PERTURBATIONS OF INTEGRABLE AND SUPERINTEGRABLE HAMILTONIAN SYSTEMS

Heinz Hanßmann
Mathematisch Instituut, Universiteit Utrecht
Postbus 80010, 3508 TA Utrecht, The Netherlands

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Abstract
Integrable systems admitting a sufficiently large symmetry group are considered. In the non–degenerate case this group is abelian and KAM theory ensures that most of the resulting Lagrangean tori persist under small non–integrable perturbations. For non–commutative symmetry groups the system is superintegrable, having additional integrals of motion that fibre Lagrangean tori into lower dimensional invariant tori. This simplifies the integrable dynamics, but renders the perturbation analysis more complicated. I review important cases where it is possible to find an "intermediate" integrable system that is non–degenerate and approximates the perturbed dynamics.

Key words
KAM theory, ramified torus bundle, perturbed rigid body, gyroscopic stabilization, proper degeneracy

1 Introduction

In Hamiltonian dynamics, integrable systems are rather the exception than the rule. Still, within this celebrated class of Hamiltonian systems one encounters a whole hierarchy of possibilities. An important aspect is always how the dynamics behave under non–integrable perturbations.

The typical or generic case (within the non–generic class of integrable Hamiltonian systems) is that of non–degenerate integrable systems ; examples in three degrees of freedom are easily constructed from a point mass in $\mathbb{R}^3$ moving in a separated potential. Almost all motion is quasi–periodic with three frequencies, in geometric language the motion is confined to invariant 3–tori in phase space. In action–angle variables $(x, y) \in \mathbb{T}^3 \times \mathbb{R}^3$ the Hamiltonian (locally) reads $H = H(y)$ and under e.g. the Kolmogorov non–degeneracy condition

$$\det D^2H(y) \quad \neq \quad 0 \tag{1}$$

most 3–tori survive a sufficiently small non–integrable perturbation.

In case there are more integrals of motion than degrees of freedom almost all motion is generically — now within this more restricted class of *superintegrable* systems — confined to invariant $(d-1)$–tori, where d denotes the number of degrees of freedom. Examples are given by a point mass in $\mathbb{R}^3$ moving in a rotationally symmetric potential $V = V(r)$. In generalized action–angle variables $(x, y, z) \in \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ the Hamiltonian only depends on y and one expects additional motion in the z–direction already under integrable perturbations. Correspondingly, the first step in studying non–integrable perturbations of (minimally) superintegrable systems is to construct an integrable approximation that *removes the degeneracy* by means of normalization with respect to the unperturbed flow. The resulting Lagrangean tori have $d-1$ frequencies of order 1 and one frequency of the order ε of the perturbation, and a further application of KAM theory yields such quasi–periodic motions also in the original perturbed system.

A maximally superintegrable system in d degrees of freedom (globally) admits $2d-1$ integrals of motion. An example is the (spatial) Kepler system of a point mass in $\mathbb{R}^3$ moving in the potential

$$V(r) \quad = \quad -\frac{1}{r} \quad ;$$

after regularization of the singularity all orbits are periodic. Note that from $d \geq 4$ on there is a whole hierarchy of superintegrable systems between the extreme cases of minimal and maximal superintegrability.

Integrable systems often admit a symmetry group. Indeed, by Noether's theorem every 1–parameter symmetry yields an integral of motion. To obtain d commuting first integrals in this way one needs a d–dimensional commutative symmetry group. For these integrals to be independent the group has to act effectively. More than d integrals in d degrees of freedom cannot all commute with each other and a corresponding symmetry group has to be non–commutative.

The dynamics of integrable Hamiltonian systems is particularly regular. To fix thoughts we concentrate on compact energy shells. Then all (but some exceptional) motion is quasi–periodic and hence confined to an invariant torus. These tori are the connected components of the level sets of the integrals of motion. In the non–degenerate case these are d–tori in the $2d$–dimensional phase space, and in the superintegrable case the $2d-r$ integrals of motion define invariant r–tori. The following result formulated in [Fassò, 2005] describes the general situation on the regular part $\mathcal{M}$ of the phase space $\mathcal{P}$ where the integrals are independent.

Theorem 1.1. *On the subset $\mathcal{M} \subseteq \mathcal{P}$ of the phase space with symplectic structure σ let $f : \mathcal{M} \longrightarrow \mathbb{R}^{2d-r}$ be a submersion with compact and connected fibres (hence, a fibration). Assume that $\{f_i, f_j\} = P_{ij} \circ f$, $i, j = 1, \ldots, 2d-r$ and that the matrix P with entries $P_{ij} : \mathcal{M} \longrightarrow \mathbb{R}$ has rank $2d-r$ at all points of $f(\mathcal{M})$. Then every fibre of f is diffeomorphic to $\mathbb{T}^r$ and the fibration f has local trivialisations which are symplectic.*

Thus, every fibre of f has a neighbourhood $\mathcal{U}$ with co–ordinates

$$(x, y, q, p) \; : \; \mathcal{U} \;\longrightarrow\; \mathbb{T}^r \times \mathbb{R}^r \times \mathbb{R}^{d-r} \times \mathbb{R}^{d-r} \qquad (2)$$

such that the level sets of f coincide with the level sets of (y, q, p) and

$$\sigma_{|\mathcal{U}} \;=\; \sum_{i=1}^{r} \mathrm{d}x_i \wedge \mathrm{d}y_i \;+\; \sum_{j=1}^{2d-r} \mathrm{d}q_j \wedge \mathrm{d}p_j \quad .$$

These co–ordinates are Nekhoroshev's generalized action–angle variables.

