CRITICAL POINTS INSIDE THE GAPS OF GROUND STATE LAMINATIONS IN STATISTICAL MECHANICS

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Abstract. We consider models of interacting particles situated in the points of a discrete set Λ. The state of each particle is determined by a real variable. The particles are interacting with each other and we are interested in ordered ground states.

Under the assumption that the set Λ and the interaction are symmetric under the action of a group G – which satisfies some mild assumption –, that the interaction is ferromagnetic, as well as periodic under addition of integers, it was shown in a previous paper that there are many ground states that satisfy an order property called selfconforming or Birkhoff.

Under the assumption that the interaction is finite range, we show that either the ground states form a one dimensional family or that there are other Birkhoff critical points which are not ground states.

In the case that the set is just a one dimensional lattice and that the interaction is just nearest neighbor, this generalizes Mather's criterion for the existence of invariant circles.

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1. Introduction

This paper is concerned with the zero temperature behavior of some of the standard models in statistical mechanics [Rue99]. Namely, we consider a discrete set Λ occupied by particles whose state is described by a real variable. These particles interact and we describe the interaction by assigning an energy to the configuration. We will assume that the set Λ and the interaction are symmetric under the action of a group and under the addition of an integer to all the variables (this is natural when the physical meaning is a spin variable or the position in a periodic potential). We will also assume that these interactions satisfy a ferromagnetism conditions that favors particles aligning with their neighbors.

As customary in statistical mechanics, [Rue99], we consider here a discrete set, Λ which one could thing as a crystal or a more complicated set such as the Bethe lattice. In each of the sites, there is a particle. We will assume that the state of each of these particles is characterized by a real number (this appears in several physical models such as models for spin waves or Frenkel-Kontorova models [BK04]. So that a configuration of our system will be just a mapping $u : \Lambda \to \mathbb{R}$.

We will assume that the particles at different sites interact. As in [Rue99], this interaction is described by assigning an energy to every finite set of variables and summing over all them. A common restriction is to assume that the interaction is just among nearest neighbors. In this paper, we will allow interactions which are not nearest neighbors but we will assume that they are finite range.

We will assume that the interactions are symmetric under the action of a group (translations or rotations) which satisfy some mild assumptions, which are also symmetries of the set Λ, of course.

Most importantly, we will assume that the interactions are ferromagnetic. This means that the interactions favor that states at two interacting points are aligned.

It has been shown in [dlLV06a] that very general assumptions are enough to construct laminations of ground state minimizers which lie at a bounded distance from any given cocycle of the group (in this generality, such laminations may be singular, i.e., their leaves may touch: see below for definitions and formal statements).

We note that in the case that $\Lambda = \mathbb{Z}$, that the interactions are nearest neighbor, these laminations established in [dlLV06a] are the celebrated minimizing Aubry-Mather sets. of [ALD83, Mat82]. (Other generalizations of this theory to higher dimensions have been considered in in [Bla89, KdlLR97, CdlL98, CdlL05].)

It is shown in [dlLV06a] that, in case the cocycle is *completely irrational*, the following alternative holds: either the above lamination consists of a *foliation* made of a continuous one parameter family of ground states, or, inside any gap of the lamination, there is a well-ordered *critical point* which is not a ground state. In the case of Aubry-Mather theory, this becomes the celebrated criterion for existence of invariant circles for twist mappings.

The goal of this paper is to extend such an alternative to any cocycle, *both rational or irrational*. Nevertheless, we will introduce several extra conditions on the interaction.

The extra conditions we assume are not very restrictive, and are satisfied, for instance, by the multidimensional Frenkel-Kontorova model and, more generally, by periodic models with finite range interactions. The paper [dlLV06a] contains a discussion of models that have been proposed in the literature. The assumptions of this paper apply to the models discussed in [dlLV06a] when the interaction is finite range.
In Section 1.1, we introduce the set-up for the models and in Section 1.3, we formulate the results of the paper.

1.1. **Description of models and assumptions.** We now formally introduce our set-up. The set-up is rather general. This allows to cover at the same time several models that have been considered in the literature (see [dlLV06a] for a discussion of models). At the same time, the generality allows to emphasize what are the essential ingredients in the argument which are mainly symmetry, ferromagnetism (as well as some regularity). The arguments do not become significantly simpler by making them in a particular case.
We consider a discrete countable set Λ, which admits an action by a group G, and a collection of potentials H_B, associated to any finite subset B of Λ. From the statistical mechanics viewpoint, Λ may be thought as a crystal, which possesses the symmetries induced by G, and each H_B describes the interaction between the particles lying in the set B.
The most customary example in statistical mechanics is when $\Lambda := \mathbb{Z}^d =: G$ and the action is given by the translations, but the present framework applies to a variety of models (see [dlLV06a] for a presentation of different models).
As customary in statistical mechanics (see, e.g., [Rue99]), the interaction potential is given by the formal sum

$$S(u) = \sum_{B \subset \Lambda} H_B(u), \tag{1}$$

where $u : \Lambda \to \mathbb{R}$ is called a *configuration* on Λ. We will be interested in equilibria and in and in particular in ground states. A fuller discussion will be done in Section 1.2.
We recall that a cocycle on G is a map

$$\sigma : G \to \mathbb{R}$$

such that

$$\sigma(\gamma\gamma') = \sigma(\gamma) + \sigma(\gamma') \,. \tag{2}$$

Since G acts on the countable set Λ, given $p \in \Lambda$ and $\gamma \in G$, we denote the action of γ on p by γp.

1.1.1. *Assumptions on the action of the group G.* We now list the hypotheses that we assume on the group G. They are the same as those in [dlLV06a].

- **G1:** The group G is *finitely generated.*
- **G2:** There is a *finite fundamental domain* for the action, i.e., a finite subset F of Λ which intersects each orbit of G in exactly one point.

G3: The group G acts on Λ *without nontrivial stabilizers*, i.e.: if $\gamma \in G$ is such that there exists $p \in \Lambda$ in such a way that $\gamma p = p$, then $\gamma = 0$.

G4: The group G is *residually finite*, i.e., for each element of G other than the identity, there is a normal, finite index subgroup of G which does not contain it.

Note that these assumptions on G are very mild (see also [CdlL98, dlLV06a] for a discussion on this point).
Up to identifying a representation class of Λ/G with the corresponding element of F, one can identify Λ/G with F.
Also, by **G1**, G is generated by a finite set of elements, say $\{g_1^\star, \dots, g_d^\star\}$. Then, given a cocycle σ, we define its norm by

$$\|\sigma\| := \sup_{1 \le k \le d} |\sigma(g_k^\star)| \,. \tag{3}$$

Moreover, any cocycle σ defines a configuration u_σ on Λ as follows. Since F is the fundamental domain, for any $p \in \Lambda$ there exists a unique $q \in F$ so that $p = \gamma q$. Then, we define

$$u_\sigma(p) = \sigma(\gamma) \,. \tag{4}$$

Note that the above is a good definition thanks to **G3**.

1.1.2. *Assumptions on the interaction potential H.* Let us now formalize the concept of interaction potential in (1).
Given $K \ge 0$ and a cocycle σ, we define $\mathcal{O}_\sigma^K$ to be the space of all configurations u such that

$$\sup_{p \in \Lambda} |u(p) - u_\sigma(p)| \le K \,. \tag{5}$$

Let $\mathcal{S}$ denote a collection of nonempty, finite subsets of Λ. Then, we consider the collection of maps

$$H = \{H_B \,, \quad B \in \mathcal{S}\} \,,$$

where each H_B is a real valued function from the configurations u of Λ.
We will be making the following assumptions on the potential, which are the same as those in [dlLV06a]. Later, we will include some other assumptions on the potential which are used in this paper.

H1: If

$$u(p) = v(p) \text{ for any } p \in B$$

then

$$H_B(u) = H_B(v) \,.$$

H2: Fixed $K \ge 0$, for any finite subset X of Λ and any configuration $u \in \mathcal{O}_\sigma^K$, the series

$$\sum_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset}} H_B(u)$$

converges.

By **H1**, we may regard any function H_B as acting on the space of sequences over the indices of B, and so we sometimes write $u_p := u(p)$, for $p \in \Lambda$, and take derivatives of H_B with respect to u_p. Also, by possibly neglecting the sets B in $\mathcal{S}$ for which $H_B \equiv 0$, without loss of generality we will suppose that $H_B \not\equiv 0$ if $B \in \mathcal{S}$.
The action of G on Λ extends to the space of configurations. That is, if $\gamma \in G$, we define

$$\mathcal{T}_\gamma u(p) := u(\gamma(p)),$$

for any $p \in \Lambda$. Given $\ell \in \mathbb{R}$, we also set

$$\mathcal{R}_\ell u := u + \ell .$$

Thus, we assume:

H3: The potential $H = \{H_B\}$ is *G-invariant*, i.e.: if $B \in \mathcal{S}$, then $\gamma B \in \mathcal{S}$ and

$$H_{\gamma B}(u) = H_B(\mathcal{T}_\gamma u)$$

for any $\gamma \in G$ and any configuration u on Λ.

H4: H has *periodic phase*, i.e.,

$$H_B(\mathcal{R}_s u) = H_B(u)$$

for any configuration u, any $B \in \mathcal{S}$ and any $s \in \mathbb{Z}$.

H5: The action of the group on the potential is *nontrivial*, i.e., we suppose that if $g \in G$ is such that $gB = B$ for some $B \in \mathcal{S}$, then g is the identity.

H6: H is *coercive*, i.e., we suppose that $H_B(u) \geq 0$ for any $B \in \mathcal{S}$ and

$$H_F(u) \geq \kappa \max_{i,j \in F} |u_i - u_j|^\theta - K,$$

for some κ, K, $\theta > 0$.

H7: There exist ϑ, $S > 0$ and $\varsigma : \mathcal{S} \to [0, +\infty)$ in such a way that

$$H_B(\sigma) \leq \varsigma(B) \left[\|\!|\sigma|\!\|^\vartheta + 1 \right],$$

for any cocycle σ and any $B \in \mathcal{S}$, and

$$\sum_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset}} \varsigma(B) \leq S \,\sharp X,$$

for any $X \subset \Lambda$.

Given $\kappa, K, S, \theta, \vartheta$ as in **H6** and **H7**, we set

$$K^\star_\sigma := \begin{cases} 9 & \text{if } \sharp(\Lambda/G) = 1, \\ 9 + \left(\dfrac{S \left(\|\!|\sigma|\!\|^\vartheta + 1 \right) \sharp F + K}{\kappa} \right)^{1/\theta} & \text{if } \sharp(\Lambda/G) \geq 2. \end{cases}$$

We also set $\mathcal{O}_\sigma := \mathcal{O}_\sigma^{K^\star_\sigma}$.
We also assume that

H8: For any $K \ge 0$ and any cocycle σ, the interaction potential H is C^2-bounded on $\mathcal{O}_\sigma^K$, that is, for any $B \in \mathcal{S}$, the map H_B is twice differentiable and

$$\sup_{u\in\mathcal{O}_\sigma^K} \sup_{p\in\Lambda} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \left| \frac{\partial H_B}{\partial u_p}(u) \right| + \sup_{u\in\mathcal{O}_\sigma^K} \sup_{p\in\Lambda} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \sum_{q\in B} \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(u) \right| < +\infty . \tag{6}$$

H9: H satisfies the *ferromagnetic* (or *twist*) condition, i.e.

$$\frac{\partial^2 H_B}{\partial u_p\, \partial u_q}(u) \le 0 ,$$

for any $u \in \mathcal{O}_\sigma^K$, any $K \ge 0$ and any p, q in Λ, with $p \ne q$.

H10: There exist $\delta : \mathcal{S} \to [0, +\infty)$ in such a way that
- $\delta(B) = \delta(gB)$, for any $B \in \mathcal{S}$ and any $g \in G$.
- For any $K \ge 0$ and any finite set $X \subset \Lambda$,

$$\lim_{R\to+\infty} \sup_{u\in\mathcal{O}_\sigma^K} \sum_{\substack{B\in\mathcal{S}\\ B\cap X\ne\emptyset\\ \delta(B)>R}} H_B(u) = 0 .$$

- Given any $R > 0$ and any finite set $X \subset \Lambda$, there exists a finite set $Y \subset \Lambda$ in such a way that

$$\bigcup_{\substack{B\in\mathcal{S}\\ B\cap X\ne\emptyset\\ \delta(B)\le R}} B \subset Y .$$

In concrete applications, one may take δ to be diameter of the set B (see, for instance, [dlLV06a] and Section 2 here below).

