SHARP HARDY TYPE INEQUALITIES ON THE CARNOT GROUP

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Abstract. In this paper we establish sharp weighted Hardy type inequalities on the Carnot group with homogeneous dimension $Q \geq 3$.

1. Introduction

The classical Hardy inequality states that for $n \geq 3$

$$\int_{\mathbb{R}^n} |\nabla \phi(x)|^2 dx \geq (\frac{n-2}{2})^2 \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^2} dx \tag{1.1}$$

where $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and the constant $(\frac{n-2}{2})^2$ is sharp. There exists a large literature dealing with the Hardy type inequalities on the Euclidean space and, in particular, sharp inequalities which have attracted considerable attention because of their application to certain singular problems .

In this paper we investigate the existence and the explicit determination of constants C and weight $q(x)$ on the Carnot group $\mathbb{G}$ such that the Hardy type inequality

$$\int_{\mathbb{G}} w(x) |\nabla_{\mathbb{G}} \phi(x)|^2 dx \geq C \int_{\mathbb{G}} q(x) |\phi(x)|^2 dx \tag{1.2}$$

holds for all $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Here we consider a special weight function $w(x)$ which is related to the fundamental solution of sub-Laplacian $\Delta_{\mathbb{G}}$ on the Carnot group $\mathbb{G}$.

It is well known that the Euclidean space $\mathbb{R}^n$ with its usual Abelian group structure is a trivial Carnot group and the weighted Hardy type inequalities have been studied extensively. We are concerned with the inequality (1.2) on the non-trivial Carnot groups.

The simplest non trivial example of the Carnot group is given by the Heisenberg group $\mathbb{H}^n$. The following Hardy type inequality on the Heisenberg group $\mathbb{H}^n$ was first proved by Garofalo and Lanconelli [10]

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} \phi|^2 dz dt \geq (\frac{Q-2}{2})^2 \int_{\mathbb{H}^n} (\frac{|z|^2}{|z|^4 + t^2}) \phi^2 dz dt \tag{1.3}$$

where $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$, $Q = 2n + 2$ and the constant $(\frac{Q-2}{2})^2$ is sharp.

The Hardy type inequalities on the Heisenberg group $\mathbb{H}^n$ have also received considerable attention in recent years. The L^p version of the inequality (1.3) has been obtained by Niu, Zhang, and Wang [16]. A different proof of (1.3) with the sharp constant $(\frac{Q-2}{2})^2$ has been given by Goldstein and Zhang [11]. In [2], D'Ambrosio obtained weighted Hardy type inequalities on the Heisenberg group $\mathbb{H}^n$.

Date: January 27, 2005.

Key words and phrases. Carnot group, Hardy inequality.

AMS Subject Classifications: 22E30, 43A80, 26D10.

Although we prove Hardy type inequality on the Carnot group with an arbitrary step, we first establish a sharp weighted Hardy type inequality on the Heisenberg group and extend this result to the H-type groups. The main point is that the fundamental solution of the sub-Laplacian on the Heisenberg group and H-type groups are known explicitly (see section 3) but not for the general Carnot group. The proof of our theorem on the Carnot group differs slightly in some steps from the the Heisenberg group and H-type group cases. The method we apply here, inspired by the work of Allegretto [1], can be applied to the Baoundi-Grushin type vector fields in that they do not arise from any Carnot group.

In order to state our theorems on the Carnot group $\mathbb{G}$, we first recall the basic properties of Carnot $\mathbb{G}$ and some well known results that will be used in the sequel. The following section is largely taken from [3], [7], [8], [9], [14] and [15].

2. CARNOT GROUP

A Carnot group is a connected, simply connected, nilpotent Lie group $\mathbb{G} \equiv (\mathbb{R}^n, \cdot)$ whose Lie algebra $\mathcal{G}$ admits a stratification. That is, there exist linear subspaces $V_1, ..., V_k$ of $\mathcal{G}$ such that

$$\mathcal{G} = V_1 \oplus ... \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad \text{for} \quad i = 1, 2, ..., k-1 \quad \text{and} \quad [V_1, V_k] = 0 \tag{2.1}$$

where $[V_1, V_i]$ is the subspace of $\mathcal{G}$ generated by the elements $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. This defines a k-step Carnot group and integer $k \geq 1$ which is called the step of $\mathbb{G}$.

