On functional linear partial differential equations in Gevrey spaces of holomorphic functions

Stéphane Malek
Université de Lille 1
UFR de Mathématiques Pures et Appliquées
Cité Scientifique - Bât. M2
59655 Villeneuve d'Ascq Cedex France
Stephane.Malek@math.univ-lille1.fr
Tel 03 20 43 42 40

Abstract

We investigate existence and unicity of global sectorial holomorphic solutions of functional linear partial differential equations in some Gevrey spaces. A version of the Cauchy-Kowalevskaya theorem for some linear partial $q-$difference-differential equations is also presented.

Nous étudions l'existence et l'unicité de solutions globales holomorphes sectorielles d'équations fonctionnelles linéaires aux dérivées partielles dans certains espaces de fonctions Gevrey. Une version du théorème de Cauchy-Kowalevskaya pour des équations linéaires aux $q-$différences-différentielles partielles est également présentée.

Key words: Cauchy problems, functional partial differential equations, Gevrey estimates, $q-$Gevrey estimates. 2000 MSC: 35C10, 35C20.

Introduction

In this paper, we study linear functional partial differential equations of the form

$$\partial_z^S \hat{X}(t,z) = \sum_{k=(k_0,k_1,l)\in\mathcal{S}} b_k(t) z^l (\partial_t^{k_0}\partial_z^{k_1}\hat{X})(\phi_k(t), zq^{-m_{1,k}}) \tag{1}$$

where $q > 1$ is a real number, $S \geq 1$ is a positive integer and $\mathcal{S}$ is a finite subset of $\mathbb{N}^3$. The coefficients b_k and the deviations ϕ_k are holomorphic functions on some open domain $G \cup D_\rho$ where G is an open sector with infinite radius centered at 0 and D_ρ an open disc centered at 0 with radius ρ and $m_{1,k} \geq 1$ are positive integers. To fix the notations, in the formula (1) the partial derivatives $\partial_t^{k_0}\partial_z^{k_1}\hat{X}$ are evaluated at the point $(\phi_k(t), zq^{-m_{1,k}})$.

Several authors have studied this kind of equations (1) in the situation where the deviations ϕ_k are shrinking maps, meaning that $|\partial_t\phi_k(t)| \leq 1$ on some domain, see for instance [2], [9],[10],[19]. Moreover, there are some results when this assumption is missing, see for instance [11],[23].

Here, we focus on the case when each deviation $(t,z) \mapsto (\phi_k(t), zq^{-m_{1,k}})$ satisfies the property that the modulus of the determinant of its jacobian matrix is smaller than 1 on $G \cup D_\rho \times \mathbb{C}$, which is equivalent to the fact that the deviations $\phi_k(t)$ satisfy

$$|\partial_t\phi_k(t)| \leq q^{m_{1,k}}$$

on $G \cup D_\rho$, for all $k = (k_0, k_1, l) \in \mathcal{S}$. In particular, if the variables t, z were real, these deviations would be volume shrinking maps.

The goal of this paper is to provide an approach to construct actual holomorphic solutions $X(t,z)$ of (1) with estimates on the l-th derivative of $X(t,z)$ with respect to t on G, for given initial conditions $(\partial_z^j \hat{X})(t,0)$, $0 \leq j \leq S-1$, be holomorphic functions of some Gevrey type on G (see Definition 2). To achieve this goal, we will have to make several additional assumptions on the form of the equation (1), see Theorem 1.

Due to the presence of $q-$difference operators in the functional equation (1), we are led to work in spaces of holomorphic functions whose l-th derivative rate of growth is like $e^{(l^2/2)(\log(q))^2}(l!)^s$ for $s \geq 0$ (see Definition 3).

Such phenomenon of fast growing derivatives appears in a natural way in the study of $q-$difference equations for which general results on $q-$Gevrey asymptotic expansions of actual holomorphic solutions have been obtained, see [6], [8], [18], [21].

The leading idea will be the same as in the paper [12], where we have dealt with linear partial differential equation, with an additional difficulty because of the presence of the deviations ϕ_k. To this aim, we will use a higher order chain rule which has been introduced by T. Yamanaka, see [20], and which is much more suitable than the classical Faà di Bruno formula to deal with Gevrey estimates. Moreover, instead of the classical Cauchy-Kowalevskaya theorem for partial differential equations, we will need a similar result for some linear partial $q-$difference-differential equations of the form

$$\partial_x^S V(t,x) = \sum_{k=(k_0,k_1,l)\in\mathcal{S}} a_k(t) x^l (\partial_t^{k_0} \partial_x^{k_1} V)(q^{m_{0,k}} t, x q^{-m_{1,k}}),$$

where $q > 1$ is a real number, $m_{0,k}, m_{1,k} \geq 1$ are positive integers and $a_k(t)$ are holomorphic functions on a neighborhood of the origin. The method of proof uses functional analysis in Banach spaces of formal series. We notice that a similar result has been obtained by C. Zhang for general linear and non-linear ordinary $q-$difference-differential equations in one complex variable, see [22].

1 A Cauchy-Kowalevskaya theorem in a class of $q-$Gevrey formal series

Definition 1 *Let q, X, T, s be real numbers such that $q > 1$, $X, T > 0$ and $s \geq 1$. We define a vector space $G^q(X,T,s)$ which is a subspace of the formal series $\mathbb{C}[[t,x]]$. A formal series $U(t,x) \in \mathbb{C}[[t,x]]$,*

$$U(t,x) = \sum_{\beta,l\geq 0} u_{\beta,l} \frac{x^\beta}{\beta!} \frac{t^l}{l!}$$

belongs to $G^q(X,T,s)$, if the series

$$\sum_{\beta,l\geq 0} \frac{|u_{\beta,l}|}{q^{P(l,\beta)}(sl+\beta)!} X^\beta T^l,$$

converges, where $P(l,\beta)$ denotes the polynomial

$$P(l,\beta) = l\beta - \frac{1}{2}(\beta^2 - \beta)$$

and where $x!$ stands for $\Gamma(1+x)$ for all $x \geq 0$ to simplify the notations. We also define a weighted L^1 norm on $G^q(X,T,s)$ as

$$||U(t,x)||_{X,T} = \sum_{\beta,l\geq 0} \frac{|u_{\beta,l}|}{q^{P(l,\beta)}(sl+\beta)!} X^\beta T^l.$$

One can easily show that $(G^q(X,T,s), ||.||_{X,T})$ is a Banach space.

