Spectral Properties of a Polyharmonic Operator with Limit Periodic Potential in Dimension Two (an announcement).

Yulia Karpeshina [1], Young-Ran Lee [1]

In memory of our colleague and friend
Robert M. Kauffman.

ABSTRACT We consider a polyharmonic operator $H = (-\Delta)^l + V(x)$ in dimension two with $l > 5$ and $V(x)$ being a limit-periodic potential. We prove that the spectrum of H contains a semiaxis and there is a family of generalized eigenfunctions at every point of this semiaxis with the following properties. First, the eigenfunctions are close to plane waves at the high energy region. Second, the isoenergetic curves in the space of momenta corresponding to these eigenfunctions have a form of a distorted circles with holes (Cantor type structure). Third, the spectrum corresponding to the eigenfunctions (the semiaxis) is absolutely continuous.

We study the operator

$$H = (-\Delta)^l + V(x) \tag{1}$$

in two dimensions, $V(x)$ being a limit-periodic potential:

$$V(x) = \sum_{n=1}^{\infty} V_n(x), \tag{2}$$

where $\{V_n\}_{n=1}^{\infty}$ is a family of periodic potentials with doubling periods and decreasing L_∞-norms, namely, V_n has orthogonal periods $2^{n-1}\vec{a_1},\ 2^{n-1}\vec{a_2}$ and $\|V_n\|_\infty < exp(-2^{\eta n})$ for some $\eta > 0$.

The one-dimensional analog of (1), (2) with $l = 1$ has been already thoroughly investigated. It is proven in [1-9] that the spectrum of the operator $H_1 u = -u'' + Vu$ is a Cantor type set. The spectrum is absolutely continuous. Generalized eigenfunctions can be represented in the form of $e^{ikx}u(x)$, $u(x)$ being limit-periodic. The case of a complex-valued potential is studied in [10]. Integrated density of states is investigated in [11-14]. Properties of eigenfunctions of discrete multidimensional limit-periodic Schrödinger operator are studied in [15]. As to the continuum multidimensional case, it is known that the integrated density of states for (1) is the limit of densities of states for periodic operators [14]. We concentrate on properties of the spectrum and eigenfunctions in the high energy region. We prove the following results for the case $d = 2$, $l > 5$.

---

[1] Research partially supported by USNSF Grant DMS-0201383.

1. The spectrum of the operator (1), (2) contains a semiaxis.

2. There are generalized eigenfunctions $\psi_\infty(\vec{k},\vec{x})$, corresponding to the semiaxis, which are close to plane waves: for every $\vec{k}$ in an extensive subset Ω_∞ of $\mathbb{R}^2$, there is a solution $\psi_\infty(\vec{k},\vec{x})$ of the equation $H\psi_\infty = \lambda_\infty\psi_\infty$ which can be described by the formula:

   $$\psi_\infty(\vec{k},\vec{x}) = e^{i\langle\vec{k},\vec{x}\rangle}\left(1+u_\infty(\vec{k},\vec{x})\right), \tag{3}$$

   $$\|u_\infty\| =_{|\vec{k}|\to\infty} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1>0, \tag{4}$$

   here $u_\infty(\vec{k},\vec{x})$ is a limit-periodic function:

   $$u_\infty(\vec{k},\vec{x}) = \sum_{n=1}^{\infty}\tilde{u}_n(\vec{k},\vec{x}), \tag{5}$$

   $\tilde{u}_n(\vec{k},\vec{x})$ being periodic with periods $2^{n-1}\vec{a_1}$, $2^{n-1}\vec{a_2}$. The eigenvalue $\lambda_\infty(\vec{k})$ corresponding to $\psi_\infty(\vec{k},\vec{x})$ is close to $|\vec{k}|^{2l}$:

   $$\lambda_\infty(\vec{k}) =_{|\vec{k}|\to\infty} |\vec{k}|^{2l} + O(|\vec{k}|^{-\gamma_2}), \quad \gamma_2>0. \tag{6}$$

   The "non-resonant" set Ω_∞ of the vectors $\vec{k}$, for which (3) – (6) hold, is a Cantor type set: $\Omega_\infty = \cap_{n=1}^{\infty}\Omega_n$, where $\{\Omega_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets with larger and larger number of holes in each bounded region, holes added on each step being of smaller and smaller size. The set Ω_∞ is proven to be sufficiently extensive:

   $$\lim_{R\to\infty}\frac{s\left(\Omega_\infty\cap\mathbf{B_R}\right)}{s\left(\mathbf{B_R}\right)} = 1, \tag{7}$$

   where $\mathbf{B_R}$ is the disc of radius R centered at the origin, $s(\cdot)$ is the Lebesgue measure in $\mathbb{R}^2$.