Via the exponential map, it is possible to induce on $\mathbb{G}$ a family of automorphisms of the group, called dilations, $\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n (\lambda > 0)$ such that

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n)$$

where $1 = \alpha_1 = ... = \alpha_m < \alpha_{m+1} \leq ... \leq \alpha_n$ are integers and $m = \dim(V_1)$.

The group law can be written in the following form

$$x \cdot y = x + y + P(x, y), \quad x, y \in \mathbb{R}^n \tag{2.2}$$

where $P : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has polynomial components and $P_1 = ... = P_m = 0$. Note that the inverse x^{-1} of an element $x \in \mathbb{G}$ has the form $x^{-1} = -x = (-x_1, ... - x_n)$.

Let $X_1, ..., X_m$ be a family of left invariant vector fields that is an orthonormal basis of $V_1 \equiv \mathbb{R}^m$ at the origin, that is, $X_1(0) = \partial_{x_1}, ..., X_m(0) = \partial_{x_m}$. The vector fields X_j have polynomial coefficients and can be assumed to be of the form

$$X_j(x) = \partial_j + \sum_{i=j+1}^{n} a_{ij}(x) \partial_i, \quad X_j(0) = \partial_j, j = 1, ..., m,$$

where each polynomial a_{ij} is homogeneous with respect to the dilations of the group, that is $a_{ij}(\delta_\lambda(x)) = \lambda^{\alpha_i - \alpha_j} a_{ij}(x)$. The horizontal gradient on the Carnot group $\mathbb{G}$ is the vector valued operator

$$\nabla_{\mathbb{G}} = (X_1, ..., X_m)$$

where $X_1, ..., X_m$ are the generators of $\mathbb{G}$. The sub-Laplacian is the second-order partial differential operator on $\mathbb{G}$ given by

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{m} X_j^2.$$

The fundamental solution u for $\Delta_{\mathbb{G}}$ is defined to be a weak solution to the equation

$$-\Delta_{\mathbb{G}}u = \delta$$

where δ denotes the Dirac distribution with singularity at the neutral element 0 of $\mathbb{G}$. In [7] Folland proved that in any Carnot group $\mathbb{G}$, there exists a homogeneous norm N such that

$$u = N^{2-Q}$$

is a fundamental solution for $\Delta_{\mathbb{G}}$ (see also [5]).

We now set $N(x) := u^{\frac{1}{2-Q}}$ if $x \neq 0$ and $N(0) := 0$. We recall that a homogeneous norm on $\mathbb{G}$ is a continuous function $N : \mathbb{G} \longrightarrow [0, \infty)$ smooth away from the origin which satisfies the conditions : $N(\delta_\lambda(x)) = \lambda N(x)$, $N(x^{-1}) = N(x)$ and $N(x) = 0$ iff $x = 0$.

The curve $\gamma : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{G}$ is called horizontal if its tangents lie in V_1, i.e, $\gamma'(t) \in span\{X_1, ..., X_m\}$ for all t. Then, the Carnot-Caréthedory distance $d_{CC}(x, y)$ between two points $x, y \in \mathbb{G}$ is defined to be the infimum of all horizontal lengths $\int_a^b \langle \gamma'(t), \gamma'(t)\rangle^{1/2} dt$ over all horizontal curves $\gamma : [a, b] \longrightarrow \mathbb{G}$ such that $\gamma(a) = x$ and $\gamma(y) = b$. Notice that d_{CC} is a homogeneous norm and satisfies the invariance property

$$d_{CC}(z \cdot x, z \cdot y) = d_{CC}(x, y), \quad \forall\, x, y, z \in \mathbb{G},$$

and is homogeneous of degree one with respect to the dilation δ_λ, i.e.