Remark: Let $U(t,x)$ be in $G(X_0,T_0,s)$ for given $X_0, T_0 > 0$ and $s \geq 1$. Then, $U(t,x)$ also belong to the spaces $G^q(X,T,s)$ for all $X \leq X_0$ and $T \leq T_0$. Moreover, the maps $X \mapsto ||U(t,x)||_{X,T}$ and $T \mapsto ||U(t,x)||_{X,T}$ are increasing functions from $[0,X_0]$ (resp. $[0,T_0]$) into $\mathbb{R}_+$.

We define the integration operator $\partial_x^{-1} : \mathbb{C}[[t,x]] \to \mathbb{C}[[t,x]]$ as

$$\partial_x^{-1} U(t,x) := \int_0^x U(t,\zeta)d\zeta.$$

By convention, given two functions $f,g : \mathbb{R} \to \mathbb{R}$, we will write $f(x) \sim g(x)$ as x tends to $+\infty$ if $\lim_{x\to+\infty} f(x)/g(x) = 1$.

Lemma 1 *Let $\mu, h_1, h_2, m_1, m_2 \geq 0$ be non negative integers. Assume that the following inequalities hold:*

$$h_2 + \mu \geq m_1, \quad m_2 \geq h_1 + h_2 + \mu, \quad h_2 \geq sh_1. \tag{2}$$

Then, there exists a constant $C > 0$ such that

$$||x^\mu(\partial_x^{-h_2}\partial_t^{h_1}U)(q^{m_1}t, q^{-m_2}x)||_{X,T} \leq CX^{\mu+h_2}T^{-h_1}||U(t,x)||_{X,T},$$

for all $U(t,x) \in G^q(X,T,s)$.

Proof For $U(t,x) \in G^q(X,T,s)$ we have

$$x^\mu(\partial_x^{-h_2}\partial_t^{h_1}U)(q^{m_1}t, q^{-m_2}x) = \sum_{\beta,l} \frac{\beta!}{(\beta-\mu)!} u_{\beta-h_2-\mu,l+h_1} q^{m_1 l} q^{-m_2(\beta-\mu)} \frac{x^\beta}{\beta!}\frac{t^l}{l!}.$$

By definition, we have

$$||x^\mu(\partial_x^{-h_2}\partial_t^{h_1}U)(q^{m_1}t, q^{-m_2}x)||_{X,T}$$
$$= X^{\mu+h_2}T^{-h_1} \sum_{\beta,l} \frac{|u_{\beta-h_2-\mu,l+h_1}|(\beta!/(\beta-\mu)!)q^{m_1 l}q^{-m_2(\beta-\mu)}}{q^{P(l,\beta)}(sl+\beta)!} X^{\beta-h_2-\mu}T^{l+h_1}$$

So that

$$||x^\mu(\partial_x^{-h_2}\partial_t^{h_1}U)(q^{m_1}t, q^{-m_2}x)||_{X,T}$$
$$= X^{\mu+h_2}T^{-h_1} \sum_{\beta,l} A(l,\beta) \frac{|u_{\beta-h_2-\mu,l+h_1}|}{q^{P(l+h_1,\beta-h_2-\mu)}(s(l+h_1)+\beta-h_2-\mu)!} X^{\beta-h_2-\mu}T^{l+h_1}$$

where

$$A(l,\beta) = \frac{(s(l+h_1)+\beta-h_2-\mu)!\beta!}{(sl+\beta)!(\beta-\mu)!} \frac{1}{q^{(P(l,\beta)-P(l+h_1,\beta-h_2-\mu)-m_1 l+m_2(\beta-\mu))}}.$$

By construction, we have

$$\begin{aligned}&P(l,\beta)-P(l+h_1,\beta-h_2-\mu)-m_1l+m_2(\beta-\mu)\\&=l(h_2+\mu-m_1)+\beta(-h_1-h_2-\mu+m_2)+h_1(h_2+\mu)+(1/2)((h_2+\mu)^2+h_2+\mu)-m_2\mu.\end{aligned}$$

We recall the classical estimates

$$\Gamma(x+b)/\Gamma(x)\sim x^b \tag{3}$$

for any $b\geq 0$, as $x>0$ tends to $+\infty$. From (3) we deduce that

$$\frac{(sl+\beta+sh_1-h_2-\mu)!\beta!}{(sl+\beta)!(\beta-\mu)!}\sim (sl+\beta)^{sh_1-h_2-\mu}\beta^{\mu},$$

as l,β tend to infinity. By these latter estimates and the assumptions (2) we get a constant $C>0$ such that $A(l,\beta)\leq C$, for all l,β. This gives the lemma. □

Lemma 2 *There exists a constant $C>0$ such that for all $\alpha\in\mathbb{N}$,*

$$||t^{\alpha}U(t,x)||_{X,T}\leq CT^{\alpha}||U(t,x)||_{X,T},$$

for all $U(t,x)\in G^q(X,T,s)$. Let $a(t)=\sum_{j\geq 0}a_jt^j$ be a holomorphic function on a neighborhood of the origin in $\mathbb{C}$. We define $|a|(t)=\sum_{j\geq 0}|a_j|t^j$. We deduce that there exists a constant $C>0$ such that

$$||a(t)U(t,x)||_{X,T}\leq C|a|(T)||U(t,x)||_{X,T},$$

for all $U(t,x)\in G^q(X,T,s)$.

