Localizations at infinity and essential spectrum of quantum Hamiltonians: I. General theory

by

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Abstract

We isolate a large class of self-adjoint operators H in $L^2(X)$ whose essential spectrum is determined by their behavior at $x \sim \infty$ and we give a canonical representation of $\sigma_{\text{ess}}(H)$ in terms of spectra of limits at infinity of translations of H. The configuration space X is an abelian locally compact not compact group.

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1 Introduction

In this paper we continue the investigation of the spectral properties of quantum Hamiltonians with C^*-algebra methods on the lines of our previous work [GI4]. More precisely, our aim is to study the essential spectrum of general classes of (unbounded) operators in $L^2(X)$, where X is a locally compact non-compact abelian group, by using crossed product techniques. For some historical remarks and comparison with other recently obtained results, see Subsections 1.2 and 1.4.

1.1. We set $\mathscr{B}(X) = B(L^2(X))$ and we denote U_x the operator of translation by $x \in X$ and V_k the operator of multiplication by the character $k \in X^*$ (our notations, although rather standard, are summarized in Section 2). We define [†]

$$\mathscr{C}(X) = \{T \in \mathscr{B}(X) \mid \lim_{k\to 0} \|[T, V_k]\| = 0 \text{ and } \lim_{x\to 0} \|(U_x - 1)T^{(*)}\| = 0\} \tag{1.1}$$

which is clearly a C^*-algebra of operators on $L^2(X)$ (without unit if X is not discrete). Besides the norm topology on $\mathscr{C}(X)$ we shall also consider on it the topology defined by the family of seminorms $\|S\|_\theta = \|S\theta(Q)\| + \|\theta(Q)S\|$ with $\theta \in \mathcal{C}_0(X)$ and we shall denote $\mathscr{C}_s(X)$ the corresponding topological space (see Remark 5.7). Here $\theta(Q)$ is the operator of multiplication by θ in $L^2(X)$.

Our main result is a description of the essential spectrum of the operators $T \in \mathscr{C}$ in terms of their "localizations at infinity". We denote δX the set of all ultrafilters on X finer than the Fréchet filter (cf. page 13).

Theorem 1.1 *If $T \in \mathscr{C}(X)$ then for each $\varkappa \in \delta X$ the limit $\lim_{x\to\varkappa} U_x T U_x^* = \varkappa.T$ exists in $\mathscr{C}_s(X)$ an*

$$\sigma_{\text{ess}}(T) = \overline{\textstyle\bigcup_\varkappa \sigma(\varkappa.T)}. \tag{1.2}$$

Note that $x \to \varkappa$ should be read "x tends to infinity along the filter $\varkappa$". The limit operator $\varkappa.T$ will be called *localization at $\varkappa$ of T*. Since an ultrafilter finer than the Fréchet filter can be thought as a point on an ideal boundary at infinity of X, the operators $\varkappa.T$ will also be called *localizations at infinity of T*.

We are mainly interested in the essential spectrum of unbounded self-adjoint operators H "affiliated" to $\mathscr{C}(X)$, but the corresponding result is an immediate consequence of Theorem 1.1. We say that H is affiliated to some C^*-algebra $\mathscr{A}$ of operators on $L^2(X)$ if $\varphi(H) \in \mathscr{A}$ for all $\varphi \in \mathcal{C}_0(\mathbb{R})$ (for this it suffices to have $(H - z)^{-1} \in \mathscr{A}$ for one $z \in \rho(H)$). For technical reasons we have to consider self-adjoint operators which are not necessarily densely defined and, in order to avoid confusions with the standard terminology, we shall call these more general objects *observables*. A more detailed presentation of this notion can be found in Subsection 2.2. For the moment we note only that an observable H is affiliated to $\mathscr{C}(X)$ if and only if

$$\lim_{k\to 0} \|[V_k, (H - z)^{-1}]\| = 0 \quad \text{and} \quad \lim_{x\to 0} \|(U_x - 1)(H - z)^{-1}\| = 0 \tag{1.3}$$

[†] We make the following convention: if a symbol like $T^{(*)}$ appears in a relation, then the relation must hold for the operator T and for its adjoint T^*.

for some $z \in \rho(H)$. This follows from the fact that if $T \in \mathscr{B}(X)$ is normal, then

$$\lim_{x\to 0} \|(U_x - 1)T\| = 0 \Longrightarrow \lim_{x\to 0} \|(U_x - 1)T^*\| = 0.$$

Theorem 1.2 *Let H be an observable on $L^2(X)$ affiliated to $\mathscr{C}(X)$. Then for each $\varkappa \in \delta X$ the limit $\varkappa.H := \lim_{x\to\varkappa} x.H$ exists in the following sense: there is an observable $\varkappa.H$ affiliated to $\mathscr{C}(X)$ such that $\lim_{x\to\varkappa} U_x \varphi(H) U_x^* = \varphi(\varkappa.H)$ in $\mathscr{C}_s(X)$ for all $\varphi \in \mathcal{C}_0(\mathbb{R})$. Moreover, we have*

$$\sigma_{\text{ess}}(H) = \overline{\bigcup_{\varkappa} \sigma(\varkappa.H)}. \tag{1.4}$$

Practically we are interested only in the case when H is a self-adjoint operator in the standard sense. However, even in this case $\varkappa.H$ could be not densely defined and quite often we have $\varkappa.H = \infty$ (i.e. the domain of $\varkappa.H$ is $\{0\}$). For example, if H has purely discrete spectrum, then $\varphi(H)$ is a compact operator and we clearly get $\varkappa.H = \infty$ for all $\varkappa$. Since $\sigma(\infty) = \emptyset$, we then obtain $\sigma_{\text{ess}}(H) = \emptyset$, as it should be.

Remark 1.3 The observable H should be thought as the Hamiltonian (energy observable) of a physical system. Thus (1.4) says that the essential spectrum of the Hamiltonian H can be computed in terms of the spectra of its localizations at infinity $\varkappa.H$. We emphasize that this notion of infinity is determined by the position observable Q. In other terms, if H satisfies (1.3) then $\sigma_{\text{ess}}(H)$ is given by its localizations in the region $Q = \infty$. This property does not hold in many situations of physical interest (e.g. if magnetic fields which do not vanish at infinity are involved) because localizations at infinity with respect to other observables must be taken into account, see [GI3].

Remark 1.4 It will be clear from the proof of Theorem 1.2 (see Lemma 5.3 and Proposition 5.10) that (1.4) remains valid if $\varkappa$ runs over sets much smaller than δX: it suffices to take $\varkappa \in \mathcal{K}$ if $\mathcal{K} \subset \delta X$ has the property: if φ is a bounded uniformly continuous function on X and $\lim_{x\to\varkappa} \varphi(x+y) = 0$ for all $y \in X$, $\varkappa \in \mathcal{K}$, then $\varphi \in \mathcal{C}_0(X)$.

Remark 1.5 We mention the following immediate consequence of (1.4): *if two observables affiliated to $\mathscr{C}(X)$ have the same localizations at infinity, then they have the same essential spectrum.* If the difference of the resolvents is a compact operator, then clearly they have the same localizations at infinity, but the converse is far from being true (e.g. see the example on page 531 from [GI4], where the essential spectrum is independent of the details of the shape of the function ω). On the other hand, one may find in [GG2] criteria which ensure the compactness of the difference of the resolvents of two self-adjoint operators under rather weak conditions, e.g. an example from [GG2, p. 26] is a general version of [LaS, Proposition 4.1].

Remark 1.6 The following remark is useful in applications: if H is an observable affiliated to $\mathscr{C}(X)$ and if $\theta : \sigma(H) \to \mathbb{R}$ is a proper continuous function, then $\theta(H)$ is affiliated to $\mathscr{C}(X)$ and we have $\varkappa.\theta(H) = \theta(\varkappa.H)$ for all $\varkappa \in \delta X$ (see page 11).

Remark 1.7 As explained in [GI4, p. 520], all our results extend trivially to the case when $L^2(X)$ is replaced with the space of L^2 functions with values in a Hilbert space

$\mathbf{E}$: it suffices to replace the algebra $\mathcal{A}$ with $\mathcal{A} \otimes K(\mathbf{E})$. For example, Theorem 1.2 remains valid without any change if $L^2(X)$ is replaced by $L^2(X;\mathbf{E})$, where $\mathbf{E}$ is finite dimensional, and $\mathscr{C}(X)$ is defined exactly as before. Thus in applications we can consider differential operators with matrix valued coefficients, like Dirac operators.

1.2. We give here the simplest applications of Theorems 1.1 and 1.2, a more detailed study and more general examples can be found in Section 4.

Assume first that X is discrete. Note that in the particularly important case $X = \mathbb{Z}^n$ Theorem 1.1 has been proved in [RRS1] (with a slightly different formulation and with quite different methods). Now we have

$$\mathscr{C}(X) = \{T \in \mathscr{B}(X) \mid \lim_{k\to 0} \|[T, V_k]\| = 0\}. \tag{1.5}$$

Since $V_k^* U_x V_k = k(x) U_x$ we see that each operator of the form $T = \sum_{a\in X} \varphi_a(Q) U_a$, with $\varphi_a \in \ell_\infty(X)$ and $\varphi_a \neq 0$ only for a finite number of a, belongs to $\mathscr{C}(X)$. Clearly, we have $\varkappa.T = \sum_{a\in X} (\varkappa.\varphi_a)(Q) U_a$ where the function $\varkappa.\varphi_a \in \ell_\infty(X)$ is defined by $(\varkappa.\varphi_a)(y) = \lim_{x\to\varkappa} \varphi(x+y)$. The Jacobi and CMV operators considered in [LaS] are particular cases of such operators T.

Now we give three examples in the case $X = \mathbb{R}^n$. We start with the Schrödinger operator. We denote by $\mathcal{H}^s$ the Sobolev space of order $s \in \mathbb{R}$ associated to $L^2(\mathbb{R}^n)$. Note that Δ is the positive Laplacian. From Proposition 4.12 we get:

Proposition 1.8 *Let W be a continuous symmetric sesquilinear form on $\mathcal{H}^1$ such that:*
(1) $W \geq -\mu\Delta - \nu$ as forms on $\mathcal{H}^1$ for some numbers $\mu < 1$ and $\nu > 0$,
(2) $\lim_{k\to 0} \|[V_k, W]\|_{\mathcal{H}^1\to\mathcal{H}^{-1}} = 0$.
Let H_0 be the self-adjoint operator associated to the form sum $\Delta + W$ and let V be a real function in $L^1_{\rm loc}(\mathbb{R}^n)$ such that its negative part is relatively bounded with respect to H_0 with relative bound < 1. Then the self-adjoint operator $H = H_0 + V(Q)$ (form sum) is affiliated to $\mathscr{C}(\mathbb{R}^n)$, hence the conclusions of Theorem 1.2 hold for it.

This can be extended to a general class of hypoelliptic operators, cf. Proposition 4.16. We present below a very particular case.

Proposition 1.9 *Let $h : \mathbb{R}^n \to \mathbb{R}$ be of class C^m for some $m \geq 1$ and such that:*
(1) $\lim_{k\to\infty} h(k) = +\infty$,
(2) the derivatives of order m of h are bounded,
(3) $\sum_{|\alpha|\leq m} |h^{(\alpha)}(k)| \leq C(1 + |h(k)|)$.
Let $\mathcal{G} = D(|h(P)|^{1/2})$ be the form domain of the operator $h(P)$ and assume that W is a symmetric continuous form on $\mathcal{G}$ such that:
(4) $W \geq -\mu h(P) - \nu$ as forms on $\mathcal{G}$ for some numbers $\mu < 1$ and $\nu > 0$,
(5) $\lim_{k\to 0} \|[V_k, W]\|_{\mathcal{G}\to\mathcal{G}^*} = 0$.
Let $H_0 = h(P) + W$ (form sum) and let $V \in L^1_{\rm loc}(\mathbb{R}^n)$ real such that its negative part is relatively bounded with respect to H_0 with relative bound < 1. Then the self-adjoint operator $H = H_0 + V(Q)$ (form sum) is affiliated to $\mathscr{C}(\mathbb{R}^n)$, hence the conclusions of Theorem 1.2 hold for it.

Remark 1.10 If X is an arbitrary group, $h : X \to \mathbb{R}$ is continuous and satisfies $|h(k)| \to \infty$ as $k \to \infty$, and if $V \in L^\infty(X)$, then obviously $h(P) + V(Q)$ is affiliated to $\mathscr{C}(X)$ and so we can apply Theorem 1.2. In order to cover unbounded V without much effort a quite weak regularity condition on h is sufficient, see Proposition 4.16 and especially relation (4.8). We shall not try to optimize on this here.

Finally, we consider a Dirac operator D. Let $\mathcal{H} = L^2(\mathbb{R}^n; \mathbf{E})$ for some finite dimensional Hilbert space $\mathbf{E}$. We only need to know that D is a symmetric first order differential operator with constant coefficients acting on $\mathbf{E}$-valued functions and which is realized as a self-adjoint operator on $\mathcal{H}$ such that the domain of $|D|^{1/2}$ is the Sobolev space $\mathcal{H}^{1/2}$. Now from Corollary 4.8 we get:

Proposition 1.11 *Let W be a continuous symmetric form on $\mathcal{H}^{1/2}$ such that:*
(1) $\pm W \le \mu|D| + \nu$ as forms on $\mathcal{H}^{1/2}$ for some numbers $\mu < 1$ and $\nu > 0$,
(2) $\lim_{k\to 0} \|[V_k, W]\|_{\mathcal{H}^{1/2}\to\mathcal{H}^{-1/2}} = 0$.
Then the self-adjoint operator $H = D + W$, defined as explained on page 23, is affiliated to $\mathscr{C}(\mathbb{R}^n)$, hence the conclusions of Theorem 1.2 hold for it.

Observe that condition (2) is trivially satisfied if W is the operator of multiplication by an operator valued function $W : \mathbb{R}^n \to B(\mathbf{E})$.

Remark 1.12 We emphasize that the conditions on the perturbation W in Propositions 1.8-1.11 is such that W can contain terms of the same order as $\Delta, h(P)$ or D respectively. For example, operators of the form

$$-\sum_{j,k} \partial_j a_{jk} \partial_k + \text{ singular lower order terms}$$

with $a_{jk} \in L^\infty$ such that the matrix $(a_{jk}(x))$ is bounded from below by a strictly positive constant are already covered by Proposition 1.8. See Example 4.13 for much more general results. These examples may be combined with the Remark 1.6 to cover functions of operators, e.g. $\sqrt{H}$ if $H \ge 0$.

1.3. Crossed products of C^*-algebras by the action of X play a fundamental rôle in our proof of Theorem 1.1 but we have to stress that they are important for two distinct reasons. First, they are in a natural sense C^*-algebras of *energy*[‡] observables (or quantum Hamiltonians), and hence they allow one to organize the Hamiltonians in classes each having some specific properties, e.g. the essential spectrum of the operators in a class is given by a "canonical" formula specific to that class (see (1.6)). On the other hand, crossed products are very efficient at a technical level, their use allows one to solve a non-abelian problem by abelian means: the problem of computing the quotient of a non-commutative C^*-algebra $\mathscr{A} \subset \mathscr{B}(X)$ with respect to the ideal $\mathscr{K}(X) \equiv K(L^2(X))$ is reduced to that of computing $\mathcal{A}/\mathcal{C}_0(X)$ where $\mathcal{A}$ is a C^*-algebra of bounded uniformly continuous functions on X.

The first reason mentioned above will be clarified by the later developments, but one may observe already now that the decomposition (1.2) is far from efficient. Indeed,

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[‡] We emphasize "energy" because algebras of observables and crossed products were frequently used in various domains of the quantum theory in the last 50 years, but with different meanings and scopes than here.

its extreme redundancy becomes clear when we realize that many $\varkappa$ give the same $\varkappa.H$ (e.g. if the filters $\varkappa$ and χ have the same envelope then $\chi.T = \varkappa.T$, see page 15) and many more give the same $\sigma(\varkappa.H)$ (e.g. $\chi.T = U_x \varkappa.T U_x^*$ if χ is the translation by $x \in X$ of $\varkappa$).

Thus at a qualitative level (1.2) is not very significant, it does not say much about $\sigma_{\text{ess}}(H)$, at least when compared with the N-body situation where the HVZ theorem has such a nice physical interpretation that you can predict it and believe it without proof.

In order to partially remediate this drawback we consider smaller classes of Hamiltonians. The following framework, introduced in [GI4], gives us more specific information about $\sigma_{\text{ess}}(H)$. Let $\mathcal{C}(X)$ be the C^*-algebra of all bounded uniformly continuous functions on X and $\mathcal{C}_\infty(X)$ that of continuous functions which have a limit at infinity (in the usual sense).

Definition 1.13 *An* algebra of interactions $\mathcal{A}$ *on* X *is a* C^**-subalgebra of* $\mathcal{C}(X)$ *which is stable under translations and which contains* $\mathcal{C}_\infty(X)$*. The* C^*-algebra of quantum Hamiltonians of type $\mathcal{A}$ *is the norm closed linear space* $\mathscr{A} \equiv \mathcal{A} \rtimes X \subset \mathscr{B}(X)$ *generated by the operators of the form* $\varphi(Q)\psi(P)$ *with* $\varphi \in \mathcal{A}$ *and* $\psi \in \mathcal{C}_0(X^*)$*.*

We have denoted $\varphi(Q)$ the operator of multiplication by φ in $L^2(X)$ and $\psi(P)$ becomes multiplication by ψ after a Fourier transformation. The Propositions 3.3 and 3.4 explain why we think of $\mathscr{A}$ as a C^*-algebra of Hamiltonians. For example, if $X = \mathbb{R}^n$, the self-adjoint operators of the form $\Delta + \sum_{k=1}^n a_k(x)\partial_k + a_0(x)$ with $a_j \in \mathcal{A}^\infty$ (functions in $\mathcal{A}$ with all derivatives in $\mathcal{A}$) generate $\mathscr{A}$. It turns out that $\mathscr{A}$ is canonically isomorphic with the crossed product of $\mathcal{A}$ by the natural action of X, which explains the notation $\mathcal{A} \rtimes X$ and the relevance of crossed products in our context.

Remark 1.14 Note that the definition and the quoted propositions tend to give the impression that the algebra $\mathscr{A}$ is rather small. But this is wrong, $\mathscr{A}$ *is much larger than expected*. For example, $\mathcal{C}(X) \rtimes X = \mathscr{C}(X)$ and we shall see in Section 4 that the set of self-adjoint operators affiliated to $\mathscr{C}(X)$ is very large. Other examples are the N-body algebra and the "bumps" algebras. In fact, we may summarize our approach as follows: we first isolate a class of *elementary Hamiltonians*, these being the simplest operators we would like to study, but our results concern all the operators affiliated to the C^*-algebra they generate, which happens to be a crossed product and is very rich.

In order to state the next consequence of Theorem 1.1 we have to introduce some new notations. Let $\sigma(\mathcal{A})$ be the space of characters of the abelian C^*-algebra $\mathcal{A}$. Then $\sigma(\mathcal{A})$ is a compact topological space which contains X as an open dense subset, so $\delta(\mathcal{A}) = \sigma(\mathcal{A}) \setminus X$ is a compact space. We shall adopt the following abbreviation: $H \in' \mathscr{A}$ means that H is either an element of the algebra $\mathscr{A}$ or an observable affiliated to $\mathscr{A}$. If H is an observable affiliated to $\mathscr{A}$ then $U_x H U_x^*$ is also an observable affiliated to $\mathscr{A}$ and we have $\varphi(U_x H U_x^*) = U_x \varphi(H) U_x^*$ for $\varphi \in \mathcal{C}_0(\mathbb{R})$. By "continuity" of a map $\sigma(\mathcal{A}) \ni \varkappa \mapsto \varkappa.H \in' \mathscr{C}_s(X)$ whose values are observables we mean that $\sigma(\mathcal{A}) \ni \varkappa \mapsto \varphi(\varkappa.H) \in \mathscr{C}_s(X)$ is continuous for all $\varphi \in \mathcal{C}_0(\mathbb{R})$.

Theorem 1.15 *If* $H \in' \mathscr{A}$ *then the map* $X \ni x \mapsto U_x H U_x^* \in' \mathscr{A}$ *extends to a continuous map* $\sigma(\mathcal{A}) \ni \varkappa \mapsto \varkappa.H \in' \mathscr{C}_s(X)$ *and we have*

$$\sigma_{\text{ess}}(H) = \overline{\bigcup\nolimits_{\varkappa\in\delta(\mathcal{A})}\sigma(\varkappa.H)}. \tag{1.6}$$

Remark: To see the connection between this and Theorems 1.1 and 1.2, we recall that an ultrafilter finer than the Fréchet filter is the same thing as a character $\varkappa$ of the algebra of all bounded functions on X such that $\varkappa(\varphi) = 0$ if $\varphi \in \mathcal{C}_0(X)$ (see Subsection 2.5). Moreover, if χ is a second such ultrafilter and $\varkappa(\varphi) = \chi(\varphi)$ for all $\varphi \in \mathcal{C}(X)$, then $\varkappa.H = \chi.H$ for all $H \in' \mathscr{C}(X)$, thus the union in (1.2) and (1.4) may be taken in fact over $\varkappa \in \delta(\mathcal{C}(X))$. We emphasize that, although Theorem 1.15 seems stronger than Theorems 1.1 and 1.2, it is in fact an immediate consequence of Theorem 1.1 (just "abstract nonsense", see Subsection 5.3 for details). Note also that (1.6) is a canonical decomposition of the essential spectrum of H, all the objects in the formula being canonically associated to $\mathcal{A}$. The representation (1.6) is further discussed in Subsection 5.3, see page 32.

Remark 1.16 We mention that, by using a more involved algebraic formalism as in [GI4], one can obtain partial, but often relevant, information concerning the essential spectrum of H as follows. Let $\mathcal{J}$ be an X-ideal such that $\mathcal{C}_0(X) \subset \mathcal{J} \subset \mathcal{A}$ and let $\mathscr{J} = \mathcal{J} \rtimes X$ (we use here notations and results from [GI4]). Then $\mathscr{K}(X) \subset \mathscr{J} \subset \mathscr{A}$ and $\mathscr{J}$ is an ideal in $\mathscr{A}$, so the image $H_{\mathcal{J}}$ of H is well defined as observable affiliated to the quotient algebra $\mathscr{A}/\mathscr{J}$. By using the natural surjection $\mathscr{A}/\mathscr{K}(X) \to \mathscr{A}/\mathscr{J}$ we clearly get $\sigma(H_{\mathcal{J}}) \subset \sigma_{\text{ess}}(H)$. In this argument $\mathscr{J}$ need not be a crossed product, but if it is, we can use $\mathscr{A}/\mathscr{J} \cong (\mathcal{A}/\mathcal{J}) \rtimes X$ to get a concrete representation of $H_{\mathcal{J}}$.

1.4. This subsection is devoted to some historical comments and a discussion of some related results from the literature.

Theorem 1.1 was announced in the preprints [Ift, GI2], see Theorems 1.3 and 4.2 in [Ift] and Theorem 4.1 and Corollary 4.2 in [GI2]. In fact, the theorem was stated in a stronger form, namely we assert that the union in (1.2) is already closed. Moreover, some nontrivial applications are stated at page 149 of [GI5]. The closedness of the union in (1.2) as well as more explicit applications of Theorem 1.1 will be discussed in the second part of this paper. However we show here that the union in (1.6) is closed for some special algebras $\mathcal{A}$ when the result is far from obvious (Section 6).

The main idea of the proof of Theorem 1.1 we had in mind at that moment is presented at [GI2, p. 30–31] and it has to be combined with the two main points of the algebraic approach we used in that paper, namely:

(1) If $\mathcal{H}$ is a Hilbert space then the quotient algebra $B(\mathcal{H})/K(\mathcal{H})$ is a C^*-algebra and, if $\widehat{T}$ is the projection of $T \in B(\mathcal{H})$ in the quotient, then $\sigma_{\text{ess}}(T) = \sigma(\widehat{T})$.

(2) We have $K(L^2(X)) = \mathcal{C}_0(X) \rtimes X$ and if $\mathcal{A}$ is an algebra of interactions then

$$\big(\mathcal{A} \rtimes X\big)/\big(\mathcal{C}_0(X) \rtimes X\big) \cong \big(\mathcal{A}/\mathcal{C}_0(X)\big) \rtimes X. \tag{1.7}$$

If $T \in \mathcal{A} \rtimes X$ the isomorphism (1.7) allows as to reduce the computation of $\widehat{T}$ to an abelian problem and hence to deduce $\widehat{T} \cong (\varkappa.T)_{\varkappa\in\delta(\mathcal{A})} \in \prod_{\varkappa\in\delta(\mathcal{A})} \mathscr{C}(X)$.

The preceding strategy requires a lot of abstract machinery and is not adapted to a purely Hilbert space setting. For example, the isomorphism (1.7) is a consequence of the fact that the functor $\mathcal{A} \mapsto \mathcal{A} \rtimes X$ transforms short exact sequences in short exact sequences, an assertion which does not even make sense if we fix the Hilbert space on which the algebras are realized.

Instead, in the present paper we decided to avoid step (2) of this strategy and to base our arguments on a beautiful theorem due to M. B. Landstad [Lan] which gives an intrinsic characterization of crossed products. We feel that this makes the argument more elementary and gives deeper insight into the matters treated here. In fact, we could now avoid completely going out from the purely Hilbert space setting (in particular, forget about the step (1) above), but this does not seem to us a natural attitude and we finally decided to adopt a median approach.

It is remarkable that $\mathscr{C}(X)$ as defined in (1.1) is precisely the crossed product $\mathcal{C}(X) \rtimes X$. Initially, this fact was proved by direct methods in the case $X = \mathbb{R}^n$ in [DG2] (because of this Theorem 4.1 from [GI2] was stated only for $X = \mathbb{R}^n$). The general case follows in fact immediately from Landstad's theorem.

We make now some comments concerning other papers with goals similar to ours. We note first that, in the particular case $X = \mathbb{Z}^n$, Rabinovich, Roch and Silberman [RRS1] discovered Theorem 1.1 before us and proved it with no C^*-algebra techniques (in Remark 3.18 we explain why (1.5) is just their algebra of "band dominated operators"). It seems that they realized the fact that their algebra in the case $X = \mathbb{Z}$ is a crossed product only in [RRR] (this fact is a particular case of [GI4, Theorem 4.1]). In [RRS1] and in subsequent works [Rab, RRS1, RRS2] (see also [RRS2] for references to earlier papers) these authors use a discretization technique in order to treat perturbations of pseudo-differential operators in $L^2(\mathbb{R}^n)$. They get relations like (1.4) and show that in some situations the union is already closed. We learned about these works quite recently thanks to a correspondence with Barry Simon who sent us a copy of the paper [Rab]; this explains why the above references were not included in our previous works on this topic.

Y. Last and B. Simon obtained in [LaS] relations like (1.4) for large classes of Schrödinger operators on $\mathbb{R}^n$ and their discrete versions (Jacobi or CMV operators). Their proofs involve "classical" geometrical methods (localization with the help of a partition of unity).

We emphasize that many people working on pseudo-differential operators have been led to consider C^*-algebras generated by such operators and to describe their quotients with respect to the ideal of compact operators: in fact, this is one of the most efficient ways to define the symbol of an operator (see [CMS] for example). Much more specific and relevant with respect to our goals is the work of H. O. Cordes (see [Cor] for a review). For example, the C^*-algebra generated by a hypoelliptic operator and by the algebra of slowly oscillating functions and the computation of its quotient with respect to the compacts seem to have been considered for the first time in M. Taylor's thesis (see [Tay, Theorem 1]). For more recent work on these lines, we refer to [Nis].

We mention that a rather different class of "C^*-algebras of Hamiltonians" appear in the work of J. Bellissard on solid state physics [Be1, Be2]: he fixes a Hamiltonian

H and considers the C^*-algebra generated by its translates. Since these algebras are not crossed product and do not contain compact operators in general, our techniques cannot be used in his setting.

The origin of our approach can be traced back to the algebraic treatment of the N-body problem from [BG1, BG2] (where the HVZ theorem and the Mourre estimate are proved in an abstract graded C^*-algebra framework for a very general class of N-body Hamiltonians). The rôle of the crossed products was pointed out in [GI2, GI3, GI4] and a treatment of the N-body problem along these lines is presented in [DG1, DG2, DG3]. Various applications and extensions of the crossed product technique can be found in [AMP, Man, MPR, Ric, Rod] and references therein.

Our interest in localizations at infinity of a Hamiltonian was initially motivated by our desire to go beyond the N-body problem and to consider general (phase space) anisotropic systems [GI1, Ift]. Indeed, in the N-body case there is a lot of supplementary structure which makes the theory simple and beautiful (cf. Subsection 6.5), but this structure has no analogue in other types of anisotropy. We first found that the C^*-algebra techniques are quite well adapted to the study of Hamiltonians with Klaus type potentials, see [GI2, GI4] and also Subsection 6.4 here for a treatment in the spirit of Theorem 1.1. We finally realized that the relation (1.7), which is the main point of the algebraic approach that we used, predicts in fact the description (1.4) of the essential spectrum of H in terms of its localizations at infinity.

The paper [HeM] played an important rôle in our understanding of this fact. Indeed, B. Helffer and A. Mohamed prove there that the essential spectrum of a magnetic Hamiltonian $(P - A)^2 + V$ is the closure of the union of the spectra of some limit Schrödinger operators. Their proof is based on hypoellipticity techniques and the result is already interesting if the magnetic field is not present. The class of potentials they consider is quite large, but the function V has to be bounded from below and to satisfy some regularity conditions. These assumptions imply that the limit operators have only polynomial electric and magnetic potentials, which is easily explained in our framework, see [GI5, Proposition 3.13].

1.5. Plan of the paper. Our purpose being to emphasize not only the power but also the simplicity of the C^*-algebra techniques, we made an effort to make the paper essentially self-contained and easy to read by people working in the spectral theory of quantum Hamiltonians and with little background in C^*-algebras.

We could have written a 20 page paper (including complete and detailed proofs) but which would have been accessible mostly to people with no interest in spectral theory and whose first question would have been "why should anybody be interested in an assertion like Theorem 1.1". Instead, we have chosen to present in some detail most of the tools which are not standard among peoples which could be interested in the subject. In particular we give in an Appendix a simple and self-contained proof of Landstad's theorem (Theorem 3.7) which plays such an important rôle in our arguments. We also give a lot of explanations concerning filters and ultrafilters, things which would seem trivial to somebody working in topology (but the chance that such a person looks through this paper is so small, that we decided to neglect it).

In Section 2 we introduce our notations and make a résumé of what we need concerning (ultra)filters and their relation with the characters of some abelian C^*-algebras. In Section 3 we introduce crossed products in the version we need and we point out several useful consequences of Landstad's theorem. This replaces the much more abstract arguments from [GI2, GI4], since we remain in a purely Hilbert space setting, but also gives stronger and more explicit results in applications. Section 4 is devoted to criteria of affiliation to the algebra $\mathscr{C}(X)$, we show there that this algebra is much larger than one would think at first sight.

