GLOBAL STRICHARTZ ESTIMATES FOR THE WAVE EQUATION WITH TIME-PERIODIC POTENTIALS

VESSELIN PETKOV

ABSTRACT. We obtain global Strichartz estimates for the solutions u of the wave equation $(\partial_t^2 - \Delta_x + V(t,x))u = F(t,x)$ for time-periodic potentials $V(t,x)$ with compact support with respect to x. Our analysis is based on the analytic properties of the cut-off resolvent $R_\chi(z) = \chi(U(T) - zI)^{-1}\psi_1$, where $U(T) = U(T,0)$ is the monodromy operator and $T > 0$ the period of $V(t,x)$. We show that if $R_\chi(z)$ has no poles $z \in \mathbb{C}$, $|z| \geq 1$, then for $n \geq 3$, odd, we have a exponential decal of local energy. For $n \geq 2$, even, we obtain also an uniform decay of local energy assuming that $R_\chi(z)$ has no poles $z \in \mathbb{C}$, $|z| \geq 1$, and $R_\chi(z)$ remains bounded for z in a small neighborhood of 0.

Keywords: Strichartz estimates, decay of energy, monodromy operator

1. INTRODUCTION

Consider the Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta_x u + V(t,x)u = F(t,x),\ (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(\tau, x) = f_0(x),\ u_t(\tau,x) = f_1(x),\ x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the potential $V(t,x) \in C^\infty(\mathbb{R}^{n+1}), n \geq 2$, satisfies the conditions:

(H_1) there exists $R_0 > 0$ such that $V(t,x) = 0$ for $|x| \geq R_0$, $\forall t \in \mathbb{R}$,

(H_2) $V(t+T, x) = V(t,x)$, $\forall (t,x) \in \mathbb{R}^{n+1}$ with $T > 0$.

Consider the homogeneous Sobolev spaces $\dot{H}^\gamma(\mathbb{R}^n) = \Lambda^{-\gamma}L^2(\mathbb{R}^n)$, where $\Lambda = \sqrt{-\Delta}$ and $-\Delta$ is the Laplacian in $\mathbb{R}^n$. Set $\dot{\mathcal{H}}_\gamma(\mathbb{R}^n) = \dot{H}^\gamma(\mathbb{R}^n) \oplus \dot{H}^{\gamma-1}(\mathbb{R}^n)$ and notice that for $\gamma < n/2$ the multiplication with smooth functions $\varphi \in C_0^\infty(\mathbb{R}^n)$ is continuous from $\dot{H}^\gamma(\mathbb{R}^n)$ to $H^\gamma(\mathbb{R}^n)$ and for functions with compact support the norms in $\dot{H}^\gamma(\mathbb{R}^n)$ and $H^\gamma(\mathbb{R}^n)$ are equivalent. The solution of (1.1) with $F = 0$ is given by the propagator

$$U(t,\tau) : \dot{\mathcal{H}}_\gamma(\mathbb{R}^n) \ni (f_0,\ f_1) \longrightarrow U(t,\tau)(f_0, f_1) = (u(t,x),\ u_t(t,x)) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$$

and we refer to [12], Chapter V, for the properties of $U(t,\tau)$. Let $U_0(t) = e^{itG_0}$ be the unitary group in $\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$ related to the Cauchy problem (1.1) with $V = 0$, $\tau = 0$ and let $U(T) = U(T,0)$. We have the representation

$$U(t,\tau)f = U_0(t-\tau)f - \int_\tau^t U(t,s)Q(s)U_0(s-\tau)f ds,$$

where

$$Q(s) = \begin{pmatrix} 0 & 0 \\ V(s,x) & 0 \end{pmatrix}.$$

By interpolation it is easy to see that

$$\|U(t,\tau)\|_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)\to\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)} \leq C_\gamma e^{\kappa_\gamma|t-\tau|},\ \kappa_\gamma \geq 0, \tag{1.2}$$

where κ_γ is bounded if γ runs in a compact interval. We say that the real numbers $1 \leq \tilde{p}, \tilde{q} \leq 2 \leq p, q \leq +\infty$, $0 \leq \gamma \leq 1$, are **admissible** for the free wave equation (see [11], [16], [3]) if the following estimate holds:

Global Minkovski Strichartz estimate. For data $(f_0, f_1) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$, $F \in L_t^{\tilde{p}}(\mathbb{R}; L_x^{\tilde{q}}(\mathbb{R}^n))$ and $u(t,x)$ solution of (1.1) with $\tau = 0$, $V = 0$ we have

$$\begin{aligned}&\|u\|_{L_t^p(\mathbb{R};\, L_x^q(\mathbb{R}^n))} + \|u(t,x)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t u(t,x)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}\\ &\leq C_0\Big(\|f_0\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|f_1\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} + \|F\|_{L_t^{\tilde{p}}(\mathbb{R};\, L_x^{\tilde{q}}(\mathbb{R}^n))}\Big)\end{aligned} \tag{1.3}$$

with a constant $C_0 = C_0(n, p, q, \tilde{p}, \tilde{q}, \gamma) > 0$ independent of $t \in \mathbb{R}$.

We refer to [7], [10], [11], [16] and to the references given there for global Strichartz estimates for the free wave equation. Notice that if q, $\tilde{q}' < \frac{2(n-1)}{n-3}$, then $p, q, \tilde{p}, \tilde{q}$, γ are admissible if the following conditions hold:

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma = \frac{1}{\tilde{p}} + \frac{n}{\tilde{q}} - 2, \tag{1.4}$$

$$\frac{1}{p} \leq \Big(\frac{n-1}{2}\Big)\Big(\frac{1}{2} - \frac{1}{q}\Big),\ \frac{1}{\tilde{p}'} \leq \Big(\frac{n-1}{2}\Big)\Big(\frac{1}{2} - \frac{1}{\tilde{q}'}\Big), \tag{1.5}$$

where $\frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1$, $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$. From the gap condition (1.4) and the admissibility conditions (1.5), we deduce

$$\frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) \leq \gamma \leq 1 - \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{\tilde{q}'}\Big).$$

In this paper we deal with the case $0 \leq \gamma \leq 1$ and for technical reasons we suppose that $\gamma \leq (n-1)/2$. The reader could consult Corollary 3.2 in [10] for more precise conditions on $p, q, \tilde{p}, \tilde{q}, \gamma$ leading to (1.3).

Let χ, ψ_1 be functions in $C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = \psi_1(x) = 1$ for $|x| \leq R_0 + T$. By a finite speed of propagation argument we can choose $\psi_1(x)$ so that

$$(1 - \psi_1)U(0,s)Q(s) = 0,\ 0 \leq s \leq T. \tag{1.6}$$

In the following we suppose that ψ_1 is fixed. Obviously, for $A > 0$ large enough and $\Im\theta \geq A$ the operator $(U(T) - e^{-i\theta}I)$ is invertible. In Section 2 we show that the cut-off resolvent

$$R_\chi(\theta) = \chi(U(T) - e^{-i\theta}I)^{-1}\psi_1 : \dot{\mathcal{H}}_1(\mathbb{R}^n) \longrightarrow \dot{\mathcal{H}}_1(\mathbb{R}^n)$$

admits a meromorphic continuation in $\mathbb{C}$ for $n \geq 3$, odd, and in

$$\mathbb{C}' = \{\theta \in \mathbb{C} :\ \theta \neq 2\pi k - i\mu,\ \mu \geq 0,\ k \in \mathbb{Z}\}$$

for n even. Introduce the following condition.