Proof Let $U(t,x)\in G^q(X,T,s)$. We have

$$t^{\alpha}U(t,x)=\sum_{\beta,l}\frac{l!}{(l-\alpha)!}u_{\beta,l-\alpha}\frac{x^{\beta}}{\beta!}\frac{t^l}{l!}.$$

By definition, we have

$$\begin{aligned}||t^{\alpha}U(t,x)||_{X,T}&=T^{\alpha}\sum_{\beta,l}\frac{|u_{\beta,l-\alpha}|(l!/(l-\alpha)!)}{q^{P(l,\beta)}(sl+\beta)!}X^{\beta}T^{l-\alpha}\\&=T^{\alpha}\sum_{\beta,l}\{\frac{(s(l-\alpha)+\beta)!l!}{(sl+\beta)!(l-\alpha)!}\frac{1}{q^{P(l,\beta)-P(l-\alpha,\beta)}}\}\frac{|u_{\beta,l-\alpha}|}{q^{P(l-\alpha,\beta)}(s(l-\alpha)+\beta)!}X^{\beta}T^{l-\alpha}\end{aligned}$$

By construction, we have that $P(l,\beta)-P(l-\alpha,\beta)=\alpha\beta$. Moreover, from the estimates (3), we also get that

$$\frac{(s(l-\alpha)+\beta)!l!}{(sl+\beta)!(l-\alpha)!}\sim (sl+\beta)^{-s\alpha}l^{\alpha}$$

as l,β tend to infinity. So that there exists a constant $C>0$ such that

$$\frac{(s(l-\alpha)+\beta)!l!}{(sl+\beta)!(l-\alpha)!}\frac{1}{q^{P(l,\beta)-P(l-\alpha,\beta)}}\leq C$$

for all l,β. This gives the lemma. □

Lemma 3 *Let $\delta_1, \delta_2 > 0$ be positive real numbers and let $U(t,x)$ be in $G^q(X_0,T_0,s)$. Then, there exist $X,T > 0$ small enough (depending on δ_1, δ_2) such that the formal series $U(\delta_1 t, \frac{x}{\delta_2})$ belongs to $G^q(X,T,s)$.*

Proof Let $U(t,x) \in G^q(X,T,s)$. We have

$$U(\delta_1 t, \frac{x}{\delta_2}) = \sum_{\beta,l\geq 0} u_{\beta,l}{\delta_1}^l {\delta_2}^{-\beta}\frac{x^\beta}{\beta!}\frac{t^l}{l!}.$$

By definition, we have

$$||U(\delta_1 t, \frac{x}{\delta_2})||_{X,T} = \sum_{\beta,l} \frac{|u_{\beta,l}|}{q^{P(l,\beta)}(sl+\beta)!}(X/\delta_2)^\beta(\delta_1 T)^l.$$

Due to the fact that $U(t,x) \in G^q(X_0,T_0,s)$, this latter series converges if $X/\delta_2 \leq X_0$ and $\delta_1 T \leq T_0$. So that, if $X/\delta_2 \leq X_0$ and $\delta_1 T \leq T_0$, then $U(\delta_1 t, \frac{x}{\delta_2}) \in G^q(X,T,s)$. □

Lemma 4 *Let $\mu \in \mathbb{N}$ and let $U(t,x)$ be in $G^q(X_0,T_0,s)$. Then, there exist $X,T > 0$ small enough (depending on μ) such that the formal series $x^\mu U(t,x)$ belongs to $G^q(X,T,s)$.*

Proof Let $U(t,x) \in G^q(X,T,s)$. From the proof of Lemma 1, we get a constant $C > 0$ such that

$$||x^\mu U(t,x)||_{X,T} \leq C(q^\mu X)^\mu \sum_{\beta,l} \frac{|u_{\beta-\mu,l}|}{q^{P(l,\beta-\mu)}(sl+\beta-\mu)!}(q^\mu X)^{\beta-\mu}T^l.$$

Due to the fact that $U(t,x) \in G^q(X_0,T_0,s)$, the series on the right hand side of the latter inequality is convergent if $q^\mu X \leq X_0$ and $T \leq T_0$. So that if $q^\mu X \leq X_0$ and $T \leq T_0$, then $x^\mu U(t,x) \in G^q(X,T,s)$. □

Lemma 5 *Let $h_1 \in \mathbb{N}$ and let $U(t,x)$ be in $G^q(X_0,T_0,s)$. Then, there exist $X,T > 0$ small enough (depending on h_1) such that the formal series $\partial_t^{h_1}U(t,x)$ belongs to $G^q(X,T,s)$.*

Proof Let $U(t,x) \in G^q(X,T,s)$. Like in the proof of Lemma 1, we get

$$||(\partial_t^{h_1}U)(t,x)||_{X,T} = \\ T^{-h_1}\sum_{\beta,l}\{\frac{(s(l+h_1)+\beta)!}{(sl+\beta)!}\frac{1}{q^{P(l,\beta)-P(l+h_1,\beta)}}\}\frac{|u_{\beta,l+h_1}|}{q^{P(l+h_1,\beta)}(s(l+h_1)+\beta)!}X^\beta T^{l+h_1}$$

By the estimates (3), we have

$$\frac{(s(l+h_1)+\beta)!}{(sl+\beta)!} \sim (sl+\beta)^{sh_1},$$

as l,β tend to infinity. By construction, we also have $P(l,\beta) - P(l+h_1,\beta) = -h_1\beta$. So that there exist $K, C_1, C_2 > 0$ such that

$$||(\partial_t^{h_1}U)(t,x)||_{X,T} \leq T^{-h_1}K\sum_{\beta,l}\frac{|u_{\beta,l+h_1}|}{q^{P(l+h_1,\beta)}(s(l+h_1)+\beta)!}(q^{h_1}C_2X)^\beta(C_1T)^{l+h_1}.$$

Due to the fact that $U(t,x) \in G^q(X_0,T_0,s)$, the series on the right hand side of the latter inequality is convergent if $q^{h_1}C_2X \leq X_0$ and $C_1T \leq T_0$. Finally, if $q^{h_1}C_2X \leq X_0$ and $C_1T \leq T_0$ then $\partial_t^{h_1}U(t,x) \in G^q(X,T,s)$. □

Let $\mathcal{D}$ be the linear operator from $\mathbb{C}[[t,x]]$ into $\mathbb{C}[[t,x]]$ defined as

$$\mathcal{D}(u(t,x)) = \partial_x^S u(t,x) - \sum_{k=(k_0,k_1,l)\in\mathcal{S}} a_k(t)x^l(\partial_t^{k_0}\partial_x^{k_1}u)(q^{m_{0,k}}t, xq^{-m_{1,k}}),$$

where $\mathcal{S}$ is a finite subset of $\mathbb{N}^3$ and $a_k(t)$ are holomorphic functions on a neighborhood of the origin in $\mathbb{C}$. Moreover, the integers $S, m_{0,k}, m_{1,k} \geq 1$ satisfy the following assumption.

Assumption (A):
i) $S - k_1 \geq sk_0, \quad S > k_1$,
ii) $S - k_1 + l \geq m_{0,k}, \quad m_{1,k} \geq k_0 + S - k_1 + l$,

for all $k = (k_0,k_1,l) \in \mathcal{S}$.