In Section 5 we prove our main result, Theorem 5.11. Finally, in Section 6 we consider three algebras of quantum Hamiltonians, those which seem the most interesting to us. The first one $\mathscr{V}(X)$ is generated by slowly oscillating potentials and is the simplest non trivial algebra of Hamiltonians since it is defined by the property that if H is affiliated to $\mathscr{V}(X)$ then all its localizations at infinity are free Hamiltonians (i.e. functions of the momentum). The second one is the algebra associated to a *sparse set* and it is remarkable because the localizations at infinity of the Hamiltonians affiliated to it are two-body Hamiltonians and thus their essential spectrum has a quite interesting structure. The third one is, of course, the N-body algebra, or rather a more general and geometrically natural algebra that we call *Grassmann algebra*, an object of a remarkable simplicity, richness and beauty. The final subsections are devoted to some remarks of a different nature on the localizations at infinity of Hamiltonians of the form $h(P) + v(Q)$ with $v(Q)$ relatively bounded with respect to $h(P)$.

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2 Preliminaries

In this section we describe our notations and recall facts needed in the rest of the paper.

2.1. If X is a locally compact topological space then $\mathcal{C}_\infty(X)$ is the C^*-algebra of continuous functions which have a limit at infinity and $\mathcal{C}_0(X)$ is the subalgebra of functions which converge to zero at infinity; thus $\mathcal{C}_\infty(X) = \mathbb{C} + \mathcal{C}_0(X)$. Let $\mathcal{C}_\mathrm{c}(X)$ be the subalgebra of functions with compact support. If $\mathscr{A}$ is a C^*-algebra then we similarly define $\mathcal{C}_0(X; \mathscr{A})$ for example, which is also a C^*-algebra. If X is compact we set $\mathcal{C}(X; \mathscr{A}) = \mathcal{C}_0(X; \mathscr{A})$ and $\mathcal{C}(X) = \mathcal{C}(X; \mathbb{C})$, which does not conflict with the notation (2.10) because the continuous functions on X are uniformly continuous.

The characteristic function of a set $S \subset X$ is denoted $\mathbf{1}_S$.

2.2. If $\mathcal{H}$ is a Hilbert space then $B(\mathcal{H})$ and $K(\mathcal{H})$ are the C^*-algebras of bounded and compact operators on $\mathcal{H}$ respectively. The resolvent set, the spectrum and the essential spectrum of an operator S are denoted $\rho(S)$, $\sigma(S)$ and $\sigma_\mathrm{ess}(S)$ respectively.

By morphism between two C^*-algebras we understand $*$-homomorphism. An ideal in a C^*-algebra is assumed to be closed and two-sided.

An *observable* is a linear operator $H : D(H) \to \mathcal{H}$ such that $HD(H) \subset \mathcal{K}$, where $\mathcal{K}$ is the closure of $D(H)$ in $\mathcal{H}$, and such that H when considered as operator in $\mathcal{K}$ is self-adjoint in the usual sense. A trivial observable which, however, is quite important, is the unique observable whose domain is equal to $\{0\}$; we shall denote it ∞. One has to think that H is equal to ∞ on $\mathcal{K}^{\perp}$ and for this reason we set $\varphi(H) = 0$ on $\mathcal{K}^{\perp}$ if $\varphi \in \mathcal{C}_0(\mathbb{R})$. Note that we keep the notation $(H-z)^{-1}$ for the resolvent of H in $\mathcal{H}$ but $(H-z)^{-1} = 0$ on $\mathcal{K}^{\perp}$.

If $\mathscr{C} \subset B(\mathcal{H})$ is any C^*-subalgebra then an observable H is said to be *affiliated* to $\mathscr{C}$ if $(H-z)^{-1} \in \mathscr{C}$ for some $z \in \rho(H)$. Then $\varphi(H) \in \mathscr{C}$ for all $\varphi \in \mathcal{C}_0(\mathbb{R})$.

It is theoretically much more convenient to define an observable affiliated to $\mathscr{C}$ as a morphism $H : \mathcal{C}_0(\mathbb{R}) \to \mathscr{C}$ and then to set $H(\varphi) \equiv \varphi(H)$. We refer to [GI4, p. 522–523] for a résumé of what we need and also to [DG3] for comments on this notion which should not be confused with that introduced by S. Baaj and S. L. Woronowicz (in [ABG, Sec. 8.1] one can find a systematic presentation of this point of view).

We recall two definitions which make the transition from Theorem 1.1 to Theorem 1.2 trivial. The *spectrum* of the observable H is the set

$$\sigma(H) = \{\lambda \in \mathbb{R} \mid \varphi \in \mathcal{C}_0(\mathbb{R}) \text{ and } \varphi(\lambda) \neq 0 \Rightarrow \varphi(H) \neq 0\}. \tag{2.8}$$

Let $\mathscr{K} = \mathscr{C} \cap K(\mathcal{H})$, this is an ideal in $\mathscr{C}$. Then the *essential spectrum* of H is the set

$$\sigma_{\text{ess}}(H) = \{\lambda \in \mathbb{R} \mid \varphi \in \mathcal{C}_0(\mathbb{R}) \text{ and } \varphi(\lambda) \neq 0 \Rightarrow \varphi(H) \notin \mathscr{K}\}. \tag{2.9}$$

We also note that any morphism $\pi : \mathscr{C} \to \mathscr{B}$ between two C^*-algebras extends in a trivial way to a map between observables affiliated to $\mathscr{C}$ to observables affiliated to $\mathscr{B}$. Indeed, it suffices to define $\pi(H)$ by the condition $\varphi(\pi(H)) = \pi(\varphi(H))$. For example, if $\pi : \mathscr{C} \to \mathscr{C}/\mathscr{K}$ is the canonical morphism of $\mathscr{C}$ onto the quotient algebra $\mathscr{C}/\mathscr{K}$, we have $\sigma_{\text{ess}}(H) = \sigma(\pi(H))$.

Finally, we mention one more immediate consequence of the definition of an observable in terms of morphisms, cf. [ABG, p. 370]. We shall use the easily proven fact that $\varphi(H)$ depends only on the restriction of φ to the closed real set $\sigma(H)$. Let $\theta : \sigma(H) \to \mathbb{R}$ be continuous and proper (i.e. $|\theta(\lambda)| \to \infty$ if $|\lambda| \to \infty$). Then the observable $\theta(H)$ is well defined by the rule $\varphi(\theta(H)) = (\varphi \circ \theta)(H)$ for $\varphi \in \mathcal{C}_0(\mathbb{R})$ (if H is a self-adjoint operator then $\theta(H)$ is just the operator defined by the usual functional calculus). Clearly: *if H is affiliated to $\mathscr{C}$, the observable $\theta(H)$ is also affiliated to $\mathscr{C}$.*

2.3. We describe now objects and notations from the harmonic analysis on groups. Everything we need can be found in [Fol] or [FeD]; see also [Wei].

Let X be an abelian locally compact group (with the operation denoted additively) equipped with a Haar measure $\mathrm{d}x$. We abbreviate $\mathscr{B}(X) = B(L^2(X))$, $\mathscr{K}(X) = K(L^2(X))$ and note that these are C^*-algebras depending on X and not on the choice of the Haar measure. Other such C^*-algebras are $L^\infty(X)$, $\mathcal{C}_\infty(X)$, $\mathcal{C}_0(X)$ and the C^*-algebra of bounded uniformly continuous functions on X, which plays the most important rôle in what follows:

$$\mathcal{C}(X) = \{\varphi : X \to \mathbb{C} \mid \varphi \text{ is bounded and uniformly continuous }\}. \tag{2.10}$$

In order to avoid ambiguities, if φ is a measurable function on X then we denote $\varphi(Q)$ the operator of multiplication by φ in $L^2(X)$ (the symbol Q has no operator meaning).

By using this map we identify the algebra $L^\infty(X)$ and its C^*-subalgebras with C^*-subalgebras of $\mathscr{B}(X)$, in particular we always embed

$$\mathcal{C}_0(X) \subset \mathcal{C}_\infty(X) \subset \mathcal{C}(X) \subset \mathscr{B}(X). \tag{2.11}$$

Note that the C^*-algebra $\ell_\infty(X)$ of all bounded functions on X cannot be embedded in $\mathscr{B}(X)$ (neither can the C^*-algebra $\mathcal{B}(X)$ of bounded Borel functions).

Let X^* be the set of characters of X (continuous homomorphisms $k : X \to \mathbb{C}$ with $|k(x)| = 1$) equipped with the locally compact group structure defined by the operation of multiplication and the topology of uniform convergence on compact sets. We denote the operation in X^* additively and its neutral element by 0, as in [Wei, ch.II, §5] (this convention looks rather strange if $X = \mathbb{Z}^n$, for example). If X is a real finite dimensional vector space then X^* is identified with the vector space dual to X as follows: let $\langle \cdot , \cdot \rangle : X \times X^* \to \mathbb{R}$ be the canonical bilinear map and take $k(x) = \mathrm{e}^{i\langle x,k\rangle}$. In fact, the field of real numbers can be replaced here by an arbitrary non-discrete locally compact field, see [Fol, page 91]) and [Wei, ch.II, §5]. We recall that the dual group $(X^*)^*$ of X^* is identified with X, each $x \in X$ being seen as a character of X^* through the formula $x(k) = k(x)$.

The Fourier transform of $u \in L^1(X)$ is the function $\mathcal{F}u \equiv \widehat{u} : X^* \to \mathbb{C}$ given by $\widehat{u}(k) = \int_X \overline{k(x)} u(x)\,\mathrm{d}x$. We equip X^* with the unique Haar measure $\mathrm{d}k$ such that $\mathcal{F}$ induces a unitary map $\mathcal{F} : L^2(X) \to L^2(X^*)$. From $\mathcal{F}^{-1} = \mathcal{F}^*$ we get $(\mathcal{F}^{-1}v)(x) = \int_{X^*} k(x)v(k)\,\mathrm{d}k$ for $v \in L^2(X^*)$. By taking into account the identification $X^{**} = X$, the Fourier transform of $\psi \in L^1(X^*)$ and the Fourier inversion formula are

$$\widehat{\psi}(x) = \int_{X^*} \overline{k(x)}\psi(k)\,\mathrm{d}k \quad \text{and} \quad \psi(k) = \int_X k(x)\widehat{\psi}(x)\,\mathrm{d}x. \tag{2.12}$$

For each measurable $\psi : X^* \to \mathbb{C}$ we define the operator $\psi(P)$ on $L^2(X)$ by $\psi(P) = \mathcal{F}^* M_\psi \mathcal{F}$, where M_ψ is the operator of multiplication by ψ in $L^2(X^*)$. In particular, the restriction to $L^\infty(X^*)$ of the map $\psi \mapsto \psi(P)$ is injective and gives us C^*-subalgebras

$$\mathcal{C}_0(X^*) \subset L^\infty(X^*) \subset \mathscr{B}(X). \tag{2.13}$$

Let $\{U_x\}_{x\in X}$ and $\{V_k\}_{k\in X^*}$ be the strongly continuous unitary representations of X and X^* in $L^2(X)$ defined by $(U_x f)(y) = f(x + y)$ and $(V_k f)(y) = k(y)f(y)$ respectively. Note that U_x and V_k satisfy the canonical commutation relations

$$U_x V_k = k(x) V_k U_x. \tag{2.14}$$

Observe that we have $U_x = x(P)$ if $x \in X$ is identified with the function $k \mapsto k(x)$ and similarly $V_k = k(Q)$. Also, we have, cf. (2.12):

$$\psi(P) = \int_X U_x \widehat{\psi}(x)\,\mathrm{d}x \quad \text{if} \quad \widehat{\psi} \in L^1(X). \tag{2.15}$$

2.4. We summarize here some facts we need concerning filters, cf. [Bou, HiS, Sam].

A *filter* on X is a family $\mathfrak{f}$ of subsets of X which does not contain the empty set, is stable under finite intersections, and has the property: $G \supset F \in \mathfrak{f} \Rightarrow G \in \mathfrak{f}$ (the empty set is a filter!). If Y is a topological space and $\theta : X \to Y$ is any map, then $\lim_{\mathfrak{f}} \theta = y$ means that $\theta^{-1}(V) \in \mathfrak{f}$ if V is a neighborhood of y. We shall often write $\lim_{x \to \mathfrak{f}} \theta(x)$ instead of $\lim_{\mathfrak{f}} \theta$ for reasons which will become clear later on.

If $\mathfrak{f}, \mathfrak{g}$ are filters and $\mathfrak{f} \subset \mathfrak{g}$ then $\mathfrak{g}$ is said to be *finer* than $\mathfrak{f}$. An *ultrafilter* is a maximal element in the set of all filters on X for this order relation. If $x \in X$ then the family of sets which contain x is the ultrafilter determined by x. A filter $\mathfrak{f}$ is an ultrafilter if and only if for each $A \subset X$ one has $A \in \mathfrak{f}$ or $A^c \equiv X \setminus A \in \mathfrak{f}$.

Ultrafilters are important because of the following property: if $\mathfrak{f}$ is an ultrafilter and $\theta : X \to Y$ is an arbitrary map with values in a compact space Y, then $\lim_{\mathfrak{f}} \theta$ exists. This fact will become clear after the explanations in Subsection 6.3.

The space γX of all ultrafilters on X is a compact space for the topology defined as follows: the map $\mathfrak{f} \mapsto \{\varkappa \in \gamma X \mid \varkappa \supset \mathfrak{f}\}$ is a bijection from the set of all filters on X onto the set of all closed subsets of γX. Thus one should think that *a filter is a closed subset of* γX. Another description of this topology will be given later on. The compact topological space γX is the *discrete Stone-Čech compactification of* X and it is characterized by the following universal property: *if* Y *is a compact space then each map* $\theta : X \to Y$ *has a unique extension to a continuous map* $\theta : \gamma X \to Y$. Since this property is important for us, we shall further discuss it in Subsection 6.3, see page 36.

The set X is identified with an open dense subset of γX (to $x \in X$ one associates the ultrafilter determined by x) and the topology induced by γX on X is the discrete topology. However, the space $\gamma X \setminus X$ is much too large for our purposes, the only ultrafilters of interest to us belong to the compact subset of γX defined by

$$\delta X = \{\varkappa \mid \varkappa \text{ is an ultrafilter finer than the Fréchet filter}\}. \tag{2.16}$$

We call *Fréchet filter* the filter consisting of the sets with relatively compact complement (this is not quite standard). This filter depends on the locally compact non compact topology given on X. In view of the standard meaning of the notation $\lim_{x\to\infty}$ it is natural to denote by ∞ the Fréchet filter. As explained above, one should think of ∞ as a certain compact subset of γX and then we have in fact $\infty = \delta X$.

2.5. We shall explain now the relation between filters and characters of certain abelian C^*-algebras. If $\mathcal{A}$ is such an algebra we denote $\sigma(\mathcal{A})$ the space of characters of $\mathcal{A}$ (a character is a non zero morphism $\mathcal{A} \to \mathbb{C}$) equipped with the weak* topology. This is a locally compact topological space which is compact if and only if $\mathcal{A}$ is unital.

Let $\mathcal{B}$ be a unital abelian C^*-algebra and let $\mathcal{A} \subset \mathcal{B}$ be a C^*-subalgebra which contains the unit of $\mathcal{B}$. Then each character of $\mathscr{B}$ restricts to a character of $\mathcal{A}$ and each character of $\mathcal{A}$ is obtained in this way. This gives a canonical map $\pi : \sigma(\mathcal{B}) \to \sigma(\mathcal{A})$ which is continuous and surjective and if we define in $\sigma(\mathcal{B})$ an equivalence relation $\varkappa \sim \chi$ by the condition $\varkappa(S) = \chi(S)\ \forall S \in \mathcal{A}$, the compact topological space $\sigma(\mathcal{A})$ is just the quotient of $\sigma(\mathcal{B})$ with respect to this relation.

In particular, *a map* $f : \sigma(\mathcal{A}) \to Y$ *is continuous if and only if* $f \circ \pi : \sigma(\mathcal{B}) \to Y$ *is continuous, where* Y *is an arbitrary topological space*.

Let $\ell_\infty(X)$ be the C^*-algebra of all bounded functions on X. Then the space of all characters of $\ell_\infty(X)$ can be identified with the space γX of all ultrafilters on X:

$$\gamma X = \sigma(\ell_\infty(X)). \tag{2.17}$$

Indeed, the map which associates to an ultrafilter $\mathfrak{f}$ the character $\varphi \mapsto \lim_{\mathfrak{f}} \varphi$ is a homeomorphism and the inverse map associates to the character $\varkappa$ the ultrafilter $\mathfrak{f} = \{F \subset X \mid \varkappa(\mathbf{1}_F) = 1\}$. From now on we shall identify $\mathfrak{f}$ and $\varkappa$, so *an ultrafilter is the same thing as a character of* $\ell_\infty(X)$, and we shall work with the interpretation which is most suited to the context. We also set $\varkappa(F) = \varkappa(\mathbf{1}_F)$ for $F \subset X$. Then

$$\delta X = \{\varkappa \in \gamma X \mid \varkappa(K) = 0 \quad \forall K \subset X \text{ compact}\}. \tag{2.18}$$

The algebras $\mathcal{A}$ that we consider are unital subalgebras of $\ell_\infty(X)$, thus their character spaces $\sigma(\mathcal{A})$ are quotients of γX. In other terms, we can view the characters of $\mathcal{A}$ as equivalence classes of ultrafilters: if $\varkappa$ is a character of $\mathcal{A}$, then there is an ultrafilter $\mathfrak{f}$ such that $\varkappa(\varphi) = \lim_{\mathfrak{f}} \varphi$ for all $\varphi \in \mathcal{A}$, and in fact there are many such ultrafilters.

For the algebras which are of interest for us we always have

$$\mathcal{C}_\infty(X) \subset \mathcal{A} \subset \mathcal{C}(X) \subset \ell_\infty(X) \tag{2.19}$$

Then X is identified with an open dense subset of $\sigma(\mathcal{A})$ and the topology induced by $\sigma(\mathcal{A})$ on X coincides with the initial topology, so $\sigma(\mathcal{A})$ is a compactification of the locally compact space X. Thus

$$\delta(\mathcal{A}) = \sigma(\mathcal{A}) \setminus X = \{\varkappa \in \sigma(\mathcal{A}) \mid \varkappa(\varphi) = 0 \quad \forall \varphi \in \mathcal{C}_0(X)\} \tag{2.20}$$

is a compact subset of $\sigma(\mathcal{A})$, the boundary of X in the compactification $\sigma(\mathcal{A})$. The *uniform compactification* $\beta_{\mathrm{u}} X$ of X is defined by the largest algebra $\mathcal{C}(X)$:

$$\beta_{\mathrm{u}} X = \sigma(\mathcal{C}(X)), \quad \delta_{\mathrm{u}} X = \beta_{\mathrm{u}} X \setminus X = \delta(\mathcal{C}(X)). \tag{2.21}$$

Later on we shall explicitly describe the equivalence relation in γX which defines β_{u}.

We are interested only in the boundary $\delta(\mathcal{A})$ of X in $\sigma(\mathcal{A})$. We show now that this is a quotient of δX.

Lemma 2.1 *Let $\mathfrak{f}$ be an ultrafilter on X and let $\varkappa$ be the character of $\mathcal{A}$ defined by $\varkappa(\varphi) = \lim_{\mathfrak{f}} \varphi$. Then $\varkappa \in \delta(\mathcal{A})$ if and only if $\mathfrak{f} \in \delta X$.*

Proof: If $\mathfrak{f}$ is an ultrafilter and $Y \subset X$ then there are only two possibilities: either $Y \notin \mathfrak{f}$, and then $X \setminus Y \in \mathfrak{f}$ hence $Y \cap Z = \emptyset$ for all $Z \in \mathfrak{f}$, or $Y \in \mathfrak{f}$, and then the sets $Y \cap Z$ with $Z \in \mathfrak{f}$ form an ultrafilter on Y. If $\mathfrak{f}$ is not finer than the Fréchet filter then there is a set with compact complement Y which does not belong to $\mathfrak{f}$, and so $Y \in \mathfrak{f}$. Since any ultrafilter on a compact set is convergent, we see that there is $y \in Y$ such that $\mathfrak{f}$ contains the filter of neighbourhoods of y. But then clearly $\lim_{\mathfrak{f}} \varphi = \varphi(y)$ for any continuous function φ, hence the character $\varkappa(\varphi) = \lim_{\mathfrak{f}} \varphi$ is just y and does not belong to $\delta(\mathcal{A})$. On the other hand, if $\mathfrak{f} \in \delta X$ then clearly $\varkappa \in \delta(\mathcal{A})$. ■

Thus the *characters $\varkappa \in \delta(\mathcal{A})$ are equivalence classes of ultrafilters* $\mathfrak{f} \in \delta X$. In general, we do not distinguish between a character and the elements of the equivalence class of ultrafilters which define it. However, when needed for the clarity of the argument, we shall use the map δ which sends an element into its equivalence class. More precisely, from (2.19) we see that there are canonical surjections

$$\delta X \to \delta_u X \to \delta(\mathcal{A}) \to \{\infty\} \tag{2.22}$$

and all of them (and their compositions) will be denoted δ. Here ∞ is the Fréchet filter and we have $\delta(\mathcal{C}_\infty(X)) = \{\infty\}$.

2.6. The space $\beta_{\mathrm{u}}X$ is the quotient of γX given by an equivalence relation that we describe now (see [Sam, p. 121]). If $\mathfrak{f}$ is a filter then its *envelope* is the filter $\mathfrak{f}^\circ$ generated by the sets $F + V$ where $F \in \mathfrak{f}$ and V belongs to the filter of neighbourhoods of the origin (observe that the sets $F+V$, with V an open neighborhood of the origin, are open and form a basis of $\mathfrak{f}^\circ$). Note that $\mathfrak{f} \supset \mathfrak{f}^\circ$ and $(\mathfrak{f}^\circ)^\circ = \mathfrak{f}^\circ$. Two filters are *u-equivalent* (uniformly equivalent) if they have the same envelope.

The quotient of γX with respect to this relation is $\beta_{\mathrm{u}}X$. We shall give a complete proof of this assertion since in [Sam] the C^*-algebra point of view is not explicitly considered. The following simple fact will be useful for other purposes too.

Lemma 2.2 *Let $\varphi : X \to \mathbb{C}$ be uniformly continuous and let $\mathfrak{f}$ be a filter on X.*
(1) $\lim_{\mathfrak{f}} \varphi$ exists if and only if $\lim_{\mathfrak{f}^\circ} \varphi$ exists and in this case they are equal.
(2) If $\lim_{x\to\mathfrak{f}} \varphi(x+y) \equiv \xi(y)$ exists for each $y \in X$ then the limit exists locally uniformly in y and ξ is a uniformly continuous function.

Proof: To prove (1) it suffices to show that $\lim_{\mathfrak{f}^\circ} \varphi = 0$ if $\lim_{\mathfrak{f}} \varphi = 0$. For $\varepsilon > 0$ let F_ε be the set of points where $|\varphi(x)| < \varepsilon$. We have $F_\varepsilon \in \mathfrak{f}$ and if we choose a neighborhood V of the origin such that $|\varphi(x) - \varphi(y)| < \varepsilon$ if $x - y \in V$, then for $x \in F_\varepsilon + V$ we have $|\varphi(x)| < 2\varepsilon$ hence $F_{2\varepsilon} \in \mathfrak{f}^\circ$.

Now we prove (2). Set $\omega_V(\varphi) = \sup_{y-z\in V} |\varphi(y) - \varphi(z)|$ if V is a neighborhood of the origin. Then φ is uniformly continuous if and only if for each $\varepsilon > 0$ there is V such that $\omega_V(\varphi) < \varepsilon$. Clearly $\omega_V(\xi) \leq \omega_V(\varphi)$, so ξ is uniformly continuous. Now let V be open and let K be a compact set. Then K is covered by the open sets $x + K$, $x \in K$, hence there is $Z \subset K$ finite such that $K \subset \bigcup_{z\in Z}(z + V)$. Thus for each $y \in K$ there is $z \in Z$ such that $y \in z + V$ and then $|\varphi(x + y) - \varphi(x + z)| \leq \omega_V(\varphi)$ for all $x \in X$. Then we have:

$$\begin{aligned} |\varphi(x+y) - \xi(y)| &\leq |\varphi(x+y) - \varphi(x+z)| + |\varphi(x+z) - \xi(z)| + |\xi(z) - \xi(y)| \\ &\leq \omega_V(\varphi) + |\varphi(x+z) - \xi(z)| + \omega_V(\xi) \\ &\leq 2\omega_V(\varphi) + |\varphi(x+z) - \xi(z)|. \end{aligned}$$

Since Z is finite, there is $F \in \mathfrak{f}$ such that $|\varphi(x + z) - \xi(z)| \leq \varepsilon/3$ for all $x \in F$ and $z \in Z$. Then we choose V such that $\omega_V(\varphi) \leq \varepsilon/3$ and we get $|\varphi(x + y) - \xi(y)| \leq \varepsilon$ for all $x \in F$ and $y \in K$. ■

Lemma 2.3 *Assume $\mathfrak{f} = \mathfrak{f}^\circ$ where $\mathfrak{f}$ is a filter on X. Then for each $F \in \mathfrak{f}$ there is an open subset $G \in \mathfrak{f}$ of F and a function $\theta \in \mathcal{C}(X)$ such that $\theta = 1$ on G and $\theta = 0$ on $F^{\mathrm{c}} \equiv X \setminus F$.*

Proof: Note first that the open sets from $\mathfrak{f}$ form a basis of $\mathfrak{f}$. Clearly there is an open $G \in \mathfrak{f}$ and an open, relatively compact neighborhood of the origin U such that $G + (U - U) \subset F$, so denoting $A = G - U$ we shall have $A + U \subset F$. We then set $\theta = |U|^{-1}\mathbf{1}_A * \mathbf{1}_U$, so for each $x \in X$ we have $\theta(x) = |U|^{-1}|A \cap (x - U)|$. For

$x \in G$, $x - U \subset A$ thus $\theta(x) = |U|^{-1}|x-U| = 1$, and for $x \notin A+U$, $A \cap (x-U) = \emptyset$ hence $\theta(x) = 0$. But $A + U \subset F$, thus $\theta = 0$ on F^{c} too. Finally, from

$$\|u * v\|_{L^\infty} \leq \|u\|_{L^1}\|v\|_{L^\infty} \quad \text{and} \quad \|x.(u*v) - u*v\|_{L^\infty} \leq \|x.u - u\|_{L^1}\|v\|_{L^\infty}$$

we get $L^1(X) * L^\infty(X) \subset \mathcal{C}(X)$, hence $\theta \in \mathcal{C}(X)$. ■

Proposition 2.4 *Let $\varkappa, \chi$ be ultrafilters on X. Then $\varkappa^\circ = \chi^\circ$ if and only if $\varkappa(\varphi) = \chi(\varphi)$ for each $\varphi \in \mathcal{C}(X)$.*

Proof: The "only if" part follows from $\varkappa(\varphi) = \lim_\varkappa \varphi = \lim_{\varkappa^\circ} \varphi = \lim_{\chi^\circ} \varphi = \lim_\chi \varphi = \chi(\varphi)$, the second and the fourth equality being consequences of Lemma 2.2. Conversely, let $\varkappa^\circ \neq \chi^\circ$. Then there is $F \in \varkappa^\circ$ such that $F \notin \chi^\circ \subset \chi$, hence $F^{\mathrm{c}} \in \chi$ because χ is an ultrafilter. Let now G and θ be as in Lemma 2.3. Since $G \in \varkappa^\circ \subset \varkappa$ we have $\varkappa(\mathbf{1}_G) = 1$, thus $\varkappa(\mathbf{1}_{G^{\mathrm{c}}}) = 0$. Hence $\varkappa(\theta) = \varkappa(\theta\mathbf{1}_G) + \varkappa(\theta\mathbf{1}_{G^{\mathrm{c}}}) = 1 + \varkappa(\theta)\varkappa(\mathbf{1}_{G^{\mathrm{c}}}) = 1$. On the other hand, $F^{\mathrm{c}} \in \chi$ implies $\chi(\mathbf{1}_{F^{\mathrm{c}}}) = 1$ and $\theta\mathbf{1}_{F^{\mathrm{c}}} = 0$, thus $0 = \chi(\theta\mathbf{1}_{F^{\mathrm{c}}}) = \chi(\theta)\chi(\mathbf{1}_{F^{\mathrm{c}}}) = \chi(\theta)$. ■

3 Crossed products

In this section we recall some facts concerning crossed products and point out some properties important for our later arguments. A locally compact non compact abelian group X is fixed in what follows.

We shall say that a C^*-algebra $\mathcal{A}$ is an *X-algebra* if a homomorphism $\alpha : x \mapsto \alpha_x$ of X into the group of automorphisms of $\mathcal{A}$ is given, such that for each $A \in \mathcal{A}$ the map $x \mapsto \alpha_x(A)$ is norm continuous[†]. An *X-subalgebra* of $\mathcal{A}$ is a C^*-subalgebra that is left invariant by all the automorphisms α_x. An *X-ideal* is an ideal stable under the α_x. If $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are two X-algebras, a morphism $\phi : \mathcal{A} \to \mathcal{B}$ is called *X-morphism* if $\phi[\alpha_x(A)] = \beta_x[\phi(A)]$ for all $x \in X$ and $A \in \mathcal{A}$.

We shall not need the abstract definition of the crossed product $\mathcal{A} \rtimes X$ of an X-algebra $\mathcal{A}$ by the action of X. We mention only that $\mathcal{A} \rtimes X$ is a C^*-algebra uniquely defined modulo a canonical isomorphism by a certain universal property (see [Rae] for example) and that the correspondence $\mathcal{A} \mapsto \mathcal{A} \rtimes X$ has certain functorial properties (see [GI5]) which play an important rôle in [GI4] but will not be used here. On the other hand, the following concrete realization of $\mathcal{A} \rtimes X$ for certain $\mathcal{A}$ will be important.

There is a natural action of X on $L^\infty(X)$ by translations $(\tau_x\varphi)(y) = \varphi(y + x)$ and it is clear that $x \mapsto \tau_x\varphi \in L^\infty(X)$ is norm continuous if and only if $\varphi \in \mathcal{C}(X)$. Thus $\mathcal{C}(X)$ becomes an X-algebra and we will be interested only in crossed products $\mathcal{A} \rtimes X$ with $\mathcal{A}$ an X-subalgebra of $\mathcal{C}(X)$, i.e. a C^*-subalgebra stable under translations. In many cases we shall slightly simplify the writing and set $x.\varphi = \tau_x\varphi$. Note that if $\varphi \in \mathcal{C}(X) \cap L^2(X)$ we have $x.\varphi = U_x\varphi$ but $(x.\varphi)(Q) = U_x\varphi(Q)U_x^*$. More generally, we shall use the notations:

$$x \in X, T \in \mathscr{B}(X) \Longrightarrow x.T \equiv \tau_x(T) = U_xTU_x^*. \tag{3.23}$$

[†] The terminology "C^*-dynamical system" used by some C^*-algebra theorists seems to us extremely confusing in our context, even if X is $\mathbb{R}$ or $\mathbb{Z}$, so we shall not use it.

The next definition describes $\mathcal{A} \rtimes X$ in what we could call the *pseudo-differential operator representation*, or ΨDO-representation.