($\mathcal{R}$) The operator $R_{\psi_1}(\theta)$ admits a holomorphic extension from $\{\theta \in \mathbb{C} : \Im\theta \geq A > 0\}$ to $\{\theta \in \mathbb{C} :\ \Im\theta \geq 0\}$, for $n \geq 3$, odd, and to $\{\theta \in C : \Im\theta \geq 0, \theta \neq 2\pi k,\ k \in \mathbb{Z}\}$ for $n \geq 2$, even. Moreover, for n even we have

$$\lim_{\lambda \to 0,\ \lambda > 0} \|R_{\psi_1}(i\lambda)\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n) \to \dot{\mathcal{H}}_1(\mathbb{R}^n)} < \infty. \tag{1.7}$$

This condition is independent on the choice of χ and ψ_1 and ($\mathcal{R}$) implies a decay of the local energy. Our main result is the following

Theorem 1. *Let the condition* ($\mathcal{R}$) *be fulfilled and let* $1 \leq \tilde{p}, \tilde{q} \leq 2 \leq p, q \leq +\infty$, $0 \leq \gamma \leq \min\{1,\ (n-1)/2\}$, $p > 2$ *be admissible for the free wave equation. Moreover, if n is even assume that* $\tilde{p} < 2$. *Then for data* $(f_0, f_1) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$, $F \in L_t^{\tilde{p}}(\mathbb{R};\ L_x^{\tilde{q}}(\mathbb{R}^n))$ *and* $u(t,x)$ *solution of* (1.1) *with* $\tau = 0$ *we have for all* $t \in \mathbb{R}$ *the estimate*

$$\begin{aligned}\|u\|_{L_t^p(\mathbb{R};\ L_x^q(\mathbb{R}^n))} &+ \|u(t,x)\|_{\dot{H}^\gamma(\mathbb{R}^n_x)} + \|\partial_t u(t,x)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n_x)} \\ &\leq C\Big(\|f_0\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|f_1\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} + \|F\|_{L_t^{\tilde{p}}(\mathbb{R};\ L_x^{\tilde{q}}(\mathbb{R}^n))}\Big)\end{aligned} \tag{1.8}$$

with a constant $C = C(n, p, q, \tilde{p}, \tilde{q}, \gamma) > 0$ *independent of* t.

Remark 1. The condition (1.7) is similar to the bound of the norm of the cut-off resolvent

$$\lim_{\lambda \to 0,\ \lambda > 0} \|\lambda P_\chi(i\lambda)\|_{L^2 \to L^2} < \infty$$

in the stationary case (see [21] for general boundary conditions and [3] for Dirichlet problem). Here $P_\chi(\lambda) = \chi(P - \lambda^2)^{-1}\chi$, $\Im\lambda > 0$, and $(P - \lambda^2)^{-1}$ is the resolvent of a self-adjoint operator P.

The decay of local energy for time dependent perturbations has been investigated in [5], [1], [12] [20], [19]. The main hypothesis is that the perturbations are *non-trapping* (see [12] and [20] for a precise definition related to the propagation of singularities). In contrast to the stationary case, the non-tapping condition is not sufficient for a local energy decay. In particular, the problem (1.1) is *non-trapping* but we may have solutions with exponentially growing local energy. To exclude the existence of such solutions, we must introduce the resonances and this explains the role of the condition ($\mathcal{R}$). For $n \geq 3$, odd, the exponential decay of local energy have been established in [1], [12] (see also [5] for moving obstacles) exploiting the spectrum of the operator $Z^\rho(T) = P_+^\rho U(T) P_-^\rho$, where $P_\pm^\rho$ are the orthogonal projectors on the Lax-Phillips spaces (see [9])

$$D_\pm^\rho = \{f \in \dot{\mathcal{H}}_1(\mathbb{R}^n) :\ U_0(t)f = 0 \text{ for } |x| \leq \pm t + \rho,\ \pm t \geq 0\},\ \rho \geq R_0.$$

The poles of $(Z^\rho(T) - zI)^{-1}$ are called resonances and their independence of ρ has been proved by Cooper and Strauss [5] (see also Chapter V in [12]). Moreover, for $n \geq 3$, odd, it was proved in [2] that the poles of $\chi(U(T) - zI)^{-1}\psi_1$ coincide with their multiplicities with the eigenvalues of the operator $Z^\rho(T)$. Thus for n, odd, the condition ($\mathcal{R}$) means that $Z^\rho(T)$ has no eigenvalues $z \in \mathbb{C}$, $|z| \geq 1$.

In [19], [20] Vainberg proposed a general analysis of problems with time-periodic perturbations including potentials, moving obstacles and high order operators, provided that the perturbations are *non-trapping.* The results of Vainberg [20] cover the case of odd and even dimensions $n \geq 2$. The analysis in [20] is based on the meromorphic continuation of an operator $R(\theta)$ (see [20] for a more precise definition). On the other hand, $R(\theta)$ has a complicated form and it seems difficult to examine its analytic continuation and to find a link between the properties of $R(\theta)$ and the behavior of the operator $Z^\rho(T)$.

The novelty in our approach is that we exploit the meromorphic continuation of $R_\chi(\theta)$. We like to mention that in the study of the time-periodic perturbations of the Schrödinger operator (see [6] and the papers cited there) the resolvent of the monodromy operator $(U(T)-z)^{-1}$ plays a central role. Moreover, the absence of eigenvalues $z \in \mathbb{C}$, $|z| = 1$ of $U(T)$, and the behavior of the resolvent near 1, are closely related to the decay of local energy as $t \to \infty$. So our results may be considered as a natural extension of those for Schrödinger operator. On the other hand, for the wave equation we may have poles $z \in \mathbb{C}$, $|z| > 1$ of the $R_\chi(\theta)$, while for the Schrödinger operator with time-periodic potentials a such phenomenon is excluded. It is interesting to raise the question when the condition $(\mathcal{R})$ holds. In this direction we have the following result for n odd which follows directly from Theorem 5.5.3 in [12] and Proposition 1 in [2].

Theorem 2. *For $n \geq 3$, odd, $(\mathcal{R})$ is equivalent to the following conditions:*

(a) *for each $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ we have*

$$\lim_{t\to\infty} \|\varphi U(t,0) f\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} = 0,\ \forall f \in \dot{\mathcal{H}}_1(\mathbb{R}^n),$$

(b) *for each $f = (0, g)$ with $g \in L^2(\mathbb{R}^n)$, supp $g \subset \{x : |x| \leq R_0\}$, there exists a sequence $m_j \to \infty$, $m_j \in \mathbb{N}$, depending on g, such that*

$$\lim_{m_j\to\infty} \|\psi(x) U(m_j T, 0) f\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} = 0,$$

where $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ is a fixed function with $\psi(x) = 1$ for $|x| \leq 3R_0$.

We would like to notice that there are many examples, where the condition (a) of the above theorem is fulfilled (see Theorem 5.1.3 and Examples 5.1.4, 5.1.5 in [12]). The same approach to the analysis of the local energy decay can be used for non-trapping moving obstacles. On the other hand, for trapping moving obstacles it seems that the cut-off resolvent $R_\chi(\theta)$ has no meromorphic continuation in $\mathbb{C}$ even for n odd. It natural to conjecture that $R_\chi(\theta)$ has a meromorphic continuation for $\Im\theta \geq \epsilon$, $\forall \epsilon > 0$ and this is an interesting open problem.