1.1.3. *Partial order among configurations.* Note that the space of configurations is endowed with a natural *partial ordering*:

Definition 1. *We say that $u \le v$ if $u(p) \le v(p)$ for any p in Λ.*

Then, generalizing a standard terminology in Aubry-Mather theory, we introduce the following order property:

Definition 2. *Given a cocycle σ of G, we say that a configuration u satisfies the* **Birkhoff property** *if the following holds: if $\gamma \in G$ and $s \in \mathbb{Z}$ are such that $\sigma(\gamma) \ge s$ (resp., $\sigma(\gamma) \le s$), then $\mathcal{T}_\gamma u \ge \mathcal{R}_s u$ (resp., $\mathcal{T}_\gamma u \le \mathcal{R}_s u$).*

The set of Birkhoff configurations in $\mathcal{O}_\sigma$ will be denoted by $\mathcal{B}_\sigma$.

1.2. **Ground states versus critical points.** Let us now introduce the equilibrium solutions we are interested in:

Definition 3. *We say that a configuration u is a* **solution of our variational equation** *(or a* **critical point***) if, for any $p \in \Lambda$,*

$$\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u) \;=\; 0\,. \tag{7}$$

Note that the series in (7) is absolutely convergent, thanks to **H8**.

Definition 4. *We say that u is a* **ground state** *(or, in the terminology of* [Mor24]*, a* **class-A minimizer***) if*

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(u) \;\leq\; \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(v)$$

for any finite subset X of Λ and any configuration v such that $u = v$ on $\Lambda - X$.

Then, the following result has been proved in Theorem 1 of [dlLV06a]:

Theorem 5. *Assume* **G1**–**G4** *and* **H1**–**H10***. Then, given any cocycle σ, there exists a ground state u of the interaction S such that*

$$|u(p) - u_\sigma(p)| \;<\; K^\star_\sigma\,. \tag{8}$$

Moreover, $u \in \mathcal{B}_\sigma$.

These ground states induce, by the natural projection, a possibly singular lamination in $F \times \mathbb{T}^1$, that is, the (closure of the) graphs of such ground states are organized into leaves that do not cross.

We now define

$$\mathcal{H} \;:=\; \{g \in G \;|\; \sigma(g) \in \mathbb{Z}\}\,.$$

In case $\mathcal{H}$ contains only the identity (that is, if the cocycle σ is "completely irrational"), it has been proved in Theorem 2 of [dlLV06a] that it is possible to find other critical points (which are not minimal) inside a gap of the lamination of ground states given by Theorem 5 here above. The situation in which $\mathcal{H}$ contains only the identity may be seen as the "typical" case of the irrational cocycles (for instance, when cocycles are just linear functions and $G = \mathbb{Z}^d$, it reduces to prescribe rationally independent slopes).

The purpose of this paper is to construct critical points inside the gaps of the ground state lamination of Theorem 5 under more relaxed conditions, which take into account "rationally dependent slopes" too. This goal will be accomplished in Theorem 6 below, which will be the main result of the present paper.

The price to pay for such generalization is that we need to make *assumptions on the range of the interaction potential.* Roughly speaking, we will want to avoid that the potential range "wraps around" the fundamental domain too much. Thus, the models we are dealing with may be seen as a generalization of finite range interaction systems. The assumptions we now introduce may in fact look quite abstract, but they are also very general: as an example, we will show later in Section 2 that multidimensional Frenkel-Kontorova models fulfill such conditions.

In further detail, the auxiliary assumptions we are going to take in this paper are the following ones. We suppose that there exists a sequence of normal, finite index subgroups $H_n \subset H_0 := H \subset \mathcal{H}$ in such a way that $(\Lambda/H_n) \subset (\Lambda/H_{n+1})$ which satisfy[1] the following conditions:

A1: Given any finite set $X \subset \Lambda$, there exists $n \in \mathbb{N}$ in such a way that $X \subset \Lambda/H_n$.

A2: There exists $k^\star \in \mathbb{N}$ such that if $B \in \mathcal{S}$ and $n \in \mathbb{N}$ are such that $n \geq k^\star$ and $B \cap (\Lambda/H_n) \neq \emptyset$, then either $B \subset \Lambda/H_n$ or

$$B \subset (\Lambda/H_{n+k^\star}) - (\Lambda/H_{n-k^\star}) .$$

Condition **A1** just says that the fundamental domains Λ/H_n exhaust the whole lattice Λ as n increases (and thus it makes **G5** more precise). We also remark that, since $\Lambda/H = \Lambda/H_0 \subset \Lambda/H_{k^\star}$, it follows from **A2** that if B intersects Λ/H, then $B \subset \Lambda/H_{2k^\star}$ (that is, roughly speaking, the sets in $\mathcal{S}$ "cannot be too large", at least "in the direction of H").

We now introduce an equivalence relation on subsets of Λ, induced by the subgroup H. Given $A, B \subset \Lambda$, we say that $A \sim_H B$ if there exists $h \in H$ in such a way that $A = hB$. The equivalence class of a set B with respect to this relation will be denoted by $[B]_H$. With this notation, we assume that:

A3: If $A, B \in \mathcal{S}$ are such that $A \sim_H B$ and $A \cap B \neq \emptyset$, then $A = B$.

A4: For any $n \in \mathbb{N}$, $n \geq k^\star$, there exist $c(n) \geq 0$ such that

$$\sum_{p \in (\Lambda/H_{n+k^\star}) - (\Lambda/H_{n-k^\star})} u_p = c(n) \sum_{p \in \Lambda/H} u_p \,, \tag{9}$$

for any configuration u such that $u_p = u_{hp}$ for any $p \in \Lambda$ and any $h \in H$.

Furthermore, the following limit holds:

$$\begin{aligned} &\lim_{n \to +\infty} \frac{c(n)}{\sharp(H/H_n)} = 0 \qquad \text{and} \\ &\lim_{n \to +\infty} \sup_{B \in \mathcal{S}} \left| \frac{\sharp\Big\{A \in [B]_H \text{ s.t. } A \subset \Lambda/H_n\Big\}}{\sharp(H/H_n)} - 1 \right| = 0 \,. \end{aligned} \tag{10}$$

Roughly speaking, condition **A3** states that the sets in $\mathcal{S}$ "cannot overlap" under the periodicity induced by H and, again, it may be seen as a requirement on the size of these sets (in the sense that a set does not overlap with respect to a periodicity notion as long as "its diameter is less than the period").

Another way of looking at condition **A3** is by saying that

$$\begin{aligned} &\text{given any } p \in \Lambda \text{ and any } [B]_H \in \mathcal{S}_H, \\ &\text{there exists at most one } A \in [B]_H \text{ such that } A \ni p. \end{aligned} \tag{11}$$

Condition **A4** is more technical and it is needed to control the combinatorics of different periodic identifications.

[1] As already mentioned on page 4, we are identifying, here and in the rest of the paper, the quotient Λ/H with a suitable fundamental domain for the action of the group H on the lattice Λ. Hence, the notation $\Lambda/H' \subset \Lambda/H''$ has to be read in the geometry of the corresponding fundamental domains, and it just means that there exist a suitable fundamental domain F' (resp., F'') for the action of the group H' (resp., H'') on Λ, in such a way that $F' \subset F''$.

We will show that all the above conditions are satisfied, for instance, in the case of finite range interactions on the lattice $\mathbb{Z}^d$, such as multidimensional Frenkel-Kontorova models (see Section 2 for details).

1.3. Statement of results.

Thus, the main result of this paper is:

Theorem 6. *Assume* **G1**–**G4**, **H1**–**H10** *and* **A1**–**A4**. *Let $u^{(0)}, u^{(1)} \in \mathcal{B}_\sigma$. Assume that the graphs of $u^{(0)}$ and $u^{(1)}$ belong to the (possibly singular) lamination of ground states induced by Theorem 5. Suppose that $u^{(1)} \geq u^{(0)}$ and $u^{(1)} \not\equiv u^{(0)}$. Then, there exists $u \in \mathcal{B}_\sigma - \{u^{(0)}, u^{(1)}\}$ such that*

$$u^{(1)} \geq u \geq u^{(0)}$$

and which is a solution of the variational equation.

Note that, the interesting case of the result is when the $u^{(0)}, u^{(1)}$ are at the boundary of a gap in the lamination, then, the solution thus produced is not included in the lamination (otherwise, we can take as new critical point some other leave in the lamination between $u^{(0)}$ and $u^{(1)}$.

It could happen that the new critical point is also a ground state. In such a case, we can add the new ground state, its translations and limits to the lamination of ground states and we can start the process again.

Hence, we conclude that either there is a lamination by ground states that covers all the fundamental domain or there is a critical point inside the gaps which is not a ground state.

Theorem 6 is a generalization of the results of [Mat86], were it is shown that, for the models of twist maps considered in Aubry-Mather theory, either there is an invariant circle (the lamination covers the fundamental domain) or there are equilibrium points which are not ground states. We will also see that there is a natural extension of the Peierls-Nabarro barrier considered in [Mat86] to the models that we consider in this paper.

The proof of Theorem 6 we will present here is based on the study a gradient flow. Similar arguments have appeared in [Ang90, CdlL98, dlLV05, dlLV06a]. In particular, the technique developed in [dlLV06a] will be extended to overcome the additional complication of the possible resonances given by the case of nontrivial $\mathcal{H}$. This will be obtained by a careful study of the group identifications naturally induced by $\mathcal{H}$.

2. Finite range potentials

As an example to which Theorem 6 applies, we consider the periodic finite range interaction potentials. Namely, we observe that:

Proposition 7. *Let $G = \Lambda := \mathbb{Z}^d$, with the sum as group action, and let the linear functions from $\mathbb{R}^d$ to $\mathbb{R}$ be the set of cocycles. Suppose that there exists $R \geq 0$ such that any $B \in \mathcal{S}$ has diameter less than R. Then, conditions* **A1**–**A4** *are fulfilled.*

Proof. Given a cocycle σ we write

$$\sigma(i) = \omega \cdot i \,,$$

for any $i \in \mathbb{Z}^d$. Then,

$$\mathcal{H} \,=\, \{i \in \mathbb{Z}^d \;|\; \omega \cdot i \in \mathbb{Z}\}\,,$$

is a linear space and so it is generated by suitable linearly independent integer vectors $e_1, \ldots, e_k \in \mathbb{Z}^d - \{0\}$ for some $k \geq 0$ (which is the "number of resonances" of ω). In fact, we will suppose $k \geq 1$, otherwise **A1**–**A4** are easily checked by choosing $H_n := H = \{0\}$.
We complete $\mathcal{H}$ into $\mathbb{Z}^d$, namely, we consider an $(n-k)$-dimensional linear space $\widetilde{\mathcal{H}} \subset \mathbb{Z}^d$ such that $\mathbb{Z}^d = \mathcal{H} \oplus \widetilde{\mathcal{H}}$.
Given any

$$a \,=\, \sum_{j=1}^{k} a_j e_j + \alpha \,\in\, \mathbb{Z}^d$$

with $\alpha \in \widetilde{\mathcal{H}}$, we define

$$|a|_* \,:=\, \sqrt{\sum_{j=1}^{k} |a_j|^2 + |\alpha|^2}\,.$$

It follows that $|\cdot|_*$ induces a norm on $\mathbb{R}^d$, which is therefore equivalent to the Euclidean norm $|\cdot|$. In particular, there exists $\lambda > 0$ in such a way that

$$|a| \,\geq\, \lambda \max\{|e_1|, \ldots, |e_k|\}\, |a|_* \,\geq\, \lambda\, |a_j|\, |e_j|\,, \qquad \text{for any } j = 1, \ldots, k. \tag{12}$$

We write

$$\mathcal{H} \,=\, \Big\{ \sum_{j=1}^{k} m_j e_j \;\;|\;\; m_j \in \mathbb{Z} \Big\}$$

and we take $\nu \in \mathbb{N}$ such that $(R+1)/\lambda < \nu$. We set

$$H \,:=\, \Big\{ \sum_{j=1}^{k} \nu m_j e_j \;\;|\;\; m_j \in \mathbb{Z} \Big\}$$

and, for $n \geq 1$,

$$H_n \,:=\, \Big\{ \sum_{j=1}^{k} \nu m_j e_j \;\;|\;\; m_j \in 2n\mathbb{Z} \Big\}\,.$$

Let us complete H into $\mathbb{Z}^d$, that is, let $\widetilde{H} \subset \mathbb{Z}^d$ be an $(n-k)$-dimensional linear space in such a way that $\mathbb{Z}^d = H \oplus \widetilde{H}$. Then, we choose as fundamental domains

$$\begin{aligned} \Lambda/H \;&=\; \widetilde{H} \qquad \text{and} \\ \Lambda/H_n \;&=\; \Big\{ \sum_{j=1}^{k} \nu m_j e_j + \beta \;\Big|\; m_j \in \mathbb{Z}\,,\; m_1, \ldots, m_k \in [-n, n)\,,\; \beta \in \widetilde{H} \Big\}\,. \end{aligned}$$

With this notation, the fact that $(\Lambda/H_n) \subset (\Lambda/H_{n+1})$ and condition **A1** are obvious.