Definition 3.1 *If $\mathcal{A}$ is an X-subalgebra of $\mathcal{C}(X)$, the* crossed product $\mathcal{A} \rtimes X \equiv \mathscr{A}$ *is the norm closed linear subspace of $\mathscr{B}(X)$ generated by the operators of the form $\varphi(Q)\psi(P)$ with $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{C}_0(X^*)$.*

The fact that $\mathscr{A}$ is a C^*-algebra follows from:

Lemma 3.2 *If $\varphi \in \mathcal{C}(X)$ and $\psi \in \mathcal{C}_0(X^*)$ then for each number $\varepsilon > 0$ there are elements $x_1, ..., x_n \in X$ and functions $\psi_1, ..., \psi_n \in \mathcal{C}_0(X^*)$ such that:*

$$\|\psi(P)\varphi(Q) - \textstyle\sum_k \varphi(Q + x_k)\psi_k(P)\| < \varepsilon. \tag{3.24}$$

For the proof, first approximate ψ by functions such that $\widehat{\psi} \in L^1(X)$ and then adapt the proof of [DG1, Lemma 2.1]. We mention two results which explain why we think of $\mathscr{A}$ as a C^*-algebra of quantum Hamiltonians. The first one is [GI4, Proposition 4.1].

Proposition 3.3 *Let $\mathcal{A}$ be an X-subalgebra of $\mathcal{C}(X)$ which contains the constants. Let $h : X^* \to \mathbb{R}$ be a continuous non-constant function such that $\lim_{k\to\infty} |h(k)| = \infty$. Then $\mathcal{A} \rtimes X$ is the C^*-algebra generated*[‡] *by the self-adjoint operators of the form $h(P + k) + v(Q)$, with $k \in X^*$ and $v \in \mathcal{A}$ real.*

The second one is [DG1, Corollary 2.4]. Here we assume $X = \mathbb{R}^n$ and denote $\mathcal{A}^\infty$ the set of functions in $\mathcal{A}$ such that all their derivatives exist and belong to $\mathcal{A}$.

Proposition 3.4 *Let h be a real elliptic polynomial of order m on X and let $\mathcal{A}$ be as in Proposition 3.3. Then $\mathcal{A} \rtimes X$ is the C^*-algebra generated by the self-adjoint operators of the form $h(P) + S$, where S runs over the set of symmetric differential operators of order $< m$ with coefficients in $\mathcal{A}^\infty$.*

Examples 3.5 We shall point out now the simplest crossed products. The smallest crossed product $\{0\} = \{0\} \rtimes X$ is, of course, of no interest.

1. The largest crossed product is $\mathscr{C}(X) = \mathcal{C}(X) \rtimes X$, see Theorem 3.10.
2. The C_0 functions of momentum: $\mathcal{C}_0(X^*) = \mathbb{C} \rtimes X$.
3. The algebra of compact operators: $\mathscr{K}(X) = \mathcal{C}_0(X) \rtimes X$.
4. The *two-body algebra*: $\mathscr{T}(X) := \mathcal{C}_\infty(X) \rtimes X = \mathcal{C}_0(X^*) + \mathscr{K}(X)$.

The name of the fifth algebra is justified by Propositions 3.3 and 3.4. Indeed, if $X = \mathbb{R}^n$ then $\mathscr{T}(X)$ is the C^*-algebra generated by the self-adjoint operators of the form $(P + k)^2 + v(Q)$ with $k \in X$ and $v \in C_{\mathrm{c}}^\infty(X)$ is real, or by those of the form $\Delta + \sum_{j=1}^n a_j\partial_j + a_0$ where a_j are C^∞ functions constant outside a compact.

Remark 3.6 Note that the only abelian crossed products are $\{0\}$ and $\mathcal{C}_0(X^*)$.

[‡] If $\mathcal{S}$ is a family of self-adjoint operators then the C^*-algebra generated by $\mathcal{S}$ is the smallest C^*-algebra of operators on $\mathcal{H}$ to which is affiliated each $H \in \mathcal{S}$.

We have defined a map $\mathcal{A} \mapsto \mathcal{A} \rtimes X$ from the set of all X-subalgebras of $\mathcal{C}(X)$ into the set of C^*-subalgebras of $\mathscr{B}(X)$ which is obviously increasing. The following theorem, which is an immediate consequence of a more general abstract result due to M.B. Landstad, cf. [Lan, Theorem 4] or [Ped], says that this map is injective and describes its range.

Theorem 3.7 *A C^*-subalgebra $\mathscr{A}$ of $\mathscr{B}(X)$ is a crossed product if and only if for each $A \in \mathscr{A}$ the following two conditions are satisfied:*

- *If $k \in X^*$ then $V_k^* A V_k \in \mathscr{A}$ and $\lim_{k\to 0} \|V_k^* A V_k - A\| = 0$,*
- *If $x \in X$ then $U_x A \in \mathscr{A}$ and $\lim_{x\to 0} \|(U_x - 1)A\| = 0$.*

In this case, there is a unique X-subalgebra $\mathcal{A} \subset \mathcal{C}(X)$ such that $\mathscr{A} = \mathcal{A} \rtimes X$, and this algebra is given by

$$\mathcal{A} = \mathscr{A}_\flat := \{\varphi \in \mathcal{C}(X) \mid \varphi(Q)^{(*)}\psi(P) \in \mathscr{A},\ \forall \psi \in \mathcal{C}_0(X^*)\}. \tag{3.25}$$

Note that, since $\mathscr{A}$ is stable under taking adjoints, if we replace $U_x A$ by AU_x and $(U_x - 1)A$ by $A(U_x - 1)$ in the second condition above we get an equivalent condition. If each element A of a C^*-subalgebra $\mathscr{A} \subset \mathscr{B}(X)$ verifies the two conditions of the theorem, we shall say that $\mathscr{A}$ *satisfies Landstad's conditions*.

The following reformulation of the second Landstad condition is useful.

Lemma 3.8 *If $T \in \mathscr{B}(X)$ then the next three assertions are equivalent:*

- $\lim_{x\to 0} \|(U_x - 1)T\| = 0$,
- *$T = \psi(P)T_0$ for some $\psi \in \mathcal{C}_0(X^*)$ and $T_0 \in \mathscr{B}(X)$,*
- *$\forall \varepsilon > 0\ \exists F \subset X^*$ with $X^* \setminus F$ compact and $\|\mathbf{1}_F(P)T\| < \varepsilon$.*

Proof: It suffices to consider only the first two conditions. If $T = \psi(P)T_0$ then

$$\|(U_x - 1)T\| \leq \|(U_x - 1)\psi(P)\|\|T_0\| \leq \|T_0\| \sup_k |k(x)\psi(k)| \to 0 \quad \text{as } x \to 0.$$

To prove the converse assertion, let $\mathscr{B}_0 = \{T \in \mathscr{B} \mid \lim_{x\to 0} \|(U_x - 1)T\| = 0\}$. This is clearly a closed subspace of $\mathscr{B}$ such that $\psi(P)\mathscr{B}_0 \subset \mathscr{B}_0$ if $\psi \in \mathcal{C}_0(X^*)$. By taking $\widehat{\psi}(k) = |K|^{-1}\mathbf{1}_K$ in (2.15), where K runs over the family of compact neighbourhoods of the origin in X^*, we easily see that each $T \in \mathscr{B}_0$ is a norm limit of operators of the form $\psi(P)T$. Now the Cohen-Hewitt factorization theorem [FeD, Theorem 9.2] shows that each $T \in \mathscr{B}_0$ can be written as $T = \psi(P)T_0$ with $\psi \in \mathcal{C}_0(X^*)$ and $T_0 \in \mathscr{B}_0$. ■

Corollary 3.9 *If $\mathscr{A}$ is a crossed product then each $A \in \mathscr{A}$ can be factorized as $A = A_1\psi_1(P) = \psi_2(P)A_2$ with $A_i \in \mathscr{A}$ and $\psi_i \in \mathcal{C}_0(X^*)$. In particular, if $A \in \mathscr{A}$ and ψ is a bounded continuous function on X^* then $A\psi(P)$ and $\psi(P)A$ belong to $\mathscr{A}$.*

Theorem 3.7 allows us to give an intrinsic description of some crossed products. By "intrinsic" we mean a description which makes no reference to the crossed product operation. Examples may be found in Section 6, here we give the description of the largest crossed product $\mathscr{C}(X)$ which makes the connection with the definition (1.1).

Theorem 3.10 *The crossed product* $\mathscr{C}(X) = \mathcal{C}(X) \rtimes X$ *is given by (1.1).*

For the proof, it suffices to note that the right hand side of (1.1) is a C^*-algebra and to apply Theorem 3.7. It is useful to view the last condition in (1.1) from the perspective of Lemma 3.8: this gives a precise meaning to the fact that the operators from $\mathscr{C}(X)$ tend to zero as $P \to \infty$.

Remark 3.11 If $X = \mathbb{R}^n$ we see that $\mathscr{C}(X)$ is the norm closed linear subspace of $\mathscr{B}(X)$ generated by the operators $\varphi(Q)\psi(P)$ with φ in the space of C^∞ functions which are bounded together with all their derivatives and ψ in the space of C^∞ functions with compact support. So $\mathscr{C}(X)$ is generated by a rather restricted class of pseudo-differential operators. In particular, $\mathscr{C}(X)$ is the norm closure of the set of pseudo-differential operators with symbols of class S^m if $m < 0$ (see [Hör, Definition 18.1.1] and use [Hör, Theorem 18.1.6]). From Proposition 3.4 it also follows that $\mathscr{C}(X)$ is generated by a rather small class of elliptic operators.

As a consequence, we get an intrinsic description of the algebras of quantum Hamiltonians, in the sense of Definition 1.13.

Proposition 3.12 *A* C^**-subalgebra* $\mathscr{A} \subset \mathscr{B}(X)$ *is a* C^**-algebra of quantum Hamiltonians if and only if* $\mathscr{A} \supset \mathscr{T}(X)$ *and*

- $x \in X, k \in X^*, A \in \mathscr{A} \Longrightarrow V_k^* A V_k$ *and* $U_x A$ *belong to* $\mathscr{A}$,
- $\lim_{k\to 0} \|[A, V_k]\| = \lim_{x\to 0} \|(U_x - 1)A\| = 0$.

Remark 3.13 Observe that the classical Riesz-Kolmogorov compactness criterion

$$\begin{aligned}\mathscr{K}(X) &= \{T \in \mathscr{B}(X) \mid \lim_{k\to 0} \|(V_k - 1)T\| = 0 \text{ and } \lim_{x\to 0} \|(U_x - 1)T\| = 0\} \\ &= \{T \in \mathscr{B}(X) \mid T = \varphi(Q)S = \psi(P)R \ \text{ with } \varphi \in \mathcal{C}_0(X), \psi \in \mathcal{C}_0(X^*) \\ &\qquad \text{and } S, R \in \mathscr{B}(X)\}\end{aligned}$$

is also an intrinsic characterization of a crossed product and follows easily from Theorem 3.7 and Lemma 3.8 together with a similar fact with the group U_x replaced by V_k. In more intuitive terms, the compact operators are characterized by the fact that they vanish when $P \to \infty$ and $Q \to \infty$.

Now we show that the set of C^*-subalgebras of $\mathscr{B}(X)$ which are crossed products is stable under arbitrary intersections and that the C^*-algebra generated by an arbitrary family of crossed products is again a crossed product. We denote by $C^*\big(\bigcup_\lambda \mathcal{B}_\lambda\big)$ the C^*-subalgebra generated by a family of C^*-subalgebras $\mathcal{B}_\lambda$.

Theorem 3.14 *If* $(\mathcal{A}_\lambda)$ *is an arbitrary family of* X*-subalgebras of* $\mathcal{C}(X)$ *then* $\bigcap_\lambda \mathcal{A}_\lambda$ *and* $C^*\big(\bigcup_\lambda \mathcal{A}_\lambda\big)$ *are* X*-subalgebras and:*

$$\bigcap\nolimits_\lambda(\mathcal{A}_\lambda \rtimes X) = \big(\bigcap\nolimits_\lambda \mathcal{A}_\lambda\big) \rtimes X, \tag{3.26}$$

$$C^*\big(\bigcup\nolimits_\lambda(\mathcal{A}_\lambda \rtimes X)\big) = C^*\big(\bigcup\nolimits_\lambda \mathcal{A}_\lambda\big) \rtimes X. \tag{3.27}$$

Proof: The fact that $\bigcap_\lambda \mathcal{A}_\lambda$ and $C^*\big(\bigcup_\lambda \mathcal{A}_\lambda\big)$ are X-subalgebras is easy to prove and the inclusions $\supset$ in (3.26) and $\subset$ in (3.27) are obvious. The proof of $\supset$ in (3.27) is elementary. Indeed, it suffices to show that $\varphi(Q)\psi(P)$ belongs to the left hand side of (3.27) if $\varphi \in C^*\big(\bigcup_\lambda \mathcal{A}_\lambda\big)$. Then we may assume that $\varphi = \varphi_L = \prod_{\lambda \in L} \varphi_\lambda$ with $\varphi_\lambda \in \mathcal{A}_\lambda$ and L a finite set. Let $\lambda \in L$ and $M = L \setminus \{\lambda\}$. Then Corollary 3.9 applied to $\varphi_\lambda(Q)\psi(P) \in \mathcal{A}_\lambda \rtimes X$ gives

$$\varphi_L(Q)\psi(P) = \varphi_M(Q)\varphi_\lambda(Q)\psi(P) = \varphi_M(Q)\psi_\lambda(P)A_\lambda$$

for some $\psi_\lambda \in \mathcal{C}_0(X^*)$ and $A_\lambda \in \mathcal{A}_\lambda \rtimes X$. Repeating the argument with φ_L replaced by φ_M we see that $\varphi_L(Q)\psi(P)$ can be written as a product of elements of $\mathcal{A}_\lambda \rtimes X$ with $\lambda \in L$. This proves (3.27).

The inclusion $\subset$ in (3.26) is a deeper fact, it depends on Theorem 3.7. Let $\mathscr{A}_\lambda = \mathcal{A}_\lambda \rtimes X$ and $\mathscr{A} = \bigcap_\lambda \mathscr{A}_\lambda$. It is easy to check that $\mathscr{A}$ satisfies the two conditions of Theorem 3.7, so $\mathscr{A} = \mathcal{A} \rtimes X$ where $\mathcal{A}$ is defined by (3.25). If $\varphi \in \mathcal{C}(X)$ has the property $\varphi(Q)^{(*)}\psi(P) \in \mathscr{A}$ for all $\psi \in \mathcal{C}_0(X^*)$ then we also have $\varphi(Q)^{(*)}\psi(P) \in \mathscr{A}_\lambda$ for all such ψ, hence $\varphi \in (\mathscr{A}_\lambda)_\flat = \mathcal{A}_\lambda$ for each λ. Thus $\varphi \in \bigcap_\lambda \mathcal{A}_\lambda$, hence $\mathcal{A} \subset \bigcap_\lambda \mathcal{A}_\lambda$. ■

Proposition 3.15 *If $\mathcal{A}, \mathcal{J}$ are X-subalgebras then $\mathcal{J}$ is an ideal of $\mathcal{A}$ if and only if $\mathcal{J} \rtimes X$ is an ideal of $\mathcal{A} \rtimes X$.*

Proof: The fact that "$\mathcal{J} \subset \mathcal{A}$ ideal $\Rightarrow \mathcal{J} \rtimes X \subset \mathcal{A} \rtimes X$ ideal" follows easily from Lemma 3.2. For the converse it suffices to show that if $\mathscr{J}, \mathscr{A}$ are crossed products and if $\mathscr{J}$ is an ideal of $\mathscr{A}$, then $\mathscr{J}_\flat$ is an ideal of $\mathscr{A}_\flat$. Let $\xi \in \mathscr{J}_\flat$ and $\varphi \in \mathscr{A}_\flat$, then by Corollary 3.9, for each $\psi \in \mathcal{C}_0(X^*)$ we can factorize $\varphi(Q)\psi(P) = \psi_0(P)S$ for some $\psi_0 \in \mathcal{C}_0(X^*)$ and $S \in \mathscr{A}$. Thus $(\xi\varphi)(Q)\psi(P) = \xi(Q)\psi_0(P)S \in \mathcal{J}$ because $\xi(Q)\psi_0(P) \in \mathscr{J}$ and $\mathscr{J}$ is an ideal of $\mathscr{A}$, hence $\xi\varphi \in \mathscr{J}_\flat$. ■

Proposition 3.16 *Assume that $\mathcal{A}, \mathcal{B}, \mathcal{J}$ are X-subalgebras of $\mathcal{C}(X)$ such that $\mathcal{A} = \mathcal{B} + \mathcal{J}$ and that $\mathcal{J}$ is an ideal in $\mathcal{A}$. Then $\mathcal{J} \rtimes X$ is an ideal in $\mathcal{A} \rtimes X$ and $\mathcal{A} \rtimes X = \mathcal{B} \rtimes X + \mathcal{J} \rtimes X$. If $\mathcal{A} = \mathcal{B} + \mathcal{J}$ is a linear direct sum, then $\mathcal{A} \rtimes X = \mathcal{B} \rtimes X + \mathcal{J} \rtimes X$ is a linear direct sum.*

Proof: We know that $\mathcal{J} \rtimes X$ is an ideal in $\mathcal{A} \rtimes X$ and that $\mathcal{B} \rtimes X \subset \mathcal{A} \rtimes X$ is a C^*-subalgebra. From [Dix, Corollary 1.8.4] we see that $\mathcal{B} \rtimes X + \mathcal{J} \rtimes X$ is closed in $\mathcal{A} \rtimes X$, and since it is clearly dense in $\mathcal{A} \rtimes X$, we have $\mathcal{A} \rtimes X = \mathcal{B} \rtimes X + \mathcal{J} \rtimes X$. Finally,

$$(\mathcal{B} \rtimes X) \cap (\mathcal{J} \rtimes X) = \big(\mathcal{B} \cap \mathcal{J}\big) \rtimes X$$

because of (3.26), and this is $\{0\}$ if $\mathcal{B} \cap \mathcal{J} = \{0\}$. . ■

We mention a fact which is useful in the explicit computations of $\mathscr{A}_\flat$.

Remark 3.17 It is clear that in (3.25) it suffices to consider only $\psi \in \mathcal{C}_{\mathrm{c}}(X^*)$. Since, by Corollary 3.9, a crossed product is a $\mathcal{C}_0(X^*)$-bimodule, we get the following simpler description of $\mathcal{A}$: *if there is $\xi \in \mathcal{C}_0(X^*)$ such that $\xi(k) \neq 0$ for all $k \in X^*$, then*

$$\mathcal{A} = \{\varphi \in \mathcal{C}(X) \mid \varphi(Q)^{(*)}\xi(P) \in \mathscr{A}\}. \tag{3.28}$$

Such a ξ exists if and only if X^* is σ-compact (i.e. a countable union of compact sets).

Remark 3.18 The following comment on the first Landstad condition is of some interest, although it does not play any rôle in our arguments. Let $C^{\mathrm{u}}(Q)$ be the set of $S \in \mathscr{B}(X)$ which verify the first Landstad condition; this is clearly a C^*-algebra. Let us say that an operator $S \in \mathscr{B}(X)$ is of *finite range* (not rank!) if there is a compact neighborhood Λ of the origin such that $S\mathbf{1}_K(Q) = \mathbf{1}_{K+\Lambda}(Q)S\mathbf{1}_K(Q)$ for any Borel set K. Clearly, the set of finite range operators is a $*$-subalgebra of $\mathscr{B}(X)$ and it can be shown that the set of finite range operators which belong to $C^{\mathrm{u}}(Q)$ is dense in $C^{\mathrm{u}}(Q)$. Moreover, under quite general conditions on X it can be shown that a finite range operator belongs to $C^{\mathrm{u}}(Q)$ (this is probably always true). Thus, if $X = \mathbb{R}^n$ or if X is a discrete group for example, then $C^{\mathrm{u}}(Q)$ is exactly the norm closure of the set of finite range operators. These questions are treated in [GG2, Propositions 4.11 and 4.12].

4 Affiliation to $\mathscr{C}(X)$

Theorem 1.2 shows that the essential spectrum of the operators affiliated to $\mathscr{C}(X)$ is determined by their localizations at infinity, so it is important to show that the class of operators affiliated to $\mathscr{C}(X)$ is large. We show in this section that this is indeed the case: singular perturbations of hypoelliptic self-adjoint pseudo-differential operators are affiliated to $\mathscr{C}(X)$. If one thinks of $\mathscr{C}(X)$ as the C^*-algebra generated by the operators of the form $\varphi(Q)\psi(P)$ with $\varphi \in \mathcal{C}(X),\ \psi \in \mathcal{C}_0(X)$, this is far from obvious.

In the rest of the section we fix a finite dimensional Hilbert space $\mathbf{E}$, we set $\mathcal{H} = L^2(X;\mathbf{E})$ and define $\mathscr{C} = \mathscr{C}(X)$ as in (1.1). Since the adjoint space[†] $\mathcal{H}^*$ is identified with $\mathcal{H}$ by using the Riesz isomorphism, if $\mathcal{G}$ is a Hilbert space with $\mathcal{G} \subset \mathcal{H}$ continuously and densely then we get a similar embedding $\mathcal{H} \subset \mathcal{G}^*$.

Let H be a self-adjoint operator on $\mathcal{H}$ and let $z \in \rho(H)$. As we saw in (1.3), H is affiliated to $\mathscr{C}$ if and only if

$$\lim_{x\to 0} \|(U_x - 1)(H-z)^{-1}\| = 0 \text{ and } \lim_{k\to 0} \|[V_k, (H-z)^{-1}]\| = 0. \tag{4.1}$$

In the next subsection we make an abstract analysis of these relations and in Subsection 4.2 we give concrete examples.

4.1. A function $\theta : X^* \to \mathbb{R}$ such that $\lim_{k\to\infty} \theta(k) = +\infty$ will be called *divergent*. Lemma 3.8 and an interpolation argument give:

Lemma 4.1 *The first condition in (4.1) is fulfilled if and only there are $s > 0$ and a continuous divergent function θ such that $D(|H|^s) \subset D(\theta(P))$. And then this property holds for all real numbers $s > 0$.*

Let $S(\mathbf{E})$ be the space of symmetric operators on $\mathbf{E}$. If $h : X^* \to S(\mathbf{E})$ is Borel, then $h(P)$ is the self-adjoint operator on $\mathcal{H}$ such that $\mathcal{F}h(P)\mathcal{F}^*$ is the operator of multiplication by h in $L^2(X^*;\mathbf{E})$. If $\lim_{k\to\infty} \operatorname{dist}(0, \sigma(h(k))) = \infty$ then we write $\lim_{k\to\infty} h(k) = \infty$. This property is equivalent to $\lim_{k\to\infty} \|(h(k) + i)^{-1}\| = 0$

[†] The adjoint space (space of antilinear continuous forms) of a Hilbert space $\mathcal{G}$ is denoted $\mathcal{G}^*$ and if $u \in \mathcal{G}$ and $v \in \mathcal{G}^*$ then we set $v(u) = \langle u, v\rangle$.

and implies $\lim_{k\to\infty}\|\varphi(h(k))\| = 0$ for all $\varphi \in \mathcal{C}_0(\mathbb{R})$. If $\mathbf{E} = \mathbb{C}$ this means $\lim_{k\to\infty}|h(k)| = \infty$.

Corollary 4.2 *If $h : X^* \to S(\mathbf{E})$ is a continuous function on X^* then $h(P)$ is affiliated to $\mathscr{C}$ if and only if $\lim_{k\to\infty} h(k) = \infty$.*

In particular, if $X = \mathbb{R}^2$ then the operator $H = \partial_1^2 - \partial_2^2$ is not affiliated to $\mathscr{C}$. A second interesting operator not affiliated to $\mathscr{C}$ is $H = (\partial_1 + ix_2)^2 + (\partial_2 + ix_1)^2$.

We now give the simplest affiliation criterion.

Proposition 4.3 *Assume that H_0 is a self-adjoint operator affiliated to $\mathscr{C}$ and that V is a bounded symmetric operator such that $\lim_{k\to 0}\|[V_k, V]\| = 0$. Then $H = H_0 + V$ is a self-adjoint operator affiliated to $\mathscr{C}$.*

Proof: If $R = (H + i)^{-1}$ and $R_0 = (H_0 + i)^{-1}$ then $R = R_0[1 + VR_0]^{-1}$ so the first condition in (4.1) is obviously satisfied. For the second one, note that $1 + VR_0$ is invertible and satisfies it, hence its inverse verifies it too. ■

From now on we consider only situations when V is not bounded.

Proposition 4.4 *Let H be a self-adjoint operator such that $V_k D(H) \subset D(H)$ for all k. Then H is affiliated to $\mathscr{C}$ if and only if $D(H) \subset D(\theta(P))$ for some continuous divergent function θ and*

$$\lim_{k\to 0}\|[V_k, H]\|_{D(H)\to D(H)^*} = 0. \tag{4.2}$$

Proof: It is clear that $V_k D(H) \subset D(H)$ for all k if and only if V_k extends to a continuous map $D(H)^* \to D(H)^*$ for each k, and then we have in $B(\mathcal{H})$:

$$[V_k, (H - z)^{-1}] = (H - z)^{-1}[H, V_k](H - z)^{-1}. \tag{4.3}$$

The operator $[H, V_k]$ belongs to $B(D(H), \mathcal{H})$ and so we can consider it as a map $D(H) \to D(H)^*$. But $(H - z)^{-1}$ is an isomorphism $\mathcal{H} \to D(H)$ and $D(H)^* \to \mathcal{H}$. To end the proof it suffices to use Lemma 4.1. ■

We shall give below three perturbative criteria of affiliation: we add to an operator affiliated to $\mathscr{C}$ an operator which is not necessarily affiliated to it. Note that functions of Q are never affiliated to $\mathscr{C}$. First we consider operator bounded perturbations.

Corollary 4.5 *Let H_0 be a self-adjoint operator affiliated to $\mathscr{C}$ such that $V_k D(H_0) \subset D(H_0)$ for all k. Let V be a symmetric operator with domain $D(H_0)$ and such that $H = H_0 + V$ is self-adjoint. Then H is affiliated to $\mathscr{C}$ if and only if*

$$\lim_{k\to 0}\|[V_k, V]\|_{D(H_0)\to D(H_0)^*} = 0. \tag{4.4}$$

Now we want to consider form bounded perturbations in a generalized sense (in order to cover not semibounded operators). Let H be a self-adjoint operator on $\mathcal{H}$. We say that a Hilbert space $\mathcal{G}$ is *adapted* to H if $D(H) \subset \mathcal{G} \subset \mathcal{H}$ continuously and densely and $H - z$ extends to an isomorphism $\mathcal{G} \to \mathcal{G}^*$ for some (hence for all) $z \in \mathbb{C}$ outside the spectrum of H. Then H extends to a continuous operator $\mathcal{G} \to \mathcal{G}^*$ and we keep the notation H for the extended map. It is not difficult to show that *if H is a semibounded operator then $\mathcal{G}$ is adapted to H if and only if $\mathcal{G} = D(|H|^{1/2})$ as topological vector spaces*, see [GG2, page 47]. But in general, for example in the case of Dirac operators, this is not the case. Observe that

$$D(H) \subset \mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^* \subset D(H)^*$$

continuously and densely, in particular $B(\mathcal{G}, \mathcal{G}^*) \subset B(D(H), D(H)^*)$. It is then clear that one has $V_k\mathcal{G} \subset \mathcal{G}$ for all k if and only if V_k extends to a continuous map $\mathcal{G}^* \to \mathcal{G}^*$ for each k, and in this case the identity (4.3) is valid in $B(\mathcal{G}^*, \mathcal{G})$. The operator $[H, V_k]$ belongs to $B(\mathcal{G}, \mathcal{G}^*)$ and so we can consider it as a map $D(H) \to D(H)^*$. But $(H - z)^{-1}$ is an isomorphism $\mathcal{H} \to D(H)$ and $D(H)^* \to \mathcal{H}$. Thus:

Proposition 4.6 *Let H be a self-adjoint operator on $\mathcal{H}$ such that $D(H) \subset D(\theta(P))$ for some continuous divergent function θ. Assume that $\mathcal{G}$ is a Hilbert space adapted to H and that $V_k\mathcal{G} \subset \mathcal{G}$ for all k. Then H is affiliated to $\mathscr{C}$ if and only if*

$$\lim_{k\to 0} \|[V_k, H]\|_{D(H)\to D(H)^*} = 0. \tag{4.5}$$

In many situations of interest in quantum mechanics the domain of the Hamiltonian is difficult to determine while its form domain is quite explicit. For this reason the following condition stronger than (4.5) is often more convenient:

$$\lim_{k\to 0} \|[V_k, H]\|_{\mathcal{G}\to\mathcal{G}^*} = 0. \tag{4.6}$$

We shall use this in the following context.

Definition 4.7 *Let H_0 be a self-adjoint operator on $\mathcal{H}$ and let $\mathcal{G}$ be a Hilbert space adapted to it. We say that V is a* standard form perturbation *of H_0 if V is a continuous symmetric sesquilinear form on $\mathcal{G}$ and there are numbers $\mu \in [0, 1)$ and $\nu \geq 0$ such that one of the following conditions is satisfied:*
(1) $\pm V \leq \mu|H_0| + \nu$ as forms on $\mathcal{G}$
(2) H_0 is bounded from below and $V \geq -\mu H_0 - \nu$ as forms on $\mathcal{G}$.

Then the operator $H = H_0 + V : \mathcal{G} \to \mathcal{G}^*$ is such that its restriction to $D(H) = \{u \in \mathcal{G} \mid Hu \in \mathcal{H}\}$ is a self-adjoint operator on $\mathcal{H}$ (and will also be denoted H) and $\mathcal{G}$ is adapted to H too (see [DG3]). Note that V is seen as a continuous operator $\mathcal{G} \to \mathcal{G}^*$.

Corollary 4.8 *Let H_0 and V as above. We assume that $\mathcal{G} \subset D(\theta(P))$ for some continuous divergent function θ, that $V_k\mathcal{G} \subset \mathcal{G}$ for all k, and $\lim_{k\to 0} \|[V_k, H]\|_{\mathcal{G}\to\mathcal{G}^*}$. Then H is affiliated to $\mathscr{C}$.*

The next result covers perturbations of H_0 which are not dominated by H_0.

Proposition 4.9 *Let H_1, H_2 be bounded from below self-adjoint operators and let us denote $\mathcal{G}_i = D(|H_i|^{1/2})$. Assume that $\mathcal{G} \equiv \mathcal{G}_1 \cap \mathcal{G}_2$ is dense in $\mathcal{H}$ and let $H = H_1 + H_2$, the sum being defined in form sense. Let us suppose that $\mathcal{G} \subset D(\theta(P))$ for some continuous divergent function θ and that for $i = 1, 2$ we have $V_k\mathcal{G}_i \subset \mathcal{G}_i$ and $\lim_{k\to 0} \|[V_k, H_i]\|_{B(\mathcal{G}_i, \mathcal{G}_i^*)} = 0$. Then H is affiliated to $\mathscr{C}$.*

Proof: Let us recall that the form sum $H = H_1 + H_2$ is defined as the unique self-adjoint operator such that $D(|H|^{1/2}) = \mathcal{G}$ and $\langle u, Hu\rangle = \langle u, H_1 u\rangle + \langle u, H_2 u\rangle$ for all $u \in \mathcal{G}$. The topology of $\mathcal{G}$ is the intersection topology of $\mathcal{G}_1$ and $\mathcal{G}_2$, so thinking in terms of sesquilinear forms we see that

$$\|[V_k, H]\|_{B(\mathcal{G},\mathcal{G}^*)} \leq C\|[V_k, H_1]\|_{B(\mathcal{G}_1,\mathcal{G}_1^*)} + C\|[V_k, H_2]\|_{B(\mathcal{G}_2,\mathcal{G}_2^*)}$$

for some constant C. Hence (4.6) is satisfied. ■

4.2. If w is a continuous divergent function on X^* let $\mathcal{H}^w \equiv \mathcal{H}^w(X) = D(w(P))$ equipped with the graph norm. We saw in Lemma 4.1 that if H is affiliated to $\mathscr{C}$ then $D(|H|^{1/2}) \subset \mathcal{H}^w$ for such a w. We consider now operators whose form domain is *equal* to some $\mathcal{H}^w$.