Global Strichartz estimates for the wave equation with non-trapping stationary perturbations have been obtained in [16], [3] and the reader may consult the references in these papers for other works. For hyperbolic equations with coefficients depending only on t, Strichartz estimates have been studied by Reissig and Yagdjiian [13], [14], [15]. To our best knowledge there are no results concerning Strichartz estimates for the wave equation with periodic in time perturbations depending on (t, x). In our analysis the non-trapping condition is replaced by $(\mathcal{R})$ and our approach was inspired by the work of Burq [3] and the recent progress related to the results of Christ and Kiselev [4]. The L^2 integrability of the local energy (see Section 4) plays an important role in the proof of Theorem 1. The investigation of the homogeneous Strichartz estimates with $F = 0$ is simpler and the corresponding results can be obtained for a larger set of indices p, q, γ.

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2. MEROMORPHIC CONTINUATION OF THE CUT-OFF RESOLVENT $\chi(U(T)-zI)^{-1}\psi_1$

Throughout this and the following sections we denote by $\|.\|$ the norm in $\dot{\mathcal{H}}_1(\mathbb{R}^n)$ and we use the same notation for the norm of the bounded operators from $\dot{\mathcal{H}}_1(\mathbb{R}^n)$ to $\dot{\mathcal{H}}_1(\mathbb{R}^n)$. Our purpose is to prove that for $\chi,\ \psi_1 \in C_0^\infty(\mathbb{R}^n)$ the cut-off resolvent

$$\chi(U(T)-e^{-i\theta}I)^{-1}\psi_1:\ \dot{\mathcal{H}}_1(\mathbb{R}^n)\to\dot{\mathcal{H}}_1(\mathbb{R}^n),$$

admits a meromorphic continuation with respect to θ in $\mathbb{C}$ for $n\geq 3$, odd, and in

$$\mathbb{C}'=\{z\in\mathbb{C}:\ z\neq 2\pi k-i\mu,\ \mu\geq 0,\ k\in\mathbb{Z}\}$$

for $n\geq 2$, even. Let $\psi\in C_0^\infty(\mathbb{R}^n)$ be a fixed cut-off such that $\psi(x)=1$ for $|x|\leq R_0+T$. By a finite speed of propagation argument we get

$$(1-\psi)U(T,s)Q(s)=0,\ Q(s)U_0(s)(1-\psi)=0,\ 0\leq s\leq T. \tag{2.1}$$

For $A>0$ large enough and $\Im\theta\geq A$ the resolvents $(U_0(T)-e^{-i\theta}I)^{-1}$, $(U(T)-e^{-i\theta}I)^{-1}$ exist and we have the equality

$$U(T)-zI=\Big[I-\psi\int_0^T U(T,s)Q(s)U_0(s)ds\psi(U_0(T)-zI)^{-1}\Big](U_0(T)-zI),\ z=e^{-i\theta}.$$

It is easy to show (see [1], [12]) that the operator

$$\psi\int_0^T U(T,s)Q(s)U_0(s)\psi ds:\ \dot{\mathcal{H}}_1(\mathbb{R}^n)\longrightarrow\dot{\mathcal{H}}_1(\mathbb{R}^n)$$

is compact and for $\Im\theta\geq A$ we have

$$(U_0(T)-zI)^{-1}=(U(T)-zI)^{-1}\Big[I-\psi\int_0^T U(T,s)Q(s)U_0(s)ds\psi(U_0(T)-zI)^{-1}\Big].$$

Now let $\psi_1\in C_0^\infty(\mathbb{R}^n)$ be a fixed cut-off function satisfying (1.6) and such that $\psi_1(x)=1$ on supp ψ. Take an arbitrary cut-off function $\chi\in C_0^\infty(\mathbb{R}^n)$ so that $\chi=1$ on supp ψ. Multiplying the above equality by χ and ψ_1, we get

$$\chi(U_0(T)-zI)^{-1}\psi_1=\chi(U(T)-zI)^{-1}\psi_1\Big[I-\psi\int_0^T U(T,s)Q(s)U_0(s)ds\psi(U_0(T)-zI)^{-1}\psi_1\Big].$$

Introduce the operator

$$K(z)=\psi\int_0^T U(T,s)Q(s)U_0(s)ds\psi(U_0(T)-zI)^{-1}\psi_1.$$

For $n\geq 3$, odd, the operator $\psi(U_0(T)-e^{-i\theta}I)^{-1}\psi_1$ admits an analytic continuation with respect to θ in $\mathbb{C}$ and this follows immediately from the Huygens principle and the expansion

$$\psi(U_0(T)-e^{-i\theta}I)^{-1}\psi_1=-\sum_{k=0}^{N(\psi,\psi_1)}\psi U_0(kT)\psi_1 e^{i(k+1)\theta}$$

which holds for $\Im\theta\geq A>0$. On the other hand, the operator $K(z)$ is compact in $\dot{\mathcal{H}}_1(\mathbb{R}^n)$ and an application of the analytic Fredholm theorem leads to a meromorphic continuation of $\chi(U(T)-e^{-i\theta}I)^{-1}\psi_1$ in $\mathbb{C}$. Notice that if z_0 is a pole of $\chi(U(T)-zI)^{-1}\psi_1$, then dim Ker $(I-K(z_0))>0$.

Inversely, assume that there exists a function $f \neq 0$ such that $f = K(z_0)f$. Then $(I - K(z))^{-1}$ is meromorphic in a neighborhood of z_0 and for $|z - z_0|$ small enough we have

$$(I - K(z))^{-1} = \sum_{j=1}^{m} \frac{A_j}{(z - z_0)^j} + B(z)$$

with analytic function $B(z)$ and finite rank operators A_j, $A_m \neq 0$. Clearly, $\mathrm{Im} A_m \subset \mathrm{Ker}(I - K(z_0))$. If $\chi(U(T) - zI)^{-1}\psi_1$ is analytic at z_0, then

$$\chi(U_0(T) - z_0 I)^{-1}\psi_1 A_m g = 0, \;\; \forall g \in \dot{\mathcal{H}}_1(\mathbb{R}^n)$$

and $\psi(U_0(T) - z_0 I)^{-1}\psi_1 A_m g = 0$. Going back to the operator $I - K(z_0)$, we conclude immediately that $A_m = 0$. Proceeding in this way, we obtain $A_j = 0$, $j = 1, ..., m$, which is a contradiction. Consequently, z_0 is a pole of $\chi(U(T) - z_0 I)^{-1}\psi_1$.

For n even we will apply the same argument in $\mathbb{C}'$ and for this purpose we must show that $\chi(U_0(T) - e^{-i\theta} I)^{-1}\psi_1$ can be continued as an analytic function in $\mathbb{C}'$. We extent $U_0(t)(\psi_1 f)$ as 0 for $t < 0$ and consider the Fourier-Block-Gelfand transform

$$g(\theta, s) = (F(U_0(t)(\psi_1 f))(\theta, s) = \sum_{k=-\infty}^{\infty} U_0(kT + s) e^{ik\theta}(\psi_1 f)$$

defined for $\Im\theta \geq A > 0$. In fact, it is easy to see that

$$g(\theta, s) = U_0(s) \sum_{k=0}^{\infty} U_0(kT) e^{ik\theta}(\psi_1 f)$$

$$= U_0(s) e^{-i\theta}(e^{-i\theta} I - U_0(T))^{-1}(\psi_1 f).$$

We refer to [20] for the properties of the Fourier-Block-Gelfand transform. We conclude that the analytic continuation of $\chi(U_0(T) - e^{-i\theta} I)^{-1}\psi_1$ is reduced to that of $\chi F(U_0(t)(\psi_1 f))(\theta, 0)$. We are in position to apply Lemma 6 and 7 in [20] saying that $\chi F(U_0(t)(\psi_1 f))(\theta, 0)$ admits an analytic continuation in $\mathbb{C}'$. In fact in [20], Lemma 7, the transformation $\chi F(\alpha(t) U_0(t)(\psi_1 f))(\theta, s)$ is treated, where $\alpha(t) \in C^\infty(\mathbb{R})$ is such that $\alpha(t) = 0$ for $t \leq t_0$, $\alpha(t) = 1$ for $t \geq t_0 + 1$, $t_0 > 0$. The analysis of the term $\chi F((1 - \alpha(t)) U_0(t)(\psi_1 f))(\theta, s)$ is trivial and we obtain the result. Moreover, in a neighborhood of 0 we have the representation

$$\chi(U_0(T) - e^{-i\theta} I)^{-1}\psi_1 = B_0(\theta)\theta^{n-1}\ln\theta + B_1(\theta), \tag{2.2}$$

where $B_0(\theta)$ and $B_1(\theta)$ are analytic for $|\theta| \leq \epsilon_0$ and $\partial_\theta^j B_0(\theta)_{|\theta=0}$, $j \geq 0$, are finite rank operators. To obtain a meromorphic continuation in $\mathbb{C}'$ of $\chi(U(T) - e^{-i\theta} I)\psi_1$, we repeat the argument for n odd and we deduce that the poles $\theta \in \mathbb{C}'$ are independent of the function χ. Thus we can introduce the following