We consider the following operator from $\mathbb{C}[[t,x]]$ into $\mathbb{C}[[t,x]]$,

$$\mathcal{D} \circ \partial_x^{-S} = \mathrm{id} - \mathcal{A},$$

where

$$\mathcal{A}(u(t,x)) = \sum_{k=(k_0,k_1,l)\in\mathcal{S}} a_k(t)x^l(\partial_t^{k_0}\partial_x^{k_1-S}u)(q^{m_{0,k}}t, xq^{-m_{1,k}}),$$

for all $u(t,x) \in \mathbb{C}[[t,x]]$, and id is the identity operator. As a consequence of Lemma 1,2 we get the following lemma.

Lemma 6 *There exist real numbers $X,T > 0$ small enough (that depend on the coefficients $a_k(t)$ for $k = (k_0,k_1,l) \in \mathcal{S}$) such that $\mathcal{A}$ is a bounded linear operator, from $(G^q(X,T,s), ||.||_{X,T})$ into itself. Moreover the estimates hold,*

$$||\mathcal{A}u(t,x)||_{X,T} \leq (1/2)||u(t,x)||_{X,T},$$

for all $u(t,x) \in G^q(X,T,s)$.

Corollary 1 *There exist real numbers $X,T > 0$ small enough (that depend on the coefficients $a_k(t)$ for $k = (k_0,k_1,l) \in \mathcal{S}$) such that $\mathcal{D} \circ \partial_x^{-S}$ is a bounded invertible operator from $(G^q(X,T,s), ||.||_{X,T})$ into itself. In particular, there exist a constant $C > 0$ such that*

$$||(\mathcal{D} \circ \partial_x^{-S})^{-1}b(t,x)||_{X,T} \leq C||b(t,x)||_{X,T},$$

for all $b(t,x) \in G^q(X,T,s)$.

The main result of this section is the following.

Theorem CK *Consider a functional partial differential equation*

$$\partial_x^S u(t,x) = \sum_{k=(k_0,k_1,l)\in\mathcal{S}} a_k(t)x^l(\partial_t^{k_0}\partial_x^{k_1}u)(q^{m_{0,k}}t, xq^{-m_{1,k}}), \tag{4}$$

where $\mathcal{S}$ is a finite subset of $\mathbb{N}^3$ and $a_k(t)$ are holomorphic functions on a neighborhood of the origin in $\mathbb{C}$.

We make the hypothesis that the integers $S, m_{0,k}, m_{1,k} \geq 1$ *satisfy the assumption* **(A).**

We impose the initial conditions : For all $0 \leq j \leq S-1$*,*

$$(\partial_x^j u)(t,0) = \phi_j(t), \tag{5}$$

where $\phi_j(t) \in G^q(X_0, T_0, s)$*, for given* $X_0, T_0 > 0$ *and* $s \geq 1$*.*

Then, there exist X, T *small enough such that the problem (4), (5) has a unique solution* $u(t,x)$ *in* $G^q(X,T,s)$*.*

Proof Every formal series $u(t,x) \in \mathbb{C}[[t,x]]$ can be written in the form

$$u(t,x) = \partial_x^{-S} v(t,x) + w(t,x),$$

where

$$w(t,x) = \sum_{j=0}^{S-1} \varphi_j(t) \frac{x^j}{j!}.$$

From the initial conditions (5) and Lemma 4, let us assume that $w(t,x)$ belongs to $G^q(X,T,s)$ for $X, T > 0$ small enough. Then, the formal series $u(t,x)$ is a solution of (4), (5) if and only if $v(t,x)$ satisfies the equation

$$\mathcal{D} \circ \partial_x^{-S} v(t,x) = -\mathcal{D} w(t,x). \tag{6}$$

From the fact that $w(t,x) \in G^q(X,T,s)$ is a polynomial in x of degree less than $S-1$, we deduce from the lemma 2,3,4,5 that $-\mathcal{D}w(t,x)$ also belongs to $G^q(X,T,s)$, for all $X, T > 0$ small enough. From the corollary 1, we deduce that for X, T small enough, there exists a unique solution $v(t,x)$ of (6) which belongs to $G^q(X,T,s)$. It remains to show that $u(t,x)$ belongs to $G^q(X,T,s)$ for X,T small enough. From Lemma 1 and 3 we get that the formal series $\nu(t,x) = u(q^S t, xq^{-S}) = \partial_x^{-S} v(q^S t, \frac{x}{q^S}) + w(q^S t, xq^{-S})$ belongs to $G^q(X,T,s)$ for X,T small enough. Finally, again by Lemma 3, we deduce that $u(t,x) = \nu(\frac{t}{q^S}, q^S x)$ belongs to $G^q(X,T,s)$ for X,T small enough. □

2 Formal series solutions of functional linear partial differential equations

In the sequel we consider a functional linear partial differential equation

$$\partial_z^S \hat{X}(t,z) = \sum_{k=(k_0,k_1,l) \in \mathcal{S}} b_k(t) z^l (\partial_t^{k_0} \partial_z^{k_1} \hat{X})(\phi_k(t), zq^{-m_{1,k}}), \tag{1}$$

where $q > 1$ is a real number, $S \geq 1$ is a positive integer and $\mathcal{S}$ is a finite subset of $\mathbb{N}^3$, the coefficients $b_k(t)$ and the functions $\phi_k(t)$ are holomorphic on a common domain $G \cup D_\rho$ where G is an open sector with infinite radius centered at 0 and D_ρ an open disc centered at 0 with radius ρ. Moreover, we assume that $\phi_k(0) = 0$ and that $\phi_k(G) \subset G$, for all $k = (k_0, k_1, l) \in \mathcal{S}$. Notice that by convention 0 does not belong to the sector G.

We study formal series $\hat{X}(t,z)$ in the variable z of the form

$$\hat{X}(t,z) = \sum_{n \geq 0} \frac{X_n(t)}{n!} z^n, \tag{7}$$

with holomorphic coefficients $X_n(t)$, $n \geq 0$, on G.

In the next lemma we describe formal series solutions of (1) in term of differential relations.

Lemma 7 *The formal series* $\hat{X}(t,z)$ *is a solution of (1) if and only if the following differential relation is satisfied,*

$$\frac{X_{m+S}(t)}{m!} = \sum_{\substack{k=(k_0,k_1,l)\in\mathcal{S}\\ l+n=m}} b_k(t)\frac{(\partial_t^{k_0}X_{n+k_1})(\phi_k(t))}{n!}\frac{1}{q^{nm_{1,k}}}, \tag{8}$$

for all $m \geq 0$.

