Decay Preserving Operators and Stability of the Essential Spectrum

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Abstract

We establish some criteria for the stability of the essential spectrum for unbounded operators acting in Banach modules. The applications cover operators acting on sections of vector fiber bundles over non-smooth manifolds or locally compact abelian groups, in particular differential operators of any order with complex measurable coefficients on $\mathbb{R}^n$, singular Dirac operators and Laplace-Beltrami operators on Riemannian manifolds with measurable metrics.

1 Introduction

The main purpose of this paper is to establish criteria which ensure that the difference of the resolvents of two operators is compact. In order to simplify later statements, we use the following definition (our notations are quite standard; we recall however the most important ones at the end of this section).

Definition 1.1 *Let A and B be two closed operators acting in a Banach space $\mathscr{H}$. We say that B* is a compact perturbation of *A if there is $z \in \rho(A) \cap \rho(B)$ such that $(A - z)^{-1} - (B - z)^{-1}$ is a compact operator.*

Under the conditions of this definition the difference $(A - z)^{-1} - (B - z)^{-1}$ is a compact operator for all $z \in \rho(A) \cap \rho(B)$. In particular, *if B is a compact perturbation of A, then A and B have the same essential spectrum*, and this for any reasonable definition of the essential spectrum, see [GW]. To be precise, in this paper we define the essential spectrum of A as the set of points $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not Fredholm.

We shall describe now a standard and simple, although quite powerful, method of proving that B is a compact perturbation of A. Note that we are interested in situations where A and B are differential (or pseudo-differential) operators with complex measurable coefficients which differ little on a neighborhood of infinity. An important point in such situations is that one has not much information about the domains of the operators. However, one often knows explicitly a generalized version of the "quadratic form domain" of the operator. Since we want to consider operators of any order (in particular Dirac operators) we shall work in the following framework, which goes beyond the theory of accretive forms.

Let $\mathscr{G}, \mathscr{H}, \mathscr{K}$ be reflexive Banach spaces such that $\mathscr{G} \subset \mathscr{H} \subset \mathscr{K}$ continuously and densely. We are interested in operators in $\mathscr{H}$ constructed according to the following procedure: let A_0, B_0 be continuous bijective maps $\mathscr{G} \to \mathscr{K}$ and let A, B be their restrictions to $A_0^{-1}\mathscr{H}$ and $B_0^{-1}\mathscr{H}$. These are closed densely defined operators in $\mathscr{H}$ and $z = 0 \in \rho(A) \cap \rho(B)$. Then in $\mathcal{B}(\mathscr{K}, \mathscr{G})$ we have

$$A_0^{-1} - B_0^{-1} = A_0^{-1}(B_0 - A_0)B_0^{-1}. \tag{1.1}$$

In particular, we get in $\mathcal{B}(\mathscr{H})$

$$A^{-1} - B^{-1} = A_0^{-1}(B_0 - A_0)B^{-1}. \tag{1.2}$$

We get the simplest compactness criterion: if $A_0 - B_0 : \mathscr{G} \to \mathscr{K}$ is compact, then B is a compact perturbation of A. But in this case we have more: the operator $A_0^{-1} - B_0^{-1} : \mathscr{K} \to \mathscr{G}$ is also compact, and this can not happen if A_0, B_0 are differential operators with distinct principal part (cf. below). This also excludes singular lower order perturbations, e.g. Coulomb potentials in the Dirac case.

The advantage of the preceding criterion is that no knowledge of the domains $\mathcal{D}(A), \mathcal{D}(B)$ is needed. To avoid the mentioned disadvantages, one may assume that one of the operators is more regular than the second one, so that the functions in its domain are, at least locally, slightly better than those from $\mathscr{G}$. Note that $\mathcal{D}(B)$ when equipped with the graph topology is such that $\mathcal{D}(B) \subset \mathscr{G}$ continuously and densely and we get a second compactness criterion by asking that $A_0 - B_0 : \mathcal{D}(B) \to \mathscr{K}$ be compact. This time again we get more than needed, because not only B is a compact perturbation of A, but also $A_0^{-1} - B_0^{-1} : \mathscr{H} \to \mathscr{G}$ is compact. However, perturbations of the principal part of a differential operator are allowed and also much more singular perturbations of the lower order terms, cf. [N1] for the Dirac case.

In this paper we are interested in situations where we have really no information concerning the domains of A and B (besides the fact that they are subspaces of $\mathscr{G}$).

The case when A, B are second order elliptic operators with measurable complex coefficients acting in $\mathscr{H} = L^2(\mathbb{R}^n)$ has been studied by Ouhabaz and Stollmann in [OS] and, as far as we know, this is the only paper where the "unperturbed" operator is not smooth. Their approach consists in proving that the difference $A^{-k} - B^{-k}$ is compact for some $k \geq 2$ (which implies the compactness of $A^{-1} - B^{-1}$). In order to prove this, they take advantage of the fact that $\mathcal{D}(A^k)$ is a subset of the Sobolev space $W^{1,p}$ for some $p > 2$, which means that we have a certain gain of local regularity. Of course, L^p techniques from the theory of partial differential equations are required for their methods to work.