Definition 1. *We say that* $z_0 \in \mathbb{C}$ *(resp.* $z_0 \in \mathbb{C}'$*) is a pole of* $\chi(U(T) - zI)^{-1}\psi_1$ *for* n *odd (resp.* n *even), if*

$$\dim \mathrm{Ker}\left(I - \psi \int_0^T U(T, s) Q(s) ds\, \psi(U_0(T) - z_0 I)^{-1}\psi_1\right) > 0.$$

Finally, to study the invertibility of the operator $(I - K(e^{-i\theta}))$ in a neighborhood of 0, we apply Theorem 8 in [20] (see also Lemma 10 in Chapter IX, [18]). Consequently, for $|\theta| \leq \epsilon_0$, $|\arg\theta - \pi/2| < \pi$, we have

$$(I - K(e^{-i\theta}))^{-1} = \theta^{-m} \sum_{j\geq 0} \Big(\frac{\theta}{P(\ln\theta)}\Big)^j P_j(\ln\theta) + C(\theta), \tag{2.3}$$

where $m \geq 0$ is an integer, P is a polynomial, P_j is a polynomial of order at most lj, $l \geq 1$, the coefficients of P, P_j are finite rank operators and $C(\theta)$ is analytic. Combining (2.2) and (2.3), we get for $|\theta| \leq \epsilon_0$, $|\arg\theta - \pi/2| < \pi$ the representation

$$\chi(U(T) - e^{-i\theta}I)^{-1}\psi_1 = \sum_{k=-m}^{\infty} \sum_{j=-m_k}^{\infty} R_{kj}\theta^k \ln^j\theta. \tag{2.4}$$

3. Decay of the local energy

In this section we will establish a decay of local energy and we assume the condition $(\mathcal{R})$ fulfilled. The results are different for n odd and n even. We fix the cut-off functions ψ, ψ_1 as in the previous section and suppose that $\chi \in C_0^\infty(\mathbb{R}^n)$ is such that $\chi(x) = 1$ on supp ψ. The argument of the previous section shows that $(\mathcal{R})$ leads to the absence of poles θ of $R_\chi(\theta)$ with $\Im\theta \geq 0$ (resp. $\Im\theta \geq 0$, $\theta \neq 2\pi k$, $k \in \mathbb{Z}$) for n odd (resp. n even). On the other hand, the representation (2.2) yields

$$\psi(U_0(T) - e^{-i\theta}I)^{-1}\psi_1 = \psi\psi_1(U_0(T) - e^{-i\theta}I)^{-1}\psi_1 = L_0 + \mathcal{O}(\theta),\ |\theta| \leq \epsilon_0$$

with a bounded operator L_0. Here and below $\mathcal{O}(\theta)$ denotes a bounded operator in $\dot{\mathcal{H}}_1(\mathbb{R}^n)$ such that $\|\mathcal{O}(\theta)\| \leq C|\theta|$, where $\|.\|$ is the norm in $\mathcal{L}(\dot{\mathcal{H}}_1(\mathbb{R}^n))$. Let

$$(I - K(e^{-i\theta}))^{-1} = \Big(I - \psi\int_0^T U(T,s)Q(s)U_0(s)ds(L_0 + \mathcal{O}(\theta))\Big)^{-1}$$

$$= \theta^{-m}\sum_{j=0}^{r} \ln^{r-j}\theta\Big(A_j + \mathcal{O}_j(\ln^{-1}\theta)\Big) + \sum_{k=1}^{M} \theta^{-m+k}\ln^{q_k}\theta(F_k + \mathcal{O}_k(\ln^{-1}\theta)) + F_0(\theta)$$

with finite rank operators A_j, $A_0 \neq 0$, F_k, $k = 1, ..., m$, $m \geq 0$, $r \geq 0$. First assume $m > 0$. Then the condition $(\mathcal{R})$ implies

$$\lim_{\lambda\to 0,\, \lambda>0} \|(L_0 + \mathcal{O}(i\lambda))((i\lambda)^{-m}\ln^r(i\lambda)A_0 + ...)\| \leq C_0,$$

and we deduce $L_0A_0 = 0$. Here ... denotes a sum of terms with lower order singularity at 0. On the other hand, for $|\theta| \leq \epsilon_0$, $\Im\theta > 0$, we have

$$\Big(I - \psi\int_0^T U(T,s)Q(s)U_0(s)ds(L_0 + \mathcal{O}(\theta))\Big)(\theta^{-m}\ln^r\theta A_0 + ...)$$

$$= \theta^{-m}\ln^r\theta A_0 + ... = I$$

and we conclude that $A_0 = 0$. The case $m > 0$, $r < 0$ can be treated in the same way and we conclude that we must have $m = 0$. Repeating the same argument with $m = 0$ and $r > 0$, we obtain that in the leading term of $(I - K(e^{-i\theta}))^{-1}$ we have $m = 0$, $r \leq 0$. Finally, $(\mathcal{R})$ implies that $(I - K(e^{-i\theta}))^{-1}$ is bounded for $|\theta| \leq \epsilon_0$ and we deduce that $R_\chi(\theta)$ is bounded for $|\theta| \leq \epsilon_0$ for every $\chi \in C_0^\infty(\mathbb{R}^n)$ having the property mentioned above.

Given a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^n)$, we will estimate the norm $\|\varphi U(t,0)f\|$ for functions $f \in \dot{\mathcal{H}}_1(\mathbb{R}^n)$, such that $f(x) = 0$ for $|x| \geq R$. For this purpose it is sufficient to estimate the norm of

$$\int_0^t \varphi U(t,s)Q(s)U_0(s)f ds$$

uniformly with respect to $f \in C_0^\infty(B(0,R)) \times \in C_0^\infty(B(0,R))$, where $B(0,R) = \{x : |x| \leq R\}$. We extend $U_0(s)f$ as 0 for $s < 0$ and consider the Fourier-Block-Gelfand transform

$$g(\theta, s) = (F(U_0(s)f))(\theta, s) = \sum_{k=-\infty}^{k=\infty} U_0(kT+s)e^{ik\theta} f$$

which is well defined for $\Im\theta \geq \alpha > 0$. Applying the inverse transform of F (see [20]), we are going to examine

$$\frac{1}{2\pi}\int_{-\infty}^t \varphi U(t,s)Q(s)\int_{d_\alpha} g(\theta,s)d\theta ds,$$

where $d_\alpha = [i\alpha - \pi, i\alpha + \pi]$ and $\alpha > 0$ will be chosen large enough below.