We now prove **A2**. For this, we take $k^\star$ to be the smallest natural number larger than $(R/\lambda)+2$. Let now suppose that $B \in \mathcal{S}$ is such that $B \cap (\Lambda/H_n) \neq \emptyset$ and $B \not\subset \Lambda/H_n$, with $n \geq k^\star$. Then, there exist two points

$$b^{(i)} = \sum_{j=1}^{k} \nu b_j^{(i)} e_j + \beta^{(i)} \in B\,, \qquad b_j^{(i)} \in \mathbb{Z}\,, \qquad \beta^{(i)} \in \widetilde{H}\,, \qquad i=1,2\,,$$

in such a way that $|b_j^{(1)}| \leq n$ for any $1 \leq j \leq k$ and $|b_{j_0}^{(2)}| \geq n$ for some $1 \leq j_0 \leq k$. Then, given any point $b \in B$, we write

$$b = \sum_{j=1}^{k} \nu b_j e_j + \beta \in H \oplus \widetilde{H}\,, \qquad b_j \in \mathbb{Z}\,, \qquad \beta \in \widetilde{H}\,,$$

and we deduce from (12) that

$$R \geq |b - b^{(i)}| \geq \lambda\, |b_j - b_j^{(i)}|$$

for any $j = 1, \dots, k$ and $i = 1, 2$. Therefore,

$$|b_j| \leq (R/\lambda) + |b_j^{(1)}| \leq (R/\lambda) + n < n + k^\star$$

and

$$|b_j| \geq |b_j^{(2)}| - |b_j - b_j^{(2)}| \geq n - (R/\lambda) > n - k^\star\,.$$

This shows that $B \subset (\Lambda/H_{n+k^\star}) - (\Lambda/H_{n-k^\star})$ and proves **A2**.

We now check condition **A3**. For this, suppose that $q \in A \cap B$, with $A = B + h$, for some $h \in H$. Let

$$h = \sum_{j=1}^{k} \nu h_j e_j$$

with $h_j \in \mathbb{Z}$.

Since $q + h \in B + h = A$, we have that both $q+h$ and q belong to A and so, from (12),

$$R \geq |(q+h) - q| = |h| \geq \lambda\nu |h_j|$$

for any $j = 1, \dots, k$. If one of the h_j were not zero, we would then have that $R \geq \lambda\nu > R$, which is a contradiction. Therefore, $h_j = 0$ for any $j = 1, \dots, k$ and so $h = 0$. This shows that $A = B$ and so proves that **A3** is fulfilled.

Let us now prove **A4**. We notice that

$$\sharp\Big(H/H_n\Big) = \sharp\Big(\mathbb{Z}/(2n\mathbb{Z})\Big)^k = (2n)^k\,. \tag{13}$$

On the other hand, given $n > m$, we observe that $p \in (\Lambda/H_n) - (\Lambda/H_m)$ if and only if

$$p = \sum_{j=1}^{k} \nu p_j e_j + \beta$$

with $\beta \in \widetilde{H} = \Lambda/H$, $p_j \in \mathbb{Z} \cap [-n, n)$ for any $j = 1, \dots, k$, and $p_{j_0} \notin [-m, m)$ for some j_0.

Note also that, if u is as requested in **A4**, we have that

$$u(p) = u\left(\sum_{j=1}^{k} \nu p_j e_j + \beta\right) = u(\beta)$$

and so (9) holds, with

$$c(n) = \sharp\Big\{p_1, \ldots, p_k \in \mathbb{Z} \cap [-(n+k^\star), n+k^\star) \;\Big|\; p_{j_0} \notin [-(n-k^\star), n-k^\star) \text{ for some } j_0\Big\}.$$

Consequently,

$$\begin{aligned}
c(n) &\le \sharp\Big\{p_1, \ldots, p_k \in \mathbb{Z} \;\Big|\; |p_j| \le n+k^\star \text{ for any } j \\
&\qquad \text{and } p_{j_0} \notin [-(n-k^\star), n-k^\star) \text{ for some } j_0\Big\} \\
&\le \sum_{j_0=1}^{k} \sharp\Big\{p_1, \ldots, p_k \in \mathbb{Z} \;\Big|\; |p_j| \le n+k^\star \text{ for any } j \\
&\qquad \text{and } p_{j_0} \notin [-(n-k^\star), n-k^\star) \Big\} \\
&\le k \cdot \sharp\Big\{p_1, \ldots, p_k \in \mathbb{Z} \;\Big|\; |p_1|, \ldots, |p_{k-1}| \le n+k^\star \\
&\qquad \text{and } n-k^\star \le |p_k| \le n+k^\star) \Big\} \\
&= k \cdot \Big(2(n+k^\star)+1\Big)^{k-1} \cdot (4k^\star+1) \\
&\le 2^{k+3} k k^\star (n+k^\star+1)^{k-1}.
\end{aligned}$$

From this and (13), the first limit in (10) follows.
We now take care of the second limit in (10). For this, we consider the map

$$\begin{aligned}
P: \quad \Lambda/H_n &\longrightarrow \Big([-n,n) \cap \mathbb{Z}\Big)^k \\
\sum_{j=1}^{k} \nu m_j e_j &\longmapsto (m_1, \ldots, m_k)
\end{aligned}$$

and we observe that if $A \in [B]_H$, then $P(A) = P(B) + h$, for some $h \in \mathbb{Z}^k$.
Since the diameter of B is less than R, if $p = (p_1, \ldots, p_k), q = (q_1, \ldots, q_k) \in P(B)$, we deduce from (12) that

$$k^\star > R \ge \lambda\nu |p_i - q_i| \ge |p_i - q_i| \qquad \text{for } i = 1, \ldots, k. \tag{14}$$

Let now $n \ge k^\star$. Given any $B \in \mathcal{S}$, fix $p \in P(B)$. Then, there exist $2(n-k^\star)$ consecutive integers $j \in \mathbb{Z}$ such that

$$p + (j, 0, \ldots, 0) \in [-n+k^\star, n-k^\star) \times \mathbb{Z}^{k-1}.$$

Thus, from (14),

$$P(B) + (j, 0, \ldots, 0) \subset [-n, n) \times \mathbb{Z}^{k-1}.$$

By repeating the above argument to any of the k coordinates of p, we get that there exist $(2(n-k^\star))^k$ integer vectors $J \in \mathbb{Z}^k$ in such a way that

$$P(B) + J \subset [-n, n)^k .$$

This shows that

$$\begin{aligned} &\sharp\Big\{A \in [B]_H \text{ s.t. } A \subset \Lambda/H_n\Big\} \\ &= \sharp\Big\{J \in \mathbb{Z}^k \text{ s.t. } P(B) + J \subset [-n,n)^k\Big\} \qquad (15) \\ &\geq \Big(2(n-k^\star)\Big)^k . \end{aligned}$$

On the other hand, obviously,

$$\sharp\Big\{J \in \mathbb{Z}^k \text{ s.t. } P(B) + J \subset [-n,n)^k\Big\} \leq \sharp\Big(\mathbb{Z}^k \cap [-n,n]^k\Big) \leq (2n+1)^k .$$

The latter estimate, (15) and (13) yield the second limit in (10), thus checking condition **A4**. □

It easily follows from Proposition 7 that Theorem 6 applies to multidimensional Frenkel-Kontorova-type models, as next result states. We note that in [dlLV06b] one can find a direct verification for some generalized Frenkel-Kontorova models that include also the possibility of interaction being by hard springs.

Corollary 8. *Let $a \in \mathbb{R}$ and $R \geq 0$. Let $V \in C^\infty(\mathbb{R})$ be such that*

$$V(r+1) = V(r) \qquad (16)$$

for any $r \in \mathbb{R}$. Then, the results of Theorems 5 and 6 apply to the system

$$\sum_{\substack{i,j \in \mathbb{Z}^d \\ |i-j| \leq R}} |u_i - u_j - a|^2 + \sum_{i \in \mathbb{Z}^d} V(u_i) \qquad (17)$$

by choosing $G = \Lambda := \mathbb{Z}^d$, with the group action of G on Λ being the sum, and the linear functions from $\mathbb{R}^d$ to $\mathbb{R}$ as the set of cocycles.

Proof. By possibly replacing $V(r)$ with $V(r) - \min V$, we may suppose that

$$V(r) \geq 0 \qquad \text{for any } r \in \mathbb{R}.. \qquad (18)$$

Conditions **G1**–**G4** are obviously fulfilled here. Let $\mathcal{S}$ be the collection of all the nonempty subsets of $\mathbb{Z}^d$ with diameter less than or equal to R containing no more than two points. That is, given $B \subset \mathbb{Z}^d$, we have that $B \in \mathcal{S}$ if and only if either B is a point or $B = \{i, j\}$ with $|i - j| \leq R$.
We define

$$H_B(u) := \begin{cases} \frac{1}{2}|u_i - u_j - a|^2 + \frac{1}{2}|u_j - u_i - a|^2 & \text{if } B = \{i,j\}, \\ V(u_i) & \text{if } B = \{i\}, \end{cases}$$

With this notation, (1) agrees with (17). Hence, condition **H1** is obvious, while **H2** follows because there are only a finite number of sets in $\mathcal{S}$ which intersect a finite set $X \subset \Lambda$. More precisely, if we set

$$c_R := 1 + \sharp\Big\{ i \in \mathbb{Z}^d \;\Big|\; |i| \le R \Big\},$$

we have that

$$\begin{aligned}
&\sharp\Big\{ B \in \mathcal{S} \;\Big|\; B \cap X \neq \emptyset \Big\} \\
&= \sharp\Big\{ i \in \mathbb{Z}^d \;\Big|\; \{i\} \cap X \neq \emptyset \Big\} \\
&\quad + \sharp\Big\{ \{i,j\} \subset \mathbb{Z}^d \;\Big|\; |i-j| \le R\,,\;\; \{i,j\} \cap X \neq \emptyset \Big\} \\
&\le \sharp X + \sum_{x \in X} \sharp\Big\{ i \in \mathbb{Z}^d \;\Big|\; |i-x| \le R \Big\} \\
&= c_R\, \sharp X\,.
\end{aligned} \tag{19}$$

Moreover, if $\gamma \in \mathbb{Z}^d$, we have that $\mathcal{T}_\gamma u(p) = u(p+\gamma)$ and so **H3** plainly follows.
The validity of **H4** is an easy consequence of (16). Condition **H6** follows from (18) and the facts that V is bounded and $F = \Lambda/G = \{0\}$.
Furthermore, recalling (3),

$$\|\sigma\| = \sup_{1 \le k \le d} |\omega_k|$$

and so

$$\begin{aligned}
H_B(\sigma) &\le |\omega \cdot (i-j) - a|^2 + \|V\|_{L^\infty} \\
&\le C\left(\|\sigma\|^2 + 1\right),
\end{aligned}$$

for a suitable $C > 0$ (possibly depending on R and a), thus yielding **H7**, via (19). Conditions **H8** and **H9** plainly follow by using (19). Condition **H10** is obviously fulfilled by taking $\delta(B)$ to be the diameter of B.
Finally, conditions **A1**–**A4** are satisfied in this case, due to Proposition 7. □

3. Proof of Theorem 6

The idea underneath the proof of Theorem 6 is the following. We introduce a reduced energy by subtracting the energy value at $u^{(0)}$ and identifying sets under the periodicity induced by H.
These tools are needed to make the reduced energy finite. However, in principle, such reduced energy might have different critical points than the original one. This is a somewhat delicate business, because the interactions are nonlocal and points inside a fundamental domain may feel the influence of points outside. On the other hand, conditions **A1**–**A4** will ensure that the variational argument patterned in [dlLV05, dlLV06a] may be adapted to the present case. Namely, we will follow any given interpolation $u^{(s)}$ of $u^{(0)}$ and $u^{(1)}$, with, say $s \in [0,1]$, under a suitable "gradient flow" (or "heat flow"). We then study the "basin of attraction" of this flow assuming, by contradiction, that no other critical point exists inside the foliation gap. The contradiction will then be

obtained by covering the interval $[0,1]$ with two disjoint open intervals, namely the basins of attraction of the points 0 and 1.