We say that w is a *weight*[†] if $w : X^* \to]0, \infty[$ is continuous and $w(k+p) \leq \omega(k)w(p)$ for some function ω and all $k, p \in X^*$. If ω is the smallest function satisfying such an estimate, then $\omega(k+p) \leq \omega(k)\omega(p)$. From now on we shall assume that ω satisfies this submultiplicativity condition. We also say *ω-weight* if we need to be more specific. If $X = \mathbb{R}^n$ then a standard choice is $w(k) = \langle k\rangle^s$ for some real s.

Lemma 4.10 *A continuous divergent function w on X^* is an ω-weight if and only if $V_k\mathcal{H}^w \subset \mathcal{H}^w$ and $\|V_k\|_{B(\mathcal{H}^w)} \leq \omega(k)$ for all k.*

Proof: We may take $\|w(P)u\|$ as norm on $\mathcal{H}^w$. From $V_k^* w(P) V_k = w(P+k)$ we see that we have $V_k\mathcal{H}^w \subset \mathcal{H}^w$ and $\|V_k\|_{B(\mathcal{H}^w)} \leq \omega(k)$ if and only if $\|w(P+k)u\| \leq \omega(k)\|w(P)u\|$ for all u, which is equivalent to $w(k+p) \leq \omega(k)w(p)$ for all k, p. ■

Proposition 4.11 *A self-adjoint operator on $\mathcal{H}$ with $D(|H|^{1/2}) = \mathcal{H}^w$ for some divergent weight w and such that $\lim_{k\to 0} \|[V_k, H]\|_{B(\mathcal{H}^w,\mathcal{H}^{w*})} = 0$ is affiliated to $\mathscr{C}$.*

This is an immediate consequence of Proposition 4.6.

Proposition 4.12 *Let H be as in Proposition 4.11 and bounded from below. Let $V \in L^1_{\mathrm{loc}}(X)$ be a real function whose negative part is form bounded with respect to H with relative bound strictly less than 1. Then the self-adjoint operator $H + V(Q)$ (form sum) is affiliated to $\mathscr{C}$.*

Proof: Let V_+, V_- be the positive and negative parts of V, then we define the sum as $(H - V_-) + V_+$ and apply successively Propositions 4.6 and 4.9. ■

Example 4.13 The most common situation is $X = \mathbb{R}^n$ and $w(k) = \langle k\rangle^s$ for some real $s > 0$. Then $\mathcal{H}^w$ is the usual Sobolev space $\mathcal{H}^s$ and typical operators satisfying the conditions of the Proposition 4.11 are the uniformly elliptic operators of order $2s$. For example, let $s = m \geq 1$ integer and

$$L = \textstyle\sum_{|\alpha|,|\beta|\leq m} P^\alpha a_{\alpha\beta} P^\beta$$

[†] The terminology is suggested by that from [Hör, Section 10.1], cf. the remark after Theorem 10.1.5.

for some measurable functions $a_{\alpha\beta} : X \to B(\mathbf{E})$ such that the operator of multiplication by $a_{\alpha\beta}$ is a continuous map $\mathcal{H}^{m-|\beta|} \to \mathcal{H}^{|\alpha|-m}$ (this is a very general assumption which allows one to give a meaning to the differential expression L). Then $L : \mathcal{H}^m \to \mathcal{H}^{-m}$ is a continuous map and $V_k^* L V_k$ is a polynomial in k. If $\langle u, Lu\rangle \geq \mu\|u\|^2_{\mathcal{H}^m} - \nu\|u\|^2$ for some $\mu, \nu > 0$, then L induces a self-adjoint operator in $\mathcal{H}$ which is affiliated to $\mathscr{C}$.

Example 4.14 We give an explicit example of physical interest in the case $s = 1$. Let

$$H = \textstyle\sum_{i,j}(P_i - A_i)G_{ij}(P_j - A_j) + V \equiv (P - A)G(P - A) + V \tag{4.7}$$

where G_{ij}, A_i, V are (the operators of multiplication by) locally integrable real functions having the following properties ($\|\cdot\|_1$ is the norm of $\mathcal{H}^1$):
(1) $G_{ij} \in L^\infty(X)$, the matrix $G(x) = (G_{ij}(x))$ is symmetric and $G(x) \geq \nu > 0$,
(2) for each $\varepsilon > 0$ there is $\delta \in \mathbb{R}$ such that $\|A_j u\| \leq \varepsilon\|u\|_1 + \delta\|u\|$ for all $u \in \mathcal{H}^1$,
(3) if V_- is the negative part of V then for each $\varepsilon > 0$ there is a real number δ such that $\langle u, V_- u\rangle \leq \varepsilon\|u\|_1^2 + \delta\|u\|^2$ for all $u \in \mathcal{H}^1$.
Note that the conditions on A_j and V_- are satisfied if there is $s < 1$ such that $\|A_j u\| \leq C\|u\|_s$ and $\langle u, V_- u\rangle \leq C\|u\|_s^2$. Then H is affiliated to $\mathscr{C}$. Indeed, observe first that $H_0 \equiv (P - A)G(P - A)$ is a self-adjoint operator with form domain equal to $\mathcal{H}^1$, because there is δ such that:

$$\langle u, H_0 u\rangle \geq \nu\|(P - A)u\|^2 \geq \frac{\nu}{2}\|Pu\|^2 - \nu\|Au\|^2 \geq \frac{\nu}{4}\|Pu\|^2 - \delta\|u\|^2$$

Hence, according to Proposition 4.12, it suffices to prove that H_0 is affiliated to $\mathscr{C}$. But

$$\begin{aligned} V_k^* H_0 V_k &= (P - A + k)G(P - A + k) \\ &= H_0 + kG(P - A) + (P - A)Gk + kGk. \end{aligned}$$

Thus $\|V_k^* H_0 V_k - H_0\|_{B(\mathcal{H}^1,\mathcal{H}^{-1})} \leq C(|k| + |k|^2)$ so we can use Proposition 4.11.

Remark 4.15 Let us consider the operator H_0 under the more general condition $A_j \in L^2_{\mathrm{loc}}(X)$. More precisely, H_0 is the positive self-adjoint operator associated to the closed quadratic form $\|(P - A)u\|^2$ whose domain is the set $\mathcal{G}$ of $u \in \mathcal{H}$ such that the distributions $(P_j - A_j)u$ belong to $\mathcal{H}$. The preceding computation shows that $V_k\mathcal{G} \subset \mathcal{G}$ and that (4.6) is satisfied. Hence H_0 is affiliated to $\mathscr{C}$ if and only if $\mathcal{G} \subset \theta(P)$ for some continuous divergent function θ. But this cannot be true without some boundedness conditions on A at infinity.

As a final example we consider singular perturbations of $h(P)$, where $h : X \to \mathbb{R}$ is a continuous divergent function and X is an arbitrary group. Let $\mathcal{G} = D(|h(P)|^{1/2})$. Two functions u, v on a neighborhood of infinity will be called *equivalent* if they satisfy $c_1|u(k)| \leq |v(k)| \leq c_2|u(k)|$ for all large k and some constants $c_1, c_2 > 0$. It is clear that $\mathcal{G} = \mathcal{H}^w$ if and only if h is equivalent to w^2. Then Proposition 4.12 implies:

Proposition 4.16 *Let $h : X \to \mathbb{R}$ be a divergent function equivalent to a weight and such that*

$$\lim_{k\to 0}\sup_p \frac{|h(p + k) - h(p)|}{1 + |h(p)|} = 0. \tag{4.8}$$

Let W be a standard form perturbation of $h(P)$ with $\lim_{k\to 0} \|[V_k, W]\|_{B(\mathcal{G},\mathcal{G}^*)} = 0$ and define $H_0 = h(P) + W$ as a form sum. Let $V \in L^1_{\text{loc}}(X)$ real and such that $V_- \le \mu H_0 + \nu$ on $\mathcal{G}$ for some $\mu < 1, \nu > 0$. Then the form sum $H = H_0 + V(Q)$ is a self-adjoint operator affiliated to $\mathscr{C}$.

Example 4.17 Let $X = \mathbb{R}^n$ and assume that h is of class C^1 and satisfies $|h'(k)| \le C(1 + |h(k)|)$. Then (4.8) is fulfilled because

$$|h(p+k) - h(p)| \le \sup_{0<\theta<1} |h'(p+\theta k)||k| \le C\big(1 + \sup_{0<\theta<1} |h(p+\theta k)|\big)|k|$$

which is $\le C'(1 + |h(p)|)|k|$ if $|k| \le 1$ because h is a equivalent to a weight. On the other hand, assume that h is of class C^m for some integer $m \ge 1$ and that we have: (1) $\lim_{k\to\infty} h(k) = +\infty$, (2) the derivatives of order m of h are bounded, (3) $\sum_{|\alpha|\le m} |h^{(\alpha)}(k)| \le C(1 + |h(k)|)$. Then from [ABG, p. 342–343] we get that h is equivalent to a weight. *Any real hypoelliptic polynomial satisfies all these conditions*, see Definition 11.1.2 and Theorem 11.1.3 in [Hör].

5 Localizations at infinity

In this section we prove our main result, Theorem 5.11, and some easy consequences.

5.1. We define first the localizations at infinity for functions in $\mathcal{C}(X)$. We denote $\mathcal{C}_s(X)$ the space $\mathcal{C}(X)$ equipped with the topology given by the seminorms $\|\varphi\|_\theta = \|\varphi\theta\|$ with $\theta \in \mathcal{C}_0(X)$ (this is the strict topology associated to the essential ideal $\mathcal{C}_0(X)$).

Lemma 5.1 *If $\varphi \in \mathcal{C}(X)$ and $\varkappa \in \delta X$ then $\varkappa.\varphi(y) := \lim_{x\to\varkappa} \varphi(x+y)$ exists locally uniformly in $y \in X$. Equivalently, we have $x.\varphi \to \varkappa.\varphi$ in $\mathcal{C}_s(X)$ if $x \to \varkappa$ in γX. The function $\varkappa.\varphi$ belongs to $\mathcal{C}(X)$ and we have $(\varkappa.\varphi)(y) = \varkappa(y.\varphi)$.*

Proof: Since φ is a bounded function, we have

$$\lim_{x\to\varkappa} \varphi(x+y) = \lim_{x\to\varkappa} (y.\varphi)(x) = \varkappa(y.\varphi)$$

by taking into account the two interpretations of $\varkappa$. Then we use Lemma 2.2. ■

Thus $\varkappa.\varphi \in \mathcal{C}(X)$ is well defined for all $\varkappa \in \gamma X$ (if $\varkappa = x \in X$, see page 16) and all $\varphi \in \mathcal{C}(X)$. The next lemma is a slight improvement of Lemma 5.1, it will allow us to give a completely elementary proof of Theorem 5.15 (see the remark after the proof of the theorem). Note that the relation $(\varkappa.\varphi)(y) = \varkappa(y.\varphi)$ remains true for all $\varkappa \in \gamma X$ if we interpret $x \in X$ as a character of $\ell_\infty(X)$. Since $y.\varphi \in \mathcal{C}(X)$ we see that $\varkappa.\varphi$ depends in fact only on the class of $\varkappa$ in $\beta_{\text{u}}X$, cf. Subsection 2.5. We shall keep the notation $\varkappa.\varphi$ even if $\varkappa \in \beta_{\text{u}}X$. Recall that $X \subset \beta_{\text{u}}X$ is an open dense subset.

Lemma 5.2 *Let $\varphi \in \mathcal{C}(X)$. Then $X \ni x \mapsto x.\varphi \in \mathcal{C}(X)$ extends to a continuous function $\beta_{\text{u}}X \ni \varkappa \mapsto \varkappa.\varphi \in \mathcal{C}_s(X)$. We have $\varkappa.\varphi(y) = \varkappa(y.\varphi)$ for all $y \in X$.*

Proof: For $\varkappa \in \beta_{\mathrm{u}}X = \sigma(\mathcal{C}(X))$ the function $\varkappa.\varphi$ is given by $\varkappa.\varphi(y) = \varkappa(y.\varphi)$, $y \in X$. It is easy to check directly that $\varkappa.\varphi$ so defined belongs to $\mathcal{C}(X)$: we have $|\varkappa(y.\varphi)| \leq \|y.\varphi\| = \|\varphi\|$ and

$$|\varkappa(y.\varphi) - \varkappa(z.\varphi)| = |\varkappa(y.\varphi - z.\varphi)| \leq \|y.\varphi - z.\varphi\| = \|(y-z).\varphi - \varphi\|.$$

It remains to prove that $\varkappa \mapsto \varkappa.\varphi\theta \in \mathcal{C}_0(X)$ is continuous for any $\theta \in \mathcal{C}_0(X)$, i.e. that for each $\chi \in \beta_{\mathrm{u}}X$, each $\varepsilon > 0$ and each $\theta \in \mathcal{C}_0(X)$ there is a neighborhood V of χ in $\beta_{\mathrm{u}}X$ such that $\|(\varkappa.\varphi - \chi.\varphi)\theta\| < \varepsilon$ if $\varkappa \in V$. Since θ is C_0, it will suffice to prove that for each χ and ε as before and each compact set $K \subset X$ there is a neighborhood V of χ such that $\varkappa \in V$ implies $|\varkappa(y.\varphi) - \chi(y.\varphi)| < \varepsilon$ for $y \in K$. But the map $y \mapsto y.\varphi \in \mathcal{C}(X)$ is norm continuous, thus $\{y.\varphi \mid y \in K\}$ is a compact subset of $\mathcal{C}(X)$. Hence there is a finite subset Z of K such that for each $y \in K$ we have $\min_{z\in Z} \|y.\varphi - z.\varphi\| < \varepsilon$. Thus for each $y \in K$ and $z \in Z$ we have

$$\begin{aligned} |\varkappa(y.\varphi) - \chi(y.\varphi)| &= |\varkappa(y.\varphi - z.\varphi) + \varkappa(z.\varphi) - \chi(z.\varphi) + \chi(z.\varphi - y.\varphi)| \\ &< 2\varepsilon + |\varkappa(z.\varphi) - \chi(z.\varphi)|. \end{aligned}$$

Now, if we take $V = \{\varkappa \in \beta_{\mathrm{u}}X \mid \sup_{z\in Z} |\varkappa(z.\varphi) - \chi(z.\varphi)| < \varepsilon\}$, then V is a neighborhood of χ in $\beta_{\mathrm{u}}X$ because Z is a finite set, and for each $\varkappa \in V$ and each $y \in K$ we have $|\varkappa(y.\varphi) - \chi(y.\varphi)| < 3\varepsilon$. ■

Lemma 5.3 *If $\varphi \in \mathcal{C}(X)$ then $\varkappa.\varphi = 0$ for all $\varkappa \in \delta X$ if and only if $\varphi \in \mathcal{C}_0(X)$.*

Proof: If $\varkappa.\varphi = 0$ for all $\varkappa \in \delta X$ then $\varkappa(\varphi) = (\varkappa.\varphi)(0) = 0$ for such $\varkappa$. If $\varphi \notin \mathcal{C}_0(X)$ then there is a number $a > 0$ such that the set $U = \{x \mid |\varphi(x)| > a\}$ is not relatively compact. Since $U \cap V \neq \emptyset$ for each V with relatively compact complement, we see that the family of sets $U \cap V$ is a filter basis and the filter $\mathfrak{f}$ it generates is finer than Fréchet and contains U. Let $\varkappa$ be any ultrafilter finer than $\mathfrak{f}$, then $\varkappa \in \delta X$ and $\varkappa(\mathbf{1}_U) = 1$. Finally, from $|\varphi| \geq a\mathbf{1}_F$ we get $|\varkappa(\varphi)|^2 = \varkappa(|\varphi|^2) \geq a^2\varkappa(\mathbf{1}_V) = a^2$, so we cannot have $\varkappa(\varphi) = 0$. ■

Definition 5.4 *If $\varphi \in \mathcal{C}(X)$ and $\varkappa \in \delta X$ then the function $\varkappa.\varphi \in \mathcal{C}(X)$ is the* localization of φ at $\varkappa$. *And $\ell(\varphi) := \{\varkappa.\varphi \mid \varkappa \in \delta X\} \subset \mathcal{C}(X)$ is the set of* localizations of φ at infinity.

For each $\varkappa \in \delta X$ let $\tau_\varkappa : \mathcal{C}(X) \to \mathcal{C}(X)$ be given by $\tau_\varkappa(\varphi) = \varkappa.\varphi$. Clearly this is a unital morphism and, since the property $x.(\varkappa.\varphi) = \varkappa.(x.\varphi)$ is easy to check, $\tau_\varkappa$ is in fact an X-morphism. By Lemma 5.3 we have

$$\textstyle\bigcap_{\varkappa\in\delta X} \ker \tau_\varkappa = \mathcal{C}_0(X). \tag{5.1}$$

Note that $\ker \tau_\varkappa$ *is the maximal X-ideal included in the maximal ideal* $\ker \varkappa$ *of* $\mathcal{C}(X)$.

Remark 5.5 In general $\tau_\varkappa \tau_\chi \neq \tau_\chi \tau_\varkappa$.

5.2. In this subsection we extend the notion of localization to operators in $\mathscr{C}(X)$.

Definition 5.6 *Let $\mathscr{C}_s(X)$ be the space $\mathscr{C}(X)$ equipped with the topology defined by the family of seminorms $\|T\|_\theta = \|T\theta(Q)\| + \|\theta(Q)T\|$ with $\theta \in \mathcal{C}_0(X)$.*

Note that if X is σ-compact then there is $\theta \in \mathcal{C}_0(X)$ with $\theta(x) > 0$ for all $x \in X$ and then $\|\cdot\|_\theta$ is a norm on $\mathscr{C}(X)$ which induces on bounded subsets of $\mathscr{C}(X)$ the topology of $\mathscr{C}_s(X)$. In any case, the topology of $\mathscr{C}_s(X)$ is finer than the strong operator topology induced by $\mathscr{B}(X)$. Note also that the topology of $\mathscr{C}_s(X)$ does not depend on any Hilbert space realization of $\mathscr{C}(X)$ because $\mathscr{C}(X)$ is a $\mathcal{C}(X)$-bimodule and $\mathcal{C}_0(X)$ is an ideal of $\mathcal{C}(X)$. Finally, observe that we could consider on $\mathscr{C}(X)$ the (intrinsically defined) strict topology associated to the ideal $\mathscr{K}(X)$; this is weaker than that of $\mathscr{C}_s(X)$ and finer than the strong operator topology (but coincides with it on bounded sets).

Remark 5.7 That this is the natural topology in our context should have been clear for us a long time ago, since it is induced by the strict topology of $\mathcal{C}(X)$, cf. [GI2, p. 31] and [GI5, p. 148]. However, we did not realize it until B. Simon, in a private communication, emphasized its importance, in relation with Proposition 3.11 and Theorem 4.5 from [LaS]. We are indebted to him for this remark. On the other hand, note that this topology does not play any rôle in our paper, the strong operator topology on $\mathscr{C}(X)$ (used in [GI2, GI5]) suffices.

We now describe some topological properties of $\mathscr{C}_s(X)$.

Lemma 5.8 *The map $T \mapsto T^*$ is continuous on $\mathscr{C}_s(X)$ and the operation of multiplication is continuous on bounded sets. If $T \in \mathscr{C}(X)$ the map $x \mapsto U_x T U_x^* \in \mathscr{C}(X)$ is norm continuous and the set $\{U_x T U_x^* \mid x \in X\}$ is relatively compact in $\mathscr{C}_s(X)$.*

Proof: The first assertion is obvious. To prove the second one, note first that *if $S \in \mathscr{C}(X)$ and $\theta \in \mathcal{C}_0(X)$ then the operators $S\theta(Q)$ and $\theta(Q)S$ are compact*. Indeed, it suffices to show this for S of the form $\varphi(Q)\psi(P)$ and then the assertion is obvious. In particular, from the Remark 3.13 it follows that there are $K \in \mathscr{K}(X)$ and $\theta' \in \mathcal{C}_0(X)$ such that $S\theta(Q) = \theta'(Q)K$, and similarly for $\theta(Q)S$. Thus for $A, B, S, T \in \mathscr{C}(X)$ we have

$$\|(BA - TS)\theta(Q)\| \le \|B\|\|(A - S)\theta(Q)\| + \|(B - T)\theta'(Q)\|\|K\|$$

from which the continuity of multiplication follows. The norm continuity of $x \mapsto U_x T U_x^*$ is obvious by (1.1). Finally, the last assertion of the lemma says that $x \mapsto U_x T U_x^* \theta(Q)$ has relatively compact range and similarly when θ is on the left side. Clearly it suffices to take $T = \varphi(Q)\psi(P)$ and then $U_x T U_x^* \theta(Q) = \varphi(Q+x)\psi(P)\theta(Q)$ and $\psi(P)\theta(Q)$ is a compact operator. Now the assertion follows from the Riesz-Kolmogorov criterion (3.13), which clearly implies: if K is a compact operator and $\varphi \in \mathcal{C}(X)$ then $\varphi(Q + x)K$ is a norm relatively compact family of operators. ■

Proposition 5.9 *If $T \in \mathscr{C}(X)$ and $\varkappa \in \delta X$ then $\varkappa.T := \lim_{x \to \varkappa} U_x T U_x^*$ exists in the topological space $\mathscr{C}_s(X)$. The map $\tau_\varkappa : \mathscr{C}(X) \to \mathscr{C}(X)$ defined by $\tau_\varkappa(T) = \varkappa.T$ is a morphism uniquely determined by the property:*

$$\varphi \in \mathcal{C}(X), \psi \in \mathcal{C}_0(X^*) \Longrightarrow \tau_\varkappa\big(\varphi(Q)\psi(P)\big) = (\varkappa.\varphi)(Q)\psi(P). \tag{5.2}$$

If $T \in \mathscr{C}(X)$ *and* $\psi : X^* \to \mathbb{C}$ *is a bounded continuous function, then*

$$\tau_\varkappa\big(T\psi(P)\big) = \tau_\varkappa(T)\psi(P) \ \text{ and } \ \tau_\varkappa\big(\psi(P)T\big) = \psi(P)\tau_\varkappa(T). \tag{5.3}$$

For each $k \in X^*$ *we have* $\tau_\varkappa\big(V_k^* T V_k\big) = V_k^* \tau_\varkappa(T) V_k$.

Proof: We must show that there is an operator $\varkappa.T \in \mathscr{C}(X)$ such that

$$\lim_{x\to\varkappa} \|(U_x T U_x^* - \varkappa.T)\theta(Q)\| = \lim_{x\to\varkappa} \|\theta(Q)(U_x T U_x^* - \varkappa.T)\| = 0$$

for all $\theta \in \mathcal{C}_0(X)$. It is clearly sufficient to consider $T = \varphi(Q)\psi(P)$ with $\varphi \in \mathcal{C}(X)$ and $\psi \in \mathcal{C}_0(X^*)$. Then we have

$$U_x T U_x^* \theta(Q) = \varphi(Q+x)\psi(P)\theta(Q) = \varphi(Q+x)\theta'(Q)K$$

for some $\theta' \in \mathcal{C}_0(X)$ and $K \in \mathscr{K}(X)$. Indeed, $\psi(P)\theta(Q)$ is a compact operator and so we can use the Remark 3.13. Now it suffices to use Lemma 5.1. The argument for $\theta(Q)U_x T U_x^*$ is even simpler. The other assertions are easy to prove, for example the last assertion follows from $V_k^* U_x T U_x^* V_k = U_x V_k^* T V_k U_x^*$. ■

Proposition 5.10 *Let* $T \in \mathscr{C}(X)$. *Then* $\varkappa.T = 0$ *for each* $\varkappa \in \delta X$ *if and only if* $T \in \mathscr{K}(X)$.

Proof: In order to prove that $\varkappa.T = 0$ if $T \in \mathscr{K}(X)$ it suffices to consider $T = \varphi(Q)\psi(P)$ with $\varphi \in \mathcal{C}_0(X)$. Then $\varkappa.T = (\varkappa.\varphi)(Q)\psi(P)$ and $\varkappa.\varphi = 0$ if $\varphi \in \mathcal{C}_0(X)$. Reciprocally, let $\mathscr{J} = \{T \in \mathscr{C}(X) \mid \varkappa.T = 0,\ \forall\, \varkappa \in \delta X\}$ and notice that $\mathscr{J}$ is a C^*-algebra and, moreover, it is a crossed product because of the last assertions of Proposition 5.9. Also, for each $S \in \mathscr{C}(X)$ we have $\varkappa.(ST) = (\varkappa.S)(\varkappa.T) = 0$ so $\mathscr{J}$ is an ideal. Thus, by Proposition 3.15, there is an ideal $\mathcal{J}$ in $\mathcal{C}(X)$ such that $\mathscr{J} = \mathcal{J} \rtimes X$. Let us show that $\mathcal{J} = \mathcal{C}_0(X)$. This will finish the proof, because then $\mathscr{J} = \mathcal{C}_0(X) \rtimes X = \mathscr{K}(X)$. From (3.25) we get

$$\mathcal{J} = \{\varphi \in \mathcal{C}(X) \mid \varkappa.(\varphi(Q)\psi(P)) = 0\ \forall\, \psi \in \mathcal{C}_0(X^*) \text{ and } \forall\, \varkappa \in \delta X\}.$$

But $\varkappa.(\varphi(Q)\psi(P)) = (\varkappa.\varphi)(Q)\psi(P)$. On the other hand, if $\theta \in \mathcal{C}(X)$ is such that $\theta(Q)\psi(P) = 0\, \forall \psi \in \mathcal{C}_0(X^*)$, then $\theta = 0$. Indeed, $V_k^*\theta(Q)\psi(P)V_k = \theta(Q)\psi(P+k)$, so if $\psi \in L^1(X^*)$ then we have in the weak operator topology

$$0 = \int_{X^*} V_k^* \theta(Q)\psi(P)V_k \,\mathrm{d}k = \theta(Q)\int_{X^*} \psi(P+k)\,\mathrm{d}k = \theta(Q)\int_{X^*} \psi\,\mathrm{d}k.$$

Thus it suffices to take ψ such that $\int_{X^*} \psi\,\mathrm{d}k \neq 0$. So we finally see that $\mathcal{J}$ is the set of $\varphi \in \mathcal{C}(X)$ such that $\varkappa.\varphi = 0$ for all $\varkappa \in \delta X$, i.e. $\mathcal{J} = \mathcal{C}_0(X)$ by Lemma 5.3. ■

The next result follows easily from Propositions 5.9 and 5.10.

Theorem 5.11 *The map* $T \mapsto (\varkappa.T)_{\varkappa\in\delta X}$ *is a morphism* $\mathscr{C}(X) \to \prod_{\varkappa\in\delta X} \mathscr{C}(X)$ *with* $\mathscr{K}(X)$ *as kernel, so we have a canonical embedding*

$$\mathscr{C}(X)/\mathscr{K}(X) \hookrightarrow \textstyle\prod_{\varkappa\in\delta X} \mathscr{C}(X). \tag{5.4}$$

Theorem 1.1 is an immediate consequence. As explained in Subsection 2.2, the morphism $\tau_\varkappa$ extends to observables affiliated to $\mathscr{A}$ and Theorem 1.2 follows easily.

Definition 5.12 *If H is an observable affiliated to $\mathscr{C}(X)$ and if $\varkappa \in \delta X$ then the observable $\varkappa.H$ affiliated to $\mathscr{C}(X)$ is called* localization of H at $\varkappa$. *The set of operators $\ell(H) := \{\varkappa.H \mid \varkappa \in \delta X\}$ is the set of* localizations of H at infinity.

Then we can write the relation (1.4) as follows:

$$\sigma_{\text{ess}}(H) = \overline{\textstyle\bigcup_{\varkappa\in\delta X}\sigma(\varkappa.H)} = \overline{\textstyle\bigcup_{K\in\ell(H)}\sigma(K)}. \tag{5.5}$$

Remark 5.13 By using the universal property of the Stone-Čech compactification γX (cf. page 13) we see that for $T \in \mathscr{B}(X)$ the following two assertions are equivalent:
(1) the set $\{x.T \mid x \in X\}$ is strongly relatively compact in $\mathscr{B}(X)$;
(2) $X \ni x \mapsto x.T$ extends to a strongly continuous map $\gamma X \ni \varkappa \mapsto \varkappa.T \in \mathscr{B}(X)$.
The set of operators having these properties is a norm closed subalgebra of $\mathscr{B}(X)$ (quite large, it contains $\mathscr{C}(X), L^\infty(X), L^\infty(X^*)$ and much more). It is easy to check that $\sigma(\varkappa.T) \subset \sigma_{\text{ess}}(T)$ if $\varkappa \in \delta X$, but in most cases the operators $\varkappa.T$ do not suffice to determine the essential spectrum of T. This fact extends to observables affiliated to this algebra. For example, if H is the Hamiltonian of a particle in 2 dimensions in a constant non-zero magnetic field, then $T = \varphi(H)$ has the properties (1)-(3) and $\varkappa.T = 0$ if $\varphi \in \mathcal{C}_0(\mathbb{R})$, i.e. $\varkappa.H = \infty$ for all $\varkappa \in \delta X$. But $\sigma_{\text{ess}}(H) \neq \emptyset$.

5.3. We fix now an algebra of interactions $\mathcal{A}$ on X, and set $\mathscr{A} = \mathcal{A} \rtimes X \subset \mathscr{C}$. Theorem 5.11 gives a description of $\mathscr{A}/\mathscr{K}(X)$ but we can make it more precise because many ultrafilters give the same character of $\mathcal{A}$.

Definition 5.14 *If $\varkappa \in \delta X$ the C^*-algebras $\mathcal{A}_\varkappa = \tau_\varkappa(\mathcal{A})$ and $\mathscr{A}_\varkappa = \tau_\varkappa(\mathscr{A}) = \mathcal{A}_\varkappa \rtimes X$ are the* localizations at $\varkappa$ *of the algebras $\mathcal{A}$ and $\mathscr{A}$ respectively.*

As explained in Subsection 2.5, and taking into account the relation $(\varkappa.\varphi)(y) = \varkappa(y.\varphi)$ (see Lemma 5.1) and Lemma 2.1, we see that $\mathcal{A}_\varkappa$ and $\mathscr{A}_\varkappa$ depend only on the restriction to $\mathcal{A}$ of the character $\varkappa$. In other terms, we have for example $\mathcal{A}_\varkappa = \mathcal{A}_\chi$ if $\delta(\varkappa) = \delta(\chi)$, where $\delta : \delta X \to \delta(\mathcal{A})$ is the canonical surjection, cf. (2.22). According to the convention made in Subsection 2.5 (see page 14) we shall use the same notations $\mathcal{A}_\varkappa$ and $\mathscr{A}_\varkappa$ if $\varkappa \in \delta(\mathcal{A})$.