In the next proposition, we give sufficient conditions for the existence and uniqueness of formal series (7) that are solutions of (1).

Proposition 1 *We assume that the finite set* $\mathcal{S}$ *satisfies*

$$\mathcal{S} \subset \{(k_0,k_1,l) \in \mathbb{N}^3 : k_1 \leq S-1\}.$$

Moreover, assume that holomorphic functions $X_\mu(t)$, *for* $0 \leq \mu \leq S-1$, *are given on* G. *Then, there exists a unique sequence* $(X_\mu(t))_{\mu\geq 0}$ *of holomorphic functions which satisfies the differential recurrence (8). As a result, for given holomorphic functions* $X_\mu(t)$, *for* $0 \leq \mu \leq S-1$ *on* G, *there exists a unique formal series (7), solution of (1).*

3 Global sectorial holomorphic solutions to functional linear partial differential equations in Gevrey spaces

First of all, we recall the definitions of the Gevrey spaces of analytic functions in which we will choose our initial conditions and expect to find our solutions.

Let Ω be an open sector centered at the origin in $\mathbb{C}$ with bisecting direction $d \in \mathbb{R}$ and with infinite radius. Consider a bounded holomorphic function $w(t)$ from Ω into $\mathbb{C}^*$ with the property that for all closed sector of infinite radius $\bar{S}_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| \leq \delta\}$ included in Ω, there exists a constant $M > 0$ such that

$$\sup_{t\in\bar{S}_{d,\delta}} \frac{|w(t)|}{|w(\phi_k(t))|} \leq M,$$

for all $k = (k_0,k_1,l) \in \mathcal{S}$.

Definition 2 *Let* $X(t)$ *be a holomorphic function on an open sector* Ω *centered at the origin in* $\mathbb{C}$ *with bisecting direction* $d \in \mathbb{R}$ *and with infinite radius. Let* s *be a real number such that* $s \geq 1$. *We say that* $X(t)$ *is of Gevrey type of order* $s-1$ *on* Ω *if the following inequalities hold.*

For all closed sector of infinite radius $\bar{S}_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| \leq \delta\}$ *included in* Ω, *there exist constants* $C, T_0 > 0$ *such that*

$$\sup_{t\in\bar{S}_{d,\delta}} |\partial_t^l X(t)||w(t)| \leq C(1/T_0)^l (sl)!$$

for all $l \geq 0$.

Definition 3 *Let* $X(t,z)$ *be a holomorphic function on* $\Omega\times\mathbb{C}$ *where* Ω *is an open sector centered at the origin in* $\mathbb{C}$ *with bisecting direction* $d \in \mathbb{R}$ *and with infinite radius. Let* s, q *be a real numbers such that* $q > 1$ *and* $s \geq 1$. *We say that* $X(t,z)$ *is of Gevrey type with double filtration of order* $s-1$ *and* q *with respect to* t *on* $\Omega \times \mathbb{C}$ *if the following inequalities hold.*

For all closed sector of infinite radius $\bar{S}_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| \leq \delta\}$ *included in* Ω *and for all closed disc* $\bar{D}_r$ *centered at 0 with radius* r*, there exist constants* $C, T_0 > 0$ *such that*

$$\sup_{t \in \bar{S}_{d,\delta}, z \in \bar{D}_r} |\partial_t^l X(t,z)||w(t)| \leq C(1/T_0)^l (sl)! e^{\frac{l^2}{2}(\log(q))^2}$$

for all $l \geq 0$*.*

In the following, we will need a rule to evaluate high order derivatives of compositions of functions which has been introduced in [20] and is compatible with Gevrey estimates. We recall this higher order chain rule (Theorem 2.1 in [20]) under stronger assumptions which will be sufficient for our scope.

Lemma 8 *Let* D, G *be open sets in* $\mathbb{C}$*. Let* $v : D \to G$ *and* $w : G \to \mathbb{C}$ *be holomorphic functions. Then, the n-th order derivative of the composite function* $w \circ v : D \to \mathbb{C}$ *is given by the formula,*

$$\partial_x^n (w \circ v)(x) = \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} (\partial_x^j w)(v(x)) \left\{ \partial_h^{n-j} [\int_0^1 (\partial_x v)(x + \theta h) d\theta]^j \right\}_{|h=0}.$$

Consider now an open set $D_1 \subset D$ and D_0 a small disc centered at 0 in $\mathbb{C}$, such that, for all $x \in D_1$, $h \in D_0$, we have $x + \theta h \in D$, for all $\theta \in [0,1]$. Let $q > 1$ be a real number. Assume that $|\partial_x v(x)| \leq q$ for all $x \in D$. Consider the function

$$\psi_{j,x}(h) = [\int_0^1 (\partial_x v)(x + \theta h) d\theta]^j,$$

for all $h \in D_0$, with $x \in D_1$ and $j \in \mathbb{N}$. We deduce that $|\psi_{j,x}(h)| \leq q^j$, for all $h \in D_0$, with $x \in D_1$ and $j \geq 0$. From the Cauchy formula, we deduce that there exist $r > 0$, such that

$$|\partial_h^k \psi_{j,x}(0)| \leq q^j k! r^k,$$

for all $k, j \in \mathbb{N}$, $x \in D_1$. From these latter estimates, we deduce

Lemma 9 *Let* D, G *be open sets in* $\mathbb{C}$ *and* $q > 1$ *a real number. Let* $v : D \to G$ *and* $w : G \to \mathbb{C}$ *be holomorphic functions, such that* $|\partial_x v(x)| \leq q$ *for all* $x \in D$*. Then, there exist* $r > 0$ *which depends only* D_0*, such that the estimates hold,*

$$|\partial_x^n (w \circ v)(x)| \leq \sum_{a_1 + a_2 = n} \frac{n!}{a_1!} |(\partial_x^{a_1} w)(v(x))| q^{a_1} r^{a_2},$$

for all $x \in D_1$ *and* $n \in \mathbb{N}$*.*

we note that similar computations have been made in the paper [10], pp. 667–668.

The next result gives sufficient conditions under which there exist holomorphic solutions $X(t,z)$ to (1) which are of Gevrey type with double filtration of order $s - 1$ and q, for given initial conditions be of Gevrey type of order $s - 1$.