Let us now start with the details of the proof of Theorem 6.

Without loss of generality, we may suppose that $u^{(0)}$ and $u^{(1)}$ are at the edge of a gap in the (possibly singular) lamination induced by the minimizers given in Theorem 5. Indeed, if the lamination has no gaps between $u^{(0)}$ and $u^{(1)}$, then infinitely many leaves lie between $u^{(0)}$ and $u^{(1)}$, thus giving the claim of Theorem 6.

Thus, we are now going to construct a new solution inside the gap bordered by $u^{(0)}$ and $u^{(1)}$.

For this, the next result (which is very similar to Lemma 36 of [dlLV06a]), will be important:

Lemma 9. *Let $u^{(0)}$ and $u^{(1)}$ be at the edge of a gap in the minimal (possibly singular) lamination of Theorem 5. Let $\tilde{H}$ be a normal subgroup of H so that $\sharp(H/\tilde{H}) < +\infty$. Then,*

$$\sum_{p\in\Lambda/\tilde{H}} |u^{(1)}(p) - u^{(0)}(p)| \,\leq\, \sharp F \cdot \sharp(H/\tilde{H})\,. \tag{20}$$

The proof of Lemma 9 is obtained if we consider the sum involved in the left hand side of (20). We can divide the range of the sum according to the class Λ/H. Then, for each set of the classes we can perform a translation and a subtraction to reduce it to the fundamental domain. Then, by the assumption that $u^{(0)}$, $u^{(1)}$ are at the edge of the gaps of a foliation, we obtain that the gaps have to add to the volume of the fundamental domain, yielding the claim of Lemma 9 (see Lemma 36 of [dlLV06a] for details).

3.1. **Some compactness properties.**

We consider the sup-norm

$$\|u\|_H \,:=\, \sup_{p\in\Lambda/H} |u(p)|\,.$$

We define

$$\mathcal{C}_\sigma(u^{(0)},u^{(1)}) \,:=\, \Big\{u\in\mathcal{B}_\sigma \quad|\quad u^{(0)}\leq u\leq u^{(1)}\Big\}\,,$$

where $u^{(0)}$ and $u^{(1)}$ are as in the statement of Theorem 6. To avoid unnecessary complications, we will use $\mathcal{C}$ as a short-hand notation for $\mathcal{C}_\sigma(u^{(0)},u^{(1)})$ in the sequel.

It is easy to see (see, e.g., Lemma 37 in [dlLV06a]) that the following compactness property holds:

Lemma 10. *Let $u_n\in\mathcal{C}$ for any $n\in\mathbb{N}$. Then, there exists a subsequence u_{n_k} and a configuration $u\in\mathcal{C}$ such that*

$$\lim_{k\to+\infty} u_{n_k}(p) \,=\, u(p)\,, \tag{21}$$

for any $p\in\Lambda$.

The idea of the proof is that, because of the a-priori bounds on $u^{(1)}-u^{(0)}$ (20) given any $\epsilon>0$, the functions differ by ϵ only on a finite set.

Pointwise and uniform convergence thus agree on $\mathcal{C}$, as next result points out:

Lemma 11. *If $u_n \in \mathcal{C}$ is such that*

$$\lim_{n\to+\infty} u_n(p) = u(p)\,,$$

for any $p \in \Lambda$, then u_n converges to u in the sup-*norm* $\|\cdot\|_H$.

Proof. Fix $\epsilon > 0$. By Lemma 9, there exists a finite set

$$X_\epsilon = \{p_1, \dots, p_m\} \subset \Lambda/H$$

in such a way that

$$|u_n(p) - u^{(0)}(p)| \,\le\, \epsilon\,,$$

for any $n \in \mathbb{N}$ and any $p \in (\Lambda/H) - X_\epsilon$. Since $\mathcal{C}$ is obviously closed under pointwise limits, $u \in \mathcal{C}$ and

$$|u(p) - u^{(0)}(p)| \,\le\, \epsilon\,,$$

for any $p \in (\Lambda/H) - X_\epsilon$.
Also, using again the pointwise convergence of u_n, we have that, for each $j = 1, \dots, m$, there exists $n(\epsilon, j)$ such that

$$|u_n(p_j) - u(p_j)| \,\le\, \epsilon\,,$$

for any $n \ge n(\epsilon, j)$. Let

$$n(\epsilon) := \max\{n(\epsilon, 1), \dots, n(\epsilon, m)\}\,.$$

Then,

$$|u_n(p) - u(p)| \,\le\, \epsilon\,,$$

for any $n \ge n(\epsilon)$ and any $p \in X_\epsilon$.
Thus, if $n \ge n(\epsilon)$,

$$\begin{aligned}
\|u_n - u\|_H \;&\le\; \sup_{p\in X_\epsilon} |u_n(p) - u(p)| \\
&\quad + \sup_{p\in(\Lambda/H)-X_\epsilon} \Big(|u_n(p) - u^{(0)}(p)| + |u^{(0)}(p) - u(p)| \Big) \\
&\le\; 3\epsilon\,,
\end{aligned}$$

proving the desired claim. □

As an easy consequence of Lemmata 10 and 11, we thus deduce that $\mathcal{C}$ is compact in the sup-norm:

Corollary 12. *Let $u_n \in \mathcal{C}$ for any $n \in \mathbb{N}$. Then, there exists a subsequence u_{n_k} and a configuration $u \in \mathcal{C}$ so u_{n_k} converges to u in the* sup-*norm* $\|\cdot\|_H$.

3.2. **The renormalized reduced energy.** We now generalize an approach, already used in [dlLV05]. That is, we introduce a *renormalized energy functional* and study its variational properties. The key point is that this renormalized energy has the same ground state and critical points that the original interaction. Nevertheless it is given by a convergent and differentiable function. Hence, we can use the direct methods of calculus of variations. In our case, we will find it convenient to use the gradient flow of this renormalized energy.

In this section, we establish some properties of the renormalized energy. We will make clear the relation of the critical points of the renormalized energy and the ground states and critical points of the original problem.

The main difference between the renormalized energy introduced here and that of [dlLV06a] is that the one introduced here has to take into account the possibility that the cocycle contains some rational components.

We recall the equivalence relation introduced on page 8 and we define

$$\mathcal{S}_H := \{ [B]_H \text{ with } B \in \mathcal{S}\} \tag{22}$$

and

$$\tilde{\mathcal{C}}_{\sigma,H} = \{u \in \mathcal{O}_\sigma \mid \mathcal{T}_g u = \mathcal{R}_{\sigma(g)} u, \text{ for any } g \in H\}\,. \tag{23}$$

It is also convenient to define $\mathcal{C}_{\sigma,H}$ to be the set of all configurations $u \in \tilde{\mathcal{C}}_{\sigma,H}$ that agree with a configuration $\tilde{v} \in \mathcal{C}$ outside a finite set (we recall that the definition of $\mathcal{C}$ was given on page 15).

Notice that if $u \in \mathcal{B}_\sigma$, then $\mathcal{T}_g u = \mathcal{R}_{\sigma(g)} u$ for any $g \in \mathcal{H}$, and so, since $\mathcal{H} \supset H$, we have that

$$\mathcal{C}_\sigma(u^{(0)}, u^{(1)}) = \mathcal{C} \subset \mathcal{B}_\sigma \subset \tilde{\mathcal{C}}_{\sigma,H}\,.$$

Concerning the definition in (22), we note that if $[B]_H \in \mathcal{S}_H$, then there exists $h \in H$ in such a way that $(hB)\cap(\Lambda/H) \neq \emptyset$, because Λ/H agrees with the fundamental domain. For any $[B]_H \in \mathcal{S}_H$ and any $u \in \tilde{\mathcal{C}}_{\sigma,H}$, we also define

$$H_{[B]_H}(u) := H_B(u)\,.$$

We remark that this is a good definition: indeed, if $[A]_H = [B]_H$, then $A = hB$ for some $h \in H$ and so

$$H_A(u) = H_{hB}(u) = H_B(\mathcal{T}_h u) = H_B(\mathcal{R}_{\sigma(h)} u) = H_B(u)\,,$$

thanks to **H3**, **H4** and (22).

With this setting, given u, $v \in \tilde{\mathcal{C}}_{\sigma,H}$, we define

$$S_H(u,v) := \sum_{[B]_H \in \mathcal{S}_H} \Big(H_{[B]_H}(u) - H_{[B]_H}(v) \Big)\,. \tag{24}$$

The renormalized energy function is then

$$S_H(u) := S_H(u, u^{(0)})\,,$$

where $u^{(0)}$ is as in the statement of Theorem 6.

We now deal with the convergence of the above series.

Lemma 13. *Let $u, v \in \tilde{\mathcal{C}}_{\sigma,H}$. Then, the following inequalities hold:*

$$\begin{aligned} |S_H(u,v)| &\leq \sum_{[B]_H \in \mathcal{S}_H} \Big| H_{[B]_H}(u) - H_{[B]_H}(v) \Big| \\ &\leq C \sum_{p \in \Lambda/H_{2k^\star}} |u_p - v_p| \,, \end{aligned}$$

where $C > 0$ here above is the quantity in (6) and $k^\star$ was introduced in **A2**.

Proof. The first inequality is obvious. We thus use **A2** and **H8** to get the second one:

$$\begin{aligned} & \sum_{[B]_H \in \mathcal{S}_H} \Big| H_{[B]_H}(u) - H_{[B]_H}(v) \Big| \\ \leq & \sum_{[B]_H \in \mathcal{S}_H} \sum_{p \in B} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1 - \tau v) \right| d\tau \, |u_p - v_p| \\ \leq & \sum_{p \in \Lambda} \sum_{\substack{B \in \mathcal{S} \\ B \ni p \\ B \cap (\Lambda/H) \neq \emptyset}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1 - \tau v) \right| d\tau \, |u_p - v_p| \\ \leq & \sum_{p \in \Lambda/H_{2k^\star}} \sum_{\substack{B \in \mathcal{S} \\ B \ni p \\ B \cap (\Lambda/H) \neq \emptyset}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1 - \tau v) \right| d\tau \, |u_p - v_p| \,, \end{aligned}$$

which yields the claim. □

By Lemmata 9 and 13, we conclude that:

Corollary 14. *Let u, $v \in \mathcal{C}_{\sigma,H}$. Then, the series in (24) is absolutely convergent.*

Closely related to the above properties is that the renormalized energy is continuous with respect to the pointwise convergence.

Lemma 15. *Let $u_n \in \mathcal{C}$ be such that*

$$\lim_{n \to +\infty} u_n(p) = u(p) \,,$$

for any $p \in \Lambda/H$. Then,

$$\lim_{n \to +\infty} \sum_{p \in \Lambda/H} |u_n(p) - u(p)| = 0 \,, \tag{25}$$

$$\lim_{n \to +\infty} S_H(u_n) = S_H(u) \tag{26}$$

and

$$\lim_{n \to +\infty} \sum_{p \in \Lambda/H} \Big(\sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \frac{\partial H_B}{\partial u_p}(u_n) \Big)^2 = \sum_{p \in \Lambda/H} \Big(\sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \frac{\partial H_B}{\partial u_p}(u) \Big)^2 . \tag{27}$$

Proof. By Lemma 9, fixed $\epsilon > 0$, we can find a finite set $X_\epsilon \subset \Lambda/H$ in such a way that

$$\sum_{p\in(\Lambda/H)-X_\epsilon} |u^{(1)} - u^{(0)}| \leq \epsilon\,.$$

Then,

$$\lim_{n\to+\infty} \sum_{p\in\Lambda/H} |u_n(p) - u(p)| \leq \lim_{n\to+\infty} \sum_{p\in X_\epsilon} |u_n(p) - u(p)| + \epsilon = \epsilon\,.$$

By taking ϵ as small as we wish, we thus obtain the proof of (25).
Furthermore, using Corollary 14, **A2** and **H8**,

$$\begin{aligned}
& |S_H(u) - S_H(u_n)| \\
\leq & \sum_{[B]_H\in\mathcal{S}_H} \Big| H_{[B]_H}(u) - H_{[B]_H}(u_n) \Big| \\
\leq & \sum_{\substack{B\in\mathcal{S} \\ B\subset(\Lambda/H_{2k^\star})}} \Big| H_B(u) - H_B(u_n) \Big| \\
\leq & \sum_{\substack{B\in\mathcal{S} \\ B\subset(\Lambda/H_{2k^\star})}} \sum_{p\in B} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1-\tau u_n)) \right| d\tau\, |u(p) - u_n(p)| \\
\leq & \sum_{p\in(\Lambda/H_{2k^\star})} \sum_{\substack{B\in\mathcal{S} \\ B\ni p}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1-\tau u_n)) \right| d\tau\, |u(p) - u_n(p)| \\
\leq & \; C \sum_{p\in(\Lambda/H_{2k^\star})} |u(p) - u_n(p)| \\
= & \; C\,\sharp(H/H_{2k^\star}) \sum_{p\in(\Lambda/H)} |u(p) - u_n(p)|\,,
\end{aligned}$$

for a suitable $C > 0$. The latter estimate and (25) yield (26).