The aim of this paper is to specify how (super)integrability structures the phase space into invariant subsets and in how far this structure is preserved under small perturbations. In the next section the non–degenerate integrable case is treated. Section 3 starts with minimally superintegrable systems, where the perturbation analysis still goes through without non–generic assumptions. Then the extreme case of maximally superintegrable systems is considered, before exemplyfying the general hierarchy of superintegrable systems.

2 Non–degenerate integrable systems

The flow on the Lagrangean tori of a Liouville integrable system is conditionally periodic. Locally around such a torus the action angle variables (2) simplify to $(x, y) \in \mathbb{T}^d \times \mathbb{R}^d$ in which the symplectic structure becomes $\sigma = \mathrm{d}x \wedge \mathrm{d}y$ and the Hamiltonian function $H = H(y)$ does not depend on the angles. The equations of motion read

$$\begin{aligned} \dot{x} &= \omega(y) \;:=\; DH(y) \\ \dot{y} &= 0 \end{aligned}$$

and where the frequency vector ω is non–resonant the quasi–periodic flow on $\mathbb{T}^n$ is dense, excluding the existence of further integrals of motion. We speak of a *non–degenerate* Liouville integrable system if almost all Lagrangean tori have dense orbits. Sufficient conditions are the Kolmogorov non–degeneracy condition (1) for almost all y or iso–energetic non–degeneracy.

The Lagrangean tori form d–parameter families and the singular fibres of the ramified d–torus bundle determine how these families fit together. At the $(d-1)$–parameter families of elliptic $(d-1)$–tori the Lagrangean tori shrink down in the same way as periodic orbits shrink down to centres in one degree of freedom. Different families of Lagrangean tori are separated by $(d-1)$–parameter families of hyperbolic $(d-1)$–tori and their (un)stable manifolds.

This picture is repeated in how the $(d-1)$–tori shrink down to $(d-2)$–parameter families of (partially) elliptic $(d-2)$–tori and are separated by $(d-2)$–parameter families of (partially) hyperbolic $(d-2)$–tori and (part of) their (un)stable manifolds. Furthermore there are $(d-2)$–parameter families of hyperbolic $(d-2)$–tori with Floquet exponents $\pm\Re \pm \mathrm{i}\Im$, together with their (un)stable manifolds these form "pinched" d–tori. In these three ways we are led to invariant tori of smaller and smaller dimension until we end up with 1–parameter families of periodic orbits and isolated equilibria.

Within the family of all $(d-1)$–tori we encounter quasi–periodic centre–saddle and frequency halving bifurcations along $(d-2)$–parameter subfamilies and more generally bifurcations of co–dimension $k \leq d-1$ along $(d-k-1)$–parameter subfamilies. Similarly invariant $(d-2)$–tori undergoing a quasi–periodic Hamiltonian Hopf bifurcation form $(d-3)$–parameter families and the n–parameter families of invariant n–tori have $(n-k)$–parameter subfamilies where bifurcations of co–dimension $k \leq n$ occur. Such bifurcations are not restricted to those of semi–local type, but may also involve coinciding stable and unstable manifolds of different invariant tori. For instance, heteroclinic orbits between hyperbolic $(d-1)$–tori form $(2d-1)$–dimensional submanifolds of the phase space $\mathcal{P}$.

2.1 Perturbation analysis

To sum up, the dynamics of a non–degenerate integrable system makes the phase space $\mathcal{P}$ a ramified torus bundle. The regular fibres are the Lagrangean invariant tori, singular fibres are invariant tori of lower dimension, together with their stable and unstable manifolds. What happens to the ramified d–torus bundle under small perturbations of the Hamiltonian ? Let us collect the partial answers that are already known and indicate possible directions of future research.

Persistence of Lagrangean tori is addressed by classical KAM theory. Most tori survive a small perturbation if the Kolmogorov condition (1) is satisfied — near such y the relative measure of surviving tori tends to 1 as the perturbation strength tends to zero. These tori form a (Whitney)–smooth Cantor family, being parametrised over a Cantor set that has the local structure

$$\mathbb{R} \times \textit{Cantor dust} \quad .$$

Where the energy level sets are transversal to the continuous direction one has persistence of most Lagrangean tori on each energy shell, parametrised by Cantor dust. The same result is obtained under the condition of iso–energetic non–degeneracy, which is independent of Kolmogorov's condition. Note that it is generic for an integrable system to satisfy both conditions almost everywhere. However, in applications it is a non–trivial task to actually check this and to determine the hypersurfaces in action space where these determinants vanish.

The Cantor set structure is defined by Diophantine conditions

$$\bigwedge_{k \in \mathbb{Z}^d \setminus \{0\}} \quad |\langle k, \omega \rangle| \;\geq\; \frac{\gamma}{|k|^\tau} \quad . \tag{3}$$

and this can be used to weaken the necessary non–degeneracy condition. Indeed, since the gaps are defined by linear inequalities the conditions on the first derivatives of the frequency mapping $y \mapsto \omega(y) = DH(y)$ can be replaced by conditions on the curvature or even higher derivatives. Such Rüssmann–like conditions still guarantee that the relative measure of surviving tori tends to 1 as the perturbation strength tends to zero, but at a price. For instance, the highest derivative $L \in \mathbb{N}$ needed in

$$< \frac{\partial^{|\ell|}\omega}{\partial y} \;\mid\; |\ell| \leq L > \;=\; \mathbb{R}^d \tag{4}$$

enters the Diophantine conditions on the frequency vector by means of the inequality $\tau > dL - 1$ on the Diophantine constant τ . For more details the reader is referred to [Broer, Huitema and Sevryuk, 1996; Rüssmann, 2001] and references therein.