In the statement of the next theorem we use the canonical identification of X (as topological space) with an open dense subset of $\sigma(\mathcal{A})$

Theorem 5.15 *If $T \in \mathscr{A}$ the norm continuous map $X \ni x \mapsto x.T \in \mathscr{A} \subset \mathscr{C}$ extends to a continuous map $\sigma(\mathcal{A}) \ni \varkappa \mapsto \varkappa.T \in \mathscr{C}_s(X)$. For each $\varkappa \in \delta(\mathcal{A})$ the map $\tau_\varkappa : \mathscr{A} \to \mathscr{C}$ defined by $\tau_\varkappa(T) = \varkappa.T$ is a morphism with $\mathscr{A}_\varkappa$ as range. One has $\varkappa.T = 0$ for all $\varkappa \in \delta(\mathscr{A})$ if and only if $T \in \mathscr{K}(X)$ which gives a canonical embedding*

$$\mathscr{A}/\mathscr{K}(X) \hookrightarrow \textstyle\prod_{\varkappa\in\delta(\mathcal{A})} \mathscr{A}_\varkappa. \tag{5.6}$$

Proof: Consider for each $T \in \mathscr{A}$ the map $F_T : \gamma X \to \mathscr{C}_s(X)$ defined by $F_T(\varkappa) = \varkappa.T$. From Lemma 5.2 it follows that F_T is continuous: indeed, it suffices to assume that $T = \varphi(Q)\psi(P)$ and to argue as in the proof of Lemma 5.8. Notice that if the characters $\varkappa, \chi \in \gamma X$ are equal on $\mathcal{A}$, then $F_T(\varkappa) = F_T(\chi)$. Indeed, for T as above we have $\varkappa.T = (\varkappa.\varphi)(Q)\psi(P) = (\chi.\varphi)(Q)\psi(P) = \chi.T$. Thus, as explained on page 13, if $\pi : \gamma X \to \sigma(\mathscr{A})$ is the canonical surjection, we shall have $F_T = f_T \circ \pi$, where $f_T : \sigma(\mathscr{A}) \to \mathscr{C}_s(X)$ is continuous. If $x \in X$ then $\pi(x) = x$ so $f_T(x) = F_T(x) = x.T$. We have both $X \subset \sigma(\mathscr{A})$ and $X \subset \gamma X$ and since the restriction of π to X is the identity mapping, π acts non-trivially only on the boundary. Let δ be the restriction of the map π to δX, hence $\delta : \delta X \to \delta(\mathscr{A})$ is a canonical surjection. Thus $f_T(\varkappa) = 0$ for all $\varkappa \in \delta(\mathscr{A})$ is equivalent to $F_T(\varkappa) = 0$ for all $\varkappa \in \delta X$ which means that $T \in \mathscr{K}(X)$. ■

Remark: By using the last assertion of Lemma 5.8 and the universal property of the space γX, cf. page 13, one may avoid the use of Lemma 5.2.

Remark 5.16 In nice situations, the localization at infinity $\mathcal{A}_\varkappa$ is simpler than $\mathcal{A}$, and $(\mathcal{A}_\varkappa)_\chi$ is still simpler, and so on, but this is not always the case. Note also that in general $\mathcal{A}_\varkappa \not\subset \mathcal{A}$. If, however, this holds for each $\varkappa \in \delta(\mathcal{A})$, then it is natural to ask whether we have $\tau_\varkappa \tau_\chi \varphi = \tau_\chi \tau_\varkappa \varphi$ for all $\varphi \in \mathcal{A}$ and all $\varkappa, \chi \in \delta(\mathcal{A})$. Although this is not true if $\mathcal{A} = \mathcal{C}(X)$, in several non-trivial and physically interesting situations this property is satisfied. See Examples 5.17 and 5.18 and Section 6.

Example 5.17 We shall consider here the localizations at infinity of the simplest algebras. If $\mathcal{A} = \mathcal{C}_\infty(X)$ then $\sigma(\mathcal{A}) = X \cup \{\infty\}$ is the Alexandroff compactification of X, we have $\delta(\mathcal{A}) = \{\infty\}$, and the localization of $\varphi \in \mathcal{A}$ at ∞ is the constant function which takes the value $\varphi(\infty) = \lim_{x\to\infty} \varphi(x)$. If $X = \mathbb{R}$ and $\mathcal{A}$ is the set of bounded continuous functions which have limits as $x \to \pm\infty$ then $\sigma(\mathcal{A}) = [-\infty, +\infty]$, $\delta(\mathcal{A}) = \{-\infty, +\infty\}$, and the localization of $\varphi \in \mathcal{A}$ at $+\infty$ is again the constant function which takes the value $\varphi(+\infty) = \lim_{x\to+\infty} \varphi(x)$ and similarly for the localization at $-\infty$. Thus in both examples we have $\mathcal{A}_\varkappa = \mathbb{C}$ for all $\varkappa \in \delta(\mathcal{A})$. In Subsection 6.2 we shall describe explicitly the largest X-subalgebra $\mathcal{A} \subset \mathcal{C}(X)$ such that $\mathcal{A}_\varkappa = \mathbb{C}$ for all $\varkappa \in \delta(\mathcal{A})$.

Example 5.18 The next two examples are relevant in the context of Remark 5.16. The first one is due to Gilles Godefroy and the second one to Yves Raynaud. We thank them for answering to our questions.
(1) Let $X = \mathbb{Z} \times \mathbb{Z}$ and let $\mathcal{A}$ be the set of $\varphi \in \ell_\infty(X)$ such that $\lim_{k\to\infty} \varphi(j,k) = 0$ for all $j \in \mathbb{Z}$. Let $\theta \in \ell_\infty(\mathbb{Z})$ and set $\varphi(j,k) = \theta(k)$ if $|k| \leq j$ and $= 0$ otherwise. Then $\varphi \in \mathcal{A}$ and $\lim_{a\to+\infty} \varphi(a+j,k) = \theta(k)$ for each j,k. It is clear now that we may construct an ultrafilter $\varkappa \in \delta X$ such that $\varkappa.\varphi = 1 \otimes \theta$ so $\varkappa.\varphi \notin \mathcal{A}$ in general.
(2) Let X be an infinite discrete group and let $A \subset X$ be infinite and such that the intersection $A \cap (x + A)$ is finite for each $x \neq 0$ (this is a sparse set, in the sense of Subsection 6.4). Let $\mathcal{A}$ be the C^*-algebra generated by the characteristic functions of the translates of A. Let $\varkappa$ be an ultrafilter on A which does not contain any finite set (i.e. is finer than the Fréchet filter of A) and denote also by $\varkappa$ the filter induced on X (see Subsection 6.3). Let us denote $\mathbf{1}_x$ the characteristic function of the set consisting

of the point x. It is clear that $\varkappa.\mathbf{1}_{x+A} = \mathbf{1}_x$ from which it follows that for $\varphi \in \ell_\infty(X)$ with finite support we have $\varkappa.(\sum_x \varphi(x)\mathbf{1}_{x+A}) = \varphi$. Thus the range of the morphism $\tau_\varkappa : \mathcal{A} \to \ell_\infty(X)$ contains the set of functions with finite support. But this is a norm dense subset of $\ell_\infty(X)$ and $\tau_\varkappa(\mathcal{A})$ is norm closed. Thus we have an example of a strict subalgebra $\mathcal{A}$ of $\mathcal{C}(X)$ and of a character $\varkappa$ of $\mathcal{A}$ such that $\tau_\varkappa(\mathcal{A}) = \mathcal{C}(X)$.

Theorem 1.15 is a corollary of Theorem 5.15. Thus, if H is an element of $\mathscr{A}$ or an observable affiliated to $\mathscr{A}$ and if we set $\varkappa = \tau_\varkappa(H)$, then

$$\sigma_{\text{ess}}(H) = \overline{\textstyle\bigcup_{\varkappa \in \delta(\mathcal{A})} \sigma(\varkappa.H)}. \tag{5.7}$$

This representation of the essential spectrum of H, although more precise than (5.5), is still quite redundant, cf. page 6, and can be improved in many situations (the most interesting one being the N-body case). To explain this, for $\varkappa \in \delta(\mathcal{A})$ let us denote

$$\mathcal{J}_\varkappa = \ker \tau_\varkappa = \{\varphi \in \mathcal{A} \mid \varkappa(x.\varphi) = 0 \quad \forall x \in X\}. \tag{5.8}$$

This is is the maximal X-ideal included in the maximal ideal $\ker \varkappa$ of $\mathcal{A}$. Although the ideals $\ker \varkappa$ for different $\varkappa$ are not comparable, it often happens that the $\mathcal{J}_\varkappa$ are comparable, i.e. we may have $\mathcal{J}_\varkappa \subset \mathcal{J}_\chi$ for $\varkappa \neq \chi$.

Lemma 5.19 *If $\mathcal{J}_\varkappa \subset \mathcal{J}_\chi$ then $\sigma(H_\chi) \subset \sigma(H_\varkappa)$. In particular, (5.7) remains true if we restrict the union to the $\varkappa$ such that the ideal $\mathcal{J}_\varkappa$ is minimal in $\{\mathcal{J}_\varkappa \mid \varkappa \in \delta(\mathcal{A})\}$.*

Proof: Here we use more abstract algebraic tools, as in [GI2, GI4]. The morphism $\tau_\varkappa : \mathcal{A} \to \mathcal{A}_\varkappa$ is surjective and has $\mathcal{J}_\varkappa$ as kernel, hence induces an isomorphism $\mathcal{A}/\mathcal{J}_\varkappa \cong \mathcal{A}_\varkappa$. If $T \in \mathcal{A}$ and if $T/\mathcal{J}_\varkappa$ is its projection in the quotient $\mathcal{A}/\mathcal{J}_\varkappa$, then $T/\mathcal{J}_\varkappa$ is sent by this isomorphism into $\varkappa.T$, hence $\sigma(T/\mathcal{J}_\varkappa) = \sigma(\varkappa.T)$. From $\mathcal{J}_\varkappa \subset \mathcal{J}_\chi$ we get a canonical surjective morphism $\mathcal{A}/\mathcal{J}_\varkappa \to \mathcal{A}/\mathcal{J}_\chi$ which sends $T/\mathcal{J}_\varkappa$ into $T/\mathcal{J}_\chi$. Finally, we recall that if Φ is a morphism then $\sigma(\Phi(S)) \subset \sigma(S)$. ■

Example 5.20 If, for $x \in X$ and $\varkappa \in \delta(\mathcal{A})$, we denote $x + \varkappa$ the character $\varkappa \circ \tau_x$, then clearly $\mathcal{J}_{x+\varkappa} = \mathcal{J}_\varkappa$, hence $\sigma((x + \varkappa).H) = \sigma(\varkappa.H)$. However, this case is trivial because clearly $(x + \varkappa).H = U_x(\varkappa.H)U_x^*$.

One further simplification may be obtained as follows.

Lemma 5.21 *Let $\mathcal{K} \subset \delta(\mathcal{A})$ such that: if $\varphi \in \mathcal{A}$ and $\varkappa(x.\varphi) = 0$ for all $\varkappa \in \mathcal{K}$ and $x \in X$, then $\varphi \in \mathcal{C}_0(X)$. Then (5.7) remains valid if $\delta(\mathcal{A})$ is replaced by $\mathcal{K}$.*

Proof: This is a consequence of the proof of Theorem 5.15 but can also be proved directly as follows. One first notices that the condition on $\mathcal{K}$ is equivalent to the density in $\delta(\mathcal{A}) = \sigma(\mathcal{A}/\mathcal{C}_0(X))$ of the set of characters of the form $\varkappa \circ \tau_x$, with $\varkappa \in \mathcal{K}$ and $x \in X$. Then one can use the following easily proven fact: if S_α is a net of operators such that $S_\alpha^{(*)} \to S^{(*)}$ strongly, then $\sigma(S)$ is included in the closure of $\bigcup_\alpha \sigma(S_\alpha)$. ■

6 Applications

After some preliminaries, we describe here three classes of C^*-algebras of Hamiltonians which seem to us particularly relevant and treat some more explicit examples.

6.1. Algebras associated to translation invariant filters. In this preliminary subsection we give an intrinsic description of a class of crossed products introduced in [GI2, GI4]. Recall that a filter $\mathfrak{f}$ is *translation invariant* if: $x \in X, F \in \mathfrak{f} \Rightarrow x+F \in \mathfrak{f}$. Note that $\mathfrak{f}^\circ$ will also be translation invariant. If $\mathfrak{f}$ is a translation invariant filter let

$$\mathcal{J}(\mathfrak{f}) = \{\varphi \in \mathcal{C}(X) \mid \lim_{\mathfrak{f}} \varphi = 0\}. \tag{6.1}$$

This is clearly an X-ideal in $\mathcal{C}(X)$ and from Lemma 2.2 we get:

$$\mathcal{J}(\mathfrak{f}) = \mathcal{J}(\mathfrak{f}^\circ). \tag{6.2}$$

Then $\mathcal{C}(\mathfrak{f}) = \mathbb{C} + \mathcal{J}(\mathfrak{f})$ is the X-algebra consisting of the bounded uniformly continuous functions φ such that $\lim_{\mathfrak{f}} \varphi$ exists. Observe that if $\mathfrak{f}$ is the Fréchet filter then $\mathcal{J}(\mathfrak{f}) = \mathcal{C}_0(X)$ and $\mathcal{C}(\mathfrak{f}) = \mathcal{C}_\infty(X)$.

Below we shall consider nets indexed by the the filter $\mathfrak{f}$ equipped with the order relation $F \leq G \Leftrightarrow F \supset G$. For example, $\lim_{F\in\mathfrak{f}} \|\mathbf{1}_F(Q)T\| = 0$ means that for each $\varepsilon > 0$ there is a Borel set $F \in \mathfrak{f}$ such that $\|\mathbf{1}_F(Q)T\| < \varepsilon$.

Proposition 6.1 $\mathcal{J}(\mathfrak{f}) \rtimes X = \{T \in \mathscr{C}(X) \mid \lim_{F\in\mathfrak{f}} \|\mathbf{1}_F(Q)T^{(*)}\| = 0\}$.

Proof: Each $T \in \mathcal{J}(\mathfrak{f}) \rtimes X$ has the property $\lim_{F\in\mathfrak{f}} \|\mathbf{1}_F(Q)T\| = 0$. Indeed, it suffices to consider operators of the form $T = \varphi(Q)\psi(P)$ with $\varphi \in \mathcal{J}(\mathfrak{f}), \psi \in \mathcal{C}_0(X^*)$. But then the set F of points x such that $|\varphi(x)| < \varepsilon$ is open and belongs to $\mathfrak{f}$, and so we have $\|\mathbf{1}_F(Q)\varphi(Q)\| \leq \varepsilon$, which is more than needed.

Conversely, let $\mathscr{J}$ be the set of $T \in \mathscr{C}(X)$ such that $\lim_{F\in\mathfrak{f}} \|\mathbf{1}_F(Q)T^{(*)}\| = 0$. This is clearly a C^*-subalgebra of $\mathscr{C}(X)$ which is stable under the morphisms $T \mapsto V_k^* T V_k$. By Theorem 3.7 we have $\mathscr{J} = \mathcal{J} \rtimes X$ for a unique X-algebra $\mathcal{J}$, namely the set of $\varphi \in \mathcal{C}(X)$ such that $\lim_{F\in\mathfrak{f}} \|\mathbf{1}_F(Q)\varphi(Q)^{(*)}\psi(P)\| = 0$ for all $\psi \in \mathcal{C}_0(X^*)$.

Thus it remains to prove the following assertion: if $\varphi \in \mathcal{C}(X)$ has the property $\lim_{F\in\mathfrak{f}} \|\mathbf{1}_F(Q)\varphi(Q)\psi(P)\| = 0$ for $\psi \in \mathcal{C}_0(X^*)$, then $\lim_{F\in\mathfrak{f}} \|\mathbf{1}_F(Q)\varphi(Q)\| = 0$. Observe that, due to (6.2) we may assume $\mathfrak{f} = \mathfrak{f}^\circ$.

Fix $f \in L^2(X), \psi \in \mathcal{C}_0(X^*)$ and let us set $\theta = \psi(P)f$ and $\theta_a(x) = (U_a^*\theta)(x) = \theta(x-a)$. Clearly $\lim_{F\in\mathfrak{f}} \|\mathbf{1}_F(Q)\varphi(Q)U_a^*\theta\| = 0$ uniformly in $a \in X$. Thus, for any $\varepsilon > 0$, there is $F \in \mathfrak{f}$ Borel such that $\|\mathbf{1}_F\varphi\theta_a\| < \varepsilon$ for all a, hence

$$|\varphi(a)|\|\mathbf{1}_F\theta_a\| \leq \|\mathbf{1}_F(\varphi(a)-\varphi)\theta_a\| + \|\mathbf{1}_F\varphi\theta_a\| \leq \|\mathbf{1}_F(\varphi(a)-\varphi)\theta_a\| + \varepsilon.$$

Since $\mathfrak{f} = \mathfrak{f}^\circ$ we may assume that $F = G + V$ where $G \in \mathfrak{f}$ and V is a a compact neighborhood of the origin. Moreover, since φ is uniformly continuous and since we may choose V as small as we wish, we may assume that $|\varphi(x)-\varphi(a)| < \varepsilon$ if $x-a \in V$. It is possible to choose f, ψ such that $\operatorname{supp}\theta \subset V$ and $\|\theta\| = 1$. Indeed, θ is equal to the convolution product $\widetilde{\psi} * f$ where $\widetilde{\psi}(x) = \widehat{\psi}(-x)$ and it suffices to choose $f, \widetilde{\psi}$

continuous, positive and not zero and such that $\operatorname{supp} f + \operatorname{supp} \widetilde{\psi} \subset V$. Then for $a \in G$ we clearly have $\operatorname{supp} \theta_a \subset F$ hence

$$|\varphi(a)| = |\varphi(a)| \, \|\theta_a\| = |\varphi(a)| \, \|\mathbf{1}_F \theta_a\| \leq \|(\varphi(a) - \varphi)\theta_a\| + \varepsilon \leq 2\varepsilon.$$

This proves that $\lim_{\mathfrak{f}} \varphi = 0$. ■

From Proposition 6.1 we easily get:

$$\mathcal{C}(\mathfrak{f}) \rtimes X = \{T \in \mathscr{C}(X) \mid \exists S \in \mathcal{C}_0(X^*) \text{ such that } \lim_{F \in \mathfrak{f}} \|\mathbf{1}_F(Q)(T - S)^{(*)}\| = 0\}.$$

The X-algebras of the form $\bigcap_\lambda \mathcal{C}(\mathfrak{f}_\lambda)$ are of some physical interest [Ric]. Indeed, one should think of a filter finer than the Fréchet filter as the set of traces on X of the filter of neighbourhoods of some closed part of the boundary of X in a compactification of X. This explains the interest of the algebras $\bigcap_\lambda \mathcal{C}(\mathfrak{f}_\lambda)$ in the present context: they consist of “potentials” which have limits at infinity when going in certain directions. One may easily deduce from Theorem 3.14 and Proposition 6.1 an intrinsic description of the crossed products $\bigcap_\lambda \mathcal{C}(\mathfrak{f}_\lambda) \rtimes X$.

6.2. The $\mathscr{V}(X)$ algebra. We shall consider now the simplest non-trivial functions in $\mathcal{C}(X)$, those all of whose localizations at infinity are constants. Our purpose is to give a simple characterization of the X-algebra $\mathcal{A}$ defined by the condition $\mathcal{A}_\varkappa = \mathbb{C}$ for all $\varkappa \in \delta X$ and of the associated crossed product. So we introduce the X-algebra:

$$\mathcal{V}(X) := \{\varphi \in \mathcal{C}(X) \mid \varkappa.\varphi \in \mathbb{C}, \ \forall \varkappa \in \delta X\} \tag{6.3}$$

Observe that the relation $\varkappa.\varphi \in \mathbb{C}$ is equivalent to $\varkappa.\varphi = \varkappa(\varphi)$.

Lemma 6.2 *We have $\varphi \in \mathcal{V}(X)$ if and only if $\varphi \in \mathcal{C}(X)$ and*

$$\lim_{x \to \infty} (\varphi(x + y) - \varphi(x)) = 0, \quad \forall y \in X. \tag{6.4}$$

Proof: The condition (6.4) is equivalent to $y.\varphi - \varphi \in \mathcal{C}_0(X)$ for all $y \in X$ and, by (5.1), this is equivalent to $\varkappa(y.\varphi - \varphi) = 0$ for all $\varkappa \in \delta X$ and all y, hence to $\varkappa.\varphi(y) = \varkappa(\varphi)$ for all $\varkappa, y$, which means $\varphi \in \mathcal{V}(X)$. ■

It is easily shown that $\varphi \in \mathcal{C}(X)$ satisfies (6.4) if and only if φ is a bounded continuous function such that $\lim_{x \to \infty}(\varphi(x + y) - \varphi(x)) = 0$ uniformly in y when y runs over a compact neighborhood of the origin. Thus the functions from $\mathcal{V}(X)$ are of *vanishing oscillation at infinity* or *slowly oscillating*, and their rôle in the theory of pseudo-differential operators was noticed a long time ago due especially to a well known theorem of H. Cordes concerning the compactness of the commutators $[\varphi(Q), \psi(P)]$ (see [ABG, p. 176–177] for a short presentation of the main ideas).

If $X = \mathbb{R}^n$ then $\mathcal{V}(X)$ is just the norm closure of the set of bounded functions of class C^1 whose derivative tends to zero at infinity. Thus results of the same nature as the embedding (6.6) may be found already in [Tay]. On the other hand, the algebra $\mathcal{V}(X)$ was systematically considered in the works [Rab, RRR, RRS1, RRS2]; see especially [RRS2] where one may find references to other earlier papers. More recently, the relevance of $\mathcal{V}(X)$ in questions related to the computation of the essential spectrum has been independently noticed in [LaS, Man].

We mention that the compactification $\sigma(\mathcal{V}(X))$ and the boundary $\upsilon X = \delta(\mathcal{V}(X))$ are called *Higson compactification* and *Higson corona* of X and play an important rôle in recent questions of topology, C^*-algebras, K-theory, etc. [Ro1, Ro2].

Finally, we note that a non-abelian version of $\mathcal{V}(X)$ appears in a natural way in the spectral analysis of Schrödinger operators on a tree X, see [GG1].

We now give an intrinsic description of the crossed product $\mathscr{V}(X) = \mathcal{V}(X) \rtimes X$ and a more specific decomposition of the essential spectrum.

Proposition 6.3 *We have*

$$\mathscr{V}(X) = \{T \in \mathscr{C}(X) \mid \varkappa.T \in \mathcal{C}_0(X^*),\ \forall \varkappa \in \delta X\}. \tag{6.5}$$

If $T \in \mathscr{V}(X)$ then the map $\varkappa \mapsto \varkappa.T \in \mathcal{C}_0(X^*)$ is norm continuous, hence (5.6) takes the more precise form

$$\mathscr{V}(X)/\mathscr{K}(X) \hookrightarrow C(\upsilon X; \mathcal{C}_0(X^*)). \tag{6.6}$$

In particular, $\ell(T) = \{\varkappa.T \mid \varkappa \in \upsilon X\} \subset \mathcal{C}_0(X^*)$ is a compact set. If H is an element of $\mathscr{V}(X)$ or is an observable affiliated to $\mathcal{V}(X)$ then:

$$\sigma_{\text{ess}}(H) = \bigcup\nolimits_{\varkappa \in \upsilon X} \sigma(\varkappa.H) = \bigcup\nolimits_{K \in \ell(H)} \sigma(K). \tag{6.7}$$

Proof: To show the inclusion $\subset$ in (6.5) and the norm continuity of the map $\varkappa \mapsto \varkappa.T \in \mathcal{C}_0(X^*)$ it suffices to consider $T = \varphi(Q)\psi(P)$ with $\varphi \in \mathcal{V}(X)$ and $\psi \in \mathcal{C}_0(X^*)$. But then $\varkappa.T = \varkappa(\varphi)\psi(P)$ and these facts become obvious. Note that the compactness of the set $\ell(T)$ implies that the union $\bigcup_{T \in \ell(T)} \sigma(T)$ is closed, hence (6.7) is true. It remains to show the inclusion $\supset$ in (6.5). Since $\varkappa.(V_k^* T V_k) = V_k^*(\varkappa.T)V_k$ and $(\varkappa.T)U_x = \varkappa.(TU_x)$ and since $\mathcal{C}_0(X^*)$ is stable under the automorphism generated by V_k and under multiplication by U_x, it is clear that the right hand side of (6.5) satisfies Landstad's conditions. Hence Theorem 3.7 shows that it suffices to prove that if $\varphi \in \mathcal{C}(X)$ has the property $(\varkappa.\varphi)(Q)\psi(P) \in \mathcal{C}_0(X^*)$ for all $\psi \in \mathcal{C}_0(X^*)$ and all $\varkappa \in \delta X$, then $\varphi \in \mathcal{V}(X)$. Thus it suffices to show that if $\xi \in \mathcal{C}(X)$ and $\xi(Q)\psi(P) \in \mathcal{C}_0(X^*)$ for all $\psi \in \mathcal{C}_0(X^*)$, then ξ is a constant. But we have

$$\xi(Q)\psi(P) = U_x \xi(Q)\psi(P) U_x^* = \xi(x+Q)\psi(P)$$

hence $(\xi(Q) - \xi(x+Q))\psi(P) = 0$ for all ψ, so $\xi(Q) = \xi(x+Q)$ for all x. ■

Remark: If the reader has any difficulty in proving that the union in (6.7) is closed, he should look at the proof of [DG2, Theorem 2.10].

Remark 6.4 $\mathscr{V}(X)$ is the largest crossed product $\mathscr{A}$ such that $\mathscr{A}/\mathscr{K}(X)$ is abelian. Indeed, $\mathscr{A}/\mathscr{K}(X) \hookrightarrow \prod_{\varkappa \in \delta(\mathcal{A})} \mathscr{A}_\varkappa$ by (5.6) and the $\mathscr{A}_\varkappa$ are crossed products, so $\mathscr{A}_\varkappa$ is abelian if and only if $\mathscr{A}_\varkappa = \{0\}$ or $\mathscr{A}_\varkappa = \mathcal{C}_0(X^*)$.

Remark 6.5 The observables affiliated to $\mathcal{C}_0(X^*)$ are functions of momentum, so that it is natural to call them *free Hamiltonians*. Then we may describe in physical terms $\mathscr{V}(X)$ as the largest C^*-algebra of energy observables such that if H is affiliated to it then all its localizations at infinity are free Hamiltonians.

Remark 6.6 We reconsider here the question of Remark 5.16 for $\mathcal{A} = \mathcal{V}(X)$. If $\varkappa \in \delta X$ then $\tau_\varkappa : \mathcal{V}(X) \to \mathbb{C}$ is just the character associated to $\varkappa$ and so if $\chi \in \delta X$ then $\tau_\chi \tau_\varkappa \varphi = \tau_\varkappa \varphi \neq \tau_\chi \varphi = \tau_\varkappa \tau_\chi \varphi$ in general.

6.3. More remarks on filters. The following general remarks will be useful in the next subsections. Let Y be a closed subspace of X (thus $K \cap Y$ is compact for each compact $K \subset X$). If $\mathfrak{f}$ is a filter on Y then $\mathfrak{f}$ can be seen as a filter basis on X and we shall denote (just for a moment) by $\mathfrak{f}^X$ the filter on X that it generates (this is the set of subsets of X which contain a set from $\mathfrak{f}$). The map $\mathfrak{f} \mapsto \mathfrak{f}^X$ is an injective map from the set of filters on Y onto the set of filters on X which contain Y. Indeed, we have $\mathfrak{f} = \{F \cap Y \mid F \in \mathfrak{f}^X\}$. It is also clear that if $\varkappa$ is an ultrafilter on Y then $\mathfrak{f}^X$ is also an ultrafilter. Finally, if $\mathfrak{f}$ is finer than Fréchet on Y then $\mathfrak{f}^X$ is finer than Fréchet on X.

Since $Y \in \mathfrak{f}^X$, if $T : X \to Z$ then $\lim_{\mathfrak{f}^X} T$ exists if and only if $\lim_{\mathfrak{f}} T|_Y$ exists and then they are equal.

From now on we shall not distinguish $\mathfrak{f}^X$ from $\mathfrak{f}$, so we use the same notation $\mathfrak{f}$ for both. In particular, we get natural embeddings

$$\gamma Y \subset \gamma X \quad \text{and} \quad \delta Y \subset \delta X. \tag{6.8}$$

It is convenient to understand this when the ultrafilters are interpreted as characters. We have an obvious embedding $\ell_\infty(Y) \subset \ell_\infty(X)$ so each character of $\ell_\infty(X)$ gives a character of $\ell_\infty(Y)$ by restriction, and reciprocally, each character of $\ell_\infty(Y)$ has a canonical extension to a character of $\ell_\infty(X)$, namely $\varkappa(\varphi) := \varkappa(\varphi \mathbf{1}_Y)$. Thus:

$$\gamma Y = \{\varkappa \in \gamma X \mid \varkappa(Y) = 1\} \quad \text{and} \quad \delta Y = \gamma Y \cap \delta X.$$

It is easy to see now that γY is a *clopen* subset of γX, equal to the closure of Y in γX.

One says that a filter on a topological space is *convergent* to some point x if it is finer than the filter of neighbourhoods of x. *Any ultrafilter on a compact space is convergent*. This is easily seen to be equivalent to any of the usual definitions of compactness [Bou, Chapter 1, §9].

It is easy now to understand the universal property of γX, cf. page 13. We first observe that γ should be considered as a functor from the category of sets into the category of compact spaces. Indeed, if X, Y are sets and $\theta : X \to Y$ then it is obvious how to define $\gamma\theta : \gamma X \to \gamma Y$ if ultrafilters are thought as characters: note first that $\varphi \mapsto \varphi \circ \theta$ is a morphism $\theta^* : \ell_\infty(Y) \to \ell_\infty(X)$ and then if $\varkappa \in \gamma X$ define $\gamma\theta(\varkappa)$ as the character of $\ell_\infty(Y)$ given by $\gamma\theta(\varkappa) = \varkappa \circ \theta^*$. The continuity of $\gamma\theta$ is clear.

Now assume Y is a compact topological space. The only thing we need to accept is that $\sigma(\mathcal{C}(Y)) = Y$, this is not difficult to prove directly. Then we have a natural continuous map $\gamma Y \ni \chi \mapsto \chi_\flat \in Y$ which associates to a character χ of $\ell_\infty(Y)$ its restriction to $\mathcal{C}(Y)$. In fact, the ultrafilter χ is convergent and $\chi_\flat$ is just its limit. Finally, $\varkappa \mapsto \gamma\theta(\varkappa)_\flat$ is the unique extension of θ to a continuous map $\gamma X \to Y$.

6.4. Sparse sets. From the point of view of the complexity of the interactions, the algebra of interactions that one should consider next is

$$\mathcal{A} = \{\varphi \in \mathcal{C}(X) \mid \varkappa.\varphi \in \mathcal{C}_\infty(X),\ \forall \varkappa \in \delta X\}. \tag{6.9}$$

The corresponding algebra of energy observables is

$$\mathscr{A} = \mathcal{A} \rtimes X = \{T \in \mathscr{C}(X) \mid \varkappa.T \in \mathscr{T}(X),\ \forall \varkappa \in \delta X\}. \tag{6.10}$$

Thus $\mathscr{A}$ is the largest C^*-algebra of energy observables such that all the localizations at infinity of a Hamiltonian H affiliated to it are two-body Hamiltonians. We shall leave for the second part of our work the study of the algebra (6.10) and we shall consider here only subalgebras corresponding to Klaus type potentials.