Moreover, by exploiting **A2** and **H8** once more, we conclude that

$$\begin{aligned}
& \left| \sum_{p\in\Lambda/H} \left(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u_n)\right)^2 - \sum_{p\in\Lambda/H} \left(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u)\right)^2 \right| \\
\leq \; & \sum_{p\in\Lambda/H} \left[\left(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \left|\frac{\partial H_B}{\partial u_p}(u_n)\right| + \left|\frac{\partial H_B}{\partial u_p}(u)\right| \right) \cdot \left(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \left|\frac{\partial H_B}{\partial u_p}(u_n) - \frac{\partial H_B}{\partial u_p}(u)\right| \right) \right] \\
\leq \; & 2C \sum_{p\in\Lambda/H} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \sum_{q\in B} \left(\int_0^1 \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(\tau u_n + (1-\tau)u)\right| d\tau \cdot |u_n(q) - u(q)| \right) \\
= \; & 2C \sum_{q\in\Lambda/H_{2k^\star}} \sum_{\substack{B\in\mathcal{S}\\ B\ni q}} \sum_{p\in B} \left(\int_0^1 \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(\tau u_n + (1-\tau)u)\right| d\tau \cdot |u_n(q) - u(q)| \right) \\
\leq \; & 2C^2 \sum_{q\in\Lambda/H_{2k^\star}} |u_n(q) - u(q)| \\
= \; & 2C^2 \, \sharp(H/H_{2k^\star}) \sum_{q\in\Lambda/H} |u_n(q) - u(q)| \,,
\end{aligned}$$

for some $C > 0$. This completes the proof of (27), by exploiting (25) once more. □

We now deal with the variational properties of S_H.

First, we observe that it is clear that the critical points of the renormalized energy are the same as the critical points in the sense of Definition 3. The reason is that the renormalized energy contains exactly the same terms that involve u than the original energy. The additional terms that we subtract depend only on the fixed ground state we have chosen. They do not contribute to the variational problem even if they make the sum convergent.

Our aim will be to show that, if $v \in \tilde{\mathcal{C}}_{\sigma,H}$ is a ground state, than the fact that $u \in \tilde{\mathcal{C}}_{\sigma,H}$ is also a ground state is equivalent to $S_H(u, v) = 0$ (see Corollary 17 below). To this goal, we need the following auxiliary result:

Lemma 16. *Let $u, v \in \mathcal{C}_{\sigma,H}$. Suppose that v is a ground state and that $u = v$ outside a finite set $X \subset \Lambda/H$. Then, $S_H(u, v) \geq 0$.*

Proof. Given $n \in \mathbb{N}$ (to be taken large in the sequel), we now perturb u into v outside the fundamental domain Λ/H_n. More explicitly, we let

$$v_p^{(n)} := \begin{cases} u_p & \text{if } p \in \Lambda/H_n, \\ v_p & \text{if } p \notin \Lambda/H_n. \end{cases}$$

Let also

$$\begin{aligned} \tilde{Y} \; &:= \; \left(\bigcup_{h\in H} (hX) \right) \\ \text{and} \quad Y \; &:= \; (\Lambda/H_n) \cap \tilde{Y} \,. \end{aligned}$$

By construction,

$$\sharp Y \;=\; \sharp(H/H_n)\,\sharp X\,,$$

thence Y is a finite subset of Λ. Moreover, $v^{(n)} = v$ outside Y. Consequently, since v is a ground state,

$$\begin{aligned}
0 \;&\le\; \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset}} \Big(H_B(v^{(n)}) - H_B(v)\Big) \\
&=\; \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset\\ B\subset(\Lambda/H_n)}} \Big(H_B(v^{(n)}) - H_B(v)\Big) + \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset\\ B\not\subset(\Lambda/H_n)}} \Big(H_B(v^{(n)}) - H_B(v)\Big) \\
&=\; \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset\\ B\subset(\Lambda/H_n)}} \Big(H_B(u) - H_B(v)\Big) + \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset\\ B\not\subset(\Lambda/H_n)}} \Big(H_B(v^{(n)}) - H_B(v)\Big)\,.
\end{aligned}$$

This inequality and **A2** yield that

$$0 \le \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset\\ B\subset(\Lambda/H_n)}} \Big(H_B(u) - H_B(v)\Big) + \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})}} \Big|H_B(v^{(n)}) - H_B(v)\Big|\,.$$

Since $u = v$ outside $\tilde{Y}$, the latter estimate reduces to

$$0 \le \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_n)}} \Big(H_B(u) - H_B(v)\Big) + \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})}} \Big|H_B(v^{(n)}) - H_B(v)\Big|\,. \tag{28}$$

We now observe that

(29)

$$\begin{aligned}
&\sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})}} \Big|H_B(v^{(n)}) - H_B(v)\Big| \\
&\le \sum_{p\in\Lambda} \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})\\ B\ni p}} \int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\tau v^{(n)} + (1-\tau)v)\right| d\tau\, |v_p^{(n)} - v_p| \\
&= \sum_{p\in(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})} \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})\\ B\ni p}} \int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\tau v^{(n)} + (1-\tau)v)\right| d\tau\, |v_p^{(n)} - v_p| \\
&\le C\, c(n) \sum_{p\in X} |u_p - v_p|\,,
\end{aligned}$$

by (6) and **A4**. Let now

$$a_n(B) \;:=\; \sharp\Big\{A \in [B]_H \text{ s.t. } A\subset \Lambda/H_n\Big\}\,.$$

Note that

$$\sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_n)}} \Big(H_B(u)-H_B(v)\Big) = \sum_{[B]_H\in\mathcal{S}_H} a_n(B)\,\Big(H_B(u)-H_B(v)\Big)\,. \tag{30}$$

Let also

$$b_n(B) \,:=\, \frac{a_n(B)}{\sharp(H/H_n)}\,.$$

We deduce from (28), (29), (30) and Lemma 9 that

$$0\,\le\,\sum_{[B]_H\in\mathcal{S}_H} b_n(B)\Big(H_{[B]_H}(u)-H_{[B]_H}(v)\Big)\,+\,\frac{\tilde C\,c(n)}{\sharp(H/H_n)}\,,$$

for a suitable $\tilde C>0$ (possibly depending on u and v). We thus take the limit as $n\to+\infty$ and we deduce from **A4** and Corollary 14 that

$$0\,\le\,\sum_{[B]_H\in\mathcal{S}_H}\Big(H_{[B]_H}(u)-H_{[B]_H}(v)\Big)\,,$$

as desired. □

Corollary 17. *Let $u,v\in\mathcal{C}_{\sigma,H}$. Suppose that v is a ground state. Then, $S_H(u,v)\ge 0$. Moreover, $S_H(u,v)=0$ if and only if u is also a ground state.*

Proof. Since $u,v\in\mathcal{C}_{\sigma,H}$, there exists a finite set $\bar X\subset\Lambda$ and two configurations $\bar u$, $\bar v\in\mathcal{C}$ in such a way that $u=\bar u$ and $v=\bar v$ outside $\bar X$.
Fix $\epsilon>0$. By Lemma 9, we find a finite set $X_\epsilon\subset\Lambda$ in such a way that

$$\sum_{p\in(\Lambda/H_{2k^\star})-X_\epsilon} |u_p^{(1)}-u_p^{(0)}|\,\le\,\frac{\epsilon}{C}\,, \tag{31}$$

where C is the quantity in (6). Without loss of generality, $X_\epsilon\supset\bar X$.
We define

$$u_p^{(\epsilon)}\,:=\,\begin{cases} u_p & \text{if } p\in X_\epsilon,\\ v_p & \text{if } p\notin X_\epsilon.\end{cases}$$

Therefore, using Corollary 14 and Lemmata 13 and 16, we get that

$$\begin{aligned} S_H(u,v) &= S_H(u,u^{(\epsilon)})\,+\,S_H(u^{(\epsilon)},v)\\ &\ge -C\sum_{p\in\Lambda/H_{2k^\star}} |u_p-u_p^{(\epsilon)}|\,+\,0\\ &= -C\sum_{p\in(\Lambda/H_{2k^\star})-X_\epsilon} |u_p-u_p^{(\epsilon)}|\\ &\ge -C\sum_{p\in(\Lambda/H_{2k^\star})-X_\epsilon} |u_p^{(1)}-u_p^{(0)}|\,. \end{aligned}$$

We conclude from (31) that $S_H(u,v) \geq -\epsilon$, and so $S_H(u,v) \geq 0$, since ϵ may be taken arbitrarily small. This proves the first claim.
Let us now suppose that both u and v are ground states. Then, by the first claim,

$$S_H(u,v) \geq 0 \quad \text{and} \quad S_H(v,u) \geq 0.$$

Since $S_H(u,v) + S_H(v,u) = 0$ thanks to Corollary 14, we conclude that $S_H(u,v) = 0$, as desired.
Let us now suppose that v is a ground state and that $S_H(u,v) = 0$. Let w be a configuration agreeing with u outside a finite set X. By **A1**, we have that $X \subset \Lambda/H_{n_0}$ for a suitable $n_0 \in \mathbb{N}$. Without loss of generality, we may assume that $n_0 \geq k^\star$. In fact, by **A2**, if $B \in \mathcal{S}$ is such that $B \cap X \neq \emptyset$, then $B \subset \Lambda/H_{n_1}$, with $n_1 := n_0 + k^\star$. Thus, we may extend w outside Λ/H_{n_1} in such a way that, if we call $\tilde{w}$ such extension, we have that $\tilde{w} \in \mathcal{C}_{\sigma, H_{n_1}}$.
Then, since v is a ground state minimizer, we know from the first claim of Corollary 17 (applied here to $S_{H_{n_1}}$ instead of S_H) that $S_{H_{n_1}}(\tilde{w}, v) \geq 0$. Also,

$$S_{H_{n_1}}(u,v) \;=\; \sharp(H/H_{n_1})\, S_H(u,v)\,,$$

since both u and v are in $\mathcal{C}_{\sigma,H}$, and so $S_{H_{n_1}}(u,v) = 0$.
Hence, by Corollary 14,

$$S_{H_{n_1}}(\tilde{w}, u) \;=\; S_{H_{n_1}}(\tilde{w}, v) - S_{H_{n_1}}(u,v) \;\geq\; 0\,.$$

Accordingly,

$$\begin{aligned} 0 \;&\geq\; \sum_{[B]_{H_{n_1}} \in \mathcal{S}_{H_{n_1}}} \Big(H_{[B]_H}(u) - H_{[B]_H}(\tilde{w}) \Big) \\ &=\; \sum_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset}} \Big(H_B(u) - H_B(w) \Big)\,, \end{aligned}$$

thence u is a ground state. □

Remark 18. The introduction of the renormalized energy is rather standard as a heuristic method in Physics when one considers the energies of excitations in a background.
The energy of the background may not be finite, but if we consider only the energy generated by the excitation, we obtain a finite sum.
When one considers the Frenkel-Kontorova model, which is included in our general framework as indicated in Section 2 the renormalized energy is called Peierls-Nabarro energy of dislocations.
The Peierls-Nabarro barrier is the smallest renormalized energy of an equilibrium point. There are heuristic arguments which indicate that this Peierls Nabarro barrier vanishes when (and only when) the dislocations are free to move. See [BK04]. Given the statements about convergence of the renormalized energy, we see that the Peierls-Nabarro barrier can be rigorously defined for the configurations in $\mathcal{C}$.

Using the remarks after Theorem 6, we see that either the lamination of minimizers is a full foliation of that there is a critical point which is not a ground state. By Corollary 17, the renormalized energy should be strictly positive.
Hence, we obtain as a corollary that the Peirls-Nabarro barrier vanishes if and only if the lamination of minimal sets does not have any gap.
For the case of one-dimensional problems with nearest neighbor interactions, this was established in [Mat86].