Theorem 1 *Consider the functional linear partial differential equation,*

$$\partial_z^S \hat{X}(t,z) = \sum_{k=(k_0,k_1,l) \in \mathcal{S}} b_k(t) z^l (\partial_t^{k_0} \partial_z^{k_1} \hat{X})(\phi_k(t), z q^{-m_{1,k}}), \tag{1}$$

where $q > 1$ is a real number, $S \geq 1$ is a positive integer and $\mathcal{S}$ is a finite subset of $\mathbb{N}^3$ which satisfies the hypothesis of Proposition 1, the numbers $m_{1,k} \geq 1$ are positive integers. The coefficients $b_k(t)$ and the functions $\phi_k(t)$ are holomorphic on a common domain $G \cup D_\rho$ where G is an open sector with bisecting direction $d \in \mathbb{R}$ with infinite radius centered at 0 and D_ρ an open disc centered at 0 with radius ρ.

1) Assume that $\phi_k(0) = 0$ and that the functions $\phi_k(t)$ satisfy

$$|\partial_t \phi_k(t)| \leq q^{m_{0,k}},$$

for integers $m_{0,k} \geq 1$, for all $k = (k_0, k_1, l) \in \mathcal{S}$, and t in $G \cup D_\rho$.

2) Assume that $\phi_k(\bar{S}) \subset \bar{S}$, for all closed sector centered at 0 of infinite radius $\bar{S} \subset G$ with bisecting direction d, for all $k = (k_0, k_1, l) \in \mathcal{S}$.

3) Assume that the quantity

$$\sup_{t \in G \cup D_\rho} \frac{|\partial_t^{l_0} b_k(t)|}{l_0!} = |b|_{k,l_0}$$

exists and that

$$|b|_k(t) = \sum_{l_0 \geq 0} |b|_{k,l_0} t^{l_0}$$

is an holomorphic function on a neighborhood of the origin in $\mathbb{C}$ for all $k = (k_0, k_1, l) \in \mathcal{S}$.

4) Let the assumption **(A)** *from section 1 be fulfilled for the integers $S, m_{0,k}, m_{1,k}$.*

5) We make the following assumption on initial conditions. Let $X_\mu(t)$, for $0 \leq \mu \leq S-1$, be given holomorphic functions which are of Gevrey type of order $s-1$ on G.

Then, the formal series

$$X(t,z) = \sum_{n \geq 0} \frac{X_n(t)}{n!} z^n$$

solution of (1) with initial conditions $(\partial_z^\mu X)(t,0) = X_\mu(t)$, $0 \leq \mu \leq S-1$, constructed in the proposition 1 is convergent for all $z \in \mathbb{C}$. Moreover, the function $X(t,z)$ is of Gevrey type with double filtration of order $s-1$ and q with respect to t on $G \times \mathbb{C}$.

Example 1: This latter theorem can be used to study q-difference-differential equations if one takes the deviations $\phi_k(t)$ to be dilations of the form $\phi_k(t) = q^{m_{0,k}} t$.

Example 2: The theorem 1 can also be applied to study difference-differential equations. Indeed, consider the equation (1) and choose the deviations $\phi_k(t)$, $k \in \mathcal{S}$ of the form $\phi_k(t) = t/(1 + \alpha_k t)$, where $\alpha_k > 0$ are positive real numbers, and assume that $b_k(t)$ are equal to zero for all $k = (k_0, k_1, l) \in \mathcal{S}$ with $k_0 \neq 0$. Assume that the sector G has bisecting direction $d = 0$ and satisfies $|1 + \alpha_k t|^2 \geq q^{-m_{0,k}}$ for all $t \in G \cup D_\rho$ with ρ small enough and $k \in \mathcal{S}$. One checks easily that the assumptions 1) and 2) of Theorem 1 are fullfilled. Under these conditions, a function $X(t,z)$ is solution of (1) on $G \times \mathbb{C}$ if and only if the function $Y(u,z) = X(1/u, z)$, with $u = 1/t$, is solution of the difference-differential equation

$$\partial_z^S Y(u,z) = \sum_{k=(0,k_1,l) \in \mathcal{S}} b_k(1/u) z^l (\partial_z^{k_1} Y)(u + \alpha_k, z q^{-m_{1,k}}),$$

on $G' \times \mathbb{C}$ where $G' = \{u \in \mathbb{C} : 1/u \in G\}$.

Proof of Theorem 1 The goal of the proof is to show that the functions $X_n(t)$ constructed from the proposition 1 satisfy the following estimates. For all closed sector of infinite radius

$\bar{S} \subset G$ centered at 0 with bisecting direction d, there exist $C, X_0, T_0 > 0$, independent of n, l, such that

$$\sup_{t\in\bar{S}} |\partial_t^l X_n(t)||w(t)| \leq C(1/X_0)^n (1/T_0)^l q^{nl-(1/2)(n^2-n)}(sl+n)!, \tag{9}$$

for all $l, n \geq 0$. Indeed, from (9) we get $X_0', T_0' > 0$ such that

$$\sup_{t\in\bar{S}} |\partial_t^l X(t,z)||w(t)| \leq \sum_{n\geq 0} \frac{\sup_{t\in\bar{S}} |\partial_t^l X_n(t)||w(t)|}{n!}|z|^n \leq C(\frac{1}{T_0'})^l (sl)! \sum_{n\geq 0} q^{-\frac{n^2}{2}}(|z|q^{\frac{1}{2}+l}/X_0')^n,$$

for all $z \in \mathbb{C}$. From Lemma 2.2 in [16], we get a constant $C' > 0$ such that

$$\sum_{n\geq 0} q^{-\frac{n^2}{2}}(|z|q^{\frac{1}{2}+l}/X_0')^n \leq C' \exp\{\frac{1}{2}(\log(2|z|q^{\frac{1}{2}+l}/X_0'))^2\}$$

for all $z \in \mathbb{C}$. So that for all $r > 0$,

$$\sup_{|z|\leq r}\sup_{t\in\bar{S}} |\partial_t^l X(t,z)||w(t)| \leq CC'(\frac{1}{T_0'})^l (sl)! \exp\{\frac{1}{2}(\log(\frac{2rq^{1/2}}{X_0'}q^l))^2\}$$

It follows from the latter inequality that $X(t,z)$ is Gevrey type with double filtration of order $s-1$ and q with respect to t on $G \times \mathbb{C}$.