3. The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\vec{k})$,

   $$\mathcal{D}_\infty(\lambda) = \left\{\vec{k}\in\Omega_\infty : \lambda_\infty(\vec{k}) = \lambda\right\},$$

   is proven to be a distorted circle with infinite number of holes. It can be described by the formula:

   $$\mathcal{D}_\infty(\lambda) = \{\vec{k} : \vec{k} = \left(\lambda^{1/2l} + h_\infty(\lambda,\vec{\nu})\right)\vec{\nu},\ \vec{\nu}\in\mathcal{B}_\infty(\lambda)\}, \tag{8}$$

   where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible propagation directions for the almost plane waves (3). The set $\mathcal{B}_\infty(\lambda)$ has a Cantor type structure and an asymptotically full measure on S_1 as $\lambda\to\infty$:

   $$|\mathcal{B}_\infty(\lambda)| =_{\lambda\to\infty} 2\pi + O\left(\lambda^{-\gamma_3}\right), \quad \gamma_3>0. \tag{9}$$

The value $h_\infty(\lambda,\vec{\nu})$ in (8) gives the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of the radius $\lambda^{1/2l}$ in the direction $\vec{\nu}$. It is proven that the deviation is asymptotically small

$$h_\infty(\lambda,\vec{\nu}) = O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4 > 0. \tag{10}$$

4. Absolute continuity of the branch of the spectrum (the semiaxis) corresponding to $\psi_\infty(\vec{k},\vec{x})$ is proven.

To prove the results listed above we develop a modification of the Kolmogorov-Arnold-Moser method. We consider a sequence of operators

$$H_0 = (-\Delta)^l, \qquad H_n = H_0 + \sum_{m=1}^{L_n} V_m, \qquad L_n \to \infty \text{ as } n \to \infty.$$

Obviously,

$$H_n = H_{n-1} + \tilde{V}_n, \qquad \tilde{V}_n = \sum_{m=L_{n-1}+1}^{L_n} V_m,$$

$$\|H - H_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Each operator H_n, $n \geq 1$, is considered as a perturbation of the previous operator H_{n-1}. Every operator H_n is periodic, but the periods go to infinity as $n \to \infty$. We show that there is a λ_0, $\lambda_0 = \lambda_0(V)$, such that the semiaxis $[\lambda_0, \infty)$ is in the spectra of all operators H_n. For every operator H_n there is a set of eigenfunctions (corresponding to the semiaxis) being close to plane waves: for every $\vec{k}$ in an extensive subset Ω_n of $\mathbb{R}^2$, there is a solution $\psi_n(\vec{k},\vec{x})$ of the differential equation $H_n\psi_n = \lambda_n\psi_n$, which can be described by the formula:

$$\psi_n(\vec{k},\vec{x}) = e^{i\langle\vec{k},\vec{x}\rangle}\left(1 + u_n(\vec{k},\vec{x})\right), \tag{11}$$

$$\|u_n\| =_{|\vec{k}|\to\infty} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1 > 0, \tag{12}$$

here $u_n(\vec{k},\vec{x})$ is a periodic function. The corresponding eigenvalue $\lambda_n(\vec{k})$ is close to $|\vec{k}|^{2l}$:

$$\lambda_n(\vec{k}) =_{|\vec{k}|\to\infty} |\vec{k}|^{2l} + O(|\vec{k}|^{-\gamma_2}), \quad \gamma_2 > 0.$$

The non-resonant set Ω_n is proven to be extensive in $\mathbb{R}^2$:

$$\lim_{R\to\infty} \frac{s\left(\Omega_n \cap \mathbf{B_R}\right)}{s\left(\mathbf{B_R}\right)} = 1. \tag{13}$$

The set $\mathcal{D}_n(\lambda)$ is defined as the level (isoenergetic) set for $\lambda_n(\vec{k})$:

$$\mathcal{D}_n(\lambda) = \left\{\vec{k} \in \Omega_n : \lambda_n(\vec{k}) = \lambda\right\}.$$

Refer the pictures of $\mathcal{D}_1(\lambda)$ and $\mathcal{D}_2(\lambda)$.