For hyperbolic n–tori the above criteria remain valid almost verbatim ; the key step is to pass to a centre manifold (and to replace d by n in the formulas). A technical difficulty is that even for analytic Hamiltonians centre manifolds may only be of finite differentiability. KAM–theorems remain true in this context, during the proof one has to intersperse an analytic approximation at each iteration step. Still, the analytic context has its advantages — for instance (4) is satisfied for some $L \in \mathbb{N}$ for an analytic frequency mapping ω if and only if $\mathrm{im}\,\omega$ does not lie within a linear hyperplane. An alternative is therefore to prove persistence of hyperbolic tori directly, see [Rüssmann, 2001; Rudnev, 2003] and references therein. This also gives a more direct hold on their stable and unstable manifolds.

Elliptic $(d-1)$–tori need one extra parameter to control the normal frequency as well. Similar to the iso–energetic case one can use time re–parametrisation and obtain Cantor families of persistent elliptic $(d-1)$–tori parametrised by Cantor dust without the use of an external parameter. Where there are more than one normal frequency to control this can no longer be done in a linear way ; a problem solved by Rüssmann–like conditions on the higher derivatives of the frequency vector, see [Broer, Huitema and Sevryuk, 1996; Rüssmann, 2001] and references therein. In case the mapping of internal frequencies satisfies Kolmogorov's condition, the higher order derivatives are only needed of normal frequencies. Now normal frequencies α_j enter the Diophantine conditions

$$|2\pi\langle k, \omega\rangle \;+\; \langle \ell, \alpha\rangle| \;\geq\; \frac{\gamma}{|k|^\tau} \tag{5}$$

only as combinations $\langle \ell, \alpha \rangle$ with $|\ell| \leq 2$. This allows to extend the result to finite–dimensional elliptic tori in infinitely many degrees of freedom, cf. [Pöschel, 1989; Kuksin, 1993]. For hypo–elliptic tori one may deal with the hyperbolic part by means of a centre manifold or use a direct approach, cf. [Huitema, 1988; Broer, Huitema and Takens, 1990; Rüssmann, 2001].

Where (lower–dimensional) n–tori undergo a semi–local bifurcation the n actions y conjugate to the toral angles x first of all have to versally unfold the bifurcation scenario. It is generic for the integrable Hamiltonian H that the n–parameter families of n–tori, $1 \leq n \leq d-1$, do not encounter bifurcations of co–dimension higher than n , so this is possible. The curvature of the frequency mapping is then used to ensure Diophanticity of most bifurcating tori, i.e. a Rüssmann–like condition with $L = 2$ is sufficient, cf. [Broer, Hanßmann and You, 2003; 2004; Hanßmann, 2003; 2004].

While the proof in the above papers is kept as simple as possible, restricting to $n = d-1$, it should be feasible to include additional elliptic and hyperbolic normal directions. On the other hand, additional violations of (5) pose a much harder problem, as in this situation even the corresponding bifurcations of equilibria have yet to be understood. Thus, if we explicitly require that the bifurcation results from violating (5) with a single normal–internal resonance, the quasi–periodic bifurcation scenario should persist for all n–tori with $2 \leq n \leq d-1$ and in fact also in infinite–dimensional Hamiltonian systems. Recall that the maximal co–dimension of occurring bifurcations is the dimension n of the bifurcating torus and not related to the number of degrees of freedom. For instance, the above curvature requirement is not necessary for 2–tori ; these may undergo the quasi–periodic analogues of the co–dimension one bifurcations of periodic orbits detailed in [Meyer, 1970; 1975]. Indeed, co–dimension two bifurcations are isolated within these 2–parameter families and cannot be prevented to disappear in resonance gaps.

Let an $(n-k)$–parameter family of n–tori that undergo a bifurcation of co–dimension k have m additional pairs of purely imaginary Floquet exponents. Then excitation of normal modes, cf. [Jorba and Villanueva, 1997; Sevryuk, 1997], leads for $l = 1, \ldots, m$ to $(n+l-k)$–parameter families of $(n+l)$–tori undergoing that co–dimension k bifurcation in the integrable system. This whole structure should persist under a (sufficiently small) non–integrable perturbation on pertinent Cantor sets. Additional hyperbolic directions augment the dimension of stable and unstable manifolds.

Up to now the reported changes of the ramified d–torus bundle under a small perturbation of the Hamiltonian were of the form *"Diophantine tori persist"* leading to a "Cantorification" of the ramified d–torus bundle — the stratification of the action space into various subfamilies parametrising the tori is replaced by a Cantor stratification. Of equal importance are those changes that make sure that the non–integrable perturbed dynamics is indeed qualitatively different from the integrable unperturbed dynamics. While the former persistence results are obtained upon genericity conditions on the unperturbed system, such changes require the perturbation to be generic.