Remark 6.7 The algebra $\mathcal{A}$ defined by (6.9) is characterized by $\mathcal{A}_\varkappa = \mathcal{C}_\infty(X)$ for each $\varkappa$, hence contains $\mathcal{C}_\infty(X)$ and is stable under all the morphisms $\tau_\varkappa$. It is also clear that $\tau_\chi \tau_\varkappa \varphi \in \mathbb{C}$ and is distinct from $\tau_\varkappa \tau_\chi \varphi$ in general, cf. Remark 5.16.

M. Klaus discovered in [Kla] the following class of Hamiltonians with nontrivial essential spectrum. Let $L \subset \mathbb{R}$ be a discrete set such that the distance between two successive points of L tends to infinity when we approach infinity. For each $l \in L$ let $V_l \in L^1(\mathbb{R})$ real such that $\|V_l\|_{L^1} \leq A$ and $\operatorname{supp} V_l \subset [-A, A]$ for a fixed finite A. Denote $H = P^2 + \sum_l V_l(Q - l)$ and $H_l = P^2 + V_l(Q)$. Then the description of $\sigma_{\text{ess}}(H)$ given in [Kla] is equivalent to:

$$\sigma_{\text{ess}}(H) = \bigcap\nolimits_{F \in \mathcal{F}(L)} \overline{\bigcup\nolimits_{l \in F^c} \sigma(H_l)} \tag{6.11}$$

where $\mathcal{F}(L)$ is the set of finite subsets of L and $F^c = L \setminus F$. One of the main examples in [GI2, GI4] consisted in an algebraic treatment of this example, treatment based on the construction of a C^*-algebra to which operators like H are affiliated. We recall below the definition of this type of algebras and then we shall give a description of $\sigma_{\text{ess}}(H)$ for the operators affiliated to them which is more in the spirit of Theorem 1.1 (description which also appears in [GI2, GI4] but which is deduced there by very different means).

If L, Λ are subsets of X we denote $L_\Lambda = L + \Lambda$ and $L_\Lambda^c = X \setminus L_\Lambda$. If L has the property $L_\Lambda \neq X$ for each compact Λ then we associate to it the filter

$$\mathfrak{f}_L = \{A \subset X \mid A \supset L_\Lambda^c \text{ for some compact } \Lambda \subset X\}. \tag{6.12}$$

This is clearly a translation invariant filter finer than the Fréchet filter and such that $\mathfrak{f}_L^\circ = \mathfrak{f}_L$. Thus

$$\mathcal{C}_L(X) = \{\varphi \in \mathcal{C}(X) \mid \lim\nolimits_{\mathfrak{f}_L} \varphi \text{ exists }\} \tag{6.13}$$

is an algebra of interactions on X. An intrinsic description of the corresponding algebra of Hamiltonians $\mathscr{C}_L(X)$ follows immediately from the results of Subsection 6.1. Let

$$\delta_L X = \delta(C_L(X)) = \sigma(\mathcal{C}_L(X)) \setminus X \tag{6.14}$$

be the boundary of X in the compactification associated to $\mathcal{C}_L(X)$. We recall that $\delta_L X$ is a quotient of δX. We set

$$\infty_L = \{\varkappa \in \gamma X \mid \varkappa \supset \mathfrak{f}_L\} = \{\varkappa \in \gamma X \mid L^{\mathrm{c}}_\Lambda \in \varkappa \text{ if } \Lambda \subset X \text{ is compact}\}. \tag{6.15}$$

This is a compact subset of δX and if $\varkappa \in \infty_L$ then $\varkappa(\varphi) = \lim_{\mathfrak{f}_L} \varphi$, so that ∞_L gives just a point in $\delta_L X$. The problem that remains to be solved is the description of the other points of $\delta_L X$.

In this subsection we consider only the case when L is a *sparse* set, in the following sense: L is locally finite and for each compact set Λ there is a co-finite set $M \subset L$ (i.e. such that $L \setminus M$ is finite) with the following property: if $m \in M$ and $l \in L$, $l \neq m$, then $(m + \Lambda) \cap (l + \Lambda) = \emptyset$.

With the conventions made in Subsection 6.3, we have $\delta L \subset \delta X$, more explicitly for $\varkappa \in \delta L$ and $\varphi \in \ell_\infty(X)$ we set

$$\varkappa(\varphi) \equiv \varkappa(\varphi \mathbf{1}_L) = \lim_{l \to \varkappa} \varphi(l).$$

Lemma 6.8 *Let* $\theta : X \times \delta L \to \delta_L X$ *be defined by* $\theta(x, \varkappa) = \varkappa \circ \tau_x$. *Then* θ *is injective and its range is* $\delta_L X \setminus \{\infty_L\}$, *which gives us an identification*

$$\delta_L X \cong (X \times \delta L) \amalg \{\infty_L\}. \tag{6.16}$$

Proof: We set $\theta(x, \varkappa) = \theta_{x,\varkappa}$ and note the more explicit formula

$$\theta_{x,\varkappa}(\varphi) = \lim_{l \to \varkappa} \varphi(l + x).$$

We first prove that θ is injective. It is clearly sufficient to show that if $x \in X$ and $\varkappa, \chi \in \delta L$ are such that $\varkappa(x.\varphi) = \chi(\varphi)$ for all $\varphi \in \mathcal{C}_L$, then $x = 0$ and $\varkappa = \chi$. Let $M \subset L$ such that $\varkappa(M) = 1$. Since $\varkappa$ is finer than the Fréchet filter, M is infinite and $\varkappa(N) = 1$ if N is a co-finite subset of M. Let $\Lambda \subset X$ be compact and such that $0, x \in \Lambda$. Eliminating if needed a finite number of points from M, we may assume that $(L \setminus M) \cap M_\Lambda = \emptyset$ and $M_\Lambda = \amalg_{l \in M}(l + \Lambda)$ where $\amalg$ means disjoint union. Choose $\varphi \in \mathcal{C}_0(X)$ with $\operatorname{supp} \varphi \subset \Lambda$ and let us define $\varphi_M = \sum_{l \in M} \tau_{-l} \varphi$. Then:

$$(\mathbf{1}_L x.\varphi_M)(y) = \textstyle\sum_{l \in M} \mathbf{1}_M(y)(\tau_{x-l}\varphi)(y) = \sum_{l \in M} \mathbf{1}_M(y)\varphi(x + y - l).$$

We must have $l, y \in M$ and $x + y \in l + \Lambda$, hence $l = y$ because $x \in \Lambda$. It follows that $\mathbf{1}_L x.\varphi_M = \mathbf{1}_M \varphi(x)$ and so, by choosing φ such that $\varphi(x) \neq 0$, we see that

$$\varkappa(x.\varphi_M) = \varkappa(\mathbf{1}_L x.\varphi_M) = \varkappa(\mathbf{1}_M)\varphi(x) = \varphi(x) \neq 0.$$

Similarly $\mathbf{1}_L \varphi_M = \mathbf{1}_M \varphi(0)$ and so $\chi(\varphi_M) = \chi(\mathbf{1}_M \varphi_M) = \chi(\mathbf{1}_M)\varphi(0)$. If $x \neq 0$ we may choose φ such that $\varphi(0) = 0$ and we see that $\varkappa(x.\varphi) \neq \chi(\varphi)$ for some $\varphi \in \mathcal{C}_L$. If $x = 0$ but $\varkappa \neq \chi$ then M can be chosen such that $\chi(M) = 0$ (because $\varkappa$ and χ are distinct ultrafilters) hence again $\varkappa(x.\varphi) \neq \chi(\varphi)$ for some $\varphi \in \mathcal{C}_L$. This proves the injectivity of the map θ.

Now we show that for any $\chi \in \delta X$ such that $\chi \notin \infty_L$ there is $(x, \varkappa) \in X \times \delta L$ such that $\chi(\varphi) = \varkappa(x.\varphi)$ for all $\varphi \in \mathcal{C}_L$. Since χ is not finer than $\mathfrak{f}_L$, there is a compact set $\Lambda \subset X$ such that $L_\Lambda^c \notin \chi$. But χ is an ultrafilter, so $L_\Lambda \in \chi$. Since χ is finer than the Fréchet filter, there is $M \subset L$ such that $\chi(M_\Lambda) = 1$ and

$$M_\Lambda = \amalg_{l \in M}(l + \Lambda) \equiv M \times \Lambda.$$

The sets $F \subset M_\Lambda$ with $\chi(F) = 1$ form a basis for χ and each such F can be uniquely written as a disjoint union $F = \amalg_{l \in N}(l + F(l))$ with $N \subset M$ and $F(l) \subset \Lambda$ non empty sets. We define surjective maps $\pi_M : M_\Lambda \to M$ and $\pi_\Lambda : M_\Lambda \to \Lambda$ with the help of the identification $M_\Lambda \equiv M \times \Lambda$. The image $\varkappa = \pi_M(\chi)$, i.e. the filter of subsets of M generated by the $\pi_M(F)$ with $F \in \chi$, is obviously an ultrafilter on M, hence on L, finer then the Fréchet filter. Similarly, $\pi_\Lambda(\chi)$ is an ultrafilter on Λ, which is a compact space, hence $\pi_\Lambda(\chi)$ converges to some point $x \in \Lambda$. If F is as above then $\pi_M(F) = N$ and $\pi_\Lambda(F) = \bigcup_{l \in N} F(l)$ and the families of these sets are bases for the filters $\varkappa$ and $\pi_\Lambda(\chi)$ respectively. In particular, since $\pi_\Lambda(\chi)$ is finer than the filter of neighbourhoods of x, for each neighborhood V of x there is F such that $\bigcup_{l \in N} F(l) \subset V$.

We prove now that $\chi(\varphi) = \varkappa(x.\varphi)$ if $\varphi \in \mathcal{C}_L$. We have $\chi(\varphi) = \lim_\chi \varphi$, thus for each $\varepsilon > 0$ there is $F \in \chi$ as above such that $|\varphi(y) - \chi(\varphi)| < \varepsilon$ for all $y \in F$. Thus $|\varphi(l+\lambda) - \chi(\varphi)| < \varepsilon$ for all $l \in N$ and $\lambda \in F(l)$. On the other hand, φ being uniformly continuous, there is a neighborhood V of x such that $|\varphi(l + \lambda) - \varphi(l + x)| < \varepsilon$ for all $l \in N$ and $\lambda \in V$. By what we said above, the preceding F may be chosen such that $\bigcup_{l \in N} F(l) \subset V$. Hence we get $|\varphi(l + x) - \chi(\varphi)| < 2\varepsilon$ for all $l \in N$. Since $N \in \varkappa$ and $\varepsilon > 0$ is arbitrary, this shows that $\lim_{l \to \varkappa} \varphi(l + x) = \chi(\varphi)$. ■

Remark: It is easy to show that $\delta_L X$ is homeomorphic with $(X \times \delta L) \amalg \{\infty_L\}$, thought as the one point compactification of $X \times \delta L$, but we do not need this.

In the next lemma we use the notation of Definition 5.14. Let $\varkappa$ be a point in δX.

Lemma 6.9 *If $\varkappa \in \infty_L$ then $\mathcal{C}_L(X)_\varkappa = \mathbb{C}$. If $\varkappa \notin \infty_L$, then $\mathcal{C}_L(X)_\varkappa = \mathcal{C}_\infty(X)$.*

Proof: If $\varkappa \in \infty_L$ then $\varkappa.\varphi(x) = \varkappa(x.\varphi) = \lim_{\mathfrak{f}_L} x.\varphi = \lim_{\mathfrak{f}_L} \varphi$ because $\mathfrak{f}_L$ is translation invariant. Thus $\varkappa.\varphi \in \mathbb{C}$ in this case. Now let $\chi \notin \infty_L$. It suffices then to show that $\chi.\varphi \in \mathcal{C}_0(X)$ if $\lim_{\mathfrak{f}_L} \varphi = 0$ and by an easy density argument we see that it suffices to assume that $\operatorname{supp} \varphi \subset L_K$ for a compact subset K of X. If $\varkappa, x$ are such that $\theta(x, \varkappa) = \chi$ then

$$\chi.\varphi(y) = \chi(y.\varphi) = \varkappa(x.(y.\varphi)) = \varkappa((x + y).\varphi) = \lim_{l \to \varkappa} \varphi(l + x + y).$$

But if $z \notin K$ then there is $M \subset L$ co-finite such that $l + z \notin L_K$ if $l \in M$, and then $\varphi(l + z) = 0$ for all such l, and so $\lim_{l \to \varkappa} \varphi(l + z)$. Hence $\operatorname{supp} \chi.\varphi \subset K - x$.

To finish the proof it remains to show that if $\chi \notin \infty_L$ and $\xi \in \mathcal{C}_\infty(X)$, then there is $\varphi \in \mathcal{C}_L$ such that $\chi.\varphi = \xi$. It suffices to show this under the assumption that ξ has compact support. Then it suffices to take $\varphi = \xi_L = \sum_{l \in L} \tau_{-l}\xi$ ■

Lemma 6.10 *If $\varphi \in \mathcal{C}_L(X)$ the map $\delta L \ni \varkappa \mapsto \varkappa.\varphi \in \mathcal{C}_\infty(X)$ is norm continuous.*

Proof: By a density argument, it suffices to show this for $\operatorname{supp} \varphi \subset M_\Lambda$, where $M \subset L$ is a co-finite set and $\Lambda \subset X$ is a compact set such that $M_\Lambda = \amalg_{l \in M}(l + \Lambda)$. If $l \in M$

let φ_l be the function defined by $\varphi_l(x) = \varphi(l+x)$ for $x \in \Lambda$ and $\varphi_l(x) = 0$ otherwise. Then clearly $\varphi_l \in \mathcal{C}_0(X)$, $\operatorname{supp}\varphi_l \subset \Lambda$, and the family $\{\varphi_l\}_{l\in M}$ is equicontinuous. Thus the set $\{\varphi_l \mid l \in M\}$ is relatively compact in $\mathcal{C}_0(X)$. From the universal property of γM, cf. page 13, there is a unique continuous map $\gamma M \ni \varkappa \mapsto \varphi_\varkappa \in \mathcal{C}_0(X)$ such that $\varphi_\varkappa = \varphi_l$ for all $l \in M$. Since $\delta M = \delta L$, it suffices to show that $\varphi_\varkappa = \varkappa.\varphi$ if $\varkappa \in \delta M$. But we have

$$\varkappa.\varphi(x) = \lim_{l\to\varkappa} \varphi(l+x) = \lim_{l\to\varkappa} \varphi_l(x) = \varphi_\varkappa(x)$$

because $\gamma M \ni \varkappa \mapsto \varphi_\varkappa(x)$ is continuous. ■

Putting all this together we obtain, for the algebra $\mathscr{A} = \mathscr{C}_L(X)$, an improvement of Theorem 5.15. If $T \in \mathscr{C}_L(X)$ then, according to that theorem, we have a continuous map $\sigma(\mathcal{C}_L(X)) \ni \varkappa \mapsto \varkappa.T \in \mathscr{C}_s(X)$ which induces an embedding

$$\mathscr{C}_L(X)/\mathscr{K}(X) \hookrightarrow \textstyle\prod_{\varkappa\in\delta_L X} \mathscr{C}_L(X)_\varkappa. \tag{6.17}$$

From Lemma 6.9 we see that the localization $\mathscr{C}_L(X)_\varkappa = \mathcal{C}_L(X)_\varkappa \rtimes X$ at $\varkappa$ is

$$\mathscr{C}_L(X)_\varkappa = \begin{cases} \mathcal{C}_0(X^*) & \text{if} \quad \varkappa = \infty_L, \\ \mathscr{T}(X) & \text{if} \quad \varkappa \neq \infty_L. \end{cases} \tag{6.18}$$

Here $\delta_L X$ is represented as in (6.16). We identify $\delta L \equiv \{0\} \times \delta L \subset X \times \delta L$ and we simplify the relation (6.17) by taking into account the discussion made on page 32. First, since $(\varkappa\circ\tau_x).T = U_x T U_x^*$, it suffices to restrict the product to the set $\delta L \cup \{\infty_L\}$. Second, we note that the contribution of the point ∞_L is already covered by the other ones. Indeed, this follows from the easy to check relation[†]

$$\infty_L(\varphi) = \infty_L(\tau_\varkappa(\varphi)) \quad \text{for all} \quad \varphi \in \mathcal{C}_L(X),$$

which implies $\mathcal{J}_\varkappa \subset \mathcal{J}_{\infty_L}$, and from Lemma 5.19. Finally, we get:

Theorem 6.11 *If $T \in \mathscr{C}_L(X)$ and $\varkappa \in \delta L$ then $\lim_{l\to\varkappa} U_l T U_l^* = \varkappa.T \equiv \tau_\varkappa(T)$ exists in $\mathscr{C}_s(X)$ and belongs to $\mathscr{T}(X)$. The map $\delta L \ni \varkappa \mapsto \varkappa.T \in \mathscr{T}(X)$ is norm continuous. The maps $\tau_\varkappa : \mathscr{C}_L(X) \to \mathscr{T}(X)$ are surjective morphisms and the intersection of their kernels is $\mathscr{K}(X)$, which gives us a canonical embedding*

$$\mathscr{C}_L(X)/\mathscr{K}(X) \hookrightarrow \mathcal{C}(\delta L; \mathscr{T}(X)). \tag{6.19}$$

If H belongs to $\mathscr{C}_L(X)$ or is an observable affiliated to $\mathscr{C}_L(X)$, then

$$\sigma_{\text{ess}}(H) = \textstyle\bigcup_{\varkappa\in\delta L} \sigma(\varkappa.H). \tag{6.20}$$

The last assertion follows from the norm continuity of the map $\varkappa \mapsto \varkappa.T$.

[†] Note that this is related to Remark 5.16: we have $\tau_\varkappa\tau_\chi = \tau_\chi\tau_\varkappa = \infty_L$ on $\mathcal{C}_L$.

Remark 6.12 Theorem 6.11 has been obtained by rather different methods in [GI2, GI4], see for example Theorems 5.5 and 5.6 in [GI4]. The point is that in these references the quotient $\mathscr{C}_L(X)/\mathscr{K}(X)$ was computed directly and the notion of localization at infinity did not play a rôle. Our purpose here was only to show that Theorem 1.1 can be effectively used even in some rather complicated situations. Our arguments in this subsection may, in fact, serve as a model for other computations.

Remark 6.13 That the class of operators affiliated to $\mathscr{C}_L(X)$ is quite large can be seen from the following result [GI4, Theorem 6.1]. Let $X = \mathbb{R}^n$ and denote $\mathcal{H}^t$ the Sobolev space of order t and $\|\cdot\|_t$ the norm in $B(\mathcal{H}^t, \mathcal{H}^{-t})$. Let $h : X \to \mathbb{R}$ be a continuous function such that $c^{-1}(1+|k|)^{2s} \leq |h(k)| \leq c(1+|k|)^{2s}$ for some constant c and all large k, and denote $H_0 = h(P)$. Let $0 \leq t < s$ reals, choose a sparse set $L \subset X$ and let $\{W_l\}_{l\in L}$ be a family of symmetric operators $W_l : \mathcal{H}^t \to \mathcal{H}^{-t}$ with the property $\sup_{l\in L} \|(1+|Q|)^\lambda W_l\|_t < \infty$ for some $\lambda > 2n$. Then the series $\sum_{l\in L} U_l^* W_l U_l \equiv W$ converges in the strong topology of $B(\mathcal{H}^t, \mathcal{H}^{-t})$. Let $H = H_0 + W$, $H_l = H_0 + W_l$ be the self-adjoint operators in $L^2(X)$ defined as form sums. Then H is affiliated to $\mathscr{C}_L(X)$. If $\varkappa \in \delta L$ then we also have $\varkappa.H = \lim_{l\to\varkappa} H_l$ in norm resolvent sense.

Remark 6.14 The preceding arguments can be simplified and everything becomes an elementary exercise for the subalgebras of $\mathscr{C}_L(X)$ corresponding to a finite number of types of bumps [GI2, p. 548]. The case of just one type is already interesting. More precisely, let $\mathcal{L}$ be a finite partition of L consisting of n infinite sets and let $\mathcal{C}_{\mathcal{L}}$ be the set of $\varphi \in \mathcal{C}_L$ such that $\lim_{a\ni l\to\infty} \varphi(l+x) \equiv a.\varphi(x)$ exists for each x and for each $a \in \mathcal{L}$. Then δL is replaced by the finite set $\mathcal{L}$ and the Hamiltonians affiliated to $\mathscr{C}_{\mathcal{L}}$ have (modulo translations) exactly $n+1$ localizations at infinity: a free one $H_0 \in' \mathcal{C}_0(X^*)$ and a two body one $a.H \in' \mathscr{T}(X)$ for each $a \in \mathcal{L}$. And $\sigma_{\text{ess}}(H) = \bigcup_{a\in\mathcal{L}} \sigma(a.H)$.

We make a final comment on the algebra $\mathcal{A}$ defined in (6.9). We saw that for any sparse set L we have $\mathcal{C}_L(X) \subset \mathcal{A}$. On the other hand, if M is a second sparse set, then $L \cup M$ is not sparse in general. However, the C^*-algebra $\mathcal{C}_{L,M}$ generated by $\mathcal{C}_L \cup \mathcal{C}_M$ is still included in $\mathcal{A}$. Note that to each Hamiltonian affiliated to $\mathscr{C}_L$ one may associate in a canonical way a free Hamiltonian, this is the localization of H at the point ∞_L. But this is not the case for Hamiltonians affiliated to $\mathscr{A}$.

6.5. Grassmann algebras. We shall construct here C^*-algebras canonically associated to finite dimensional vector spaces and which allow one to consider a very general version of N-body Hamiltonians. This algebras have first been pointed out in [DG1] and the spectral theory of the operators affiliated to them (essential spectrum and the Mourre estimate) has been studied in detail in [DG2]. Our approach here is rather different, the graded algebra structure so important in the quoted works does not play a big rôle anymore.

If Y is a closed subgroup of a locally compact abelian group X then X/Y is also a locally compact abelian group and we have a continuous surjective group morphism $\pi_Y : X \to X/Y$. Then the map defined by $\varphi \mapsto \varphi \circ \pi_Y$ gives us a natural embedding $\mathcal{C}(X/Y) \hookrightarrow \mathcal{C}(X)$. In fact

$$\mathcal{C}(X/Y) = \{\varphi \in \mathcal{C}(X) \mid y.\varphi = \varphi \quad \forall y \in Y\}. \tag{6.21}$$

Note that we shall denote just 0 the group $\{0\}$ and then $\mathcal{C}(0) = \mathcal{C}_0(0) = \mathbb{C}$, hence $\mathcal{C}(X/X) = \mathcal{C}_0(X/X) = \mathbb{C}$. On the other hand, if $0 \subset Y \subset Z \subset X$ are closed subgroups then $X/Z \cong (X/Y)/(Y/Z)$ and we have natural maps

$$X \to X/Y \to X/Z \to 0 \tag{6.22}$$

hence we get embeddings

$$\mathbb{C} \subset \mathcal{C}(X/Z) \subset \mathcal{C}(X/Y) \subset \mathcal{C}(X). \tag{6.23}$$

In the rest of this subsection *we shall consider only finite dimensional real vector spaces*, although much of the theory can be extended to more general groups. We shall consider the algebra generated by the C^*-subalgebras $\mathcal{C}_0(X/Y) \subset \mathcal{C}(X/Y) \subset \mathcal{C}(X)$. We recall that the *Grassmannian* $\mathbb{G}(X)$ is the set of all vector subspaces of X and the *projective space* $\mathbb{P}(X)$ is the set of all one dimensional subspaces of X.

Definition 6.15 *The* (classical) Grassmann algebra *of the vector space X is the X-subalgebra $\mathcal{G}(X) \subset \mathcal{C}(X)$ defined by*

$$\mathcal{G}(X) = \text{norm closure of } \textstyle\sum_Y \mathcal{C}_0(X/Y) \tag{6.24}$$

where Y runs over $\mathbb{G}(X)$. The quantum Grassmann algebra *of X is the C^*-algebra $\mathscr{G}(X) \subset \mathscr{B}(X)$ defined by*

$$\mathscr{G}(X) = \mathcal{G}(X) \rtimes X = \text{norm closure of } \textstyle\sum_Y \mathcal{C}_0(X/Y) \rtimes X. \tag{6.25}$$

Remarks 6.16 The fact that $\mathcal{G}(X)$ is a C^*-algebra follows from the obvious relation

$$\mathcal{C}_0(X/Y) \cdot \mathcal{C}_0(X/Z) \subset \mathcal{C}_0(X/(Y \cap Z)). \tag{6.26}$$

The second equality from (6.25) follows from Theorem 3.14.

Let $G(X)$ be the set of finite unions of strict vector subspaces of X:

$$G(X) = \{L \subset X \mid \exists \mathbb{F} \subset \mathbb{G}(X) \setminus \{X\} \text{ finite such that } L = \textstyle\bigcup_{Y\in\mathbb{F}} Y\}.$$

If L is as above and $\Lambda \subset X$ is compact then $L_\Lambda = L + \Lambda = \bigcup_{Y\in\mathbb{F}}(Y + \Lambda)$. Thus L_Λ is a closed set, $L_\Lambda \neq X$, and we have $L_\Lambda \cup M_\Lambda = (L \cup M)_\Lambda$ and $L_{\Lambda'} \subset L_{\Lambda''}$ if $\Lambda' \subset \Lambda''$. This justifies the next definition.

Definition 6.17 *The* Grassmann filter $\mathfrak{g} = \mathfrak{g}_X$ *on X is the filter generated by the family of open sets $L^c_\Lambda = X \setminus L_\Lambda$ where L runs over $G(X)$ and Λ over the set of compact subsets of X. If Y is a subspace of X, then we denote also by $\mathfrak{g}_Y$ the filter on X generated by the Grassmann filter of Y.*

Clearly, $\mathfrak{g}_X$ is translation invariant, finer than the Fréchet filter, and $\mathfrak{g}^\circ_X = \mathfrak{g}_X$. If X is one dimensional, then $\mathfrak{g}_X$ is just the Fréchet filter.

Remark 6.18 For $L \in G(X)$ we may consider the filter $\mathfrak{f}_L$ defined as in (6.12). Then $\mathfrak{g}_X$ is just the filter generated by $\bigcup_L \mathfrak{f}_L$. This can be expressed in other terms as follows: (1) $\mathfrak{g}_X$ is the upper bound of the set of filters $\mathfrak{f}_L$; (2) when seen as compact subsets of δX (cf. page 13), $\mathfrak{g}_X$ is the intersection of the compact sets $\mathfrak{f}_L$.

Remark 6.19 If we equip X with an Euclidean norm $|\cdot|$ and denote π_Y the orthogonal projection onto $Y^\perp \cong X/Y$, then $\delta_L(x) \equiv \text{dist}(x, L) = \min_{Y \in \mathbb{F}} |\pi_Y x|$ (with $L, \mathbb{F}$ as before). Then the sets $L_r^c = \{x \in X \mid \delta_L(x) > r\}$, with $L \in G(X)$ and $r > 0$ real, form a basis of the filter $\mathfrak{g}_X$. Note that L has empty interior and if x is outside it then $\delta_L(tx) = |t|\delta_L(x) \to \infty$ as $t \to \infty$.

Lemma 6.20 *Let* $Y, Z \in \mathbb{G}(X)$. *If* $Y \subset Z$ *then* $\lim_{\mathfrak{g}_Y} \pi_Z = 0$. *If* $Y \not\subset Z$ *then* $\lim_{\mathfrak{g}_Y} \pi_Z = \infty$.

Proof: Since $Y \in \mathfrak{g}_Y$ the above limits involve only the restriction of π_Z to Y. In the first case, if $y \in Y$ then $\pi_Z(y) = 0$, so the assertion is clear. If $Y \not\subset Z$ then $E = Y \cap Z$ is a strict subspace of Y. Let E' be a supplementary subspace for E in Y. Then $\pi_Z : E' \to X/Z$ is injective, hence if $K \subset X/Z$ is compact then the set Λ of $y \in E'$ such that $\pi_Z y \in K$ is a compact in E' and thus in Y. If $y \in Y \setminus E_\Lambda \in \mathfrak{g}_Y$ then $y = e + e'$ with $e \in E$ and $e' \in E' \setminus \Lambda$ so $\pi_Z y = \pi_z e' \notin K$. ■

Corollary 6.21 *If* $\varphi \in \mathcal{G}(X)$ *and* $Y \in \mathbb{G}(X)$ *then* $\lim_{\mathfrak{g}_Y} \varphi$ *exists. If* $\varphi = \sum_Z \varphi^Z$ *is a finite sum of* $\varphi^Z \in \mathcal{C}_0(X/Z)$, *then* $\lim_{\mathfrak{g}_Y} \varphi = \sum_{Z \supset Y} \varphi^Z(0)$.

We see that each filter $\mathfrak{g}_Y$ defines a character of $\mathcal{G}(X)$ and we could proceed as in the proof of Lemma 6.8 and describe $\delta_{\mathbb{G}} X \equiv \delta(\mathcal{G}(X))$ in terms of couples (Y, y) with $y \in X/Y$. We shall not do it explicitly, but this is hidden in what follows. We only note that the rôle of ∞_L is now played by $\mathfrak{g}_X$.

Proposition 6.22 *For each* $\varphi \in \mathcal{G}(X)$ *the limit*

$$\tau_Y \varphi = \lim_{y \to \mathfrak{g}_Y} y.\varphi \tag{6.27}$$

exists locally uniformly on X. *If* φ *is a finite sum* $\varphi = \sum_Z \varphi^Z$ *with* $\varphi^Z \in \mathcal{C}_0(X/Z)$, *then* $\tau_Y \varphi = \sum_{Z \supset Y} \varphi^Z$. *If* $Y \in \mathbb{P}(X)$, $y \in Y \setminus 0$, *then* $\tau_Y \varphi(x) = \lim_{t \to \infty} \varphi(x + ty)$.

Proof: We have to show that $\lim_{y \to \mathfrak{g}_Y} \varphi(x + y)$ exists locally uniformly in x. But this is an immediate consequence of the Corollary 6.21 and Lemma 2.2. ■

According to the conventions we made at the beginning of this subsection, we have $\mathcal{G}(X/Y) \subset \mathcal{C}(X/Y) \subset \mathcal{C}(X)$ if Y is a subspace of X.

Proposition 6.23 *We have* $\mathcal{G}(X/Y) \subset \mathcal{G}(X)$. *Moreover, there is a unique morphism* $\tau_Y : \mathcal{G}(X) \to \mathcal{G}(X/Y)$ *such that* τ_Y *is a projection (in the sense of linear spaces). The map* τ_Y *is given by (6.27). If* $Y \subset Z \subset X$ *then*

$$\mathbb{C} = \mathcal{G}(0) \subset \mathcal{G}(X/Z) \subset \mathcal{G}(X/Y) \subset \mathcal{G}(X) \tag{6.28}$$

and $\tau_Y \tau_Z = \tau_Z \tau_Y = \tau_Z$. *More generally, for any* $Y, Z \in \mathbb{G}(X)$ *we have*

$$\mathcal{G}(X/Y) \bigcap \mathcal{G}(X/Z) = \mathcal{G}(X/(Y + Z)) \tag{6.29}$$

and $\tau_Y \tau_Z = \tau_Z \tau_Y = \tau_{Y+Z}$.