We shall explain now in the most elementary situation the main ideas of our approach to these questions. Let $\mathscr{H} = L^2(\mathbb{R})$ and $P = -i\frac{d}{dx}$. We consider operators of the form $A_0 = PaP + V$ and $B_0 = PbP + W$ where a, b are bounded operators on $\mathscr{H}$ such that Re a and Re b are bounded below by strictly positive numbers. V and W are assumed to be continuous operators $\mathscr{H}^1 \to \mathscr{H}^{-1}$, where $\mathscr{H}^s$ are Sobolev spaces associated to $\mathscr{H}$. Then $A_0, B_0 \in \mathcal{B}(\mathscr{H}^1, \mathscr{H}^{-1})$ and we put some conditions on V, W which ensure that A_0, B_0 are invertible (e.g. we could include the constant z in them). Thus we are in the preceding abstract framework with $\mathscr{G} = \mathscr{H}^1$ and $\mathscr{K} = \mathscr{H}^{-1} \equiv \mathscr{G}^*$. Then from (1.2) we get

$$A^{-1} - B^{-1} = A_0^{-1} P(b-a) P B^{-1} + A_0^{-1}(W - V) B^{-1}. \tag{1.3}$$

Let R be the first term on the right hand side and let us see how we could prove that it is a compact operator on $\mathscr{H}$. Note that the second term should be easier to treat since we expect V and W to be operators of order less than 2.

We have $R\mathscr{H} \subset \mathscr{H}^1$, so we can write $R = \psi(P)R_1$ for some $\psi \in B_0(\mathbb{R})$ (bounded Borel function which tends to zero at infinity) and $R_1 \in \mathcal{B}(\mathscr{H})$. This is just half of the conditions needed for compactness, in fact R will be compact if and only if one can also find $\varphi \in B_0(\mathbb{R})$ and $R_2 \in \mathcal{B}(\mathscr{H})$ such that $R = \varphi(Q)R_2$, where $\varphi(Q)$ is the operator of multiplication by φ. Of course, the only factor which can help to get such a decay is $b - a$. So let us suppose that we can write $b - a = \xi(Q)U$ for some $\xi \in B_0(\mathbb{R})$ and a bounded operator U on $\mathscr{H}$. We denote $S = A_0^{-1}P$ and note that this is a bounded operator on $\mathscr{H}$, because $P : \mathscr{H} \to \mathscr{H}^{-1}$ and $A_0^{-1} : \mathscr{H}^{-1} \to \mathscr{H}^1$ are bounded. Then $R = S\xi(Q)UPB^{-1}$ and $UPB^{-1} \in \mathcal{B}(\mathscr{H})$, hence R will be compact if the operator $S \in \mathcal{B}(\mathscr{H})$ has the following property: for each $\xi \in B_0(\mathbb{R})$ there are $\varphi \in B_0(\mathbb{R})$ and $T \in \mathcal{B}(\mathscr{H})$ such that $S\xi(Q) = \varphi(Q)T$.

An operator S with the property specified above will be called *decay preserving*. Thus we see that the compactness of R follows from the fact that S preserves decay and our main point is that it is easy to check this property under very general assumptions on A, cf. Corollary 2.22 and Proposition 2.32 for abstract criteria, Lemmas 3.6, 5.2 and 7.10 and Theorems 7.5 and 7.6 for more concrete examples. Note that the perturbative technique used in the proof of Lemma 7.10 shows that in many cases it suffices to prove the decay preserving property only for operators with smooth coefficients.

An abstract formulation of the ideas described above (see Proposition 2.16) allows one to treat situations of a very general nature, like pseudo-differential operators on finite dimensional vector spaces over a local[1] (for example p-adic) field, in particular differential operators of arbitrary order with irregular coefficients on $\mathbb{R}^n$, the Laplace operator on manifolds with locally L^∞ Riemannian metrics, and operators acting on sections of vector bundles over locally compact spaces. Sections 4-7 are devoted to such applications. We stress once again that, in the applications to differential operators, we are interested only in situations where the coefficients are not smooth and the lower order terms are singular.

Plan of the paper: In Section 2 we introduce an algebraic formalism which allows us to treat in a unified and simple way operators which have an algebraically complicated structure, e.g. operators acting on sections of vector fiber bundles over a locally compact space. The class of decay improving (or vanishing at infinity) operators is defined through an a priori given algebra of operators on a Banach space $\mathscr{H}$, that we call *multiplier algebra of* $\mathscr{H}$, and this allows us to define the notion of decay preserving operator in a natural and general context, that of Banach modules. Several examples of multiplier algebras are given Subsections 2.1, 2.4 and 7.1. We stress that Section 2 is only an accumulation of definitions and straightforward consequences.