One of the effects of a small generic perturbation is that stable and unstable manifolds of hyperbolic tori no longer coincide, but split and intersect transversely, cf. [Robinson, 1970a; 1970b; Delshams, de la Llave and Seara, 2003a; 2003b]. Where this concerns heteroclinic orbits between two different families of hyperbolic tori this leads to drastic changes of the connection bifurcation scenario. Indeed, heteroclinic orbits exist in the integrable system only at $\mu = 0$ for an appropriately chosen transversal parameter μ . For a sufficiently small generic perturbation there is a whole interval of μ–values containing a Cantor subset of relative measure near 1 for which there are heteroclinic orbits between surviving hyperbolic tori. Similar observations apply to stable and unstable manifolds of parabolic and other bifurcating tori.

Completely new phenomena are also to be expected in the gaps of the Cantor sets parametrising persistent tori. Disintegrating Lagrangean tori lead to invariant n–tori, where $d-n$ is the number of independent resonances $\langle k, \omega \rangle = 0$ of the (internal) frequencies. Most of these lower dimensional tori will be elliptic or hyperbolic, cf. [Treshchëv, 1991]. The new hyperbolic tori lie at the basis of the example in [Arnol'd, 1964] of dynamical instablility. This approach to Arnol'd diffusion relies on the splitting of separatrices which also leads to transversal intersections of stable and unstable manifolds of neighbouring hyperbolic tori in the same energy shell. These hyperbolic tori form a Cantor family, and one of the main problems is to make sure that the transition chain of hyperbolic tori and their heteroclinic connections bridges the occuring gaps, cf. [Delshams, de la Llave and Seara, 2003a; 2003b] and references therein.

The dynamics in the gaps of Cantor families of hyperbolic tori can already be studied in the perturbation near resonant singular fibres of the ramified d–torus bundle. On the centre manifold these become again (resonant) regular fibres, but the full perturbed motion is superposed by the hyperbolic dynamics in the symplectic normal directions. In particular, secondary hyperbolic tori — maximal tori on the centre manifold that appear in the resonance gap — are used in [Delshams, de la Llave and Seara, 2003a] together with hyperbolic tori of even lower dimension to continuate a transition chain through the resonance gap.

A Lagrangean torus with $d-1$ independent resonances consists of periodic orbits. When the torus breaks up under the perturbation, only finitely many of these are expected to survive. At the same time the trivial normal behaviour of these periodic orbits changes, resulting in hyperbolic and elliptic periodic orbits. The latter can serve as starting points for the construction of solenoids, cf. [Markus and Meyer, 1980]. This construction should carry over to elliptic tori, where the "encircling" tori emerge from the normal–internal resonances studied in [Broer, Hanßmann, Jorba, Villanueva and Wagener, 2003]. This might also result in solenoids that are limits of tori with varying dimension.

The nature of the gaps where (5) is not satisfied for elliptic tori is twofold. Internal resonances $\langle k, \omega \rangle = 0$ lead again to the destruction of the torus. The study [Broer, Hanßmann, Jorba, Villanueva and Wagener, 2003] of normal–internal resonances relates boundary points of the resulting gaps to quasi–periodic bifurcations. In particular resonance gaps

$$| \; 2\pi \langle k, \omega \rangle \; + \; 2\alpha \; | \; < \; \frac{\gamma}{|k|^{\tau}}$$

are completely filled by hyperbolic tori (in accordance with [Bourgain, 1994; 1997; Xu and You, 2001]) that terminate in frequency halving bifurcations. One may speculate that resonance gaps

$$| \; 2\pi \langle k, \omega \rangle \; + \; \alpha_1 + \alpha_2 \; | \; < \; \frac{\gamma}{|k|^{\tau}}$$

are similarly filled by hyperbolic tori obtained in quasi–periodic Hamiltonian Hopf bifurcations generated by the perturbation.

The results in [Broer, Hanßmann and You, 2003; 2004; Hanßmann, 2003; 2004] address persistence of Diophantine tori involved in a bifurcation and the corresponding gaps trigger again new phenomena. A first step has been made in [Litvak–Hinenzon, 2001; Litvak–Hinenzon and Rom–Kedar, 2002a; 2002b; 2004] where (internally) resonant parabolic tori involved in a quasi–periodic Hamiltonian pitchfork bifurcation are considered. This may result in large dynamical instabilities, especially where multiple parabolic resonances are encountered. The effect is further amplified for tangent (or flat) parabolic resonances, which fail to satisfy the iso–energetic non–degeneracy condition.

2.2 The Lagrange top

The rigid body with a fixed point is a mechanical system with three degrees of freedom, the phase space $\mathcal{P} = T^*SO(3) \cong SO(3) \times \mathbb{R}^3$ being the cotangent bundle of the group $SO(3)$ of three–dimensional rotations g . An example of a non–degenerate integrable system on $\mathcal{P}$ is the Lagrange top, an axially symmetric rigid body subject to a constant vertical force field, cf. [Cushman and Bates, 1997]. Next to the energy

$$H(\rho, \ell) \;=\; I_1 \frac{\ell_1^2 + \ell_2^2}{2} \;+\; I_3 \frac{\ell_3^2}{2} \;+\; \chi g_{33}$$

both the component ℓ_3 of the angular momentum along the figure axis and the component

$$\mu_3 \;=\; g_{31}\ell_1 \;+\; g_{32}\ell_2 \;+\; g_{33}\ell_3$$

of the angular momentum along the vertical axis are (commuting) integrals of motion. These two integrals generate the rotations about the figure axis and the vertical axis, respectively. When the top is standing upright or hanging upside down these two S^1–actions coincide and correspondingly the motion is periodic[1] and consists of rotation about that common axis. In case X_H lies within the plane spanned by X_{μ_3} and X_{ℓ_3} the motion is called regular precession — a superposition of the rotation about the figure axis and the precession of the figure axis about the vertical axis — and takes place on a 2–torus. For regular values of the energy–momentum mapping $\mathcal{EM} = (H, \ell_3, \mu_3)$ we obtain the Lagrangean 3–tori as the figure axis starts to nutate up and down as well.