Choose an integer $m \in \mathbb{Z}$ so that $t' = t - mT \in [0, T[$ and fix m. Changing the variable $s = s' + mT$ and using the property $U(t+mT, s+mT) = U(t,s)$, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{t'} \varphi U(t',s')Q(s')\int_{d_\alpha} e^{-im\theta} g(\theta,s')d\theta ds'$$

$$= \frac{1}{2\pi}\int_0^{t'} \varphi U(t',s')Q(s')U_0(s')\int_{d_\alpha} e^{-im\theta} g(\theta,0)d\theta ds'$$

$$+\frac{1}{2\pi}\sum_{k=0}^{\infty}\int_{-kT-T}^{-kT} \varphi U(t',s')Q(s')\int_{d_\alpha} e^{-im\theta} g(\theta,s')d\theta ds' = I_1(t) + I_2(t).$$

The integral $I_1(t)$ can be estimated following the argument given below and we will deal with the infinite sum. Changing the variable $s' = -T - kT + \xi$, we get the series

$$\sum_{k=0}^{\infty}\int_0^T \int_{d_\alpha} \varphi U(t'+T,0)U(kT)U(0,\xi)e^{i(k+1)\theta}Q(\xi)e^{-im\theta}g(\theta,s')d\theta d\xi. \tag{3.1}$$

Here we have used the fact that

$$U(t'+T+kT,\xi) = U(t'+T,0)U(kT)U(0,\xi).$$

Now choose $\alpha > 0$ so that the series

$$\sum_{k=0}^{\infty} U(kT)e^{i(k+1)\theta} = (e^{-i\theta}I - U(T))^{-1}$$

is convergent in the operator norm for $\theta \in d_\alpha$. For $0 \leq \xi \leq T$ we can find a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ so that $\chi(x) = 1$ on supp ψ and

$$\varphi U(t'+T,0)(1-\chi) = 0.$$

Notice that the function χ depends on φ. According to the properties (1.6), (2.1) of ψ_1, ψ, it is clear that (3.1) can be written in the form

$$\int_{d_\alpha}\int_0^T \varphi U(t'+T,0)\chi(e^{-i\theta}I-U(T))^{-1}\psi_1 U(0,\xi)Q(\xi)U_0(\xi)e^{-im\theta}\psi g(\theta,0)d\xi d\theta. \tag{3.2}$$

First assume that $n \geq 3$ is odd. Then $(\mathcal{R})$ implies that $R_\chi(\theta)$ has no poles with $\Im\theta \geq 0$ and we can choose $\delta > 0$ so that $R_\chi(\theta)$ has no poles θ with $\Im\theta \geq -\delta T$, $-\pi < \Re\theta \leq \pi$. Let $d_{-\delta T} = [-i\delta T - \pi, -i\delta T + \pi]$. Recall that $t = mT + t'$, so

$$e^{-m\delta T} \leq Ce^{-\delta t}$$

with $C > 0$ independent on m and t. On the other hand,

$$\psi g(\theta,0) = \psi\sum_{k=0}^{\infty} U_0(kT)e^{ik\theta}f = e^{-i\theta}\psi(e^{-i\theta}I-U_0(T))^{-1}f,\ \Im\theta > 0$$

and we conclude that $\psi g(\theta,0)$ admits an analytic continuation in $\mathbb{C}$. We shift the contour of the integration from d_α to $d_{-\delta T}$ and we obtain

$$\|I_2(t)\| \leq C_1e^{-\delta t}\|f\|,\ t \geq 0,$$

where $C_1 > 0$ depends on φ and R. To estimate $I_1(t)$, we shift again the contour of integration to $d_{-\delta T}$ and we obtain the same estimate as that for $I_2(t)$. Combining these estimates, we get

$$\|\varphi U(t,0)f\| \leq C_2e^{-\delta t}\|f\|. \tag{3.3}$$

Next let $0 \leq s \leq t$ and let $s - jT \in [0,T[$, $j \in \mathbb{N}$. Then

$$\begin{aligned}\|\varphi U(t,s)f\| &= \|\varphi U(t-jT,0)U(0,s-jT)f\|\\ &\leq C_3e^{-\delta(t-jT)}\|U(0,s-jT)f\| \leq C_4e^{-\delta(t-s)}\|f\| \end{aligned}\tag{3.4}$$

with a constant C_4 depending on φ and $R+T$. Here we have used the fact that $U(0,s-jT)f$ has a compact support independent of s.

Passing to the case n even, we will estimate the integral (3.2). Choose again $\delta > 0$ so that $R_\chi(\theta)$ has no poles θ lying in

$$\{\theta : \Im\theta \geq -\delta T,\ -\pi \leq \pm\Re\theta < 0\}.$$

Next choose $\delta \geq \epsilon_0 > 0$ so that $R_\chi(\lambda)$ is bounded for $|\theta| \leq \epsilon_0$ and consider the contour $\gamma = \Gamma_1 \cup \omega \cup \Gamma_2$, where

$$\Gamma_1 = [-i\delta T - \pi, -i\delta T - \nu],\ \Gamma_2 = [-i\delta T + \nu, -i\delta T + \pi]$$

and $0 < \nu < \epsilon_0$ is sufficiently small. The contour ω is a curve connecting $-i\delta T - \nu$ and $-i\delta T + \nu$, symmetric with respect to the axis $\Re\theta = 0$. The part of ω lying in $\{\theta : \Im\theta \geq 0\}$ is a half-circle with radius ϵ_0 and $\omega \cap \{\pm\Re\theta > 0,\ \Im\theta \leq 0\}$ is formed by line segments. Thus ω is included in the region where we have no poles of $R_\chi(\theta)$. We shift the integration from d_α to the contour γ. The integrals on Γ_k, $k = 1, 2$, can be estimated as in the case n odd. The integral over ω can be handled following Lemma 7 in Chapter IX, [18]. In fact we must estimate only the integral over a part of the circle $|\theta| = \epsilon_0$. Since $(I - K(e^{-i\theta}))^{-1}$ is bounded for $|\theta| \leq \epsilon_0$, the leading term of the singular part of $(I - K(e^{-i\theta}))^{-1}$ is given by

$$A_0 + \sum_{k=j}^{l} \ln^{-k} \theta A_k + o(|\ln \theta|^{-l}),\ j \geq l \geq 1,$$

where A_j are finite rank operators. Then

$$\int_\omega e^{-im\theta} \ln^{-l} \theta d\theta = m^{-1} \sum_{j=1}^{M} a_j \ln^{-l-j} m + \mathcal{O}(m^{-1} \ln^{-l-M-1} m),\ m \to \infty.$$

On the other hand, according to (2.2), the leading term of the singular part of $\chi(U_0(T) - e^{-i\theta})^{-1}\psi_1$ is $\theta^{n-1} \ln \theta B_0(0)$ and

$$\int_\omega e^{-im\theta} \theta^{n-1} \ln \theta d\theta = a_0 m^{-n} + \mathcal{O}(m^{-n} \ln^{-1} m),\ m \to \infty.$$

The integrals of the terms analytic for $|\theta| \leq \epsilon_0$ are trivially bounded and summing up all contributions, we get

$$\|\varphi U(t, 0) f\| \leq C_5 t \ln^{-2} t \|f\|,\ t \geq t_0 > 1.$$

In the same way, as in the case n odd, we obtain

$$\|\varphi U(t, s) f\| \leq C_6 (t - s)^{-1} \ln^{-2}(t - s) \|f\|,\ t - s \geq t_0 > 1. \tag{3.5}$$

