A Sufficient Condition for the Stochastic inequality

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Abstract

We will propose a new sufficient condition for the stochastic inequality. This condition differs in an essential way from the usual lattice condition. We also show an example, which satisfies the new condition, but does not satisfy the lattice condition.

1 Introduction

Let E be a finite set, let $\Omega = \{0,1\}^E$ be a configuration space, and let $\mathcal{F}$ be the collection of all subsets of Ω. Let μ_1 and μ_2 be strictly positive probability measures on $(\Omega, \mathcal{F})$, that is $\mu_1(\omega) > 0$ and $\mu_2(\omega) > 0$ for arbitrary $\omega \in \Omega$. Holley showed that if these measures satisfy the lattice condition

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2), \quad \omega_1, \omega_2 \in \Omega, \tag{1}$$

then they satisfy the stochastic inequality

$$\mu_1(X) \leq \mu_2(X) \text{ for an arbitrary increasing function } X, \tag{2}$$

where $\omega_1 \vee \omega_2 = (\omega_1 \vee \omega_2(e) : e \in E)$, $\omega_1 \vee \omega_2(e) = \max\{\omega_1(e), \omega_2(e)\}$ and $\omega_1 \wedge \omega_2 = (\omega_1 \wedge \omega_2(e) : e \in E)$, $\omega_1 \wedge \omega_2(e) = \min\{\omega_1(e), \omega_2(e)\}$, and we denote $\mu(X)$ as

$$\mu(X) = \sum_{\omega \in \Omega} X(\omega)\mu(\omega). \tag{3}$$

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We note that the lattice condition (1) is equivalent to the following relation (see [1]):

$$\begin{aligned}\mu_2(\omega(e) = 1 | \omega(f) = \zeta(f) \text{ for all } f \in E \setminus \{e\}) \\ \geq \mu_1(\omega(e) = 1 | \omega(f) = \xi(f) \text{ for all } f \in E \setminus \{e\})\end{aligned} \tag{4}$$

for all $e \in E$, and all pairs ξ, $\zeta \in \Omega$ satisfying $\xi \leq \zeta$. We emphasize that the inequality (4) is essential for proving the stochastic inequality (see [1], [3] and [4]). In this paper, we propose another sufficient condition, which implies the stochastic inequality. In Section 2, we give its exact statement. In Section 3, we give the proof of the theorem. In section 4, we give an example, which satisfies the new sufficient condition (this example also satisfies the stochastic inequality), but it does not satisfy the lattice condition (1).

2 Statement of the Main Result

In this section, we propose a new sufficient condition for the stochastic inequality. The statement is as follows:

Theorem 2.1 Let μ_1 and μ_2 be probability measures on the finite space $(\Omega, \mathcal{F})$, which satisfy

$$\mu_2(\omega^e) + \mu_1(\omega_e) \geq \mu_1(\omega^e) + \mu_2(\omega_e), \quad \omega \in \Omega, e \in E, \tag{5}$$

and, whenever ω_1, $\omega_2 \in \Omega$ satisfy

$$\{\mu_2(\omega_1) - \mu_1(\omega_1)\}\{\mu_2(\omega_2) - \mu_1(\omega_2)\} \leq 0,$$

$$\begin{aligned}\{\mu_2(\omega_1 \vee \omega_2) - \mu_1(\omega_1 \vee \omega_2)\}\{\mu_2(\omega_1 \wedge \omega_2) - \mu_1(\omega_1 \wedge \omega_2)\} \\ \leq \{\mu_2(\omega_1) - \mu_1(\omega_1)\}\{\mu_2(\omega_2) - \mu_1(\omega_2)\}.\end{aligned} \tag{6}$$

Then they satisfy the stochastic inequality:

$$\mu_1(X) \leq \mu_2(X) \text{ for increasing function } X : \Omega \to \mathbf{R}. \tag{7}$$

We remark that $\mu_1(X) \leq \mu_2(X)$ if and only if there exists a coupling κ of μ_1 and μ_2 satisfying $\kappa(S) = 1$, where $S = \{(\omega_1, \omega_2) \in \Omega^2 | \omega_1 \leq \omega_2\}$ (see [1]). To construct a coupling, μ_1 and μ_2 may need to satisfy strict positivity and the lattice condition (1), equivalently (4). We emphasize that our sufficient condition assumes neither strict positivity nor the lattice condition (1).

3 Proof of Theorem 2.1

3.1 Preparation

In this section, we give a proof of Theorem 2.1. To prove Theorem 2.1, we shall prepare the following notations. Let m be the signed measure on $(\Omega, \mathcal{F})$, given by

$$m(\omega) = \mu_2(\omega) - \mu_1(\omega). \tag{8}$$

We note that this measure satisfies $m(\Omega) = 0$ and is increasing, that is, $m(\eta) \geq m(\pi)$ for arbitrary $\eta \geq \pi$ in Ω, because of the condition (5) . Let P, N be subsets of Ω such that

$$P = \{\omega \in \Omega : m(\omega) \geq 0\}, \tag{9}$$
$$N = \{\omega \in \Omega : m(\omega) < 0\}. \tag{10}$$

We know that

$$m(P) = -m(N). \tag{11}$$

We define probability measures m_+, m_- as follows:

$$m_+(\omega) = \frac{1}{M}\{m(\omega) \vee 0\}, \quad \omega \in \Omega, \tag{12}$$
$$m_-(\omega) = \frac{1}{M}\{-m(\omega) \vee 0\}, \quad \omega \in \Omega, \tag{13}$$

where $M = m(P) = -m(N)$. We remark that m_+ is increasing, while m_- is decreasing, that is, $m_-(\eta) \leq m_-(\pi)$ for arbitrary $\eta \geq \pi$ in Ω. We also note that $m_+(\omega) > 0$ when $\omega \in P$, and $m_-(\omega) > 0$ when $\omega \in N$. To prove Theorem 2.1, it will be enough to show the following proposition.