$$d_{CC}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_{CC}(x, y), \quad \forall\, x, y, z \in \mathbb{G}, \forall\, \lambda > 0.$$

The Carnot-Carethédory balls are defined by $B(x, R) = \{y \in \mathbb{G} | d_{CC}(x, y) < R\}$. By left-translation and dilation, it is easy to see that the Haar measure of $B(x, R)$ is proportional by R^Q. More precisely

$$|B(x, R)| = R^Q |B(x, 1)| = R^Q |B(0, 1)|$$

where

$$Q = \sum_{j=1}^{k} j(\dim V_j)$$

is the homogeneous dimension of $\mathbb{G}$.

3. Hardy type inequality on the Carnot group of step 2

Among the Carnot groups of step two, the Heisenberg group and Heisenberg type (H-type) groups are of particular significance. These groups appear naturally in analysis, geometry, representation theory and mathematical physics. In this section, we first prove Hardy type inequality on the Heisenberg group and we extend this result to the H-type group.

Heisenberg group. The Heisenberg group $\mathbb{H}^n$ is an example of a noncommutative Carnot group. Denoting points in $\mathbb{H}^n$ by (z, t) with $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $t \in \mathbb{R}$ we have the group law given as

$$(z, t) \circ (z', t') = (z + z', t + t' + 2\sum_{j=1}^{n} Im(z_j \bar{z}'_j))$$

With the notation $z_j = x_j + iy_j$, the horizontal space V_1 is spanned by the basis

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}.$$

The one dimensional center V_2 is spanned by the vector field $T = \frac{\partial}{\partial t}$. We have the commutator relations $[X_j, Y_j] = -4T$, and all other brackets of $\{X_1, Y_1, ..., X_n, Y_n\}$ are zero. The sub-elliptic gradient is the $2n$ dimensional vector field given by

$$\nabla_{\mathbb{H}^n} = (X_1, ..., X_n, Y_1, ..., Y_n)$$

and the Kohn Laplacian on $\mathbb{H}^n$ is the operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n}(X_j^2 + Y_j^2).$$

A homogeneous norm on $\mathbb{H}^n$ is given by

$$\rho = |(z,t)| = (|z|^4 + t^2)^{1/4}$$

and the homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n + 2$.

A remarkable analogy between Kohn Laplacian and the classical Laplace operator has been obtained by Folland [6]. He found that the fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole zero is given by

$$\Gamma(z,t) = \frac{c_Q}{\rho(z,t)^{Q-2}} \quad \text{where} \quad c_Q = \frac{2^{(Q-2)/2}\Gamma((Q-2)/4)^2}{\pi^{Q/2}}.$$

We now prove the following theorem on the Heisenberg group $\mathbb{H}^n$ (See [2] for a different proof). In the various integral inequalities below (Section 3 and Section 4), we allow the values of the integrals on the left-hand sides to be $+\infty$.

Theorem 3.1. *Let* $\alpha \in \mathbb{R}$ *and* $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0,0\})$. *Then we have :*

$$\int_{\mathbb{H}^n} \rho^\alpha |\nabla_{\mathbb{H}^n}\phi|^2 dzdt \geq (\frac{Q+\alpha-2}{2})^2 \int_{\mathbb{H}^n} \rho^\alpha \frac{|z|^2}{\rho^4}\phi^2 dzdt$$

where $\rho = (|z|^4 + l^2)^{1/4}$ *is the homogeneous norm on* $\mathbb{H}^n$. *Moreover, the constant* $(\frac{Q+\alpha-2}{2})^2$ *is sharp.*

Proof. Let $\phi = \rho^\beta\psi$ where $\beta \in \mathbb{R}\setminus\{0\}$ and $\psi \in C_0^\infty(\mathbb{H}^n \setminus \{0,0\})$. A direct calculation shows that

$$\rho^\alpha|\nabla_{\mathbb{H}^n}\phi|^2 = \beta^2\rho^{\alpha+2\beta-2}|\nabla_{\mathbb{H}^n}\rho|^2\psi^2 + 2\beta\rho^{\alpha+2\beta-1}\psi\nabla_{\mathbb{H}^n}\rho \cdot \nabla_{\mathbb{H}^n}\psi + \rho^{\alpha+2\beta}|\nabla_{\mathbb{H}}\psi|^2. \tag{3.1}$$