Finally, for $0 \leq s \leq t$ we get

$$\|\varphi U(t, s) f\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} \leq C(n,\ \varphi, R) p(t - s) \|f\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}, \tag{3.6}$$

where for $t \geq t_0 > 1$ we have

$$p(t) = \begin{cases} e^{-\delta t},\ n \geq 3, \text{ odd}, \\ t^{-1} \ln^{-2} t,\ n \geq 2, \text{ even}. \end{cases} \tag{3.7}$$

4. L^2-INTEGRABILITY OF THE LOCAL ENERGY

We start with the following

Proposition 1. *Assume the condition* $(\mathcal{R})$ *fulfilled and* $0 \leq \gamma \leq \min\{1, (n-1)/2\}$. *Let* $(f_0,\ f_1) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$ *and let* $F \in L^2_t(\mathbb{R};\ \dot{H}^\gamma_x(\mathbb{R}^n))$ *be supported in* $\{(t, x) :\ |x| \leq R\}$. *Then for every fixed* $\varphi \in C_0^\infty(\mathbb{R}^n)$ *the solution* $u(t, x)$ *of* (1.1) *with* $\tau = 0$ *satisfies the estimate*

$$\int_{-\infty}^{\infty} \|(\varphi u(t, x), \varphi \partial_t u(t, x))\|^2_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)} dt$$
$$\leq C\Big(\|f_0\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|f_1\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} + \|F\|_{L^2_t(\mathbb{R}; \dot{H}^\gamma_x(\mathbb{R}^n))}\Big)^2 \tag{4.1}$$

with a constant $C = C(n, \gamma, \varphi, R) > 0$ *depending only on* n, γ, φ *and* R.

Proof. First notice that for the free wave equation and $f = (f_0, f_1) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$ we have

$$\int_{-\infty}^{\infty} \|\varphi U_0(t) f\|^2_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)} dt \leq C_1(n, \gamma, \varphi) \|f\|^2_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)}. \tag{4.2}$$

This result is well known for the energy space $\dot{\mathcal{H}}_1(\mathbb{R}^n)$ and n odd. To obtain it for $\gamma \leq (n-1)/2$, we can apply a result of Smith and Sogge.

Lemma 1. ([16], *Lemma* 2.2) *For* $\gamma \leq (n-1)/2$ *the following estimate holds*

$$\int_{-\infty}^{\infty} \|\varphi\Big(e^{\pm it\Lambda} f\Big)\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} dt \leq C_{n,\gamma,\varphi} \|f\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}.$$

In [16] the authors consider only odd dimensions $n \geq 3$, but the proof of this lemma goes without any change for even dimensions. Setting $(u_0(t,x),\ \partial u_0(t,x)) = (U_0(t)f)$, we have the representation

$$u_0(t,x) = \frac{\sin(t\Lambda)}{\Lambda} f_1(x) + \cos(t\Lambda) f_0(x)$$

and we obtain immediately (4.2).

Passing to the estimate of $\varphi U(t,0)f$, we write

$$U(t,0)f = U_0(t)f - \int_0^t U(t,s)Q(s)U_0(s)f ds$$

and we get

$$\|\varphi U(t,0)f\|_{\dot{\mathcal{H}}_\gamma} \leq \|\varphi U_0(t)f\|_{\dot{\mathcal{H}}_\gamma} + \Big\|\int_{t-t_0}^t \varphi U(t,s)Q(s)U_0(s)f ds\Big\|_{\dot{\mathcal{H}}_\gamma}$$
$$+ \Big\|\int_0^{t-t_0} \varphi U(t,s)Q(s)U_0(s)f ds\Big\|_{\dot{\mathcal{H}}_\gamma}.$$

The estimate (1.2) of $\|U(t,s)\|_{\dot{\mathcal{H}}_\gamma \to \dot{\mathcal{H}}_\gamma}$ for $|t-s| \leq t_0$ and $0 \leq \gamma \leq 1$ implies

$$\Big\|\int_{t-t_0}^t \varphi U(t,s)Q(s)U_0(s)f ds\Big\|_{\dot{\mathcal{H}}_\gamma} \leq Ce^{k_1 t_0} \|f\|_{\dot{\mathcal{H}}_\gamma},$$

where $k_1 > 0$ is independent of t. On the other hand, for $f \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$ we get

$$Q(s)U_0(s)f \in \dot{\mathcal{H}}_{\gamma+1}(\mathbb{R}^n) \subset \dot{\mathcal{H}}_1(\mathbb{R}^n),\ 0 \leq \gamma \leq 1,$$

and choosing a cut-off function $\beta \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $\operatorname{supp}_x V(t,x)$, we get

$$\Big\|\int_0^{t-t_0} \varphi U(t,s)Q(s)U_0(s)f ds\Big\|_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)} \leq \Big\|\int_0^{t-t_0} \varphi U(t,s)\beta Q(s)U_0(s)f ds\Big\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}$$
$$\leq \int_0^{t-t_0} \|\varphi U(t,s)\beta\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n) \to \dot{\mathcal{H}}_1(\mathbb{R}^n)} \|Q(s)U_0(s)f\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} ds$$
$$\leq C_2\Big(Y(t-t_0)p(t) * \|Y(t)Q(t)U_0(t)f\|_{\dot{\mathcal{H}}_1(\mathbb{R}^n)}\Big).$$

Here $Y(t)$ denotes the Heaviside function and we have used (3.6) with $p(t)$ given by (3.7). It is clear that Lemma 1 implies

$$\int_0^\infty \|Q(t)U_0(t)f\|^2_{\dot{\mathcal{H}}_1(\mathbb{R}^n)} dt = \int_0^\infty \|V(t,x)u_0(t,x)\|^2_{L^2(\mathbb{R}^n)} dt \leq C_3 \|f\|^2_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)}. \tag{4.3}$$

Since $Y(t-t_0)p(t) \in L^1(\mathbb{R})$, an application of the Young inequality for the convolution combined with (4.3) yield (4.1) with $F = 0$.

In the general case ($F \neq 0$) consider the solution $v(t,x)$ of the problem (1.1) with $\tau = 0,\ f_0 = f_1 = 0,\ F \in L^2_t(\mathbb{R}; \dot{H}^\gamma_x(\mathbb{R}^n))$. Then

$$(\varphi v(t,x), \varphi \partial_t v(t,x)) = \int_0^t \varphi U(t,s)\chi(x)(0, F(s,x))ds$$

with $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ for $|x| \leq R$. Notice that we have

$$(0, F(t,x)) \in L^2_t(\mathbb{R}; \dot{\mathcal{H}}_{\gamma+1}(\mathbb{R}^n)) \subset L^2_t(\mathbb{R}; \dot{\mathcal{H}}_1(\mathbb{R}^n)).$$

Exploiting the local energy decay of $\|\varphi U(t,s)\chi\|_{\dot{\mathcal{H}}_1 \to \dot{\mathcal{H}}_1}$ and repeating the above argument, we get for $\varphi v(t,x)$ the estimate (4.1) with $f_0 = f_1 = 0$. This completes the proof.

Remark 2. It is natural to obtain the estimate (4.1) under the condition $F \in L^2_t(\mathbb{R}; \dot{H}^{\gamma-1}_x(\mathbb{R}^n))$. To do this, we must use a local energy decay of $\|\varphi U(t,s)\chi\|_{\dot{\mathcal{H}}_\gamma \to \dot{\mathcal{H}}_\gamma}$ which can be deduced from a decay of $\|\varphi U(t,s)\chi\|_{\dot{\mathcal{H}}_0 \to \dot{\mathcal{H}}_0}$ and interpolation.

We need also a result concerning the non-homogeneous free wave equation.