Proof: The algebra $\mathcal{G}(X/Y)$ is generated the $\mathcal{C}_0((X/Y)/E)$ with $E \subset X/Y$ subspace. If $Z = \pi_Y^{-1}(E)$ then $Y \subset Z \subset X$, $E = Z/Y$ and $(X/Y)/E \cong X/Z$ allows us to identify $\mathcal{C}_0((X/Y)/E) = \mathcal{C}_0(X/Z)$ and thus to get the first assertion of the proposition. Observe that

$$(6.30) \qquad \begin{aligned} \mathcal{G}(X/Y) &= \text{norm closure of } \textstyle\sum_{Z\supset Y} \mathcal{C}_0(X/Z) \\ &= \{\varphi \in \mathcal{G}(X) \mid y.\varphi = \varphi \quad \forall y \in Y\}. \end{aligned}$$

The other assertions of the proposition are easy to check. ■

Proposition 6.24 *If $\varphi \in \mathcal{G}(X)$ and $\tau_Y\varphi = 0$ for all $Y \in \mathbb{P}(X)$, then $\varphi \in \mathcal{C}_0(X)$.*

Proof: This follows from Theorem 3.2 and Lemma 4.1 of [DG1], but we give a self-contained proof here. Consider first a finite set $\mathbb{F} \subset \mathbb{G}(X)$ which is stable under intersections and such that $0 \in \mathbb{F}$ and let $\mathcal{A} = \sum_{Y\in\mathbb{F}} \mathcal{C}_0(X/Y)$. Then $\mathcal{A}$ is a $*$-algebra because of (6.26) and $\mathcal{C}_0(X) \subset \mathcal{A}$. Clearly $\|\tau_Y\varphi\| \le \|\varphi\|$ for all $Y \in \mathbb{F}, \varphi \in \mathcal{A}$. Let us write $\varphi = \sum_Y \varphi^Y$ with $\varphi^Y \in \mathcal{C}_0(X/Y)$. From Proposition 6.22 we get $\tau_Y\varphi = \sum_{Z\supset Y} \varphi^Z$, so if Y is a maximal element of $\mathbb{F}$ then $\tau_Y\varphi = \varphi^Y$, hence $\|\varphi^Y\| \le \|\varphi\|$. By induction,we easily see that there is a constant c such that

$$\|\varphi^Y\| \le c\|\varphi\| \quad \text{for all} \quad Y \in \mathbb{F}, \varphi \in \mathcal{A}.$$

This clearly implies that $\mathcal{A}$ is a C^*-algebra and that $\sum_{Y\in\mathbb{F}} \mathcal{C}_0(X/Y)$ is a topological direct sum. If φ is as above and $\tau_Y\varphi = 0$ for all $Y \neq 0$ then $\sum_{Z\supset Y} \varphi^Z = 0$ if $Y \neq 0$ hence, the sum being direct, we get $\varphi^Z = 0$ for all $Z \neq 0$, thus $\varphi \in \mathcal{C}_0(X)$.

It follows that the map $\varphi \mapsto (\tau_Y\varphi)_{Y\neq 0}$ is a morphism from $\mathcal{A}$ into $\prod_{Y\neq 0} \mathcal{G}(X/Y)$ with kernel equal to $\mathcal{C}_0(X)$. In particular, the induced map $\mathcal{A}/\mathcal{C}_0(x) \to \prod_{Y\neq 0} \mathcal{G}(X/Y)$ is an isometry, so that if $\psi \in \mathcal{A}$ is such that $\|\tau_Y\psi\| \le \varepsilon$ for all $Y \neq 0$ then there is $\psi_0 \in \mathcal{C}_0(X)$ such that $\|\psi - \psi_0\| \le 2\varepsilon$ (just by definition of the quotient norm).

Let now $\varphi \in \mathcal{G}(X)$ such that $\tau_Y\varphi = 0$ for all $Y \in \mathbb{P}(X)$. From Proposition 6.23 it follows that this property remains true for all $Y \in \mathbb{G}(X), Y \neq 0$. From the definition (6.24) it follows easily that for each $\varepsilon > 0$ there is $\mathcal{A}$ as above and there is $\psi \in \mathcal{A}$ such that $\|\varphi - \psi\| \le \varepsilon$. Then clearly we have $\|\tau_Y\psi\| \le \varepsilon$ for all $Y \neq 0$, so by what we proved above there is $\psi_0 \in \mathcal{C}_0(X)$ such that $\|\psi - \psi_0\| \le 2\varepsilon$, and hence $\|\varphi - \psi_0\| \le 3\varepsilon$. This clearly implies $\varphi \in \mathcal{C}_0(X)$. ■

The next theorem is now an immediate consequence of Theorem 5.15, Propositions 6.23 and 6.24, and of Lemma 5.21. We denote

$$(6.31) \qquad \mathscr{G}_Y(X) = \mathscr{G}(X/Y) \rtimes X = \text{norm closure of } \textstyle\sum_{Z\supset Y} \mathcal{C}_0(X/Z) \rtimes X.$$

We mention that we have non canonical isomorphisms $\mathscr{G}_Y(X) \simeq \mathscr{G}(X/Y) \otimes \mathcal{C}_0(Y^*)$.

Theorem 6.25 *If $T \in \mathscr{G}(X)$ and $Y \in \mathbb{G}(X)$ then $\tau_Y T = \lim_{y\to\mathfrak{g}_Y} U_x T U_x^*$ exists in $\mathscr{C}_s(X)$ and belongs to $\mathscr{G}_Y(X)$. The map $\tau_Y : \mathscr{G}(X) \to \mathscr{G}_Y(X)$ is a morphism and a linear projection and is uniquely characterized by these properties. We have $\tau_Y\tau_Z = \tau_Z\tau_Y = \tau_{Y+Z}$. If $Y \in \mathbb{P}(X)$ and $y \in Y, y \neq 0$ then $\tau_Y T = \lim_{t\to\infty} U_{ty} T U_{ty}^*$. We have $T \in \mathscr{K}(X)$ if and only if $\tau_Y T = 0$ for all $Y \in \mathbb{P}(X)$, which gives us*

$$(6.32) \qquad \mathscr{G}(X)/\mathscr{K}(X) \hookrightarrow \textstyle\prod_{Y\in\mathbb{P}(X)} \mathscr{G}_Y(X).$$

From (6.32) we get that the the essential spectrum of an observable H affiliated to $\mathscr{G}(X)$ is equal to the closure of the union $\sigma(\tau_Y H)$ with $Y \in \mathbb{P}(X)$. But now we can prove more: as in the situations considered in Theorem 6.11 and Proposition 6.3, the union is already closed (although it is not finite, as in the usual N-body problem).

Theorem 6.26 *If $T \in \mathscr{G}(X)$ then $\{\tau_Y T \mid Y \in \mathbb{P}(X)\}$ is a compact set in $\mathscr{G}(X)$. In particular, if H belongs to $\mathscr{G}(X)$ or is an observable affiliated to $\mathscr{G}(X)$, then*

$$\sigma_{\text{ess}}(H) = \textstyle\bigcup_{Y \in \mathbb{P}(X)} \sigma(\tau_Y H). \tag{6.33}$$

One should note that the map $Y \mapsto \tau_Y T$ is not continuous: if $T \in \mathcal{C}_0(X/Z) \rtimes X$ then $\tau_Y T = T$ if $Y \subset Z$ and $\tau_Y T = 0$ if $Y \not\subset Z$.

Theorem 6.26 is a corollary of Theorem 4.2 and Proposition 5.4 from [DG2]. We shall give below a slightly improved proof. Note that only some general properties of the lattice $\mathbb{G}(X)$ and of the graded algebra structure of $\mathscr{G}(X)$ are really needed. The next two lemmas imply the first assertion of Theorem 6.26 (hence the second).

Lemma 6.27 *If $T \in \mathscr{G}(X)$ then for each $Z \in \mathbb{G}(X)$, $Z \neq 0$ there is $Y \in \mathbb{P}(X)$ such that $\tau_Z T = \tau_Y T$.*

Proof: Let $\mathbb{E} \subset \mathbb{G}(X)$ be countable. Then $\{E \cap Z \mid E \in \mathbb{E}, E \cap Z \neq Z\}$ is a countable set of strict subspaces of Z, so its union is not Z. Let $Y \in \mathbb{P}(Z)$ such that $Y \cap E \cap Z = 0$ if E is in the preceding set. Then from $E \in \mathbb{E}$ and $E \supset Y$ we get $E \supset Z$. Now if $T_{\mathbb{E}}$ is a finite sum $\sum_{E \in \mathbb{E}} T^E$ with $T^E \in \mathcal{C}_0(X/E) \rtimes X$ then clearly $\tau_Z T = \tau_Y T$. Finally, if T is arbitrary, then there is $\mathbb{E}$ as above such that T be a norm limit of operators of the form $T_{\mathbb{E}}$, so we have $\tau_Z T = \tau_Y T$. ■

Lemma 6.28 *Let $\{Y_n\}_{n \geq 0}$ be a sequence of linear subspaces of X and let us define $Y = \bigcap_{n \geq 0} \sum_{m \geq n} Y_m$. If k is the dimension of Y, then there is N such that for all $n \geq N$ and all $T \in \mathscr{G}(X)$:*

$$\|(\tau_{Y_n} - \tau_Y)T\| \leq k \sup_{m \geq n} \|(\tau_{Y_n} - \tau_{Y_m})T\|.$$

Proof: Since a decreasing sequence of subspaces is eventually constant, there is N such that $Y = \sum_{m \geq n} Y_m$ for all $n \geq N$. The dimension of Y being k, for each $n \geq N$ there are $n < n_1 < \cdots < n_k$ such that $Y = Y_n + Y_{n_1} + \cdots + Y_{n_k}$. From Theorem 6.25 we get $\tau_Y = \tau_{Y_n} \tau_{Y_{n_1}} \ldots \tau_{Y_{n_k}}$. Let $\mathcal{P} = \tau_{Y_n}$, $\mathcal{P}_i = \tau_{Y_{n_i}}$, and $\mathcal{P}'_i = 1 - \mathcal{P}_i$. Then:

$$\mathcal{P} - \tau_Y = \mathcal{P}[1 - \mathcal{P}_1 \ldots \mathcal{P}_k] = \sum_{i=1}^{k-1} \mathcal{P}\mathcal{P}'_i \mathcal{P}_{i+1} \ldots \mathcal{P}_k + \mathcal{P}\mathcal{P}'_k.$$

Since the morphisms $\mathcal{P}_i$ commute, we get $\|(\mathcal{P} - \tau_Y)T\| \leq \sum_{i=1}^k \|\mathcal{P}\mathcal{P}'_i T\|$. Now it suffices to note that $\mathcal{P}\mathcal{P}'_i = \mathcal{P}(\mathcal{P} - \mathcal{P}_i)$. ■

Proof of Theorem 6.26: If $\{\tau_{Y_n} T\}$ is a norm Cauchy sequence and Y is as in the Lemma 6.28 then $\|(\tau_{Y_n} - \tau_Y)T\| \to 0$. Observe that we do not have $k = 0$ because this would imply $Y_n = 0$ for large n. Thus we can use Lemma 6.27 and find $E \in \mathbb{P}(X)$ such that $\tau_Y T = \tau_E T$, which proves the first assertion of the theorem. ■

Remark 6.29 The usual form of the HVZ theorem for N-body Hamiltonians follows easily from Theorem 6.26. Indeed, in the Agmon-Froese-Herbst formalism [ABG] one is given a finite lattice $\mathcal{L}$ and an injective map $\mathcal{L} \ni a \mapsto X_a \in \mathbb{G}(X)$ such that $X_{a \wedge b} = X_a \cap X_b$, $X_{\max \mathcal{L}} = X$ and $X_{\min \mathcal{L}} = 0$. The N-body Hamiltonians are observables H affiliated to the C^*-algebra $\mathscr{C} = \sum_{a \in \mathcal{L}} \mathcal{C}_0(X^a) \rtimes X \subset \mathscr{G}(X)$, where $X^a = X/X_a$. Let $\tau_a = \tau_{X_a}$, then τ_a is a morphism and a linear projection of $\mathscr{C}$ onto the C^*-subalgebra $\mathscr{C}_a = \sum_{b \geq a} \mathcal{C}_0(X^b) \rtimes X$. Let us set $H_a = \tau_a H$. Then (a generalized version of) the HVZ theorem says that

$$\sigma_{\text{ess}}(H) = \textstyle\bigcup_{a \in \mathcal{M}} \sigma(H_a), \tag{6.34}$$

where $\mathcal{M}$ is the set of atoms of $\mathcal{L}$. To get this from (6.33), note that for each $Y \in \mathbb{P}(X)$ there is a smallest b in $\mathcal{L}$ such that $Y \leq X_b$, so we have $Y \subset X_c$ if and only if $b \leq c$. Then for $T \in \mathscr{C}$ we have $\tau_Y T = \tau_b T$. On the other hand, there is an atom a such that $a \leq b$, and then $\tau_b T = \tau_b \tau_a T$. Thus $\sigma(\tau_Y T) \subset \sigma(\tau_a T)$. Reciprocally, if $Z \in \mathbb{P}(X)$ and $Z \subset X_a$ then $\tau_a T = \tau_a \tau_Z T$ and so $\sigma(\tau_a T) \subset \sigma(\tau_Z T)$.

Example 6.30 The simplest application of Theorem 6.26 is obtained by taking $X = \mathbb{R}^n$ and $H = \Delta + V(x)$ where $V \in \mathcal{G}(X)$. Although simple, this situation is, however, not trivial because the union in (6.33) contains an infinite number of distinct terms in general. For example, the construction of V may involve an infinite number of subspaces Y whose union is dense in X.

Example 6.31 We show here that in an N-body type situation (i.e. involving only a finite number of subspaces Y) the class of Hamiltonians for which (6.33) applies is very large. We use the setting of Remark 6.29 and, to simplify notations, we equip X with an Euclidean structure, so that X is identified with X^* and $X^a = X_a^\perp$. For real s let $\mathcal{H}^s$ be the usual Sobolev spaces, set $\mathcal{H} = \mathcal{H}^0 = L^2(X)$, and embed as usual $\mathcal{H}^s \subset \mathcal{H} \subset \mathcal{H}^{-s}$ if $s > 0$. Fix $s > 0$ and denote $\| \cdot \|_s$ the norm in $B(\mathcal{H}^s, \mathcal{H}^{-s})$. Let $h : X \to \mathbb{R}$ be continuous and such that $c'(1 + |k|)^{2s} \leq h(k) \leq c''(1 + |k|)^{2s}$ outside a compact, for some constants c', c''. Then $H(\max \mathcal{L}) := h(P)$ is a self-adjoint operator with domain $\mathcal{H}^{2s}$ and form domain $\mathcal{H}^s$. Then for each $a \neq \max \mathcal{L}$ let $H(a) : \mathcal{H}^s \to \mathcal{H}^{-s}$ be a symmetric continuous operator such that the following properties hold:
(1) $U_x H(a) U_x^* = H(a)$ if $x \in X_a$,
(2) $\| V_k H(a) V_k^* - H_a \|_s \to 0$ as $k \to 0$ in X,
(3) $\| (V_k - 1) H(a) \|_s \to 0$ as $k \to 0$ in X^a,
(4) $H_a := \sum_{b \geq a} H(b) \geq \mu h(P) - \nu$ as forms on $\mathcal{H}^s$, for some $\mu, \nu > 0$.
Then $H \equiv H(\min \mathcal{L})$ is affiliated to $\mathscr{C}$ and $\tau_a H = H_a$, so (6.34) holds. See [DG2, Theorem 4.6] for the details of the computation and for more general results.

6.6. On the operators $\varkappa.H$. We observed after Theorem 1.2 that if H is a self-adjoint operator affiliated to $\mathscr{C}(X)$ then its localizations at infinity $\varkappa.H$ are not necessarily densely defined. We shall make in this subsection some comments on this question and we shall give conditions which allow one to compute $\varkappa.H$ directly in terms of $x.H$ and so to avoid considering the resolvent of H. This is not possible if $H = h(P) + v(Q)$ if the operator $V = v(Q)$ is not relatively bounded with respect to $h(P)$, so we shall consider here only more elementary situations which are of some physical interest.

To fix the ideas we consider here only the case $X = \mathbb{R}^n$ and take $\mathcal{H} = L^2(X; \mathbf{E})$, where $\mathbf{E}$ a finite dimensional Hilbert space, cf. Section 4. For simplicity we consider only operators whose form domain is a Sobolev space $\mathcal{H}^s$ with $s > 0$ (everything extends with no difficulty to hypoelliptic operators). Set $\langle k\rangle = (1+|k|^2)^{1/2}$ and denote $S(\mathbf{E})$ the space of symmetric operators on $\mathbf{E}$ and $|S|$ the absolute value of $S \in S(\mathbf{E})$.

Let $h : X \to S(\mathbf{E})$ be locally Lipschitz and such that $c'\langle k\rangle^{2s} \leq |h(k)| \leq c''\langle k\rangle^{2s}$ and $|h'(k)| \leq c\langle k\rangle^{2s}$ outside a compact, where c, c', c'' are constants. We set $H_0 = h(P)$ and observe that $D(|H_0|^{1/2}) = \mathcal{H}^s$.

Let $v : X \to S(\mathbf{E})$ be a locally integrable function such that the operator $V = v(Q)$ satisfies $V\mathcal{H}^s \subset \mathcal{H}^{-s}$ and $\pm V \leq \mu|H_0| + \nu$ for some real numbers μ, ν with $\mu < 1$. Then the self-adjoint operator $H = H_0 + V$ (form sum) is affiliated to $\mathscr{C}(X)$, cf. Corollary 4.8. Note that $x.V = U_x V U_x^* = v(x + Q)$ satisfies the same estimates as V and that $x.H = H_0 + x.V$. We mention that the next lemma is valid under the much more general conditions of Definition 4.7.

Lemma 6.32 *Let us assume that for each $\mathcal{C}^\infty$ function with compact support f the set $\{x.Vf \mid x \in X\}$ is relatively compact in $\mathcal{H}^{-s}$. Then for each $\varkappa \in \delta X$ the limit $\lim_{x\to\varkappa} x.V = \varkappa.V$ exists in the strong operator topology in $B(\mathcal{H}^s, \mathcal{H}^{-s})$, we have $\pm\varkappa.V \leq \mu|H_0| + \nu$ as forms on $\mathcal{H}^s$, and we have $\varkappa.H = H_0 + \varkappa.V$ if $\varkappa.H$ is defined as in Theorem 1.2.*

Proof: Let $z \in \rho(H)$ and $R = (H - z)^{-1} \in \mathscr{C}(X)$. Then $\varkappa.H$ is defined by the operator $\varkappa.R = \lim_{x\to\varkappa} x.R$ where the limit exists in $\mathscr{C}_s(X)$. Note that we know that the limit exists but we do not yet know whether $\varkappa.R$ is injective or not. On the other hand, the existence of $\varkappa.V$ follows from the fact that the set of operators $x.V$ is bounded in $B(\mathcal{H}^s, \mathcal{H}^{-s})$: thus it suffices to show the existence of the limit $\lim_{x\to\varkappa} x.Vf$ in $\mathcal{H}^{-s}$ for f a $\mathcal{C}^\infty$ function with compact support, and this is obvious by the universal property of the Stone-Čech compactification γX of the discrete space X and our assumption. Note that $\varkappa.V$ is the operator of multiplication by a distribution which could not be a function, but clearly the estimate verified by V remains valid in the limit. Hence if *we define* $\varkappa.H = H_0 + \varkappa.V$ as form sum, we get a densely defined self-adjoint operator such that $\varkappa.H - z$ extends to an isomorphism $\mathcal{H}^s \to \mathcal{H}^{-s}$. Now it suffices to prove that $\varkappa.R = (\varkappa.H - z)^{-1}$. Since $x.H - z : \mathcal{H}^s \to \mathcal{H}^{-s}$ is also an isomorphism, one can easily justify the equality

$$(x.H - z)^{-1} - (\varkappa.H - z)^{-1} = (x.H - z)^{-1}(\varkappa.V - x.V)(\varkappa.H - z)^{-1}$$

in $B(\mathcal{H}^{-s}, \mathcal{H}^s)$. Then for $f \in \mathcal{H}^{-1}$ we have

$$\|[(x.H - z)^{-1} - (\varkappa.H - z)^{-1}]f\|_{\mathcal{H}^s} \leq C\|(\varkappa.V - x.V)(\varkappa.H - z)^{-1}f\|_{\mathcal{H}^{-s}}$$

where

$$C = \|(H - z)^{-1}\|_{B(\mathcal{H}^{-s},\mathcal{H}^s)} = \|(x.H - z)^{-1}\|_{B(\mathcal{H}^{-s},\mathcal{H}^s)}.$$

This clearly finishes the proof. ■

This lemma gives a rather concrete method of computing $\varkappa.H$ and also shows that this operator is densely defined. The most elementary way of checking the relative compacity assumption from the lemma is described below.

Proposition 6.33 *Assume that for each* $\mu > 0$ *there is* ν *such that* $|V| \le \mu\langle P\rangle^{2s} + \nu$. *Then* $\lim_{x\to\varkappa} x.V = \varkappa.V$ *exists strongly in* $B(\mathcal{H}^s, \mathcal{H}^{-s})$ *for each* $\varkappa \in \delta X$*, for each* $\mu > 0$ *there is* ν *such that* $\pm\varkappa.V \le \mu|H_0| + \nu$ *as forms on* $\mathcal{H}^s$*, and* $\varkappa.H = H_0 + \varkappa.V$ *if* $\varkappa.H$ *is defined as in Theorem 1.2. In particular, we have*

$$\sigma_{\text{ess}}(H) = \overline{\bigcup_{\varkappa\in\delta X}\sigma(\varkappa.H)}.$$

Proof: We only have to show that the set $\{x.Vf \mid x \in X\}$ is relatively compact in $\mathcal{H}^{-s}$ if $f \in \mathcal{C}_c^\infty(X)$, i.e. if f is a $\mathcal{C}^\infty$ function with compact support. This is equivalent to the relative compactness in $\mathcal{H}$ of the set $\{\langle P\rangle^{-s}x.Vf \mid x \in X\}$. Let $\psi, \xi \in \mathcal{C}_c^\infty(X)$ with $\xi(x) = 1$ on $\operatorname{supp} f$ and let $S = \langle P\rangle^{-s}\xi(Q)\langle P\rangle^s$ and $T = \langle P\rangle^{-s}V\langle P\rangle^{-s}$. Then:

$$\psi(P)\langle P\rangle^{-s}x.Vf = \psi(P)\langle P\rangle^{-s}\xi(Q)x.Vf = \psi(P)SU_xTU_x^*\langle P\rangle^s f \equiv \psi(P)Sf_x.$$

The set $\{f_x \mid x \in X\}$ is bounded in $\mathcal{H}$ and the operator $\psi(P)S$ is compact in $\mathcal{H}$, so the set $\{\psi(P)\langle P\rangle^{-s}x.Vf \mid x \in X\}$ is relatively compact in $\mathcal{H}$. Thus it suffices to prove the following assertion: for each $\varepsilon > 0$ there is $\psi \in \mathcal{C}_c^\infty(X)$ such that $\|\psi(P)^\perp\langle P\rangle^{-s}x.Vf\| \le \varepsilon$ for all $x \in X$, where $\psi(P)^\perp = 1-\psi(P)$. Let $V_\pm$be the positive and negative parts of V, so that $V = V_+ - V_-$ and $|V| = V_+ + V_-$, then it is clearly sufficient to prove this assertion with V replaced by $V_\pm$. If $T_\pm = \langle P\rangle^{-s}V_\pm\langle P\rangle^{-s}$ then

$$\|\psi(P)^\perp\langle P\rangle^{-s}x.V_\pm f\| = \|\psi(P)^\perp U_xT_\pm U_x^*\langle P\rangle^s f\| \le \|\psi(P)^\perp T_\pm\|\|\langle P\rangle^s f\|$$

and if we set $C_\pm = \|T_\pm\|^{1/2}$ then

$$\|\psi(P)^\perp T_\pm\| \le C_\pm\|\psi(P)^\perp T_\pm^{1/2}\| = C_\pm\|\psi(P)^\perp T_\pm\psi(P)^\perp\|^{1/2}.$$

On the other hand, from $|V| \le \mu\langle P\rangle^{2s} + \nu$ we get $V_\pm \le \mu\langle P\rangle^{2s} + \nu$ and then $T_\pm \le \mu + \nu\langle P\rangle^{-2s}$ and so

$$\psi(P)^\perp T_\pm\psi(P)^\perp \le [\mu + \nu\langle P\rangle^{-2s}](1 - \psi(P))^2.$$

Since μ can be chosen as small as we wish, it is clear that the right hand side above can be made $\le \varepsilon$ for any $\varepsilon > 0$ by choosing ψ conveniently. Since the left hand side is positive we then get $\|\psi(P)^\perp T_\pm\psi(P)^\perp\| \le \varepsilon$. ■

Corollary 6.34 *If the conditions of Proposition 6.33 are satisfied and if for each* C^∞ *function* f *with support in the unit ball we have* $\lim_{x\to\infty}\|x.Vf\|_{\mathcal{H}^{-s}} = 0$*, then the essential spectrum of* H *is given by* $\sigma_{\text{ess}}(H) = \sigma(H_0)$.

Example 6.35 There are at least three physically interesting situations covered by Proposition 6.33:
(1) The Schrödinger operator $H = P^2 + V = \Delta + V(x)$. Then $s = 1$ and the assumptions of the proposition are satisfied if V is of Kato class, so we get Theorem 4.5 from [LaS]. Corollary 6.34 is similar to [LaS, Theorem 4.3].
(2) The relativistic spin zero operator $H = (P^2 + m^2)^{1/2} + V(x)$, then $s = 1/2$. Here m is any real number.
(3) The Dirac operator $H = D + V(x)$. Here D is the free Dirac operator of mass $m \ge 0$, $s = 1/2$, $\mathbf{E} = \mathbb{C}^N$ is not trivial, and $V(x)$ is matrix valued. The last two situations are also considered in [Rab].

6.7. Cocompact subgroups. We consider now a situation similar to that from [LaS, Section 5]. In a C^*-algebra setting such examples and generalizations appear in [Man].

Throughout this subsection X is an abelian locally compact non compact group and Y a closed subgroup such that X/Y is compact. Since Y is fixed, we shall abbreviate $\pi_Y = \pi$. We embed $\mathcal{C}(X/Y) \subset \mathcal{C}(X)$ as explained on page 41, so we think of $\mathcal{C}(X/Y)$ as a translation invariant C^*-subalgebra of $\mathcal{C}(X)$ containing the constants, explicitly described by (6.21). More generally, we identify any function v defined on X/Y with the function $v \circ \pi$ defined on X.

The full justification of the class of functions introduced below will become clear later on, cf. Lemma 6.40.

Definition 6.36 *If $\theta : X \to X$ then write $\theta(a+x) \sim a + \theta(x)$ if*

$$\lim_{x\to\infty} [\theta(a+x) - a - \theta(x)] = 0 \quad \forall a \in X. \tag{6.35}$$

If θ is uniformly continuous then this is equivalent to having $\theta(x) = x + \xi(x)$ where $\xi : X \to X$ is uniformly continuous and slowly oscillating in a sense similar to (6.4).

The next proposition is completely elementary but we state it separately because the main idea of the proof is very clear in this context.

Proposition 6.37 *Let $h : X^* \to \mathbb{R}$ be a continuous function such that $|h(k)| \to \infty$ as $k \to \infty$. Assume that $\theta : X \to X$ is uniformly continuous and $\theta(a+x) \sim a + \theta(x)$. If $v : X/Y \to \mathbb{R}$ is continuous, $H = h(P) + v(Q)$ and $H_\theta = h(P) + v \circ \theta(Q)$, then $\sigma_{\text{ess}}(H_\theta) = \sigma(H)$.*

Proof: This will be a consequence of Theorem 1.2. The self-adjoint operator H_θ is affiliated to $\mathscr{C}(X)$ because of Proposition 4.3. It remains to compute the localizations $\varkappa.H$ for $\varkappa \in \delta X$. The image $\pi\circ\theta(\varkappa)$ is an ultrafilter on the compact space X/Y, hence it converges to some unique point $\widehat{\varkappa} \in X/Y$. Let $z \in X$ such that $\widehat{\varkappa} = \pi(z)$. Since $v \equiv v\circ\pi$ we may define the translated function $\widehat{\varkappa}.v(s) = v(\widehat{\varkappa} + s) = v\circ\pi(z+x)$ for $s = \pi(x) \in X/Y$. We shall prove that $\varkappa.H_\theta = \widehat{\varkappa}.H$ where $\widehat{\varkappa}.H = h(P) + \widehat{\varkappa}.v(Q)$. Note that $\widehat{\varkappa}.H = U_z H U_z^*$, so $\sigma(\widehat{\varkappa}.H) = \sigma(H)$, which finishes the proof.

Observe that $D(H) = D(h(P))$ is stable under translations, so it suffices to prove the much stronger fact: $\operatorname{s-lim}_{x\to\varkappa} x.H_\theta f = \widehat{\varkappa}.Hf$ if $f \in D(H)$. But this follows from $\operatorname{s-lim}_{x\to\varkappa} x.v\circ\theta(Q) = \widehat{\varkappa}.v(Q)$. This means that for each $f \in L^2(X)$ we have

$$\lim_{x\to\varkappa} \int_X |v\circ\pi(\theta(x+y)) - v\circ\pi(z+y)|^2 |f(y)|^2 \,\mathrm{d}y = 0 \tag{6.36}$$

where z is as above. Now for large x we have $\pi(\theta(x+y)) \sim \pi(\theta(x)) + \pi(y)$ and $\lim_{x\to\varkappa} \pi(\theta(x)) = \widehat{\varkappa} = \pi(z)$ and since v is uniformly continuous we have

$$\lim_{x\to\varkappa} \sup_{y\in K} |v\circ\pi(\theta(x+y)) - v\circ\pi(z+y)| = 0$$

for each compact $K \subset X$. This clearly implies (6.36). ■

The extension of Proposition 6.37 to bounded measurable functions v seems to require further conditions. Indeed, one could say that it suffices to use the dominated convergence theorem in (6.36). But this requires some care because $\varkappa$ is a filter, we did not assume X separable, and the dominated convergence theorem is not true if sequences are replaced by nets. We indicate below two situations where these problems can be avoided.

Proposition 6.38 *If X/Y is separable or if θ is such that $\theta^{-1}(N)$ is of measure zero whenever $N \subset X$ is of measure zero, then Proposition 6.37 remains valid if v is a bounded measurable function.*

Proof: If X/Y is separable then the point $\widehat{\varkappa}$ has a countable fundamental system of neighbourhoods $\{G_n\}$. For each n choose $F_n \in \varkappa$ such that $\pi(\theta(F_n)) \subset G_n$ and then choose points $x_n \in X$ such that $x_n \in F_n$. Clearly we shall have $\pi(\theta(x_n)) \to \widehat{\varkappa}$ and $\text{s-lim}_{n\to\infty} x_n.H_\theta = \varkappa.H_\theta$ if the left hand side exists. Now the rest of the proof of Proposition 6.37 works after replacing x by x_n and $x \to \varkappa$ by $n \to \infty$, this time we can use the dominated convergence theorem directly in (6.36).