It is easy to see that

$$|\nabla_{\mathbb{H}^n}\rho|^2 = \frac{|z|^2}{\rho^2}$$

and integrating (3.1) over $\mathbb{H}^n$, we get

$$\begin{aligned}\int_{\mathbb{H}^n} \rho^\alpha|\nabla_{\mathbb{H}^n}\phi|^2 dzdt = \int_{\mathbb{H}^n} \beta^2\rho^{\alpha+2\beta-4}|z|^2\psi^2 dzdt + \int_{\mathbb{H}^n} 2\beta\rho^{\alpha+2\beta-1}\psi\nabla_{\mathbb{H}^n}\rho \cdot \nabla_{\mathbb{H}^n}\psi dzdt \\ + \int_{\mathbb{H}^n} \rho^{\alpha+2\beta}|\nabla_{\mathbb{H}^n}\psi|^2 dzdt\end{aligned} \tag{3.2}$$

Applying integration by parts to the middle integral on the right-hand side of (3.2), we obtain

$$\begin{aligned}\int_{\mathbb{H}^n}\rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi|^2dzdt=&\int_{\mathbb{H}^n}\beta^2\rho^{\alpha+2\beta-4}|z|^2\psi^2dzdt-\frac{\beta}{\alpha+2\beta}\int_{\mathbb{H}^n}\Delta_{\mathbb{H}^n}(\rho^{\alpha+2\beta})\psi^2dzdt\\&+\int_{\mathbb{H}^n}\rho^{\alpha+2\beta}|\nabla_{\mathbb{H}^n}\psi|^2dzdt.\end{aligned}\tag{3.3}$$

One can show that

$$\Delta_{\mathbb{H}^n}(\rho^{\alpha+2\beta})=|z|^2\rho^{\alpha+2\beta-4}(\alpha+2\beta)(\alpha+2\beta+Q-2).\tag{3.4}$$

Substituting (3.4) into (3.3) gives the following

$$\begin{aligned}\int_{\mathbb{H}^n}\rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi|^2dzdt&=(\beta^2-\beta(\alpha+2\beta+Q-2))\int_{\mathbb{H}^n}\rho^{\alpha+2\beta-4}|z|^2\psi^2dzdt+\int_{\mathbb{H}^n}\rho^{\alpha+2\beta}|\nabla_{\mathbb{H}^n}\psi|^2dzdt\\&\geq(-\beta^2-\beta(\alpha+Q-2))\int_{\mathbb{H}^n}\rho^{\alpha+2\beta-4}|z|^2\psi^2dzdt.\end{aligned}$$

We now choose $\beta=\frac{2-\alpha-Q}{2}$ (Note that the quadratic equation $-\beta^2-\beta(\alpha+Q-2)$ reaches its maximum value at $\beta=\frac{2-\alpha-Q}{2}$) and noting that $\psi=\rho^{-\beta}\phi$, we have the following inequality

$$\int_{\mathbb{H}^n}\rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi|^2dzdt\geq(\frac{Q+\alpha-2}{2})^2\int_{\mathbb{H}^n}\rho^{\alpha}\frac{|z|^2}{\rho^4}\phi^2dzdt.\tag{3.5}$$

□

Heisenberg type group. Another important model of Carnot groups are the H-type (Heisenberg type) groups which were introduced by Kaplan [13] as direct generalizations of the Heisenberg group $\mathbb{H}^n$. An H-type group is a Carnot group with a two-step Lie algebra $\mathcal{G}=V_1\oplus V_2$ and an inner product $\langle,\rangle$ in $\mathcal{G}$ such that the linear map

$$J:V_2\longrightarrow \mathrm{End}V_1,$$

defined by the condition

$$\langle J_z(u),v\rangle=\langle z,[u,v]\rangle,\quad u,v\in V_1,z\in V_2$$

satisfies

$$J_z^2=-||z||^2\mathbf{Id}$$

for all $z\in V_2$, where $||z||^2=\langle z,z\rangle$.