3.3. **Some remarks on the notion of periodicity.**

Lemma 19. *For any $u \in \mathcal{O}_\sigma$, $s \in \mathbb{Z}$, $\gamma \in G$, $p \in B \in \mathcal{S}$, we have that*

$$\frac{\partial H_B}{\partial u_p}(\mathcal{R}_s u) \;=\; \frac{\partial H_B}{\partial u_p}(u) \;=\; \frac{\partial H_{\gamma^{-1}B}}{\partial u_{\gamma^{-1}p}}(\mathcal{T}_\gamma u)\,.$$

Proof. The first equality plainly follows from **H4**. The second one is a consequence of **H3** by means of the following argument. Since

$$H_{\gamma^{-1}B}(\mathcal{T}_\gamma u) = H_{\gamma(\gamma^{-1}B)}(u) = H_B(u)\,,$$

we have that

$$\frac{\partial H_B}{\partial u_p}(u) \;=\; \sum_{q\in\gamma^{-1}B} \frac{\partial H_{\gamma^{-1}B}}{\partial u_q}(\mathcal{T}_\gamma u)\,\frac{\partial(\mathcal{T}_\gamma u)_q}{\partial u_p}$$

$$=\; \sum_{q\in\gamma^{-1}B} \frac{\partial H_{\gamma^{-1}B}}{\partial u_q}(\mathcal{T}_\gamma u)\,\frac{\partial u_{\gamma q}}{\partial u_p} \;=\; \frac{\partial H_{\gamma^{-1}B}}{\partial u_{\gamma^{-1}p}}(\mathcal{T}_\gamma u)\,,$$

as desired. □

Corollary 20. *If $u \in \tilde{\mathcal{C}}_{\sigma,H}$ and $h \in H$,*

$$\frac{\partial H_B}{\partial u_p}(u) \;=\; \frac{\partial H_{hB}}{\partial u_{hp}}(u)\,.$$

Proof. Since $\gamma := h^{-1} \in H \subset \mathcal{H}$, we have that $s := \sigma(\gamma) \in \mathbb{Z}$, and so we use Lemma 19 twice to gather that

$$\frac{\partial H_B}{\partial u_p}(u) = \frac{\partial H_{hB}}{\partial u_{hp}}(\mathcal{T}_\gamma u) = \frac{\partial H_{hB}}{\partial u_{hp}}(\mathcal{R}_s u) = \frac{\partial H_{hB}}{\partial u_{hp}}(u)\,,$$

as desired. □

Recalling (11), we now introduce the following notation: given $p \in \Lambda$ and $[B]_H \in \mathcal{S}_H$, we denote by B_p the unique (if any) set $A \in [B]_H$ such that $A \ni p$. With this setting, we get the following result:

Corollary 21. *Let $u \in \tilde{\mathcal{C}}_{\sigma,H}$ and $v \in \mathcal{O}_\sigma$. Suppose that*

$$\mathcal{T}_h v \;=\; v \tag{32}$$

for any $h \in H$. Then,

$$\sum_{p\in B} \frac{\partial H_B}{\partial u_p}(u)\, v_p \;=\; \sum_{p\in\Lambda/H} \frac{\partial H_{B_p}}{\partial u_p}(u)\, v_p \tag{33}$$

for any $B \in \mathcal{S}$. Also, the above are sums of finite number of terms.

Proof. That only a finite number of terms are involved in the sum in the left hand side of (33) follows from the fact that $\sharp B < +\infty$ for any $B \in \mathcal{S}$ (recall the definition of $\mathcal{S}$ given on page 4).
Concerning the sum on the right hand side of (33), suppose that there are $p_1, \dots, p_\ell \in \Lambda/H$ in such a way B_{p_j} is defined, i.e., $h_j B \ni p_j$ for some $h_j \in H$, with $j = 1, \dots, \ell$. Let $q_j := h_j^{-1} p_j$. Then, all the q_j's are different if so are the p_j's, since the p_j's lie in the fundamental domain Λ/H.
By construction, all the q_j's lie in B and so $\ell \le \sharp B < +\infty$, which shows that the sum in the right hand side of (33) is made of finitely many terms.
We now prove (33). Given $p \in B$, we write $p = hq$, with $h \in H$ and $q \in \Lambda/H$. While such a q is obviously unique, because Λ/H is a fundamental domain, the uniqueness of h is here a consequence of **H5** and **A3**. Indeed, if $p = hq = \tilde{h}q$, with $h, \tilde{h} \in H$, we set $A := h^{-1}B$ and $\tilde{A} := \tilde{h}^{-1}B$. We observe that $A \sim_H \tilde{A}$ and $q \in A \cap \tilde{A} \ne \emptyset$. Accordingly, by **A3**, $A = \tilde{A}$ and so, by **H5**, $h = \tilde{h}$. This shows that the notation $p = hq$ with $h \in H$ and $q \in \Lambda/H$ is uniquely determined.
Then, we use (32) and Corollary 20 to gather that

$$\begin{aligned}
\sum_{p \in B} \frac{\partial H_B}{\partial u_p}(u)\, v_p &= \sum_{\substack{h \in H \\ q \in \Lambda/H \\ hq \in B}} \frac{\partial H_B}{\partial u_{hq}}(u)\, v_{hq} \\
&= \sum_{\substack{h \in H \\ q \in (\Lambda/H) \cap (h^{-1}B)}} \frac{\partial H_{h(h^{-1}B)}}{\partial u_{hq}}(u)\, v_q = \sum_{\substack{h \in H \\ q \in (\Lambda/H) \cap (h^{-1}B)}} \frac{\partial H_{h^{-1}B}}{\partial u_q}(u)\, v_q \,. \qquad (34)
\end{aligned}$$

Recalling **A3**, we conclude that the union in

$$\bigcup_{h \in H} (hB)$$

is disjoint and so the latter term in (34) equals

$$\sum_{q \in \Lambda/H} \frac{\partial H_{B_q}}{\partial u_q}(u)\, v_q \,,$$

which gives the desired result. $\square$

3.4. **The heat flow.** We now consider the solution of a "heat flow" problem, which, indeed, is the "gradient flow" of our renormalized energy functional.
Given $u \in \mathcal{C}$ and $t > 0$, we denote by $\Phi^t(u)$ the configuration $p \mapsto U(p,t)$ that satisfies

$$\begin{cases} \partial_t U(p,t) = -\sum\limits_{\substack{B \in \mathcal{S} \\ B \ni p}} \dfrac{\partial H_B}{\partial u_p}(U(\cdot,t)) & \text{for any } p \in \Lambda \text{ and } t > 0\,, \\ U(p,0) = u(p) & \text{for any } p \in \Lambda\,. \end{cases} \qquad (35)$$

Notice that the series in (35) converges, thanks to **H8**. Moreover, the following properties hold:

Lemma 22. *If $u \in \mathcal{C}$, then Φ^t is well-defined for any $t > 0$. Also, $\Phi^t(\mathcal{C}) \subset \mathcal{C}$,*

$$\mathcal{T}_h \Phi^t(u) = \mathcal{R}_{\sigma(h)} \Phi^t(u) \,, \tag{36}$$

$$\partial_t \Big(\mathcal{T}_h \Phi^t(u) \Big) = \partial_t \Phi^t(u) \tag{37}$$

for any $h \in H$, and

$$-\frac{d}{dt} S_H \Big(\Phi^t(u) \Big) = \sum_{p \in \Lambda / H} \left(\sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \frac{\partial H_B}{\partial u_p} (\Phi^t(u)) \right)^2 \geq 0 \,. \tag{38}$$

Furthermore, if $u \in \mathcal{C}$, the map $t \mapsto \Phi^t(u)$ belongs to $C\Big([0,+\infty), \mathcal{C}\Big) \cap C^1\Big((0,+\infty), \mathcal{C}\Big)$. Moreover, the following result on the dependence on initial data holds: fixed any $T > 0$, the map

$$\mathcal{C} \ni u \mapsto \Phi^t(u)$$

is continuous, and its modulus of continuity is uniform for any $t \in [0, T]$.

Proof. Fixed any $p \in \Lambda$,

$$\begin{aligned}
\sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \left| \frac{\partial H_B}{\partial u_p}(u) - \frac{\partial H_B}{\partial u_p}(v) \right| &\leq \sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \sum_{q \in B} \int_0^1 \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q} (\tau u + (1-\tau) v) \right| d\tau \cdot \|u - v\|_H \\
&\leq C \, \|u - v\|_H \,,
\end{aligned}$$

for any u, $v \in \mathcal{C}$, due to **H8**. Accordingly, the map

$$\mathcal{C} \ni u \longmapsto \sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \frac{\partial H_B}{\partial u_p}(u)$$

is uniformly Lipschitz continuous with respect to the sup-norm. Then, the short time existence of the heat flow with continuous dependence on the initial data follows from the theory of ODE's in Banach spaces (see, e.g., [Hal80]). The short time flow may be continued provided that it does not exit from $\mathcal{C}$ (see, e.g., the Prolongation Lemma in [Hal80]).

Using the ferromagnetism assumption **H9**, it is possible to show that the gradient flow satisfies a a comparison principle and that it preserves the order among configurations. We recall that the (weak) comparison principle, i.e., the fact that $u \geq v$ yields $\Phi^t(u) \geq \Phi^t(v)$ for any $t \geq 0$, is a consequence of the ferromagnetic property and it is based on the fact that the off-diagonal part of the linearized equation has a sign This is established in see, e.g., Lemma 4.3 in [KdlLR97]. See also [dlLV06b] for a sharper version in a particular case. We remark that this is the only place in this paper where we use the ferromagnetism assumption.

Since the set $\mathcal{C}$ is defined by the order properties and by the translations, which commute with the flow, we conclude that $\Phi^t(\mathcal{C}) \subset \mathcal{C}$ and since we have uniform bounds for

the time of definition for all the initial data in $\mathcal{C}$, we conclude, applying the Prolongation Lemma in [Hal80] that the flow Φ^t in $\mathcal{C}$ is defined for all time.
We now prove (36) and (37). In fact, since (37) plainly follows from (36), we focus on the proof of (36). To this extent, if $U(\cdot,t) = \Phi^t(u)$, we have that

$$U(hp,0) = u(hp) = u(p) + \sigma(h) = U(p,0) + \sigma(h)$$

since $u \in \mathcal{C}$, and

$$\begin{aligned}
-\partial_t U(hp,t) &= \sum_{\substack{B\in\mathcal{S}\\ h^{-1}B\ni p}} \frac{\partial H_{h(h^{-1}B)}}{\partial u_{hp}}\Big(U(\cdot,t)\Big)\\
&= \sum_{\substack{B\in\mathcal{S}\\ h^{-1}B\ni p}} \frac{\partial H_{h^{-1}B}}{\partial u_{p}}\Big(U(\cdot,t)\Big)\\
&= \sum_{\substack{A\in\mathcal{S}\\ A\ni p}} \frac{\partial H_{A}}{\partial u_{p}}\Big(U(\cdot,t)+\sigma(h)\Big)\\
&= -\partial_t\Big(U(p,t)+\sigma(h)\Big),
\end{aligned}$$

where Lemma 19 and Corollary 20 have been utilized. Therefore, by the uniqueness of the solution of the Cauchy problem, $U(hp,t) = U(p,t) + \sigma(h)$, yielding (36).
It now remains to prove (38). To this extent, let $t > 0$ and $\delta > 0$ (the case $\delta < 0$ with $t + \delta > 0$ is analogous). We observe that (32) is fulfilled here with $v := \partial_t(\Phi^t(u))$, thanks to (37). Therefore, we use Corollary 21 and (38) to obtain that

$$\begin{aligned}
\frac{d}{ds} H_{[B]_H}\Big(\Phi^{t+\delta s}(u)\Big) &= \frac{d}{ds} H_B\Big(\Phi^{t+\delta s}(u)\Big)\\
&= \delta \sum_{p\in B} \frac{\partial H_B}{\partial u_p}\Big(\Phi^{t+\delta s}(u)\Big)\ \partial_\tau\Big(\Phi^{\tau}(u)\Big)_p\Big|_{\tau=t+\delta s}\\
&= -\delta \sum_{p\in\Lambda/H} \frac{\partial H_{B_p}}{\partial u_p}\Big(\Phi^{t+\delta s}(u)\Big) \sum_{\substack{A\in\mathcal{S}\\ A\ni p}} \frac{\partial H_A}{\partial u_p}\Big(\Phi^{t+\delta s}(u)\Big).
\end{aligned}$$