To complete this description of the ramified torus bundle note that unstable rotations about the upright standing figure axis are accompanied by asymptotic motions forming the (un)stable manifold, this turns the level set of $\mathcal{EM}$ into a pinched 3–torus. When the magnitude of $\ell_3 = \mu_3$ increases these periodic orbits get gyroscopically stabilized through a periodic Hamiltonian Hopf bifurcation. Corresponding to the two rotation senses two such bifurcations take place at $\ell_3 = \mu_3 = \pm 2\sqrt{I_1 \chi}$.

[1] There is a whole S^1 of equilibria when furthermore μ_3 and ℓ_3 vanish.

While the periodic orbits survive a small perturbation by means of the implicit mapping theorem, the two bifurcations serve as organizing centres for the Cantorification of the family of invariant 2–tori, see [Pacha, 2002]. Furthermore, the monodromy around the pinched 3–tori ensures that the Kolmogorov condition (1) is satisfied almost everywhere.

Let us discuss two procedures to generalize this result to more degrees of freedom. A weak coupling with a quasi–peirodic oscillator of n frequencies is considered in [Hoo, 2005; Broer, Hanßmann and Hoo, 2004]. This turns the Lagrangean tori into $(n + 3)$–tori, the elliptic tori into $(n + 2)$–tori and the periodic orbits into $(n + 1)$–tori. Again the quasi–periodic Hamiltonian Hopf bifurcations serve as organizing centres for the Cantorification of the ramified torus bundle, see [Hoo, 2005; Broer, Hanßmann and Hoo, 2004]. In particular the $(n + 1)$–parameter families of elliptic and hyperbolic tori persist on Cantor sets, as does the n–parameter family of $(n + 1)$–tori in $1{:}{-1}$ resonance.

One may also weakly couple two (or even more) Lagrange tops. Where 3–periodic motion of one body is superposed with the ramified torus bundle defined by the other body this resembles the weak coupling with a 3–dimensional oscillator and the results of [Hoo, 2005; Broer, Hanßmann and Hoo, 2004] still apply. Superposition with elliptic 2–periodic motion yields 5–tori, 4–tori and 3–tori. Persistence of the elliptic 5–tori and 4–tori follows from [Huitema, 1988; Pöschel, 1989; Broer, Huitema and Takens, 1990; Rüssmann, 2001], and the same holds true for the elliptic and hyperbolic 3–tori. To prove persistence of a normally elliptic quasi–periodic Hamiltonian Hopf bifurcation one would have to drag the Diophantine conditions (5) through the computations in [Hoo, 2005; Broer, Hanßmann and Hoo, 2004].

The above applies mutatis mutandi for the superposition with elliptic or hyperbolic rotations about the vertical figure axis. The coupling of two periodic Hamiltonian Hopf bifurcations is a much more difficult problem. Still, one may take [Hoo, 2005; Broer, Hoo and Naudot, 2004] as a starting point where persistence of the resulting 2–tori in normal $1{:}1{:}{-1}{:}{-1}$ resonance has been proven.

3 Perturbations of superintegrable systems

We speak of a superintegrable system if the regular fibres of the ramified torus bundle are isotropic tori of dimension $< d \, (= \frac{1}{2} \dim \mathcal{P})$. Determined by the dimension of these "maximal" tori this defines a whole hierarchy, starting with the minimally superintegrable systems where the regular fibres are $(d-1)$–tori (and almost all of them have dense quasi–periodic orbits) up to maximally superintegrable systems where almost all orbits are periodic. In the case of $d = 2$ degrees of freedom all these notions coincide. According to Theorem 1.1 the Hamiltonian of a superintegrable system only depends on the r actions conjugate to the toral angles, so the $2(d-r)$ "extra integrals" are mute parameters and a family of n–tori still encounters only bifurcations up to co–dimension n — although these are no longer isolated but form $2(d-r)$–parameter families.

Again we want to know what happens to the ramified r–torus bundle under small perturbations of the Hamiltonian. The strategy is to find an "intermediate" system that is also integrable, but non–degenerately so.

Definition 3.1. *The perturbation εP of a superintegrable Hamiltonian N removes the degeneracy if the perturbed Hamiltonian $H = N + \varepsilon P$ can be written in the form*

$$H \;=\; N \;+\; \varepsilon S \;+\; \varepsilon^2 R$$

where $N + \varepsilon S$ is a non–degenerate integrable Hamiltonian.