Proposition 3.1 Let m_+ be an increasing probability measure on $(\Omega, \mathcal{F})$, and m_- be a decreasing probability measure on $(\Omega, \mathcal{F})$. If m_+ and m_- satisfy the lattice condition

$$m_+(\omega_1 \vee \omega_2)m_-(\omega_1 \wedge \omega_2) \geq m_-(\omega_1)m_+(\omega_2), \tag{14}$$

then we have

$$m_-(X) \leq m_+(X) \text{ for increasing functions } X : \Omega \to \mathbf{R}. \tag{15}$$

To prove Proposition 3.1, we prepare probability measures as follows:

$$m_+^\epsilon = \frac{1}{M + |\Omega|\epsilon}\{m_+(\omega) + \epsilon\}, \quad \omega \in \Omega, \tag{16}$$
$$m_-^\epsilon = \frac{1}{M + |\Omega|\epsilon}\{m_-(\omega) + \epsilon\}, \quad \omega \in \Omega, \tag{17}$$

where ϵ is a positive real. We note that m_+^ϵ and m_-^ϵ are strictly positive probability measures, and satisfy the following proposition.

Proposition 3.2 The pair m_+^ϵ and m_-^ϵ satisfy the lattice condition

$$m_+^\epsilon(\omega_1 \vee \omega_2) m_-^\epsilon(\omega_1 \wedge \omega_2) \geq m_-^\epsilon(\omega_1) m_+^\epsilon(\omega_2), \quad \omega_1, \omega_2 \in \Omega. \tag{18}$$

Proof: Let $\omega_1, \omega_2 \in \Omega$. It is sufficient to show

$$m_+(\omega_1 \vee \omega_2) - m_+(\omega_2) \geq 0, \;\; m_-(\omega_1 \wedge \omega_2) - m_-(\omega_1) \geq 0. \tag{19}$$

But this is obvious by the monotonicity properties of m_+ and m_-. □

Then, by Holley's theorem [3], we obtain that

$$m_+^\epsilon(X) \geq m_-^\epsilon(X). \tag{20}$$

So, we obtain the following inequality

$$m_+(X) \geq m_-(X). \tag{21}$$

Proposition 3.1 follows.

Proof of Theorem 2.1: Assume that the probability measures $\mu_1(\omega)$, $\mu_2(\omega)$ satisfy inequalities (5) and (6). Then, the probability measure $m_+(\omega)$ is increasing, and $m_-(\omega)$ is decreasing, and they satisfy the lattice condition (14). For an arbitrary increasing function X, we obtain

$$m_+(X) \geq m_-(X). \tag{22}$$

So it follows that

$$\mu_2(X) - \mu_1(X) = m(X) \geq M\{m_+(X) - m_-(X)\} \geq 0. \tag{23}$$

□

4 Example

In this section, we will provide an example of μ_1 and μ_2, which does not satisfy the lattice condition (1), but it satisfies inequalities (5) and (6). We remark that this example also satisfies the following inequalities ([1] and [2]):

$$\mu_2(\omega^e)\mu_1(\omega_e) \geq \mu_1(\omega^e)\mu_2(\omega_e), \qquad \omega \in \Omega, e \in E, \tag{24}$$

$$\mu_2(\omega^{ef})\mu_1(\omega_{ef}) \geq \mu_1(\omega_f^e)\mu_2(\omega_e^f), \qquad \omega \in \Omega, e, f \in E. \tag{25}$$

Let μ_1 and μ_2 be probability measures on $\Omega = \{0,1\}^3$ defined by

$$\begin{aligned}
&\mu_2(111) = \frac{1}{8}, \quad \mu_2(000) = \frac{1}{8} - \epsilon, \\
&\mu_2(110) = \mu_2(101) = \mu_2(011) = \frac{1}{8} + \frac{\epsilon}{3}, \\
&\mu_2(100) = \mu_2(001) = \mu_2(010) = \frac{1}{8},
\end{aligned} \tag{26}$$

and define μ_1 symmetrically by $\mu_1(xyz) = \mu_2([1-x][1-y][1-z])$.

Claim : The pair μ_1, μ_2 of strictly positive probability measures as defined above satisfies the inequalities (5) and (6), however, μ_1 and μ_2 do not satisfy the lattice condition (1).

Proof: It is easy to check the required properties of μ_1 and μ_2. For example, we take $\omega_1 = (001)$, and $\omega_2 = (110)$, then, we have the following inequality:

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) < \mu_1(\omega_1)\mu_2(\omega_2).$$

□

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