If θ has the property $|N| = 0 \Rightarrow |\theta^{-1}(N)| = 0$, we argue as follows. Since v is bounded, it is sufficient to prove (6.36) for f the characteristic function of a compact set. Then we approximate v in $L^2(X/Y)$ by functions $w \in \mathcal{C}(X/Y)$, for such w the relation (6.36) being obvious. The only problem which appears is to estimate the term

$$\int_K |v \circ \eta(x+y) - w \circ \eta(x+y)|^2 \, \mathrm{d}y$$

where $K \subset X$ is a compact and $\eta = \pi \circ \theta$. The map $\eta : X \to X/Y$ is continuous and has the property $|N| = 0 \Rightarrow |\eta^{-1}(N)| = 0$, by hypothesis and [Fol, Theorem 2.9]. But this implies that there is an integrable function $g \geq 0$ on X/Y such that the preceding integral be $\leq \int_{X/Y} |v - w|^2 g \, \mathrm{d}s$, and this can be made as small as we wish. ■

In the case $X = \mathbb{R}^n$ and under stronger assumptions on θ we may extend Proposition 6.37 to unbounded functions v, in particular we may recover Theorem 5.1 of the revised version of [LaS]. In order to be specific, *we assume that h is as in Subsection 6.6 and that $s \leq 1$.* In particular our assumptions below imply those of Proposition 6.33. Then we easily obtain:

Proposition 6.39 *Let Y be the additive subgroup of $X = \mathbb{R}^n$ generated by n linearly independent vectors. Let $\theta : X \to X$ be a homeomorphism such that θ and θ^{-1} are Lipschitz and such that $\theta(a + x) \sim a + \theta(x)$. Assume that $v : X \to B(\mathbf{E})$ is a locally integrable Y-periodic function and that $V = v(Q)$ has the property: for each $\mu > 0$ there is ν such that $|V| \leq \mu\langle P\rangle^{2s} + \nu$. Then the operator $V_\theta = v \circ \theta(Q)$ has the same property and if we set $H = h(P) + V$ and $H_\theta = h(P) + V_\theta$ then $\sigma_{\text{ess}}(H_\theta) = \sigma(H)$.*

We shall now consider the questions treated above in this subsection from the point of view of Theorem 1.15. As in the proof of Proposition 6.3, we get from (6.21) and from Theorem 3.7:

$$\mathcal{C}(X/Y) \rtimes X = \{T \in \mathscr{C}(X) \mid y.T = T \quad \forall y \in Y\}.$$

Clearly $\mathcal{C}_0(X) \cap \mathcal{C}(X/Y) = \{0\}$, from which it follows easily that $\mathcal{A} = \mathcal{C}_0(X) + \mathcal{C}(X/Y)$ is an algebra of interactions and that we are in the conditions of Proposition 3.16, hence we have a topological direct sum

$$\begin{aligned} \mathscr{A} &\equiv \mathcal{A} \rtimes X = \mathscr{K}(X) + \mathcal{C}(X/Y) \rtimes X \\ &= \{T \in \mathscr{C}(X) \mid y.T - T \in \mathscr{K}(X) \quad \forall y \in Y\}. \end{aligned} \tag{6.37}$$

This is a rather trivial algebra but things become less trivial when we look at the image of $\mathscr{A}$ under an automorphism of $\mathscr{C}(X)$.

If $\theta : X \to X$ is a uniformly continuous homeomorphism then $\theta^* : \mathcal{C}(X) \to \mathcal{C}(X)$ is the injective morphism defined by $\theta^*\varphi = \varphi \circ \theta$. Clearly $\theta^*\mathcal{C}_0(X) = \mathcal{C}_0(X)$ but in nontrivial situations the image through θ^* of an X-algebra is not an X-algebra. However, we are interested only in algebras of interactions (which contain $\mathcal{C}_0(X)$), and the property of θ isolated in Definition 6.36 will be sufficient. The next lemma and its corollary are obvious.

Lemma 6.40 *If $\theta : X \to X$ is uniformly continuous and $\theta(a + x) \sim a + \theta(x)$, then for each $a \in X$ the map $\tau_a\theta^* - \theta^*\tau_a$ sends $\mathcal{C}(X)$ into $\mathcal{C}_0(X)$.*

Corollary 6.41 *Let $\theta : X \to X$ be a uniformly continuous homeomorphism such that $\theta(a + x) \sim a + \theta(x)$. Then, if $\mathcal{A}$ is an algebra of interactions, $\mathcal{A}^\theta := \theta^*\mathcal{A}$ is also an algebra of interactions. Moreover, θ^* leaves $\mathcal{C}_0(X)$ invariant and so it induces a canonical isomorphism of X-algebras $\mathcal{A}/\mathcal{C}_0(X) \cong \mathcal{A}^\theta/\mathcal{C}_0(X)$.*

We apply this to the situation (6.37). Since X/Y is compact, we have

$$\delta(\mathcal{A}) = \sigma(\mathcal{A}/\mathcal{C}_0(X)) = \sigma(\mathcal{C}(X/Y)) = X/Y$$

and thus we get $\delta(\mathcal{A}^\theta) \cong X/Y$. Since X acts transitively on X/Y we see that, modulo a unitary equivalence, we have only one localization at infinity for an observable affiliated to $\mathscr{A}_\theta := \mathcal{A}^\theta \rtimes X$. This assertion can be made more precise as follows.

Proposition 6.42 *If $\theta : X \to X$ is a uniformly continuous homeomorphism such that $\theta(a + x) \sim a + \theta(x)$, then there is a unique morphism $\mathcal{P} : \mathscr{A}_\theta \to \mathcal{C}(X/Y) \rtimes X$ such that*

$$\mathcal{P}(\varphi \circ \theta(Q)\psi(P)) = \begin{cases} 0 \quad \text{if} \quad \varphi \in \mathcal{C}_0(X), \\ \varphi(Q)\psi(P) \quad \text{if} \quad \varphi \in \mathcal{C}(X/Y). \end{cases} \tag{6.38}$$

This morphism is surjective and has $\mathscr{K}(X)$ as kernel. If $\varkappa$ is a filter on X finer than the Fréchet filter and such that $\lim_\varkappa \pi \circ \theta = 0$, then $\mathcal{P}(T) = \lim_{x \to \varkappa} U_x T U_x^*$, where the limit exists in $\mathscr{C}_s(X)$.

Proof: The uniqueness of $\mathcal{P}$ follows from the fact that the operators $\varphi \circ \theta(Q)\psi(P)$ with $\varphi \in \mathcal{A}$ generate $\mathscr{A}_\theta$, and the surjectivity holds for a similar reason. A filter $\varkappa$ as in the statement of the proposition exists because $\pi \circ \theta : X \to X/Y$ is surjective. If $T = \varphi \circ \theta(Q)\psi(P)$ then $U_x T U_x^* = \varphi \circ \theta(x + Q)\psi(P)$ and $\varphi \circ \theta(x + Q)\psi(P)\xi(Q)$

converges strongly to zero or to $\varphi(Q)\psi(P)\xi(Q)$ as $x \to \varkappa$ if $\varphi \in \mathcal{C}_0(X)$ or $\varphi \in \mathcal{C}(X/Y)$ respectively. Here $\xi \in \mathcal{C}_0(X)$ and Remark 3.13 is used. Thus we can define $\mathcal{P}(T)$ by the last assertion of the proposition. If $\mathcal{P}(T) = 0$ then the beginning of the proof of Proposition 6.37 shows that $\tau_\varkappa T = 0$ for all $\varkappa \in \delta X$, so T is compact. ■

Remark 6.43 Thus, if H is an element of $\mathscr{A}_\theta$ or an observable affiliated to $\mathscr{A}_\theta$, then

$$\sigma_{\text{ess}}(H) = \sigma(\mathcal{P}(H)). \tag{6.39}$$

Note that [DG3, Theorem 2.8] gives a large class of operators affiliated to $\mathscr{A}_\theta$ which allows one to recover Propositions 6.37–6.39.

> Je n'ajoute pas la raison d'où j'infère que, s'il y avait un triangle rectangle de cette nature, il y en aurait un autre de même nature moindre que le premier, parce que le discours en serait trop long et que c'est là tout le mystère de ma méthode. Je serai bien aise que les Pascal et les Roberval et tant d'autres savants la cherchent sur mon indication.
>
> Pierre de Fermat (lettre à Carcavi, août 1659)

7 Appendix

7.1. We give here a detailed proof of Theorem 3.7. We follow rather closely Landstad's arguments, but we use the characterization of $\mathcal{A} \rtimes X$ taken as Definition 3.1, which makes the proof more transparent. We mention that the space $\mathscr{B}_2^\circ(X)$ is suggested by Kato's theory of smooth operators, cf. [RS]. We shall not discuss the uniqueness of $\mathcal{A}$ because the proof of [Lan, Lemma 3.1] can hardly be simplified (if X is discrete we have $\mathcal{A} = \mathscr{I}(\mathscr{A})$ so uniqueness is trivial, see Remark 7.8).

We begin with some heuristic comments which will make the rigorous proof quite natural. The first question is, given $\mathscr{A}$, how to determine $\mathcal{A}$. Observe that if we know $A\psi(P + k)$ for all k then we can recuperate the operator A by integrating over k, because this operation will give $A\langle\psi\rangle$ with $\langle\psi\rangle := \int_{X^*} \psi(k)\,\mathrm{d}k$. On the other hand, $\psi(P + k) = V_k^*\psi(P)V_k$ so that if A commutes with V_k then we get $A\langle\psi\rangle = \int_{X^*} V_k^* A\psi(P)V_k\,\mathrm{d}k$. Thus if $T = \sum_j \varphi_j(Q)\psi_j(P)$ then

$$\sum_j \varphi_j(Q)\langle\psi_j\rangle = \int_{X^*} V_k^* T V_k\,\mathrm{d}k =: \mathscr{I}(T)$$

If the group X is discrete, so that X^* is compact, this formal argument can easily be made rigorous, the map $\mathscr{I}$ is well defined on all $\mathscr{B}(X)$ and we have $\mathcal{A} = \mathscr{I}(\mathscr{A})$ (we strongly advise the reader to first prove Landstad's theorem for discrete X; this is a really pleasant exercise). In general, one can give a meaning to $\mathscr{I}(T)$ for a sufficiently large class of T for the rest of the proof to work. Anyway, the preceding formula shows how to extract the part in $\mathcal{A}$ of the operator $T \in \mathscr{A}$. The second point is that one can reconstruct T from such quantities by using the formally obvious relation

$$T = \int_X \mathscr{I}(TU_x^*)U_x \, \mathrm{d}x$$

which is just the Fourier inversion formula, see (7.11). But the right hand side here is, again formally, in $\mathcal{A} \rtimes X$.

To make all this rigorous demands some preliminary constructions that we expose in Subsection 7.2 in a more general context. We set $\mathcal{H} = L^2(X)$ and we abbreviate $\mathscr{B} = \mathscr{B}(X) = B(\mathcal{H})$. We recall that we have unitary representations U_x and V_k of X and X^* in $\mathcal{H}$ which satisfy the canonical commutation relations

$$U_x V_k = k(x) V_k U_x. \tag{7.1}$$

Most of the next arguments do not depend on the explicit form of the operators U_x, V_k.

7.2. We first introduce the space of "smooth operators" with respect to the unitary representation V_k:

$$\mathscr{B}_2^\circ := \big\{T \in \mathscr{B} \mid \int_{X^*} \|TV_k f\|^2 \, \mathrm{d}k < \infty, \ \forall f \in \mathcal{H}\big\}. \tag{7.2}$$

Lemma 7.1 *$\mathscr{B}_2^\circ$ is a left ideal (not closed in general) in $\mathscr{B}$. If $T \in \mathscr{B}_2^\circ$ then*

$$|\!|\!|T|\!|\!| := \sup_{\|f\|=1} \left(\int_{X^*} \|TV_k f\|^2 \, \mathrm{d}k \right)^{1/2} < \infty \tag{7.3}$$

and $\big(\mathscr{B}_2^\circ, |\!|\!|\cdot|\!|\!|\big)$ is a Banach space such that $|\!|\!|ST|\!|\!| \le \|S\| \cdot |\!|\!|T|\!|\!|$ for all $S \in \mathscr{B}$. Finally, if $x \in X$ and $T \in \mathscr{B}_2^\circ$ then $TU_x \in \mathscr{B}_2^\circ$ and $|\!|\!|TU_x|\!|\!| = |\!|\!|T|\!|\!|$.

For the proof of (7.3) we have only to remark that the map which sends $f \in \mathcal{H}$ into $(TV_k f)_{k \in X^*} \in L^2(X^*; \mathcal{H})$ is clearly closed and linear, hence it is continuous. The last assertion of the lemma follows from (7.1).

The map $x \mapsto TU_x \in \mathscr{B}_2$ is not norm continuous in general. For this reason it will be convenient to consider the following left ideal in $\mathscr{B}$ and closed subspace of $\mathscr{B}_2^\circ$

$$\mathscr{B}_2 := \{T \in \mathscr{B}_2^\circ \mid \lim_{x \to 0} |\!|\!|TU_x - T|\!|\!| = 0\}. \tag{7.4}$$

The following property of $\mathscr{B}_2$ will be important in what follows: if $S^*S \le \sum T_j^* T_j$ for some $T_j \in \mathscr{B}_2$, then $S \in \mathscr{B}_2$.

Lemma 7.2 *If $\psi \in L^\infty(X^*)$ then $\psi \in \mathscr{B}_2^\circ$ if and only if $\psi \in L^2(X^*)$. In this case we have $\psi \in \mathscr{B}_2$ and $|\!|\!|\psi(P)|\!|\!| = \|\psi\|_{L^2}$.*

Proof: We have

$$\begin{aligned} \int_{X^*} \|\psi(P)V_k f\|^2 \, \mathrm{d}k &= \int_{X^*} \|V_k^* \psi(P) V_k f\|^2 \, \mathrm{d}k = \int_{X^*} \|\psi(P+k) f\|^2 \, \mathrm{d}k \\ &= \int_{X^*} \mathrm{d}k \int_{X^*} |\psi(p+k)|^2 |\widehat{f}(p)|^2 \, \mathrm{d}p \\ &= \|\psi\|_{L^2(X^*)} \|\widehat{f}\|_{L^2(X^*)} = \|\psi\|_{L^2(X^*)} \|f\|_{L^2(X)}. \end{aligned}$$

Then $\|\psi(P)U_x - \psi(P)\| = \|x\psi - \psi\|_{L^2}$ where x is identified with the map $k \mapsto k(x)$ on X^*, we clearly get $\psi \in \mathscr{B}_2$. ■

Definition 7.3 *$\mathscr{B}_1$ is the linear subspace of $\mathscr{B}$ generated by the operators of the form S^*T with $S, T \in \mathscr{B}_2$.*

The polarization identity

$$4S^*T = \sum_{m=0}^{3} i^m (i^m S + T)^* (i^m S + T) \tag{7.5}$$

shows that $\mathscr{B}_1$ is linearly generated by the operators of the form S^*S with $S \in \mathscr{B}_2$.

We recall that a subset $\mathscr{C} \subset \mathscr{B}$ is called hereditary if: $0 \le S \le T \in \mathscr{C} \Rightarrow S \in \mathscr{C}$.

Lemma 7.4 *$\mathscr{B}_1$ is an hereditary $*$-subalgebra of $\mathscr{B}$. If $S \ge 0$ then $S \in \mathscr{B}_1$ if and only if $\sqrt{S} \in \mathscr{B}_2$. If $S \in \mathscr{B}_1$ then $U_x S U_y \in \mathscr{B}_1$ for all $x, y \in X$.*

Proof: The fact that $\mathscr{B}_1$ is a linear space and that $S^* \in \mathscr{B}_1$ if $S \in \mathscr{B}_1$ is obvious. $\mathscr{B}_1$ is stable under multiplication because for $S, T \in \mathscr{B}_2$ we have $S^*ST^*T = S^* \cdot ST^*T \in \mathscr{B}_1$ the space $\mathscr{B}_2$ being a left ideal.

We prove now that if $S \ge 0$ and $S \in \mathscr{B}_1$ then $\sqrt{S} \in \mathscr{B}_2$ (the reverse implication being obvious). Since $S \in \mathscr{B}_1$ we have $S = \sum_{j=1}^n \lambda_j S_j^* S_j$ with $\lambda_j \in \mathbb{C}$ and $S_j \in \mathscr{B}_2$. If $S = S^*$ then by taking the real parts we may assume that $\lambda_j \in \mathbb{R}$. Then

$$S = \Big(\sum_{\lambda_j > 0} + \sum_{\lambda_j < 0} \Big) \lambda_j S_j^* S_j \le \sum_{\lambda_j > 0} \lambda_j S_j^* S_j,$$

which implies $\sqrt{S} \in \mathscr{B}_2$ by the property mentioned after (7.4).

Finally, if $0 \le S \le T \in \mathscr{B}_1$ then $\sqrt{T} \in \mathscr{B}_2$, so $S \in \mathscr{B}_1$ by the same property. ■

Let $T \in \mathscr{B}_1$ and let us write $T = \sum_{j=1}^n S_j^* T_j$ with $S_j, T_j \in \mathscr{B}_2$. Then if $f, g \in \mathcal{H}$:

$$\begin{aligned} \int_{X^*} |\langle V_k f, T V_k g \rangle| \, \mathrm{d}k &\le \sum_{j=1}^n \Big(\int_{X^*} \|S_j V_k f\|^2 \, \mathrm{d}k \Big)^{1/2} \Big(\int_{X^*} \|T_j V_k g\|^2 \, \mathrm{d}k \Big)^{1/2} \\ &\le \sum_{j=1}^n \|S_j\| \|T_j\| \|f\| \, \|g\| < \infty. \end{aligned}$$

From the operator version of the Riesz lemma it follows that there is a unique operator $\mathscr{I}(T) \in \mathscr{B}(X)$ such that

$$\langle f, \mathscr{I}(T) g \rangle = \int_{X^*} \langle V_k f, T V_k g \rangle \, \mathrm{d}k \ \text{ for all } f, g \in \mathcal{H}.$$

In other terms, we see that the strongly continuous map $k \mapsto V_k^* T V_k$ is such that the integral

$$\mathscr{I}(T) = \int_{X^*} V_k^* T V_k \, \mathrm{d}k \tag{7.6}$$

exists in the weak operator topology of $\mathscr{B}(X)$. It is clear that for all $T \in \mathscr{B}_2$ we have

$$\|\mathscr{I}(T^*T)\|^{1/2} = |\!|\!|T|\!|\!|. \tag{7.7}$$

Moreover, the computation done above gives for $S, T \in \mathscr{B}_2$:

$$\|\mathscr{I}(S^*T)\| \leq |\!|\!|S|\!|\!| \, |\!|\!|T|\!|\!|. \tag{7.8}$$

Example 7.5 If $S \in \mathscr{B}(X)$ and $\xi, \eta \in L^\infty(X^*) \cap L^2(X^*)$ then $\xi(P)S\eta(P) \in \mathscr{B}_1$ and

$$\| \mathscr{I}\big(\xi(P)S\eta(P)\big) \| \leq \|S\| \, \|\xi\|_{L^2(X^*)} \|\eta\|_{L^2(X^*)}. \tag{7.9}$$

Indeed, we write $\xi(P)S\eta(P) = \big(S^*\overline{\xi}(P)\big)^*\eta(P)$ and use (7.8) and Lemma 7.2.

Lemma 7.6 *If $T \in \mathscr{B}_1$ then $(x, y) \mapsto \mathscr{I}(U_xTU_y)$ is bounded and norm continuous.*

Proof: Due to (7.8) it suffices to assume that $T = S^*S$ for some $S \in \mathscr{B}_2$ and to show continuity at $x = y = 0$ of the map $(x, y) \mapsto \mathscr{I}(S_x^*S_y)$ with $S_x = SU_x$. Then

$$\begin{aligned} \|\mathscr{I}(S_x^*S_y) - \mathscr{I}(S^*S)\| &\leq \|\mathscr{I}\big((S_x - S)^*S_y\big)\| + \|\mathscr{I}\big(S^*(S_y - S)\big)\| \\ &\leq |\!|\!|S_x - S|\!|\!| |\!|\!|S|\!|\!| + |\!|\!|S|\!|\!| |\!|\!|S_y - S|\!|\!| \end{aligned}$$

because of the estimate (7.8). ■

Proposition 7.7 *If $T \in \mathscr{B}_1$ then $\mathscr{I}(T) \in \mathcal{C}(X)$ and $U_x^*\mathscr{I}(T)U_x = \mathscr{I}(U_x^*TU_x)$ for all $x \in X$. The map $\mathscr{I} : \mathscr{B}_1 \to \mathcal{C}(X)$ is linear and positive.*

Proof: We clearly have $V_k\mathscr{I}(T)V_k^* = \mathscr{I}(T)$ for all $k \in X^*$. Since the Von Neumann algebra generated by $\{V_k\}_{k\in X^*}$ is just $L^\infty(X)$, we get $\varphi(Q)\mathscr{I}(T) = \mathscr{I}(T)\varphi(Q)$ for all $\varphi \in L^\infty(X)$. But $L^\infty(X)$ is maximal abelian in $\mathscr{B}$, thus $\mathscr{I}(T) \in L^\infty(X)$. From (7.1) we get $U_x^*V_k^*TV_kU_x = V_k^*U_x^*TU_xV_k$ hence $\mathscr{I}(U_x^*TU_x) = U_x^*\mathscr{I}(T)U_x$. Since $\varphi \in \mathcal{C}(X)$ if and only if $\varphi \in L^\infty(X)$ and $x \mapsto U_x^*\varphi(Q)U_x$ is norm continuous, we get $\mathscr{I}(T) \in \mathcal{C}(X)$. The last assertion of the proposition is obvious. ■

Remark 7.8 In a similar way we can associate an hereditary $*$-subalgebra $\mathscr{B}_1^\circ$ to $\mathscr{B}_2^\circ$ and define an extension of $\mathscr{I}$ to it, but then we only have $\mathscr{I} : \mathscr{B}_1^\circ \to L^\infty(X)$. If X is a discrete group, then $\mathscr{B}_1 = \mathscr{B}$ and $\mathscr{I}$ is a conditional expectation.

For $T \in \mathscr{B}_1$ and for $x \in X$ we set $\widetilde{T}(x) := \mathscr{I}(TU_x^*)$, so that we associate to T a function $\widetilde{T} : X \to \mathcal{C}(X)$ which, by Lemma 7.6, is bounded and norm continuous. Let $\widehat{T}$ be the Fourier transform of the function $k \mapsto V_k^*TV_k$, more precisely

$$\widehat{T}(x) = \int_{X^*} \overline{k(x)} V_k^*TV_k \, \mathrm{d}k$$

where the integral exists in the weak operator topology. From (7.1) we get

$$\widetilde{T}(x) = \widehat{T}(x)U_x^*. \tag{7.10}$$

so that $\widetilde{T}$ is a kind of twisted Fourier transform. Now the inversion formula for the Fourier transform gives us a *formal relation*

$$T = \int_X \widetilde{T}(x) U_x \, \mathrm{d}x \tag{7.11}$$

whose rigorous meaning is given below.

Lemma 7.9 *For each $T \in \mathscr{B}_1$ and $\theta \in L^1(X)$ we have*

$$\int_X \widetilde{T}(x) U_x \theta(x) \, \mathrm{d}x = \int_{X^*} V_k^* T V_k \widehat{\theta}(k) \, \mathrm{d}k \tag{7.12}$$

where both integrals exist in the weak operator topology.

Proof: For each $f \in \mathcal{H}$,

$$\begin{aligned} \int_X \langle f, \widetilde{T}(x) U_x f \rangle \theta(x) \, \mathrm{d}x &= \int_X \Big(\int_{X^*} \langle V_k f, T \overline{k(x)} V_k f \rangle \, \mathrm{d}k \Big) \theta(x) \, \mathrm{d}x \\ &= \int_X \Big(\int_{X^*} \overline{k(x)} \langle V_k f, T V_k f \rangle \, \mathrm{d}k \Big) \theta(x) \, \mathrm{d}x. \end{aligned}$$

Since $\theta \in L^1(X)$ and the function $k \mapsto \langle V_k f, T V_k f \rangle$ is in $L^1(X^*)$, we can apply the Fubini theorem and get thus get (7.12). ■

Let us remark that the l.h.s. of the identity (7.12) always exists in the strong operator topology, and the same is true for the r.h.s. if $\widehat{\theta} \in L^1(X^*)$. We recall the following result (see e.g. [Fol, Lemma 4.19]).

Lemma 7.10 *Let $\Lambda \subset X^*$ be a neighborhood of the neutral element in X^* and let $\varepsilon > 0$. Then there is $\theta \in \mathcal{C}_\mathrm{c}(X)$ such that $\widehat{\theta} \geq 0$, $\widehat{\theta} \in L^1(X^*)$, $\int_{X^*} \widehat{\theta}(k) \, \mathrm{d}k = 1$, and $\int_{X^* \setminus \Lambda} \widehat{\theta}(k) \, \mathrm{d}k \leq \varepsilon$.*

The next version of the Fourier inversion formula is an easy consequence of Lemmas 7.9 and 7.10:

Proposition 7.11 *If $T \in \mathscr{B}_1$ and $k \mapsto V_k^* T V_k$ is norm continuous, then T is a norm limit of operators of the form $\int_X \widetilde{T}(x) U_x \theta(x) \, \mathrm{d}x$ with $\theta \in \mathcal{C}_\mathrm{c} X$ and $\widehat{\theta} \in L^1(X^*)$*

Lemma 7.12 *Let $\psi \in \mathcal{C}_0(X^*)$ and $T \in \mathscr{B}_1$. Then if $\theta \in L^1(X)$ and $\widehat{\theta} \in L^1(X^*)$ the integral $\int_X \widetilde{T}(x) U_x \psi(P) \theta(x) \, \mathrm{d}x$ exists in the norm operator topology and $\mathscr{I}(T)\psi(P)$ is a norm limit of such integrals.*

Proof: The map $x \mapsto U_x \psi(P)$ is norm continuous if $\psi \in \mathcal{C}_0(X^*)$, hence the integrand is norm continuous. The last assertion follows by choosing θ as in Lemma 7.10 but with the rôles of X and X^* inverted. ■

7.3. We are now ready to prove Landstad's theorem (Theorem 3.7). From now on, $\mathscr{A}$ and $\mathcal{A}$ are as in that theorem.

Lemma 7.13 *$\mathscr{A}$ is a non-degenerate $\mathcal{C}_0(X^*)$-bimodule. More precisely, if $A \in \mathscr{A}$ and $\psi \in \mathcal{C}_0(X^*)$ then $A\psi(P) \in \mathscr{A}$, $\psi(P)A \in \mathscr{A}$ and A is a limit of operators of the form $A\psi(P)$ and of operators of the form $\psi(P)A$.*

Proof: It is clearly sufficient to consider only the right action and, since each $\psi \in \mathcal{C}_0(X^*)$ is limit in the sup norm of functions whose Fourier transform is integrable, we may assume $\widehat{\psi} \in L^1(X)$. Then $A\psi(P) = \int_X AU_x\widehat{\psi}(x)\,\mathrm{d}x$, we have $AU_x \in \mathscr{A}$ and the integral converges in norm by the second assumption of Theorem 3.7, so $A\psi(P) \in \mathscr{A}$. By taking $\widehat{\psi} = |K|^{-1}\mathbf{1}_K$, where K runs over the set of compact neighbourhoods of the origin in X, and by taking into account the norm continuity of the map $x \mapsto AU_x$, we see that A is a norm limit of operators of the form $A\psi(P)$. ■

Lemma 7.14 *$\mathcal{A}$ is an X-subalgebra of $\mathcal{C}(X)$.*

Proof: It is clear that $\mathcal{A}$ is a norm closed subspace of $\mathcal{C}(X)$ stable under conjugation and stable under translations (note that $(\tau_x\varphi)(Q) = U_x\varphi(Q)U_x^*$). To show that it is stable under multiplication, let $\varphi_1, \varphi_2 \in \mathcal{A}$ and $\psi \in \mathcal{C}_0(X^*)$. Since $\varphi_2(Q)\psi(P) \in \mathscr{A}$ we can write it as a norm limit of operators of the form $\xi(P)A$ with $\xi \in \mathcal{C}_0(X^*)$ and $A \in \mathscr{A}$, so that $\varphi_1(Q)\varphi_2(Q)\psi(P)$ is a norm limit of operators of the form $\varphi_1(Q)\xi(P)A$ which belong to $\mathscr{A}$. ■

Now we may consider the crossed product $\mathcal{A} \rtimes X$, this is the norm closed subspace of $\mathscr{B}(X)$ generated by the operators $\varphi(Q)\psi(P)$ with $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{C}_0(X^*)$. We clearly have $\mathcal{A} \rtimes X \subset \mathscr{A}$ and it remains to prove the reverse inclusion.

Lemma 7.15 *If $T \in \mathscr{A} \cap \mathscr{B}_1$ then $\mathscr{I}(T) \in \mathcal{A}$*

Proof: Due to Proposition 7.7 it suffices to show that $\mathscr{I}(T)\psi(P) \in \mathscr{A}$ if $\psi \in \mathcal{C}_0(X^*)$. Because of Lemma 7.12, it is enough to prove that $\int_X \widetilde{T}(x)U_x\psi(P)\theta(x)\,\mathrm{d}x \in \mathscr{A}$ if $\theta \in L^1(X)$ and $\widehat{\theta} \in L^1(X^*)$. But (7.12) implies:

$$\int_X \widetilde{T}(x)U_x\psi(P)\theta(x)\,\mathrm{d}x = \int_X V_k^*TV_k\psi(P)\widehat{\theta}(k)\,\mathrm{d}k.$$

Since $V_k^*TV_k\psi(P) \in \mathscr{A}$ and is a norm continuous function of k the last integral belongs to $\mathscr{A}$. ■

Lemma 7.16 *If $T \in \mathscr{A} \cap \mathscr{B}_1$ then $T\psi(P) \in \mathcal{A} \rtimes X$ for all $\psi \in \mathcal{C}_0(X^*)$.*

Proof: We shall have $TU_x^* \in \mathscr{A} \cap \mathscr{B}_1$, hence $\widetilde{T}(x) = \mathscr{I}(TU_x^*) \in \mathcal{A}$, and thus the map $\widetilde{T} : X \to \mathcal{A}$ is bounded and norm continuous. On the other hand, Proposition 7.11 shows that for each $\psi \in \mathcal{C}_0(X^*)$ the operator $T\psi(P)$ is a norm limit of integrals $\int_X \widetilde{T}(x)U_x\psi(P)\theta(x)\,\mathrm{d}x$. But $U_x\psi(P) \in \mathcal{C}_0(X^*)$ and the map $x \mapsto U_x\psi(P)$ is norm continuous, thus the preceding integral converges in norm. Also, we have $\widetilde{T}(x)U_x\psi(P) \in \mathcal{A} \rtimes X$ for each x, thus the integral belongs to $\mathcal{A} \rtimes X$. ■

Now we prove $\mathscr{A} \subset \mathcal{A} \rtimes X$. For this it suffices to find a dense subset of $\mathscr{A}$ which is included in $\mathcal{A} \rtimes X$. The Example 7.5 and Lemma 7.13 imply that $\mathscr{A} \cap \mathscr{B}_1$ is a dense subspace of $\mathscr{A}$. Thus *it suffices to show that $\mathscr{A} \cap \mathscr{B}_1 \subset \mathcal{A} \rtimes X$*. But this follows from Lemma 7.16 because each $T \in \mathscr{A}$ is a norm limit of operators of the form $T\psi(P)$ with $\psi \in \mathcal{C}_0(X^*)$.

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