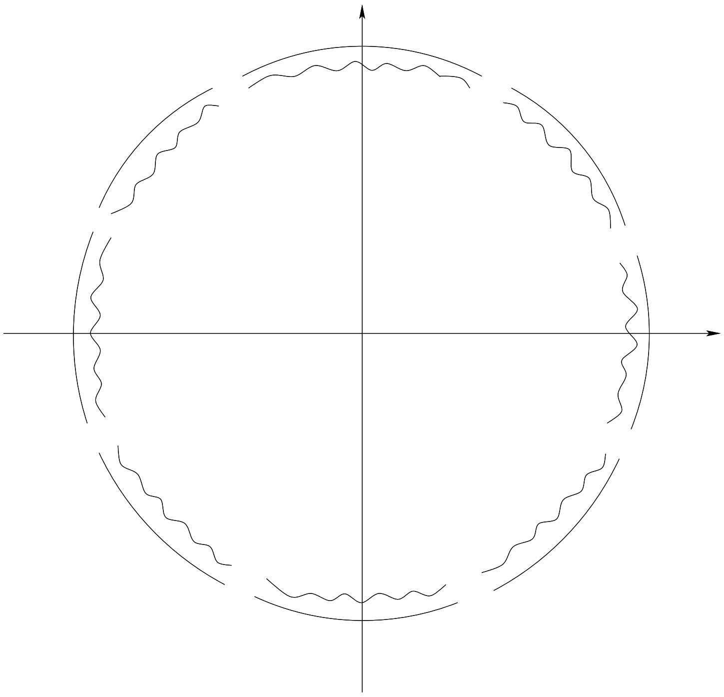

Figure 1: Distorted circle with holes, $\mathcal{D}_1(\lambda)$

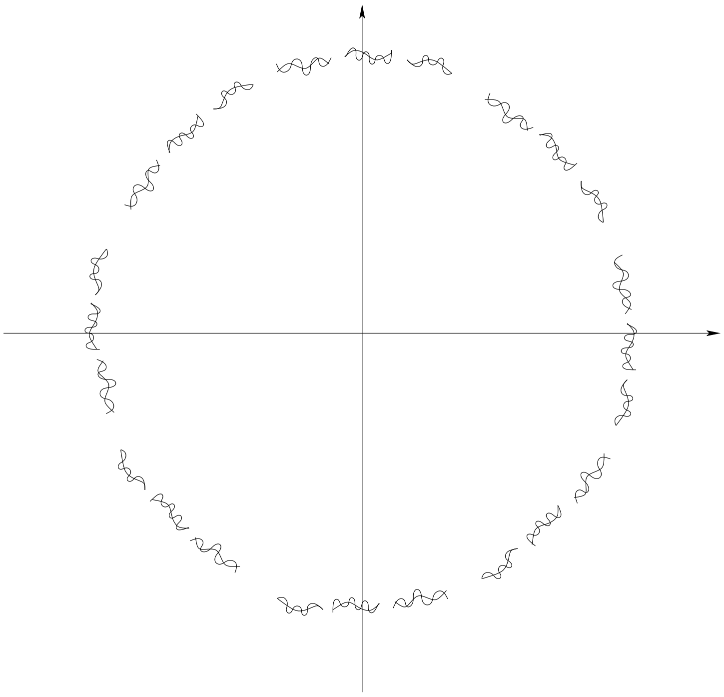

Figure 2: Distorted circle with holes, $\mathcal{D}_2(\lambda)$

This set is proven to be a distorted circle with a finite number of holes. It can be described by the formula:

$$\mathcal{D}_n(\lambda) = \{\vec{k} : \vec{k} = \left(\lambda^{1/2l} + h_n(\lambda, \vec{\nu})\right) \vec{\nu},\ \vec{\nu} \in \mathcal{B}_n(\lambda)\}, \tag{14}$$

where $\mathcal{B}_n(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_n(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (11). It has an asymptotically full measure on S_1 as $\lambda \to \infty$:

$$|\mathcal{B}_n(\lambda)| =_{\lambda \to \infty} 2\pi + O\left(\lambda^{-\gamma_3}\right), \quad \gamma_3 > 0. \tag{15}$$