Let (x, q, y, p) complete the action angle variables (x, y) of $N = N(y)$ to action angle variables of $N(y) + \varepsilon S(y, p)$. If N satisfies "its" Kolmogorov condition $\det D^2N(y) \neq 0$ for almost all y then

$$\det \begin{pmatrix} \frac{\partial^2 S}{\partial p_1^2} & \cdots & \frac{\partial^2 S}{\partial p_1 \partial p_{d-r}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 S}{\partial p_1 \partial p_{d-r}} & \cdots & \frac{\partial^2 S}{\partial p_{d-r}^2} \end{pmatrix} \;\neq\; 0$$

ensures that the integrable Hamiltonian $N + \varepsilon S$ is non–degenerate. Under this condition most invariant tori

$$\mathbb{T}^r \;\times\; \mathbb{T}^{d-r} \;\times\; \{(y, p)\}$$

persist under the perturbation of the intermediate system by $\varepsilon^2 R(x, q, y, p)$, cf. [Arnol'd, 1963b]. If N is iso–energetically non–degenerate in y then this holds true on every energy shell.

3.1 Minimally superintegrable systems

One way to put $\varepsilon P(x, q, y, p)$ into the form $\varepsilon S(y, p) + \varepsilon^2 R(x, q, y, p)$ is to compute a normal form of εP with respect to N . This results in an intermediate Hamiltonian $\bar{H} = N + \varepsilon \bar{P}$ where $\bar{P}$ is the average of P along the fibres of the ramified torus bundle defined by N . On the regular part of this bundle this defines a $\mathbb{T}^r$–symmetry and regular reduction makes $\bar{P}$ a Hamiltonian in $d - r$ degrees of freedom.

In the minimally superintegrable case $r = d - 1$ this is a one–degree–of–freedom system and always integrable. Furthermore, it is generic for $\bar{P}$ to have non–trivial dynamics in one degree of freedom, so P removes the degeneracy with $S = \bar{P}$. The remainder term $\varepsilon^2 R$ is given by

$$R \;=\; \frac{1}{\varepsilon}\left(P \circ \Psi \;-\; \bar{P}\right)$$

where Ψ is the normalizing transformation. Note that the dynamics defined by N is fast with respect to the ε–slow one–degree–of–freedom dynamics defined by $\varepsilon \bar{P}$.

From the ramified $(d-1)$–torus bundle and the one–degree–of–freedom dynamics we now construct the ramified d–torus bundle defined by the intermediate system. The Lagrangean tori consist of the regular $(d-1)$–tori superposed with the slow periodic one–degree-of–freedom dynamics. To obtain singular fibres we can proceed in two different ways.

One the one hand, the (relative) equilibria of the one–degree–of–freedom system lead to singular fibres. This is already true for the $(d-1)$–tori of the "fast" ramified torus bundle and even more so for its singular fibres. A further hierarchical structure is imposed by the co–dimensions of the various equilibria of the $(d-1)$–parameter family $S_y = \bar{P}(y, ..)$, starting at saddles and centres of co–dimension zero and generically ranging to bifurcations up to co–dimension $d - 1$.

On the other hand, superposing singular fibres of the (fast) ramified $(d-1)$–torus bundle with the slow periodic one–degree–of–freedom dynamics leads to singular fibres of the ramified d–torus bundle as well. In this way the intermediate system has four kinds of motion :

- (i) The regular fibres correspond to conditionally periodic motions with $d - 1$ fast frequencies and one slow frequency.
- (ii) Singular fibres with periodic slow motion correspond on resulting $(n+1)$–tori, $0 \leq n \leq d-2$, to conditionally periodic motions with n fast and one slow frequencies. The symplectic normal behaviour is fast as well, this is in particular true for the asymptotic motion on existing (un)stable manifolds.
- (iii) Singular fibres constructed from regular $(d-1)$–tori have fast conditionally periodic motion and slow symplectic normal behaviour. In particular, the motion on existing (un)stable manifolds combines a fast rotational motion with a slow approximation of the invariant $(d-1)$–torus.
- (iv) The superposition of one–degree–of–freedom equilibria and singular fast fibres leads to conditionally periodic motion with n fast frequencies, $0 \leq n \leq d-2$, while the symplectic normal behaviour is a combination of $d - n - 1$ fast degrees of freedom and one slow degree of freedom.

It remains to understand what happens to the ramified d–torus bundle defined by $\bar{H} = N + \varepsilon \bar{P}$ under perturbation by $\varepsilon^2 R$. For the regular fibres (i) the result in [Arnol'd, 1963b] yields persistence of a Cantor family of Lagrangean tori. The proof relies on initial normalizing transformations, using the ultraviolet cut–off introduced in [Arnol'd, 1963a]. The lower–dimensional tori (ii) also have the slow dynamics encoded in one of the internal frequencies, the symplectic normal behaviour is of the same magnitude as the fast frequencies. This should allow to obtain their persistence along the same lines.

For the $(d-1)$–tori (iii) the two time scales distinguish the internal from the normal dynamics. In the hyperbolic case the normal hyperbolicity of the centre manifold is of order ε and thus sufficiently large with respect to the perturbation strength ε^2 to yield persistence. During local bifurcations in the slow dynamics one has the alternative between a scaling argument [Broer, Hanßmann and You, 2003] and a direct incorporation in the KAM–iteration [Hanßmann, 1998]. In the elliptic case (treated in [Lieberman, 1971; 1972]) the Diophantine conditions again involve both the ε–small normal and the "large" internal frequencies.

