in \mathbb{R}$, $Q + \alpha - 4 > 0$. Then the following inequality is valid*

$$\int_{\mathbb{G}} \frac{N^\alpha}{|\nabla_{\mathbb{G}}N|^2}|\Delta_{\mathbb{G}}\phi|^2dx \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_{\mathbb{G}} N^\alpha\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\phi^2dx \tag{4.9}$$

Remark 4.2. In the abelian case, when $\mathbb{G} = \mathbb{R}^n$ with the ordinary dilations, one has $\mathcal{G} = V_1 = \mathbb{R}^n$ so that $Q = n$. Now it is clear that the inequality (4.9) with the homogeneous norm $N(x) = |x|$ and $\alpha = 0$ recovers the Rellich inequality (4.8).

Polarizable Carnot group. The second main result of this section is to establish weighted Hardy-type inequality including the weight function $|\nabla_{\mathbb{G}}N|^{\gamma}$ as in the Section 3. We now establish such a inequality on polarizable Carnot groups. This class of groups were introduced by Balogh and Tyson [2] and admit the analogue of polar coordinates.

A Carnot group $\mathbb{G}$ is said to be polarizable if the homogeneous norm $N = u^{1/(2-Q)}$, associated to Folland's solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$, satisfies the following ∞-sub-Laplace equation,

$$\Delta_{\mathbb{G},\infty}N := \frac{1}{2}\langle\nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}}N|^2), \nabla_{\mathbb{G}}N\rangle = 0, \qquad \text{in} \quad \mathbb{G}\setminus\{0\}. \tag{4.10}$$

It has been proved that the homogeneous norm (3.9) satisfies the equation (4.10) (see [2] and [11]) . This result implies that H-type groups are polarizable Carnot groups. Unfortunately, at the present time, it is unknown an example of polarizable Carnot group which is not of H-type.

Balogh and Tyson [2] proved that the homogeneous norm $N = u^{1/(2-Q)}$, associated to Folland's solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$, enters also in the expression of the fundamental solution of the sub-elliptic p-Laplacian

$$\Delta_{\mathbb{G},p}u = \sum_{i=1}^{m} X_i(|Xu|^{p-2}X_iu), \quad 1 < p < \infty. \tag{4.11}$$

More precisely, Balogh and Tyson [2] proved that for every $1 < p < \infty, p = Q$ there exists $c_p > 0$ such that the fundamental solution of $-\Delta_{\mathbb{G},p}$ is given by

$$u_p = \begin{cases} c_p N^{\frac{p-Q}{p-1}}, & \text{if} \quad p \neq Q, \\ -c_Q \log N, & \text{if} \quad p = Q. \end{cases} \tag{4.12}$$

We are now ready to state our the second main theorem in this section.

Theorem 4.2. *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let $\phi \in C_0^{\infty}(\mathbb{G}\setminus\{0\})$, $\alpha \in \mathbb{R}$,$\gamma > -1$, $Q \geq 3$, $Q + \alpha - 2 > 0$. Then the following inequality is valid*

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}N|^{\gamma}|\nabla_{\mathbb{G}}\phi|^2dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^{\gamma+2}}{N^2}\phi^2dx. \tag{4.13}$$

Furthermore, the constant $C(Q,\alpha) = (\frac{Q+\alpha-2}{2})^2$ is sharp in the sense that if it is replaced by an grater number the inequality fails.

Proof. Let $\phi = N^{\beta}\psi$ where $\psi \in C_0^{\infty}(\mathbb{G}\setminus\{0\})$ and $\beta \in \mathbb{R}\setminus\{0\}$. A direct calculation shows that

$$\begin{aligned}\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}N|^{\gamma}|\nabla_{\mathbb{G}}\phi|^2dx = \beta^2 &\int_{\mathbb{G}} N^{\alpha+2\beta-2}|\nabla_{\mathbb{G}}N|^{\gamma+2}\psi^2dx \\ &+ 2\beta\int_{\mathbb{G}} N^{\alpha+2\beta-1}|\nabla_{\mathbb{G}}N|^{\gamma}\psi\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\psi dx \\ &+ \int_{\mathbb{G}} N^{\alpha+2\beta}|\nabla_{\mathbb{G}}N|^{\gamma}|\nabla_{\mathbb{G}}\psi|^2dx.\end{aligned} \tag{4.14}$$

Applying integration by parts to the middle term:

$$
\begin{aligned}
\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}N|^{\gamma}|\nabla_{\mathbb{G}}\phi|^2dx = \beta^2 & \int_{\mathbb{G}} N^{\alpha+2\beta-2}|\nabla_{\mathbb{G}}N|^{\gamma+2}\psi^2dx \\
& - \beta \int_{\mathbb{G}} \psi^2 \mathrm{div}\big(N^{\alpha+2\beta-1}|\nabla_{\mathbb{G}}N|^{\gamma}\nabla_{\mathbb{G}}N\big)dx \\
& + \int_{\mathbb{G}} N^{\alpha+2\beta}|\nabla_{\mathbb{G}}N|^{\gamma}|\nabla_{\mathbb{G}}\psi|^2dx.
\end{aligned} \tag{4.15}
$$

We now choose $\gamma = p-2 > 1$ and $\alpha + 2\beta - 1 = 1 - Q$, we get

$$
\begin{aligned}
\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}N|^{\gamma}|\nabla_{\mathbb{G}}\phi|^2dx &= \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^{\alpha-2}|\nabla_{\mathbb{G}}N|^{\gamma+2}\phi^2dx \\
& - \beta c_p^{1-p}\int_{\mathbb{G}}(\Delta_{\mathbb{G},p}(u_p))N^{-2\beta}\phi^2dx + \int_{\mathbb{G}} N^{2-Q}|\nabla_{\mathbb{G}}N|^{\gamma}|\nabla_{\mathbb{G}}\psi|^2dx.
\end{aligned} \tag{4.16}
$$

Since u_p is the fundamental solution of sub-p-Laplacian $-\Delta_{\mathbb{G},p}$, we get

$$
-\int_{\mathbb{G}}(\Delta_{\mathbb{G},p}(u_p))N^{Q+\alpha-2}\phi^2dx = N^{Q+\alpha-2}(0)\phi^2(0) = 0.
$$

We now obtain the desired inequality

$$
\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}N|^{\gamma}|\nabla_{\mathbb{G}}\phi|^2dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^{\alpha-2}|\nabla_{\mathbb{G}}N|^{\gamma+2}\phi^2dx. \tag{4.17}
$$

To show that the constant $\left(\frac{Q+\alpha-2}{2}\right)^2$ is sharp, we use the following family of functions

$$
\phi_{\epsilon}(x) = \begin{cases} 1 & \text{if} \quad N(x) \in [0,1], \\ N^{-(\frac{Q+\alpha-2}{2}+\epsilon)} & \text{if} \quad N(x) > 1, \end{cases} \tag{4.18}
$$

and pass to the limit as $\epsilon \longrightarrow 0$. Here we notice that $|\nabla_{\mathbb{G}}N|$ is uniformly bounded and polar coordinate integration formula holds on $\mathbb{G}$ (see [2], [15]).

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