The set $\mathcal{B}_n$ has only a finite number of holes, however their number is growing with n. More and more holes of a smaller and smaller size are added at each step. The value $h_n(\lambda, \vec{\nu})$ gives the deviation of $\mathcal{D}_n(\lambda)$ from the perfect circle of the radius $\lambda^{1/2l}$ in the direction $\vec{\nu}$. It is proven that the deviation is asymptotically small:

$$h_n(\lambda, \vec{\nu}) = O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4 > 0. \tag{16}$$

On each step more and more points are excluded from the non-resonant sets Ω_n, thus $\{\Omega_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. The set Ω_∞ is defined as the limit set: $\Omega_\infty = \cap_{n=1}^{\infty} \Omega_n$. It has an infinite number of holes, but nevertheless satisfies the relation (7). For every $\vec{k} \in \Omega_\infty$ and every n, there is a generalized eigenfunction of H_n of the type (11), (12). It is proven that the sequence $\psi_n(\vec{k}, \vec{x})$ has a limit in $L_\infty(\mathbb{R}^2)$ when $\vec{k} \in \Omega_\infty$. The function $\psi_\infty(\vec{k}, x) = \lim_{n \to \infty} \psi_n(\vec{k}, \vec{x})$ is a generalized eigenfunction of H. It can be written in the form (3)–(5), the functions $\tilde{u}_n$ being related to u_n in (11) by the formula $\tilde{u}_n = u_n - u_{n-1}$. Naturally, the corresponding eigenvalue $\lambda_\infty(\vec{k})$ is the limit of $\lambda_n(\vec{k})$ as $n \to \infty$.

Obviously, $\{\mathcal{B}_n(\lambda)\}_{n=1}^{\infty}$ is a decreasing sequence of sets, since on each step more and more directions are excluded. We consider the limit $\mathcal{B}_\infty(\lambda)$ of $\mathcal{B}_n(\lambda)$:

$$\mathcal{B}_\infty(\lambda) = \bigcap_{n=1}^{\infty} \mathcal{B}_n(\lambda).$$

This set has a Cantor type structure on the unit circle. It is proven that $\mathcal{B}_\infty(\lambda)$ has an asymptotically full measure on the unit circle (see (9)). We prove that the sequence $h_n(\lambda, \vec{\nu})$, $n = 1, 2, ...$, describing the deviations of the isoenergetic surfaces D_n from the perfect circle, converges. Hence, $\mathcal{D}_\infty(\lambda)$ can be described as the limit of $\mathcal{D}_n(\lambda)$ in the sense (8), where $h_\infty(\lambda, \vec{\nu}) = \lim_{n \to \infty} h_n(\lambda, \vec{\nu})$, $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. It is shown that the derivatives of the functions $h_n(\lambda, \vec{\nu})$ (with respect to the angle variable on the unit circle) have a limit as $n \to \infty$ for every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. Thus, the limit surface $\mathcal{D}_\infty(\lambda)$ has some smoothness properties in spite of its Cantor type structure. It looks as a distorted circle with infinite number of holes.

Absolute continuity of the branch of the spectrum (the semiaxis) corresponding to the functions $\psi_\infty(\vec{k},\vec{x})$, $\vec{k} \in \Omega_\infty$ follows from the convergence of the spectral projections corresponding to $\psi_n(\vec{k},\vec{x})$, $\vec{k} \in \Omega_\infty$, to spectral projections of H (in a strong sense uniformly in λ) and smoothness properties of the level curves $D_\infty(\lambda)$, $\lambda > \lambda_0$.