The fast–slow dynamics is contained in the symplectic normal behaviour for lower–dimensional tori (iv). Again the hypo–elliptic case does not pose new problems. When incorporating additional elliptic and hyperbolic directions into the KAM iteration one should furthermore be able to let one of them be slow — where the bifurcation scenario is developing in the slow dynamics the above alternatives still apply. More interesting is the combination of two bifurcations in both the fast and the slow (symplectic normal) dynamics. Indeed, with two different time scales e.g. the dynamics triggered by two simultaneous violations of (5) appears to be of $(1+1)$–degree–of–freedom rather than having truly 2 degrees of freedom. This might help to obtain more detailed results.

3.2 Systems in three degrees of freedom

For superintegrable systems with regular r–tori with $r \leq d-2$ it is no longer automatic for the average $\varepsilon\bar{P}$ of the perturbation εP to reduce to an r–parameter family of integrable systems, the remaining number of degrees of freedom being $d-r \geq 2$. It seems therefore unlikely that a given perturbation removes the degeneracy, but see [Arnol'd, 1963b] for a treatment of the planetary system as a perturbation of the superintegrable superposition of 9 Keplerian systems that does remove the degeneracy. The ensuing problems can already be illustrated in three degrees of freedom.

The Euler–Poinsot system is a "free" rigid body not subject to any external force or torque. This makes the spatial components μ_1, μ_2, μ_3 of the angular momentum three non–commuting integrals of motion next to the (kinetic) energy

$$N \;=\; \frac{\ell_1}{2I_1} + \frac{\ell_2}{2I_2} + \frac{\ell_3}{2I_3}$$

and replacing one of them by the sum $\mu^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$ of their squares yields a second integral (next to the energy) that commutes with all other integrals. In this way the phase space $\mathcal{P}$ becomes a ramified 2–torus bundle with a complicated singular set at $\mu^2 = 0$. Redefining

$$\mathcal{P} \;=\; T^*SO(3)^{\setminus SO(3)}$$

by taking out the zero section (where no dynamics takes place, $SO(3)$ consists of equilibria) simplifies this situation and also allows to replace μ^2 by $|\mu| = \sqrt{\mu^2}$.

In the dynamically symmetric case $I_1 = I_2$ of two equal moments of inertia the conditionally periodic motion along the regular 2–tori becomes particularly transparent. Indeed, for such an Euler top the precession of the figure axis about the angular momentum is superposed by a rotation of the body about the figure axis. At $\ell_3 = 0$ the 2–tori become (internally) 1:0 resonant as the precession consists of the body rotating about any axis perpendicular to the figure axis. Note that the Hamiltonian

$$N \;=\; \frac{|\mu|^2}{2I_1} - \frac{I_3 - I_1}{I_1 I_3}\frac{\ell_3^2}{2}$$

can be expressed as a function of $|\mu|$ and ℓ_3 whence the additional integral ℓ_3 does not lead to topological changes of the regular fibres of the ramified 2–torus bundle. This will change below when we replace the additional S^1–symmetry by an additional $SO(3)$–symmetry.

For the general free rigid body with three different moments of inertia the above 1:0 resonant 2–tori break up and the (un)stable manifolds of the rotation about the "middle" axis of inertia separate four families of regular 2–tori. Depending on which family a 2–torus belongs to, the rôle of the figure axis is played by the "longest" or the "shortest" axis of inertia, which still precesses regularly about the direction of the angular momentum. However, the rotational motion of the body about this figure axis looks more complicated as the sines and cosines of the dynamically symmetric case have to be replaced by elliptic functions. The free rigid body is a minimally superintegrable system.

The torque exerted by a perturbing external force field causes the angular momentum to slowly move in space. For the intermediate system this motion is periodic and superposed to the fast precessional–rotational motion of the free rigid body. In [Mazzocco, 1997] persistence of the resulting 3–tori is explicitly proven. The structure of the ramified 3–torus bundle defined by the intermediate system depends on the precise form of (the average of) the perturbation. The case of an affine[2] force field is detailed in [Hanßmann, 1995; 1997] for the dynamically symmetric case.

A free rigid body with three equal moments $I_1 = I_2 = I_3$ of inertia has only periodic motions (we still exclude the zero section $SO(3)$ from the phase space) since every axis through the fixed point is a principal axis of

[2]The linear part is needed to break the rotational symmetry of the constant part of the force field.

inertia. Correspondingly, one has five independent integrals of motion by choosing next to the energy

$$N = \frac{|\mu|^2}{2I_1}$$

two of the three components ℓ_1, ℓ_2, ℓ_3 of the angular momentum about a body set of axes and two out of the μ_1, μ_2, μ_3 in a spatial frame. The free rigid body with trivial tensor of inertia is a maximally superintegrable system.

The effect of the torque of the average of a perturbing external force field is now that the direction of the angular momentum moves both in the spatial and the body frame. Fixing $|\mu|$, regular reduction of the S^1–symmetry generated by $|\mu|$ yields a two–degree–of–freedom system on $S^2_{|\mu|} \times S^2_{|\mu|}$. This system may of course be integrable, e.g. because the force field is S^1–symmetric. Note that the external force field has to "detect" the asymetries of the rigid body, whence an affine force field is no longer sufficiently general, leading to an S^1–symmetric system. But already a generic quadratic force field has an average that cannot be used to remove the degeneracy.