Proposition 2. *Assume* $1 \leq \tilde{p} < 2$. *Let* $f = (f_0, f_1) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$, $F \in L^{\tilde{p}}_t(\mathbb{R};\ L^{\tilde{q}}_x(\mathbb{R}^n))$ *and let* $u(t,x)$ *be the solution of* (1.1) *with* $V = 0$, $\tau = 0$. *Then for every* $\varphi \in C_0^\infty(\mathbb{R}^n)$ *we have*

$$\int_{-\infty}^{\infty} \|(\varphi u(t,x), \varphi \partial_t u(t,x))\|^2_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)} dt$$
$$\leq C(n, \tilde{p}, \tilde{q}, \gamma, \varphi)\Big(\|f_0\|_{\dot{H}^\gamma} + \|f_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\tilde{p}}_t(\mathbb{R}; L^{\tilde{q}}_x(\mathbb{R}^n))}\Big)^2. \tag{4.4}$$

Proof. It is sufficient to consider the case $f_0 = f_1 = 0$. The solution $u(t,x)$ has the form

$$(u(t,x), \partial_t u(t,x)) = \int_0^t e^{i(t-s)G_0}(0, F(s,x))ds.$$

Given a fixed $t_0 > 0$, we will estimate the norm

$$\Big\| \int_0^{t_0} \varphi e^{i(t-s)G_0}(0, F(s,x))ds \Big\|_{L^2_t(\mathbb{R}^+; \dot{\mathcal{H}}_\gamma(\mathbb{R}^n))}$$

uniformly with respect to t_0. Without loss of the generality we may suppose that $F(t,x) = 0$ for $t < 0$. First, according to (4.2), we have

$$\Big\| \varphi e^{itG_0} \int_0^{t_0} e^{-isG_0}(0, F(s,x))ds \Big\|_{L^2_t(\mathbb{R}^+; \dot{\mathcal{H}}_\gamma(\mathbb{R}^n))} \leq C_0 \Big\| \int_0^{t_0} e^{-isG_0}(0, F(s,x))ds \Big\|_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)},$$

with a constant $C_0 > 0$ independent of t_0. Since $e^{it_0 G_0}$ is unitary in $\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$, it is clear that

$$\Big\| \int_0^{t_0} e^{-isG_0}(0, F(s,x))ds \Big\|_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)}$$
$$= \Big\| \int_0^{t_0} e^{i(t_0-s)G_0}(0, F(s,x))ds \Big\|_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)} = \|(u(t_0,x), \partial_t u(t_0,x))\|_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)}.$$

Second, the estimate (1.3) yields

$$\|u(t_0,x)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t u(t_0,x)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \leq C_1 \|F\|_{L^{\tilde{p}}_t(\mathbb{R}; L^{\tilde{q}}_x(\mathbb{R}^n))}$$

with a constant $C_1 > 0$ independent of t_0. Thus we obtain

$$\Big\|\int_0^{t_0} \varphi e^{i(t-s)G_0}(0, F(s,x))ds\Big\|_{L^2(\mathbb{R}^+;\dot{\mathcal{H}}_\gamma(\mathbb{R}^n))} \leq C_0C_1\|F\|_{L_t^{\tilde{p}}(\mathbb{R};L_x^{\tilde{q}}(\mathbb{R}^n))}. \tag{4.5}$$

We will apply a version of Christ-Kiselev lemma [4] given in [8] and for the sake of completeness we state it below (see also [16], Lemma 3.1 and [17], Lemma 3.1).

Lemma 2. ([8], *Lemma* 8.1) *Let X and Y be Banach spaces, and for all $s, t \in \mathbb{R}^+$ let $K(s,t) : X \longrightarrow Y$ be an operator-values kernel from X to Y. Suppose we have*

$$\Big\|\int_{0\leq s<t_0} K(s,t)g(s)ds\Big\|_{L^q([t_0,\infty);\, Y)} \leq A\|g\|_{L^p(\mathbb{R}^+;\, X)}$$

for some $A > 0$, $1 \leq p < q \leq \infty$, and all $t_0 \in \mathbb{R}^+$ and $g \in L^p(\mathbb{R}^+;\, X)$. Then we have

$$\Big\|\int_{0\leq s<t} K(s,t)g(s)ds\Big\|_{L^q(\mathbb{R}^+;\, Y)} \leq C_{p,q}A\|g\|_{L^p(\mathbb{R}^+;\, X)},$$

where $C_{p,q} > 0$ depends only on p, q.

In [8] the above result is formulated with $\mathbb{R}$ instead of $\mathbb{R}^+$ and $s,\ t,\ t_0 \in \mathbb{R}$, but, as it was mentioned in [8], the same proof works for other intervals and in particular for $\mathbb{R}^+$. By hypothesis $\tilde{p} < 2$, so taking into account (4.5), we deduce from Lemma 2 the estimate

$$\Big\|\int_0^t \varphi e^{i(t-s)G_0}(0, F(s,x))ds\Big\|_{L^2(\mathbb{R}^+;\dot{\mathcal{H}}_\gamma(\mathbb{R}^n))} \leq C_2\|F\|_{L_t^{\tilde{p}}(\mathbb{R};L_x^{\tilde{q}}(\mathbb{R}^n))}.$$

In the same way we treat the norm

$$\Big\|\int_0^t \varphi e^{i(t-s)G_0}(0, F(s,x))ds\Big\|_{L_t^2(\mathbb{R}^-;\dot{\mathcal{H}}_\gamma(\mathbb{R}^n))}$$

and the proof is complete.

Remark 3. The estimate (4.4) has been proved in [16] for $n \geq 3$, odd, and $1 \leq \tilde{p} \leq 2$. The restriction $\tilde{p} < 2$ in Proposition 2 is related to the application of Lemma 2 and it is an open problem to see if this estimate remains valid for n even and $\tilde{p} = 2$.

Corollary 1. *Assume $1 \leq \tilde{p} < 2$. Let $f = (f_0, f_1) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$, $F \in L_t^{\tilde{p}}(\mathbb{R};\, L_x^{\tilde{q}}(\mathbb{R}^n))$ and let $u(t,x)$ be the solution of* (1.1) *with $\tau = 0$. Then for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have*

$$\begin{aligned}\int_{-\infty}^{\infty} &\|(\varphi u(t,x), \varphi\partial_t u(t,x))\|^2_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)}dt \\ \leq A(n,\tilde{p},\tilde{q},\gamma,\varphi)&\Big(\|f_0\|_{\dot{H}^\gamma} + \|f_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L_t^{\tilde{p}}(\mathbb{R}:\, L_x^{\tilde{q}}(\mathbb{R}^n))}\Big)^2.\end{aligned} \tag{4.6}$$

Proof. We write $u = u_0 + v$, where u_0 is the solution of the problem

$$\begin{cases}(\partial_t^2 - \Delta)u_0 = F, \\ u_0|_{t=0} = f_0,\ \partial_t u_0|_{t=0} = f_1,\end{cases}$$

while v is the solution of the problem

$$\begin{cases}(\partial_t^2 - \Delta + V)v = -Vu_0, \\ v|_{t=0} = \partial_t v|_{t=0} = 0.\end{cases}$$

Applying Proposition 2 for Vu_0, we obtain the estimate

$$\|Vu_0\|_{L^2_t(\mathbb{R};\, \dot{H}^\gamma_x(\mathbb{R}^n))} \leq C_0\Big(\|f_0\|_{\dot{H}^\gamma} + \|f_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\tilde{p}}_t(\mathbb{R};\, L^{\tilde{q}}_x(\mathbb{R}^n))}\Big). \tag{4.7}$$

In fact, choosing a function $\beta \in C_0^\infty(\mathbb{R}^n)$ such that $\beta = 1$ on $\operatorname{supp}_x V(t,x)$, we have

$$\|V(t,x)u_0\|_{\dot{H}^\gamma_x(\mathbb{R}^n)} \leq C_{\gamma,V}\|\beta u_0\|_{\dot{H}^\gamma_x(\mathbb{R}^n)}.$$

Next we write

$$(\varphi v(t,x), \varphi\partial_t v(t,x)) = -\int_0^t \varphi U(t,s)(0, Vu_0(s,x))ds.$$

Since $Vu_0 \in L^2_t(\mathbb{R}; \dot{H}^\gamma_x(\mathbb{R}^n))$, repeating the argument of the proof of Proposition 1, we get (4.6).