In the last term here above, the first sum consists only in a finite numbers of terms, due to Corollary 21, and the second sum is absolutely convergent, due to **H8**. Therefore, for any $B \in \mathcal{S}$,

$$
\begin{aligned}
& \frac{H_{[B]_H}\Big(\Phi^{t+\delta}(u)\Big) - H_{[B]_H}\Big(\Phi^t(u)\Big)}{\delta} \\
= \;& \frac{1}{\delta}\int_0^1 \frac{d}{ds} H_{[B]_H}\Big(\Phi^{t+\delta s}(u)\Big)\, ds \\
= \;& -\sum_{p\in\Lambda/H}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}} \int_0^1 \frac{\partial H_{B_p}}{\partial u_p}\Big(\Phi^{t+\delta s}(u)\Big)\frac{\partial H_{A_p}}{\partial u_p}\Big(\Phi^{t+\delta s}(u)\Big)\, ds \\
= \;& -\sum_{p\in\Lambda/H}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}} \frac{\partial H_{B_p}}{\partial u_p}\Big(\Phi^{t}(u)\Big)\frac{\partial H_{A_p}}{\partial u_p}\Big(\Phi^{t}(u)\Big) \;+\; R_{\delta,B}\,,
\end{aligned}
\tag{39}
$$

where **A3** has been used to replace A with A_p and $R_{\delta,B}$ satisfies

$$
\begin{aligned}
|R_{\delta,B}| \;\le\; \sum_{p\in\Lambda/H}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}} \int_0^1 \quad & \left|\frac{\partial H_{B_p}}{\partial u_p}\Big(\Phi^{t+\delta s}(u)\Big)\right|\cdot\left|\frac{\partial H_{A_p}}{\partial u_p}\Big(\Phi^{t+\delta s}(u)\Big) - \frac{\partial H_{A_p}}{\partial u_p}\Big(\Phi^{t}(u)\Big)\right| \\
& + \left|\frac{\partial H_{A_p}}{\partial u_p}\Big(\Phi^{t}(u)\Big)\right|\cdot\left|\frac{\partial H_{B_p}}{\partial u_p}\Big(\Phi^{t+\delta s}(u)\Big) - \frac{\partial H_{B_p}}{\partial u_p}\Big(\Phi^{t}(u)\Big)\right|\, ds\,.
\end{aligned}
\tag{40}
$$

Moreover, for any $B \in \mathcal{S}$,

$$
\begin{aligned}
& \left|\frac{\partial H_{B}}{\partial u_p}\Big(\Phi^{t+\delta s}(u)\Big) - \frac{\partial H_{B}}{\partial u_p}\Big(\Phi^{t}(u)\Big)\right| \\
\le \;& \sum_{q\in B}\int_0^1 \left|\frac{\partial^2 H_B}{\partial u_p \partial u_q}\Big(\tau\Phi^{t+\delta s}(u) + (1-\tau)\Phi^t(u)\Big)\right|\,\left|\Big(\Phi^{t+\delta s}(u) - \Phi^t(u)\Big)_q\right|\, d\tau
\end{aligned}
$$

Accordingly, from (40),

$$
\begin{aligned}
&\sum_{[B]_H \in \mathcal{S}_H} |R_{\delta,B}| \\
&\leq \sum_{[B]_H \in \mathcal{S}_H} \sum_{p \in \Lambda/H} \sum_{\substack{A \in \mathcal{S} \\ A \ni p}} \int_0^1 \\
&\quad \left| \frac{\partial H_{B_p}}{\partial u_p} \Big(\Phi^{t+\delta s}(u) \Big) \right| \cdot \left| \frac{\partial H_{A_p}}{\partial u_p} \Big(\Phi^{t+\delta s}(u) \Big) - \frac{\partial H_{A_p}}{\partial u_p} \Big(\Phi^{t}(u) \Big) \right| \\
&+ \left| \frac{\partial H_{A_p}}{\partial u_p} \Big(\Phi^{t}(u) \Big) \right| \cdot \left| \frac{\partial H_{B_p}}{\partial u_p} \Big(\Phi^{t+\delta s}(u) \Big) - \frac{\partial H_{B_p}}{\partial u_p} \Big(\Phi^{t}(u) \Big) \right| \, ds \\
&\leq \sum_{p \in \Lambda/H} \sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \sum_{\substack{A \in \mathcal{S} \\ A \ni p}} \int_0^1 \\
&\quad \left| \frac{\partial H_{B}}{\partial u_p} \Big(\Phi^{t+\delta s}(u) \Big) \right| \cdot \left| \frac{\partial H_{A}}{\partial u_p} \Big(\Phi^{t+\delta s}(u) \Big) - \frac{\partial H_{A}}{\partial u_p} \Big(\Phi^{t}(u) \Big) \right| \\
&+ \left| \frac{\partial H_{A}}{\partial u_p} \Big(\Phi^{t}(u) \Big) \right| \cdot \left| \frac{\partial H_{B}}{\partial u_p} \Big(\Phi^{t+\delta s}(u) \Big) - \frac{\partial H_{B}}{\partial u_p} \Big(\Phi^{t}(u) \Big) \right| \, ds \\
&\leq 2C_1 \sum_{p \in \Lambda/H} \sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \int_0^1 \left| \frac{\partial H_{B}}{\partial u_p} \Big(\Phi^{t+\delta s}(u) \Big) - \frac{\partial H_{B}}{\partial u_p} \Big(\Phi^{t}(u) \Big) \right| \, ds \\
&\leq 2C_1 \sum_{p \in \Lambda/H} \sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \sum_{q \in B} \int_0^1 \int_0^1 \\
&\quad \left| \frac{\partial^2 H_{B}}{\partial u_p \partial u_q} \Big(\tau \Phi^{t+\delta s}(u) + (1-\tau) \Phi^{t}(u) \Big) \right| \left| \Big(\Phi^{t+\delta s}(u) - \Phi^{t}(u) \Big)_q \right| \, d\tau \, ds \\
&\leq 2C_1 C_2 \sum_{q \in \Lambda/H_{2k^\star}} \int_0^1 \left| \Big(\Phi^{t+\delta s}(u) - \Phi^{t}(u) \Big)_q \right| ds \,,
\end{aligned}
\tag{41}
$$

where $C_1 > 0$ (resp., $C_2 > 0$) is the first (resp., the second) derivative bound given by **H8**, and **A2** has also been used.
Note also that

$$
\lim_{\delta \to 0} \left| \Big(\Phi^{t+\delta s}(u) - \Phi^{t}(u) \Big)_q \right| \;=\; 0 \tag{42}
$$

and that

$$
\left| \Big(\Phi^{t+\delta s}(u) - \Phi^{t}(u) \Big)_q \right| \;\leq\; u_q^{(1)} - u_q^{(0)} \tag{43}
$$

for any $q \in \Lambda$, since we have already shown that $\Phi^t(\mathcal{C}) \subset \mathcal{C}$. Also, from Lemma 9,

$$
\sum_{q \in \Lambda/H_{2k^\star}} \int_0^1 u_q^{(1)} - u_q^{(0)} \, ds \;<\; +\infty \,. \tag{44}
$$

The Dominated Convergence Theorem, (41), (42), (43) and (44) imply that

$$\lim_{\delta\to 0} \sum_{[B]_H \in \mathcal{S}_H} |R_{\delta,B}| \;=\; 0\,.$$

Consequently, from Corollary 14, (39), and **A2**,

$$\begin{aligned}
&\lim_{\delta\to 0} \frac{S_H(\Phi^{t+\delta}(u)) - S_H(\Phi^t(u))}{\delta}\\
=\;& \lim_{\delta\to 0} \sum_{[B]_H\in\mathcal{S}_H} \frac{H_{[B]_H}(\Phi^{t+\delta}(u)) - H_{[B]_H}(\Phi^t(u))}{\delta}\\
=\;& \lim_{\delta\to 0} - \sum_{[B]_H\in\mathcal{S}_H} \sum_{p\in\Lambda/H} \sum_{\substack{A\in\mathcal{S}\\ A\ni p}} \frac{\partial H_{B_p}}{\partial u_p}\Big(\Phi^t(u)\Big)\frac{\partial H_{A_p}}{\partial u_p}\Big(\Phi^t(u)\Big) \;+\; \sum_{[B]_H\in\mathcal{S}_H} R_{\delta,B}\\
=\;& -\sum_{p\in\Lambda/H} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \sum_{\substack{A\in\mathcal{S}\\ A\ni p}} \frac{\partial H_B}{\partial u_p}\Big(\Phi^t(u)\Big)\frac{\partial H_A}{\partial u_p}\Big(\Phi^t(u)\Big) \;+\; 0\,.
\end{aligned}$$

From this, (38) plainly follows, thus completing the proof of Lemma 22. □

Given any $s\in[0,1]$, we now consider the linear interpolation between $u^{(0)}$ and $u^{(1)}$, by defining

$$u^{(s)} \;:=\; (1-s)\,u^{(0)} \;+\; s\,u^{(1)}\,.$$

It is easily seen that $u^{(s)}\in\mathcal{C}$. Furthermore, the following compactness result holds:

Corollary 23. *Fixed $s\in[0,1]$, there exists a sequence $t_{n,s}$ and a configuration $u_\star^{(s)}\in\mathcal{C}$, which is a solution of our variational equation* (7)*, such that*

$$\begin{aligned}
&\lim_{n\to+\infty} t_{n,s} = +\infty\\
\textit{and}\qquad &\lim_{n\to+\infty} \Big\|\Phi^{t_{n,s}}(u^{(s)}) - u_\star^{(s)}\Big\|_H = 0\,.
\end{aligned}$$

Proof. Let $f(t) := S_H(\Phi^t(u))$. By Lemmata 14 and 22, we know that f is decreasing and bounded by below. Thus,

$$\lim_{t\to+\infty} f(t) = \inf_{[0,\infty)} f\,.$$

Then, there must exists a divergent sequence of times $t_{n,s}$ such that

$$\lim_{n\to+\infty} f'(t_{n,s}) = 0\,. \tag{45}$$

If not, one would have that $f'(t)\le -c$ for some $c>0$ and any t large enough, but then

$$0 = \lim_{t\to+\infty} f(t+1) - f(t) \le -c\,,$$

which is a contradiction.

Moreover, by Lemma 22 and Corollary 12, we may suppose, possibly taking subsequences, that $\Phi^{t_{n,s}}(u^{(s)})$ converges as $t_{n,s}\to+\infty$ in the sup-norm $\|\cdot\|_H$, say to a suitable $u_\star^{(s)}\in\mathcal{C}$. By Lemma 15, (45), (38) and (7), we have that $u_\star^{(s)}$ is a solution of our variational equation (7). □

3.5. **The basin of attraction of the heat flow.** The rest of the proof of Theorem 6 is now very close to the variational argument of [dlLV05, dlLV06a]. We provide full details for the reader's facility.
We consider the "basin of attraction" of the configurations $u^{(0)}$ and $u^{(1)}$ with respect to the heat flow Φ^t. Namely, for $i \in \{0,1\}$, we define

$$\beta_i := \left\{ s \in [0,1] \quad | \quad \lim_{t \to +\infty} \left\| \Phi^t(u^{(s)}) - u^{(i)} \right\|_H = 0 \right\}. \tag{46}$$

Note that, since $u^{(i)}$ is a solution of our variational equation (7), then $\Phi^t(u^{(i)}) = u^{(i)}$, and so $i \in \beta_i$, for $i \in \{0,1\}$. We also consider the configurations in $\mathcal{C}$ with zero renormalized energy, that is the set

$$\mu_\sigma := \left\{ u \in \mathcal{C} \quad | \quad S_H(u) = 0. \right\}.$$

We now introduce the set of the solutions of our variational equation which belong to $\mathcal{C}$ and whose S_H-value is less than r. More precisely, given $r > 0$, we define

$$\nu_\sigma(r) := \left\{ u \in \mathcal{C} \quad | \quad u \text{ is a solution of our variational equation (7) and } S_H(u) \le r \right\}.$$

By Lemma 14, we have that

$$u^{(0)},\ u^{(1)} \in \mu_\sigma \subset \nu_\sigma(r),$$

for any $r > 0$. We now investigate the properties of the basins of attraction when μ_σ and $\nu_\sigma(r)$ are "as small as possible".

Lemma 24. *Assume that*

$$\mu_\sigma - \{u^{(0)},\ u^{(1)}\} = \emptyset. \tag{47}$$

Then, there exists $r_0 > 0$ in such a way that the following holds. Let $r > 0$ and suppose that there exists $\gamma \in C([0,1], \mathcal{C})$ such that $\gamma(0) = u^{(0)}$, $\gamma(1) = u^{(1)}$ and $S_H(\gamma(t)) \le r$ for any $t \in [0,1]$. Then, $r \ge r_0$.