An interesting phenomenon appears in some applications involving the Kepler system. Indeed, the regularized spatial Kepler problem is a maximally superintegrable three–degree–of–freedom system with Hamiltonian $N(K) = K$ and has a "first" normal form

$$\bar{H} = K + \varepsilon S(K, L)$$

for the lunar problem, the Rydberg (or hydrogen) atom in crossed fields and the problem of orbiting dust, cf. [van der Meer and Cushman, 1986; 1987; Cushman, 1992; Cushman and Sadovskiĭ, 2000; Sommer, 2003; Efstathiou, 2004]. Here L is the third component of the angular momentum and in the two former cases $S(K, L)$ is a multiple of $K \cdot L$, while it can be brought into this form by an additional transformation in the latter case as well. A second normalization yields

$$\overline{\overline{H}} = K + \varepsilon S(K, L) + \varepsilon^2 T(K, L, I)$$

with an appropriately chosen third action I .

In all these cases the conditionally periodic motion defined by $\overline{\overline{H}}$ has three time scales, while the rates of change of these frequencies are only of the two orders ε and ε^2 of magnitude. This is not a coincidence since for a maximally superintegrable system with nowhere vanishing periodic flow the first action can always be chosen to be the unperturbed Hamiltonian. While a perturbation

$$H_\varepsilon(J, \phi) = J_1 + \varepsilon S(J_1, J_2) + \varepsilon^2 T(J, \varepsilon) + \textit{h.o.t.}$$

does not fulfil Definition 3.1, it could nonetheless be shown in [Sommer, 2003] that most regular fibres of the ramified 3–torus bundle defined by $\overline{\overline{H}}$ persists as a Cantor family, provided that appropriate non–degeneracy conditions hold true.

3.3 The hierarchy of superintegrable systems

One still obtains a minimally superintegrable system when coupling an Euler top with one or several Lagrange tops. Note that the Euler–Poinsot system can be realized even in the presence of constant gravity by letting the fixed point coincide with the centre of mass. Similarly, the weak coupling of m tops appropriately chosen among the Lagrange top, the Euler top and the dynamically spherically symmetric Euler top yields a superintegrable system in $d = 3m$ degrees of freedom with the dimension r of the regular fibres of the resulting ramified torus bundle satisfying $m \leq r \leq 3m$.

A class of examples where the dimension r may assume any number between the number of degrees of freedom $r = d$ and the maximally superintegrable case $r = 1$ is given by the C.Neumann system, cf. [Dullin and Hanßmann, 2005]. A point moves on a sphere S^d under the influence of a linear force field. Only the differences between the coefficients of the force field have dynamical consequences. In particular, when all coefficients are equal to each other the C.Neumann system becomes the geodesic flow, a maximally superintegrable system with symmetry group $SO(d + 1)$. A subgroup of $SO(d + 1)$ of product form

$$SO(m_0) \times \cdots \times SO(m_s)$$

is the symmetry group of the degenerate C.Neumann system with $s+1$ groups of m_ς equal coefficients. Note that $m_\varsigma = 2$ equal coefficients do not yet lead to superintegrability. Indeed, the angular momentum defining the corresponding $SO(2)$–action merely replaces one of the Uhlenbeck integrals, see [Dullin and Hanßmann, 2005] for more details. For $m_\varsigma \geq 3$ the factor $SO(m_\varsigma)$ is non–commutative and does lead to superintegrability. Thus, the degenerate C.Neumann system is minimally superintegrable if and only if there is one group of three equal coefficients and all other coefficients equal at most one more coefficient.

As formulated in Theorem 1.1, the regular part $\mathcal{M}$ of the ramified torus bundle determined by a superintegrable system is a fibration by isotropic invariant tori $\mathbb{T}^r$. Since the Poisson bracket of two first integrals $\{f_i, f_j\} = P_{ij} \circ f$ is again a first integral this isotropic fibration admits a polar foliation which is co–isotropic. In the local co–ordinates (2) the co–isotropic leaves are co–ordinized by (x, q, p) while y is fixed. Where superintegrability is coming from a non–commutative symetry group the quotient of a co–isotropic leaf by $\mathbb{T}^r$ is the co–adjoint orbit, with local co–ordinates (q, p) . On the other hand, for a non–degenerate integrable system both the isotropic fibration and the co–isotropic foliation coincide with the fibration by Lagrangean tori.

A superintegrable Hamiltonian N depends only on y and in case the co–isotropic foliation is in fact a fibration $c : \mathcal{M} \longrightarrow \mathcal{A}$ — as holds true in all examples in the present paper — one may write $N \circ c$ for this Hamiltonian. Similarly, a normal form $\bar{P}$ of the perturbation P with respect to N may be written as $\bar{P} \circ i$ where $i : \mathcal{M} \longrightarrow \mathcal{B}$ denotes the isotropic fibration. We have already seen that $N \circ c + \varepsilon \bar{P} \circ i$ is integrable and may serve as an intermediate system for minimally superintegrable N.

But even when not integrable (and thus not helpful for KAM–like results) normal forms $N \circ c + \varepsilon \bar{P} \circ i$ show that the perturbed motion has three time scales. Next to the fast motion

$$\dot{x} \;=\; \frac{\partial N}{\partial y} \;+\; \mathcal{O}(\varepsilon)$$

there is an ε–slow motion in q and p . Moreover, the motion in y can be made very slow by considering a high order of the normal form. Along these lines one may obtain Nekhoroshev–like results, for more details see [Fassò, 2005] and references therein.

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