Sub-Laplacian is defined in terms of a fixed basis $X_1,...,X_m$ for V_1:

$$\Delta_{\mathbb{G}}=\sum_{i=1}^{m}X_i^2.\tag{3.6}$$

The exponential mapping of a simply connected Lie group is an analytic diffeomorphism. One can then define analytic mappings $v:\mathbb{G}\longrightarrow V_1$ and $z:\mathbb{G}\longrightarrow V_2$ by

$$x=\exp(v(x)+z(x))$$

for every $x\in\mathbb{G}$. In [13] Kaplan proved that there exists a constant $c>0$ such that the function

$$\Phi(x)=c\Big(|v(x)|^4+16|z(x)|^2\Big)^{\frac{2-Q}{4}}$$

is a fundamental solution for the operator $-\Delta_{\mathbb{G}}$. We note that

$$K(x) = \Big(|v(x)|^4 + 16|z(x)|^2\Big)^{\frac{1}{4}} \tag{3.7}$$

defines a homogeneous norm and $Q = m + 2k$ is the homogeneous dimension of $\mathbb{G}$ where $m = \text{dim}V_1$ and $k = \text{dim}V_2$. This result generalized Folland's fundamental solution for the Heisenberg group $\mathbb{H}^n$ [6]. Note that Kaplan's results provides us an explicit fundamental solution for $\Delta_{\mathbb{G}}$ on H-type groups. We should also mention that the explicit fundamental solutions for the sub-elliptic p-Laplacian on the H-type groups were obtained by Capogna, Danielli and Garofalo [4], Heinonen and Holopainen [12].

We cite, without proof of the following, useful formulas which can be found in [4]: Let u be a radial function, i.e., $u(x) = f(K(x))$ where $f \in C(\mathbb{R})$ then

$$|\nabla_{\mathbb{G}} u|^2 = \frac{|v|^2}{K^2}|f'(K)|^2.$$

Moreover if $u(x) = f(K(x))$ and $f \in C^2(\mathbb{R})$ then

$$\begin{aligned} \Delta_{\mathbb{G}} u &= |\nabla_{\mathbb{G}} K(x)|^2\Big[f''(K) + \frac{Q-1}{K}f'(K)\Big] \\ &= \frac{v^2}{K^2}\Big[f''(K) + \frac{Q-1}{K}f'(K)\Big] \end{aligned} \tag{3.8}$$

at every point $x \in \mathbb{G} \setminus \{0\}$ where $f'(K(x)) \neq 0$.

We now have the following theorem on the H-type group :

Theorem 3.2. *Let $\mathbb{G}$ be an H-type group with homogeneous dimension $Q = m + 2k$ and let $\alpha \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Then the following inequality is valid :*

$$\int_{\mathbb{G}} K^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx \geq (\frac{Q+\alpha-2}{2})^2 \int_{\mathbb{G}} K^\alpha \frac{|v|^2}{K^4}\phi^2 dx \tag{3.9}$$

where $K(x) = (|v(x)|^4 + 16|z(x)|^2)^{1/4}$. Moreover, the constant $(\frac{Q+\alpha-2}{2})^2$ is sharp.

Proof. The proof is identical to the Heisenberg group case. ☐

4. HARDY-TYPE INEQUALITY ON THE CARNOT GROUP OF ARBITRARY STEP

In this section, we consider the Carnot group $\mathbb{G}$ of any step k with the homogeneous norm $N = u^{1/(2-Q)}$ associated to Folland's solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$ [7]. We have the following theorem:

Theorem 4.1. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and let $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $\alpha \in \mathbb{R}$, $Q + \alpha - 2 > 0$. Then the following inequality is valid*

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx \geq (\frac{Q+\alpha-2}{2})^2 \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^2}\phi^2 dx. \tag{4.1}$$

Here $N = u^{1/(2-Q)}$ is the homogeneous norm associated with the fundamental solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$. Furthermore, the constant $C(Q,\alpha) = (\frac{Q+\alpha-2}{2})^2$ is sharp.