We consider now the sequence of quantities

$$w_{n_0,m_1} = \sup_{t\in\bar{S}} |\partial_t^{n_0} X_{m_1}(t)||w(t)|,$$

for $n_0, m_1 \geq 0$, where $\bar{S}$ is a closed sector centered at 0 of infinite radius in G with bisecting direction d. We show that this sequence is in fact real valued and satisfies a multivariate recursion inequality.

From the Leibniz formula, we have that

$$w(t)\partial_t^{n_0}\left(b_k(t)\frac{(\partial_t^{k_0}X_{n+k_1})(\phi_k(t))}{n!q^{nm_{1,k}}}\right) = \sum_{h_1+h_2=n_0} \frac{n_0!}{h_2!h_1!}\partial_t^{h_2}b_k(t)\frac{w(t)\partial_t^{h_1}((\partial_t^{k_0}X_{n+k_1})(\phi_k(t)))}{n!q^{nm_{1,k}}}.$$

From Lemma 9, the assumption 1) in Theorem 1 and the definition of $w(t)$, we deduce that there exist $M > 0$ and $r_k > 0$ for all $k = (k_0, k_1, l) \in \mathcal{S}$ such that

$$\begin{aligned}|w(t)||\partial_t^{h_1}((\partial_t^{k_0}X_{n+k_1})(\phi_k(t)))| &\leq \sum_{a_1+a_2=h_1}\frac{h_1!}{a_1!}|w(t)||(\partial_t^{a_1+k_0}X_{n+k_1})(\phi_k(t))|q^{a_1m_{0,k}}r_k^{a_2}\\ &\leq \sum_{a_1+a_2=h_1}\frac{h_1!}{a_1!}M|w(\phi_k(t))||(\partial_t^{a_1+k_0}X_{n+k_1})(\phi_k(t))|q^{a_1m_{0,k}}r_k^{a_2}.\end{aligned}$$

From the assumptions 2) and 3) in Theorem 1 we deduce

$$\begin{aligned}&\sup_{t\in\bar{S}}|w(t)|\left|\partial_t^{n_0}\left(b_k(t)\frac{(\partial_t^{k_0}X_{n+k_1})(\phi_k(t))}{n!q^{nm_{1,k}}}\right)\right| \leq\\ &\sum_{h_1+h_2=n_0}\frac{n_0!M|b|_{k,h_2}}{q^{nm_{1,k}}}\sum_{a_1+a_2=h_1}\frac{\sup_{t\in\bar{S}}|(\partial_t^{a_1+k_0}X_{n+k_1})(\phi_k(t))||w(\phi_k(t))|}{a_1!n!}q^{a_1m_{0,k}}r_k^{a_2}\\ &\leq \sum_{h_1+h_2=n_0}\frac{n_0!M|b|_{k,h_2}}{q^{nm_{1,k}}}\sum_{a_1+a_2=h_1}\frac{\sup_{t\in\bar{S}}|(\partial_t^{a_1+k_0}X_{n+k_1})(t)||w(t)|}{a_1!n!}q^{a_1m_{0,k}}r_k^{a_2}\end{aligned}$$

From the latter estimates, we get inequalities for the number w_{n_0,n_1},

$$\frac{w_{n_0,m_1+S}}{n_0!m_1!} \leq \sum_{\substack{k=(k_0,k_1,l)\in\mathcal{S}\\ l+n=m_1}} \sum_{h_1+h_2=n_0} \frac{M|b|_{k,h_2}}{q^{nm_{1,k}}} \sum_{a_1+a_2=h_1} \frac{w_{a_1+k_0,n+k_1}}{a_1!n!} q^{a_1 m_{0,k}} r_k^{a_2}, \tag{10}$$

for all $n_0, m_1 \geq 0$.

We define the functions,

$$R_k(t) = \sum_{j\geq 0} r_k^j t^j,$$

where r_k are positive real numbers for all $k = (k_0, k_1, l) \in \mathcal{S}$. Consider now the auxiliary functional partial differential equation,

$$\partial_z^S V(t,z) = \sum_{k=(k_0,k_1,l)\in\mathcal{S}} M|b|_k(t) R_k(t) z^l (\partial_t^{k_0}\partial_z^{k_1} V)(q^{m_{0,k}}t, \frac{z}{q^{m_{1,k}}}), \tag{11}$$

constructed from (1). Due to the assumption 4) in Theorem 1, we deduce that the hypotheses of Theorem CK in section 1 are satisfied for the latter equation (11). From Theorem CK, we deduce the existence of a unique formal series $V(t,z) \in G^q(X,T,s)$, for $X, T > 0$ small enough, solution of (11), if $(\partial_z^j V)(t,0)$ are prescribed formal series which belong to $G^q(X_0,T_0,s)$, for all $0 \leq j \leq S-1$, for given $X_0, T_0 > 0$.

If we expand $V(t,z)$ into a Taylor series in the variables t, z,

$$V(t,z) = \sum_{n_0,n_1\geq 0} v_{n_0,n_1} \frac{t^{n_0} z^{n_1}}{n_0! n_1!},$$

we get that

$$R_k(t)(\partial_t^{k_0}\partial_z^{k_1} V)(q^{m_{0,k}}t, \frac{z}{q^{m_{1,k}}}) = \sum_{h_1,n_1} (\sum_{a_1+a_2=h_1} r_k^{a_2} q^{a_1 m_{0,k}} \frac{v_{a_1+k_0,n_1+k_1}}{a_1!n_1!q^{n_1 m_{1,k}}}) t^{h_1} z^{n_1},$$

and

$$M|b|_k(t) z^l R_k(t)(\partial_t^{k_0}\partial_z^{k_1} V)(q^{m_{0,k}}t, \frac{z}{q^{m_{1,k}}}) = \sum_{n_0,m_1} \mathcal{A}_{n_0,m_1} t^{n_0} z^{m_1},$$

with

$$\mathcal{A}_{n_0,m_1} = \sum_{\substack{h_1+h_2=n_0\\ l+n_1=m_1}} \frac{M|b|_{k,h_2}}{q^{n_1 m_{1,k}}} \sum_{a_1+a_2=h_1} \frac{v_{a_1+k_0,n_1+k_1}}{a_1!n_1!} q^{a_1 m_{0,k}} r_k^{a_2},$$

for all $n_0, m_1 \geq 0$. From this, we deduce a multivariate linear recurrence satisfied by the coefficients v_{n_0,n_1},