5. Global Strichartz estimates

In this section we establish the estimate (1.8) and complete the proof of Theorem 1. We present the solution of (1.1) as a sum $u = u_0 + v$, where u_0 and v are the same as in the proof of Corollary 1. The estimate of $\|u_0\|_{L^p_t(\mathbb{R};\, L^q_x(\mathbb{R}^n))}$ follows form (1.3). Next we have

$$v(t,x) = -\int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda}(Vu_0 + Vv)(s,x)ds.$$

As in the previous section, for Vu_0 we have the estimate (4.7). We apply Proposition 1 for Vv and deduce

$$\|Vu_0 + Vv\|_{L^2_t(\mathbb{R};\, \dot{H}^\gamma_x(\mathbb{R}^n))} \leq C_1\Big(\|f_0\|_{\dot{H}^\gamma} + \|f_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\tilde{p}}_t(\mathbb{R};\, L^{\tilde{q}}_x(\mathbb{R}^n))}\Big). \tag{5.1}$$

We wish to prove that

$$\Big\|\int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda}(Vu_0 + Vv)(s,x)ds\Big\|_{L^p_t(\mathbb{R}^+;\, L^q_x(\mathbb{R}^n))} \leq C_2\|Vu_0 + Vv\|_{L^2_t(\mathbb{R}^+;\, \dot{H}^\gamma_x(\mathbb{R}^n))}. \tag{5.2}$$

Let $\beta \in C_0^\infty(\mathbb{R}^n)$ be the same as in the proof of Corollary 1. An application of Lemma 1 shows that the operator

$$T : \dot{H}^{-\gamma}(\mathbb{R}^n) \ni g \mapsto \beta e^{\pm it\Lambda}g \in L^2_t(\mathbb{R}^+; \dot{H}^{-\gamma}_x(\mathbb{R}^n))$$

is bounded. The adjoint operator

$$(T^*G)(x) = \int_0^\infty e^{\mp is\Lambda}\beta G(s,x))ds$$

is bounded as an operator from $L^2_t(\mathbb{R}^+; \dot{H}^\gamma_x(\mathbb{R}^n))$ to $\dot{H}^\gamma_x(\mathbb{R}^n)$ and this yields

$$\Big\|\int_0^\infty e^{\pm is\Lambda}\beta h(s,x)(s,x)ds\Big\|_{\dot{H}^\gamma(\mathbb{R}^n)} \leq C_2\|h\|_{L^2_t(\mathbb{R}^+;\, \dot{H}^\gamma_x(\mathbb{R}^n))}. \tag{5.3}$$

Consider the integral operators

$$J : L^2_t(\mathbb{R}^+; \dot{H}^\gamma_x(\mathbb{R}^n)) \ni h(t,x) \longrightarrow \int_0^t K(s,t)h(s,x)ds \in L^p_t(\mathbb{R}^+;\ L^q_x(\mathbb{R}^n)),$$

where $K(s,t) = \Lambda^{-1}\sin((t-s)\Lambda)\beta$. To apply Christ-Kiselev lemma [4], it is sufficient to have an estimate for

$$\Big\|\int_0^\infty \frac{\sin((t-s)\Lambda)}{\Lambda}\beta h(s,x)ds\Big\|_{L^p_t(\mathbb{R}^+; L^q_x(\mathbb{R}^n))}.$$

By (1.3) and (5.3), we get

$$\Big\|e^{\pm it\Lambda}\Lambda^{-1}\int_0^\infty e^{\pm is\Lambda}\beta h(s,x)ds\Big\|_{L^p_t(\mathbb{R}^+;\,L^q_x(\mathbb{R}^n))}$$

$$\leq C_3\Big\|\int_0^\infty e^{\pm is\Lambda}\beta h(s,x)ds\Big\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \leq C_2C_3\|h\|_{L^2_t(\mathbb{R}^+;\,\dot{H}^\gamma_x(\mathbb{R}^n))}.$$

We take $h = Vu_0 + Vv$ and we use the addition formula for $\sin((t-s)\Lambda)$ to conclude that

$$\Big\|\int_0^\infty \frac{\sin((t-s)\Lambda)}{\Lambda}(Vu_0+Vv)ds\Big\|_{L^p_t(\mathbb{R}^+;L^q_x(\mathbb{R}^n))} \leq C_4\|Vu_0+Vv\|_{L^2_t(\mathbb{R}^+;\,\dot{H}^\gamma_x(\mathbb{R}^n))}. \tag{5.4}$$

By hypothesis $p > 2$, and an application of Christ-Kiselev lemma [4] yields immediately (5.2). Consequently, (5.1) implies an estimate for $\|v\|_{L^p_t(\mathbb{R}^+;L^q_x(\mathbb{R}^n))}$. Similarly, we deal with the norm $\|v\|_{L^p_t(\mathbb{R}^-;L^q_x(\mathbb{R}^n))}$. To estimate $\|v(t_0,x)\|_{\dot{H}^\gamma(\mathbb{R}^n_x)}$ uniformly with respect to t_0, notice that

$$\Big\|e^{\pm it\Lambda}\Lambda^{-1}\int_0^{t_0} e^{\pm is\Lambda}(Vu_0+Vv)(s,x)ds\Big\|_{\dot{H}^\gamma(\mathbb{R}^n)} \leq C_5\Big\|\int_0^{t_0} e^{\pm is\Lambda}(Vu_0+Vv)(s,x)ds\Big\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}$$

with a constant $C_5 > 0$ independent of t_0. As above, we can estimate the right hand part by $\|Vu_0+Vv\|_{L^2_t(\mathbb{R};\,\dot{H}^\gamma_x(\mathbb{R}^n))}$ uniformly with respect to t_0 and apply (5.1). A similar argument works for $\|\partial_t v(t_0,x)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n_x)}$. Thus the proof of Theorem 1 is complete.

To obtain homogeneous Strichartz estimates, we need to apply Proposition 1 combined with the estimate (4.2). Moreover, γ is related only to n, p, q.

Theorem 3. *Let the condition* $(\mathcal{R})$ *be fulfilled. Suppose that* $2 \leq p, q \leq +\infty$, $0 \leq \gamma \leq (n-1)/2$, $p > 2$ *are such that the solution* $u_0(t,x)$ *of the problem* (1.1) *with* $V=0$, $F=0$, $\tau=0$ *satisfies the estimate*

$$\|u_0\|_{L^p_t(\mathbb{R};\,L^q_x(\mathbb{R}^n))} \leq C\Big(\|f_0\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|f_1\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}\Big).$$

Then the solution $u(t,x)$ *of the problem* (1.1) *with* $F=0$, $\tau=0$ *satisfies the following estimate*

$$\|u\|_{L^p_t(\mathbb{R};\,L^q_x(\mathbb{R}^n))} \leq C_1\Big(\|f_0\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|f_1\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}\Big).$$

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Département de Mathématiques Appliquées, Université Bordeaux I, 351, Cours de la Libération, 33405 Talence, France

E-mail address: `petkov@math.u-bordeaux1.fr`