Proof. Let $\phi = N^\beta \psi$ where $\psi \in C_0^\infty(\mathbb{G} \setminus \{0\})$ and $\beta \in \mathbb{R} \setminus \{0\}$. A direct calculation shows that

$$|\nabla_{\mathbb{G}}(N^\beta \psi)|^2 = \beta^2 N^{2\beta-2}|\nabla_{\mathbb{G}} N|^2 \psi^2 + 2\beta N^{2\beta-1}\psi \nabla_{\mathbb{G}} N \cdot \nabla_{\mathbb{G}} \psi + N^{2\beta}|\nabla_{\mathbb{G}}\psi|^2. \tag{4.2}$$

Multiplying both sides of (4.2) by the N^α and applying integration by parts over $\mathbb{G}$ gives

$$\begin{aligned}\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx &= \beta^2 \int_{\mathbb{G}} N^{\alpha+2\beta-2}|\nabla_{\mathbb{G}} N|^2 \psi^2 dx - \frac{\beta}{\alpha+2\beta}\int_{\mathbb{G}} \Delta_{\mathbb{G}}(N^{\alpha+2\beta})\psi^2 dx \\ &\quad + \int_{\mathbb{G}} N^{\alpha+2\beta}|\nabla_{\mathbb{G}}\psi|^2 dx \\ &\geq \beta^2 \int_{\mathbb{G}} N^{\alpha+2\beta-2}|\nabla_{\mathbb{G}} N|^2 \psi^2 dx - \frac{\beta}{\alpha+2\beta}\int_{\mathbb{G}} \Delta_{\mathbb{G}}(N^{\alpha+2\beta})\psi^2 dx.\end{aligned} \tag{4.3}$$

A straightforward calculation shows that

$$-\frac{\beta}{\alpha+2\beta}\Delta_{\mathbb{G}}(N^{\alpha+2\beta}) = -\beta(\alpha+2\beta+Q-2)N^{\alpha+2\beta-2}|\nabla_{\mathbb{G}} N|^2 - \frac{\beta}{2-Q}N^{\alpha+2\beta+Q-2}\Delta_{\mathbb{G}} u. \tag{4.4}$$

Substituting (4.4) into (4.3) and using the fact that $\psi^2 = N^{-2\beta}\phi^2$, we get the following :

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx \geq (-\beta^2 - \beta(\alpha+Q-2))\int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^2}\phi^2 dx - \frac{\beta}{2-Q}\int_{\mathbb{G}} (\Delta_{\mathbb{G}} u)N^{\alpha+Q-2}\phi^2 dx.$$

Since u is the fundamental solution of sub-Laplacian $\Delta_{\mathbb{G}}$ on the Carnot group $\mathbb{G}$, we get

$$\int_{\mathbb{G}} (\Delta_{\mathbb{G}} u)N^{\alpha+Q-2}\phi^2 dx = N^{\alpha+Q-2}(0)\phi^2(0) = 0.$$

This implies that

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx \geq (-\beta^2 - \beta(\alpha+Q-2))\int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^2}\phi^2 dx.$$

Choosing

$$\beta = \frac{2-Q-\alpha}{2}$$

gives the following sharp inequality

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx \geq (\frac{Q+\alpha-2}{2})^2 \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^2}\phi^2 dx. \tag{4.5}$$

□

Remark. In the abelian case, when $\mathbb{G} = \mathbb{R}^n$ with the ordinary dilations, one has $\mathcal{G} = V_1 = \mathbb{R}^n$ so that $Q = n$. Now it is clear that the inequality (4.1) with the homogeneous norm $N(x) = |x|$ and $\alpha = 0$ recovers the Hardy's inequality (1.1).

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