The main technical difficulty to overcome concerns the non-resonance sets Ω_n. The set Ω_n is obtained by deleting a "resonant" part R_n of Ω_{n-1}: $\Omega_n = \Omega_{n-1} \setminus R_n$. The definition of R_n includes Bloch eigenvalues of H_{n-1}; and not only non-resonant eigenvalues of the type (6), but also resonant eigenvalues for which no suitable formula exists. The absence of formulas cause difficulties in estimating the size of R_n. To overcome this problem we show that the resonant set R_n can be described as a neighborhood of zeros of a determinant of the type $Det(I + S_n)$, $S_n = S_n(t)$ being a nuclear type operator expressed in terms of $H_{n-1}(t)$ and $H_{n-2}(t)$. Further we use the fact that $H_n(t)$ and, hence, $S_n(t)$ are analytic with respect to each component of quasimomentum t. Considering that R_n is a neighborhood of the zeros of the determinant, we reduce the problem of estimating the size of R_n to a problem in complex analysis. We use theorems for analytic functions to count the zeros of the determinant and to estimate the size of their neighborhoods. Proving that R_n is small, we obtain estimates (13) for Ω_n and (15) for B_n.

Proofs of the described results (except absolute continuity) are presented in [16]. A preprint containing all the proofs is in preparation.

References

[1] J.Avron, B.Simon *Almost Periodic Schrödinger Operators. I: Limit Periodic Potentials.* Commun. Math. Physics **82** (1981), 101–120.

[2] J.Avron, B.Simon *Cantor Sets and Schrödinger Operators: Transient and Recurrent Spectrum.* J. Func. Anal. **43** (1981), 1–31.

[3] V.A. Chulaevski *On perturbation of a Schrödinger Operator with Periodic Potential.* Russian Math. Surv. **36(5)**, (1981), 143 – 144.

[4] J. Moser *An Example of the Scrödinger Operator with Almost-Periodic Potentials and Nowhere Dense Spectrum.* Comment. Math. Helv. **56** (1981), 198 – 215.

[5] B.Simon *Almost Periodic Schrödinger Operators. A Review.* Advances in Applied Mathematics, **3** (1982), 463 – 490.

[6] L.A. Pastur, V.A. Tkachenko *On the Spectral Theory of the One-Dimensional Scrödinger Operator with a Limit-Periodic Potentials.* Dokl. Akad. Nauk SSSR, **279** (1984), 1051 – 1054.

[7] S.A. Molchanov and V.A. Chulaevskii *Structure of the Spectrum of Lacunary Limit-Periodic Schrödinger Operator.* Func. Anal. Appl. **18** (1984), **18**, 91 – 92.

[8] L.A. Pastur, V.A. Tkachenko *Spectral Theory of a Class of One-Dimensional Scrödinger Operator with a Limit-Periodic Potentials.* Trans. Moscow Math. Soc. **51** (1989), 115 –166.

[9] L.Pastur, A.Figotin *Spectra of Random and Almost-Periodic Operators.* Springer-Verlag, 1992, 583pp.

[10] L. Zelenko "On a Generic Topological Structure of the Spectrum to One-Dimensional Schrödinger Operators with Complex Limit-Periodic Potentials " *Integral Equations and Operator Theory* , **50** (2004), 393-430.

[11] M.A. Shubin *Density of States for Selfadjoint Elliptic Operators with Almost Periodic Coefficients.* Trudy sem. Petrovskii (Moscow University), **3** (1978), 243–281.

[12] M.A. Shubin *Spectral Theory and Index of Elliptic Operators with Almost Periodic Coefficients.* Russ. Math. Surveys, **34** (1979), 109 – 157.

[13] J.Avron, B.Simon *Almost Periodic Schrödinger Operators. II: The Integrated Density of States.* Duke Math. J.,**50** (1983), 1, 369 – 391.

[14] G.V. Rozenblum, M.A. Shubin, M.Z.Solomyak *Spectral Theory of Differential Operators.* in series Encyclopaedia of Mathematical Sciences, **64**, 1994.

[15] Yu. P. Chuburin *Multidimensional Discrete Schrödinger Equation with Limit Peridic Potential.* Theoretical and Mathematical Physics, **102** (1995), 1 53 – 59.

[16] Young-Ran Lee "Spectral Properties of a Polyharmonic Operator with Limit Periodic Potential in Dimension Two ", PhD thesis, defended at UAB, Dept. of Mathematics, on May 12, 2004.