Proof. We argue by contradiction. Suppose that, for any $n \in \mathbb{N}$, there exists

$$\gamma_n \in C([0,1], \mathcal{C}) \tag{48}$$

such that

$$\gamma_n(0) = u^{(0)},\ \gamma_n(1) = u^{(1)} \text{ and} \tag{49}$$

$$S_H(\gamma_n(t)) \le 1/n \text{ for any } t \in [0,1]. \tag{50}$$

Let us define

$$g_n(t) := \|\gamma_n(t) - u^{(0)}\|_H .$$

Then, $g_n \in C([0,1])$, thanks to (48). Also, $g_n(0) = 0$ and $g_n(1) = \|u^{(1)} - u^{(0)}\|_H$, due to (49). Thus, there exists $t_n \in [0,1]$ in such a way

$$g_n(t_n) = \frac{\|u^{(1)} - u^{(0)}\|_H}{2} =: c .$$

That is,

$$\|\gamma_n(t_n) - u^{(0)}\|_H = c . \tag{51}$$

We recall that $c > 0$, because $u^{(0)} \not\equiv u^{(1)}$.
Moreover, by Corollary 12, perhaps taking subsequences, we have that $\gamma_n(t_n)$ converges in the sup-norm to some $\bar{u} \in \mathcal{C}$. By (51), $\|\bar{u} - u^{(0)}\|_H = c$ and thus $\bar{u} \notin \{u^{(0)}, u^{(1)}\}$. Thus, by (47), $S_H(\bar{u}) > 0$. On the other hand, by (50) and Lemma 15, $S_H(\bar{u}) = 0$. This contradiction yields the desired result. $\square$

We remark the fact that r_0 in Lemma 24 is independent of the path γ.

Lemma 25. *Assume that*

$$\nu_\sigma(r_0) - \{u^{(0)}, u^{(1)}\} = \emptyset\,, \tag{52}$$

where r_0 is as in Lemma 24. Then, there exists $r_1 > 0$ such that the following holds. If $v \in \mathcal{C}$ and $\|v - u^{(0)}\|_H \le r_1$, then

$$\lim_{t \to +\infty} \|\Phi^t(v) - u^{(0)}\|_H = 0\,.$$

Analogously, if $v \in \mathcal{C}$ and $\|v - u^{(1)}\|_H \le r_1$, then

$$\lim_{t \to +\infty} \|\Phi^t(v) - u^{(1)}\|_H = 0\,.$$

Proof. We prove the first assertion, the second one following analogously. The idea of the proof is the following: let us start the flow close to $u^{(0)}$; we know by Corollary 23 that this flow has to converge to a solution of our variational equation, up to subsequence. Since the flow is energy decreasing, (52) yields that any subsequence of the above flow approaches either $u^{(0)}$ or $u^{(1)}$. If all the subsequences approaches $u^{(0)}$ we are done, thence we have to exclude the existence of subsequences approaching $u^{(1)}$. But if any of such subsequences existed, a path joining $u^{(0)}$ and $u^{(1)}$ with small energy would be constructed, in contradiction with Lemma 24.
We now give the proof in full detail. The proof is by contradiction: assume that there exists $v \in \mathcal{C}$ such that

$$\|v - u^{(0)}\|_H \le r_1 \tag{53}$$

in such a way

$$\Phi^t(v) \text{ does not converge to } u^{(0)} \text{ in the norm } \|\cdot\|_H. \tag{54}$$

Here, r_1 will be a suitably small quantity, defined as follows. By Lemma 9, we can find a finite set $X_{r_0} \subset \Lambda/H_{2k^\star}$ in such a way that

$$\sum_{p \in (\Lambda/H_{2k^\star}) - X_{r_0}} |u_p^{(1)} - u_p^{(0)}| \le \frac{r_0}{4C}\,, \tag{55}$$

where C is the quantity in (6). We then set

$$r_1 := \min\left\{ r_0\,,\ \frac{r_0}{4C\, \sharp X_{r_0}\, \sharp(H/H_{2k^\star})} \right\}.$$

We claim that

$$\sup_{\substack{w \in \mathcal{C} \\ \|w - u^{(0)}\|_H \le r_1}} S_H(w) \le \frac{r_0}{2}\,. \tag{56}$$

Indeed, if w is as here above, we get from **H8** and (55) that

$$\begin{aligned}
S_H(w) &\leq \sum_{\substack{B\in\mathcal{S}\\ B\subset\Lambda/H_{2k^\star}}} \sum_{p\in B} \int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\tau w+(1-\tau)u^{(0)})\right| d\tau\cdot|w_p-u_p^{(0)}| \\
&\leq \sum_{p\in\Lambda/H_{2k^\star}} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\tau w+(1-\tau)u^{(0)})\right| d\tau\cdot|u_p^{(1)}-u_p^{(0)}| \\
&\leq C \sum_{p\in\Lambda/H_{2k^\star}} |u_p^{(1)}-u_p^{(0)}| \\
&\leq C \sum_{p\in X_{r_0}} |u_p^{(1)}-u_p^{(0)}| + \frac{r_0}{4} \\
&\leq C\,\|u^{(1)}-u^{(0)}\|_H\,\sharp X_{r_0}\,\sharp(H/H_{2k^\star}) + \frac{r_0}{4} \\
&\leq \frac{r_0}{2},
\end{aligned}$$

thus proving (56)
From this and (53),

$$S_H(v)\leq\frac{r_0}{2}, \tag{57}$$

provided that r_1 is small enough with respect to r_0.
Also, recalling Corollary 23, possibly extracting a subsequence, we have that $\Phi^{t_n}(v)$ converges to a suitable configuration $\bar{u}\in\mathcal{C}$ in the norm $\|\cdot\|_H$, and that

$$\bar{u}\text{ is a solution of our variational equation (7).} \tag{58}$$

Furthermore, it follows from (57) and Lemma 22 that

$$S_H(\Phi^t(v))\leq\frac{r_0}{2}\text{ for any }t\geq 0. \tag{59}$$

Consequently, by Lemma 15,

$$S_H(\bar{u})\leq\frac{r_0}{2}. \tag{60}$$

Note that $\bar{u}$ cannot be equal to $u^{(0)}$, because of (54); thus, (58), (60) and (52) imply that $\bar{u}=u^{(1)}$.
Therefore, there exists $T>0$ such that

$$\|\Phi^T(v)-u^{(1)}\|_H=\|\Phi^T(v)-\bar{u}\|_H\leq r_1\,. \tag{61}$$

Let now

$$\tilde{\gamma}(t):=\Phi^{2T(t-1/4)}(v)\,, \tag{62}$$

for any $t\in[1/4,\,3/4]$.

Then, $\tilde\gamma \in C([1/4,\,3/4],\mathcal{C})$, thanks to Lemma 22. Also, $\tilde\gamma(1/4) = v$ and $\tilde\gamma(3/4) = \Phi^T(v)$. Let

$$\gamma(t) := \begin{cases} u^{(0)} + 4t(v - u^{(0)}) & \text{if } t \in [0,\,1/4), \\ \tilde\gamma(t) & \text{if } t \in [1/4,\,3/4], \\ \Phi^T(v) + (4t-3)(u^{(1)} - \Phi^T(v)) & \text{if } t \in (3/4,\,1]. \end{cases}$$

Then,

$$\gamma \in C([0,\,1],\mathcal{C}),\ \gamma(0) = u^{(0)} \text{ and } \gamma(1) = u^{(1)}. \tag{63}$$

Moreover, if $t \in [0,\,1/4)$,

$$\|\gamma(t) - u^{(0)}\|_H \le \|v - u^{(0)}\|_H \le r_1\,,$$

due to (53), and, if $t \in (3/4,\,1]$

$$\|\gamma(t) - u^{(1)}\|_H \le \|\Phi^T(v) - u^{(1)}\|_H \le r_1\,,$$

due to (61). Therefore, making use of (56), we gather that $S_H(\gamma(t)) \le r_0/2$, for $t \in [0,\,1/4) \cup (3/4,\,1]$, provided that r_1 is small enough. In fact, recalling (59) and (62),

$$S_H(\gamma(t)) \le r_0/2 \text{ for any } t \in [0,\,1]. \tag{64}$$

But the existence of a path γ satisfying (63) and (64) is in contradiction with Lemma 24. Notice indeed that the assumptions of Lemma 24 are fulfilled here, since (52) implies (47).
This contradiction ends the proof of Lemma 25. □

Lemma 26. *Assume* (52). *Then, β_0 and β_1, as defined in* (46)*, are open in the relative topology of* $[0,1]$.

Proof. We prove that β_0 is open, the argument for β_1 being analogous. Let $\bar s \in \beta_0$ and take

$$s \in [0,\,1] \cap [\bar s - \epsilon,\ \bar s + \epsilon]\,.$$

We show that $s \in \beta_0$, provided that $\epsilon > 0$ is sufficiently small. For this, assume, by contradiction, that $s \notin \beta_0$. Then, there exists a sequence $t_n \to +\infty$ in such a way

$$\Phi^{t_n}(u^{(s)}) \text{ does not converge to } u^{(0)} \text{ in the norm } \|\cdot\|_H. \tag{65}$$

Now, since $\bar s \in \beta_0$, there exists $\bar n \in \mathbb{N}$ such that

$$\|\Phi^{t_{\bar n}}(u^{(\bar s)}) - u^{(0)}\|_H \le \frac{r_1}{2}\,,$$

where r_1 is the quantity given in Lemma 25. By the continuity with respect to the initial data (recall Lemma 22), we thus have that

$$\|\Phi^{t_{\bar n}}(u^{(s)}) - u^{(0)}\|_H \le r_1\,,$$

provided that ϵ is small enough.
Therefore, by Lemma 25, it follows that $\Phi^t(\Phi^{t_{\bar n}}(u^{(s)}))$ converges to $u^{(0)}$ in the norm $\|\cdot\|_H$. That is, $\Phi^{t_n}(u^{(s)})$ converges to $u^{(0)}$ in the norm $\|\cdot\|_H$. Since the latter assertion is in contradiction with (65), the proof of the desired result is complete. □

Corollary 27. *Suppose that* (52) *holds. Then, there exists* $s \in [0,1] - (\beta_0 \cup \beta_1)$.

Proof. Assume, by contradiction, that $[0,1] = \beta_0 \cup \beta_1$. Let

$$\bar{s} := \sup\{s \in \beta_0\}\,. \tag{66}$$

By Lemma 26, β_0 and β_1 are nonempty, disjoint, relatively open subsets of $[0,1]$. Since $1 \in \beta_1$ and β_1 is open, we have that

$$\bar{s} < 1\,. \tag{67}$$

Analogously, since $0 \in \beta_0$ and β_0 is open, we have that

$$\bar{s} > 0\,. \tag{68}$$

Thus, in the light of (66), there are two cases to distinguish: either $\bar{s} \in \beta_0$ or $\bar{s} \in \beta_1$.
If $\bar{s} \in \beta_0$, then, since β_0 is relatively open, it follows from (67) that $\bar{s}+\epsilon \in \beta_0$, provided that $\epsilon > 0$ is sufficiently small. But this contradicts (66).
If, on the other hand, $\bar{s} \in \beta_1$, the fact that β_1 is open and (68) imply that $\bar{s} - \epsilon \in \beta_1$, for any $\epsilon > 0$ sufficiently small. But this again contradicts (66).
These contradictions give the proof of Corollary 27. □

3.6. **End of the proof of Theorem 6.** We distinguish two alternatives: either (52) is violated or it does hold.
If (52) is violated, take

$$u \in \nu_\sigma(r_0) - \{u^{(0)},\, u^{(1)}\}\,.$$

Then, by construction, $u \in \mathcal{B}_\sigma$, u is a solution of our variational equation (7), $u^{(0)} \le u \le u^{(1)}$ and u does not coincide with either $u^{(0)}$ or $u^{(1)}$. Thus, u fulfills the claim of Theorem 6.
If, on the other hand, (52) holds, then there exists

$$s \in [0,1] - (\beta_0 \cup \beta_1)\,,$$

thanks to Corollary 27. Therefore, by Corollary 12, there exists a sequence $t_n \to +\infty$ in such a way that $\Phi^{t_n}(u^{(s)})$ converges in the norm $\|\cdot\|_H$ to a suitable $u \in \mathcal{C}$, which does not coincide with either $u^{(0)}$ or $u^{(1)}$. Then, the fact that $u \in \mathcal{C}$ gives that $u \in \mathcal{B}_\sigma$, and that $u^{(0)} \le u \le u^{(1)}$. Also, Corollary 23 implies that u is a solution of our variational equation (7). Hence u fulfills the claim of Theorem 6 in this case too. □

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