$$\frac{v_{n_0,m_1+S}}{n_0!m_1!} = \sum_{\substack{k=(k_0,k_1,l)\in\mathcal{S}\\ l+n=m_1}} \sum_{h_1+h_2=n_0} \frac{M|b|_{k,h_2}}{q^{nm_{1,k}}} \sum_{a_1+a_2=h_1} \frac{v_{a_1+k_0,n+k_1}}{a_1!n!} q^{a_1 m_{0,k}} r_k^{a_2}, \tag{12}$$

for all $n_0, m_1 \geq 0$. From the fact that $V(t,z) \in G^q(X,T,s)$, we deduce the following estimates for the coefficients v_{n_0,n_1}. There exists $C > 0$ such that

$$|v_{n_0,n_1}| \leq C(1/X)^{n_1}(1/T)^{n_0} q^{n_1 n_0 - (1/2)(n_1^2 - n_1)} (sn_0 + n_1)!, \tag{13}$$

for all $n_0, n_1 \geq 0$. From Theorem CK, we see that the sequence $(v_{n_0,m_1})_{n_0,m_1}$ is uniquely determined by the sequences $(v_{n_0,m_1})_{n_0,m_1}$, $0 \leq m_1 \leq S-1$. We choose now the latter sequences as follows.

$$v_{n_0,m_1} := w_{n_0,m_1},$$

for all $0 \leq m_1 \leq S-1$, $n_0 \geq 0$.

From the assumption 5) in Theorem 1, we deduce that the series

$$V(t,0), \ldots, (\partial_z^{S-1} V)(t,0)$$

are formal series with Gevrey order $s-1$, which means in particular that they belong to $G^q(X_0, T_0, s)$ for $X_0, T_0 > 0$. Moreover, due to the relations (10) and (12), we deduce that

$$w_{n_0,m_1} \leq v_{n_0,m_1},$$

for all $n_0, m_1 \geq 0$.

From the estimates (13), we finally deduce that there exist $C > 0$, such that the sequence w_{n_0,n_1} satisfies

$$w_{n_0,n_1} \leq C(1/X)^{n_1}(1/T)^{n_0} q^{n_1 n_0 - (1/2)(n_1^2 - n_1)}(sn_0 + n_1)!,$$

for all $n_0, n_1 \geq 0$, which yields the result. □

References

[1] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems.* Monographs in Mathematics, 96. Birkhäuser, 2001.

[2] A. Augustynowicz, H. Leszczyński, W. Walter, *Cauchy-Kovalevskaya theory for equations with deviating variables.* Dedicated to Janos Aczel on the occasion of his 75th birthday. Aequationes Math. 58 (1999), no. 1-2, 143–156.

[3] W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Springer-Verlag, New-York, 2000.

[4] W. Balser, S. Malek, *Formal solutions of the complex heat equation in higher spatial dimensions*, RIMS, 1367, (2004), 95-102.

[5] J.P. Bézivin, *Sur les équations fonctionnelles aux q-différences*, Aequationes Math. 43 (1993), 159–176.

[6] L. Di Vizio, J.-P. Ramis, J. Sauloy, C. Zhang, *Équations aux q-différences.* Gaz. Math. No. 96 (2003), 20–49.

[7] J. Écalle, *Les fonctions résurgentes.* Publications Mathématiques d'Orsay, (1981).

[8] A. Fruchard, C. Zhang, *Remarques sur les développements asymptotiques.* Ann. Fac. Sci. Toulouse Math. (6) 8 (1999), no. 1, 91–115.

[9] M. Kawagishi, T. Yamanaka, *The heat equation and the shrinking.* Electron. J. Differential Equations 2003, No. 97, 14 pp.

[10] M. Kawagishi, T. Yamanaka, *On the Cauchy problem for non linear PDEs in the Gevrey class with shrinkings.* J. Math. Soc. Japan **54** (2002), no. 3, 649–677.

[11] T. Kato, *Asymptotic behavior of solutions of the functional differential equation* $y'(x) = ay(\lambda x) + by(x)$. Delay and functional differential equations and their applications (Proc. Conf., Park City, Utah, 1972), pp. 197–217. Academic Press, New York, 1972.

[12] S. Malek, *On the summability of formal solutions of linear partial differential equations*, J. Dynam. Control. Syst. 11 (2005), No. 3.

[13] B. Malgrange, *Sommation des séries divergentes.* Exposition. Math. **13** (1995), no. 2-3, 163–222.

[14] J. Prüss, *Evolutionary integral equations and applications.* Monographs in Mathematics, 87. Birkhäuser Verlag, Basel, 1993.

[15] J.P. Ramis, *Dévissage Gevrey.* Journées Singulières de Dijon (Univ. Dijon, 1978), pp. 4, 173-204, Astérisque, 59-60, Soc. Math. France, Paris, 1978.

[16] J.P. Ramis, *About the growth of entire functions solutions of linear algebraic q-difference equations.* Ann. Fac. Sci. Toulouse Math. (6) 1 (1992), no. 1, 53–94.

[17] J.P Ramis, *Séries divergentes et théories asymptotiques.* Bull. Soc. Math. France 121 (1993), Panoramas et Syntheses, suppl., 74 pp.

[18] J. P. Ramis, C. Zhang, *Développement asymptotique $q-$Gevrey et fonction thêta de Jacobi.* C. R. Math. Acad. Sci. Paris 335 (2002), no. 11, 899–902.

[19] L. E. Rossovskii, *On the boundary value problem for the elliptic functional-differential equation with contractions.* International Conference on Differential and Functional Differential Equations (Moscow, 1999). Funct. Differ. Equ. 8 (2001), no. 3-4, 395–406.

[20] T. Yamanaka, *A new higher order chain rule and Gevrey class.* Ann. Global Anal. Geom. **7** (1989), no.3, 179–203.

[21] C. Zhang, *Développements asymptotiques $q-$Gevrey et séries $Gq-$sommables.* Ann. Inst. Fourier (Grenoble) 49 (1999), no.1, 227–261.

[22] C. Zhang, *Sur un théorème du type de Maillet-Malgrange pour les équations $q-$différences-différentielles.* Asymptot. Anal. 17 (1998), no. 4, 309–314.

[23] O. Zubelevich, *Functional-differential equation with dilation and compression of argument*, preprint mp_arc 05-136.