We mention that this algebraic framework allows one to study differential operators in L^p or more general Banach spaces. Since these extensions are rather obvious and the examples are not particularly interesting, we shall not consider explicitly such situations.

Section 3 contains several abstract compactness criteria which formalize in the context of Banach modules the ideas involved in the example discussed above.

In Section 4 we give our first concrete applications of the abstract theory. In Subsection 4.1 we discuss operators in divergence form on $\mathbb{R}^n$, hence of order $2m$ with $m \geq 1$ integer, with coefficients of a rather general form (they do not have to be functions, for example). In the next subsection we consider “hypoelliptic” operators on abelian groups and treat as an example the Dirac operators on $\mathbb{R}^n$.

Perturbations of the Laplace operator on a Riemannian manifold with locally L^∞ metric are considered in Section 5. We introduce and study an abstract model of this situation which fits very naturally in our algebraic framework and covers the case of Lipschitz manifolds with measurable metrics. We consider in more detail

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[1] See [Sa, Ta] for the corresponding pseudo-differential calculus.

the case when the manifold is C^1 (but the metric is only locally L^∞) and establish stability of the essential spectrum under certain perturbations of the metric, see Theorems 5.4 and 5.5. We also consider, in an abstract setting and without going into technical details, the Laplace operator acting on differential forms.

In Section 7 we present several results concerning the case when the coefficients of the operator $A - B$ vanish at infinity only in some weak sense. This question has been studied before, for example in [He, LV, OS, We]. We present the notion of *weakly vanishing at infinity* functions in terms of filters finer than the Fréchet filter, a natural idea in our context being to extend the standard notion of neighborhood of infinity.

If X is a locally compact space, it is usual to define the filter of neighborhoods of infinity as the family of subsets of X with relatively compact complement; we shall call this the *Fréchet filter*. If $\mathcal{F}$ is a filter on X finer than the Fréchet filter then a function $\varphi : X \to \mathbb{C}$ such that $\lim_{\mathcal{F}} \varphi = 0$ can naturally be thought as convergent to zero at infinity in a generalized sense (recall that $\lim_{\mathcal{F}} \varphi = 0$ means that for each $\varepsilon > 0$ the set of points x such that $|\varphi(x)| < \varepsilon$ belongs to $\mathcal{F}$). In Subsection 7.1 we consider three such filters and describe corresponding classes of decay preserving operators in Theorem 7.1, Proposition 7.8 and Theorem 7.5.

We note that the proof of Theorem 7.5 requires nontrivial tools from the modern theory of Banach spaces. As a preparation, we prove in Section 6 a version of a factorization theorem due to B. Maurey which does not seem to be covered by the results existing in the literature. Theorem 7.11 is a last application of our formalism: we prove a compactness result for operators of order $2m$ in divergence form assuming that the difference between their coefficients vanishes at infinity in a weak sense. Such results were known before only in the case $m = 1$, see especially Theorem 2.1 in [OS]. We assume that the coefficients are bounded, thus their Theorem 3.1 is not covered unless we add an implicit assumption, as is done in [OS] (or in our Theorem 5.5). In fact, our main abstract compactness result Theorem 3.2 is stated such as to apply to situations when the coefficients of principal part of the operators are locally unbounded, as in [Ba1, Ba2], but we have not developed this idea here.

In an Appendix we collect some general facts concerning operators acting in scales of spaces which are often used without comment in the rest of the paper.

Notations: If $\mathscr{G}$ and $\mathscr{H}$ are Banach spaces then $\mathcal{B}(\mathscr{G}, \mathscr{H})$ is the space of bounded linear operators $\mathscr{G} \to \mathscr{H}$, the subspace of compact operators is denoted $\mathcal{K}(\mathscr{G}, \mathscr{H})$, and we set $\mathcal{B}(\mathscr{H}) = \mathcal{B}(\mathscr{H}, \mathscr{H})$ and $\mathcal{K}(\mathscr{H}) = \mathcal{K}(\mathscr{H}, \mathscr{H})$. The domain and the resolvent set of an operator S will be denoted by $\mathcal{D}(S)$ and $\rho(S)$ respectively. The norm of a Banach space $\mathscr{G}$ is denoted by $\| \cdot \|_{\mathscr{G}}$ and we omit the index if the space plays a central rôle. The adjoint space (space of antilinear continuous forms) of a

Banach space $\mathscr{G}$ is denoted $\mathscr{G}^*$ and if $u \in \mathscr{G}$ and $v \in \mathscr{G}^*$ then we set $v(u) = \langle u, v\rangle$. The embedding $\mathscr{G} \subset \mathscr{G}^{**}$ is realized by defining $\langle v, u\rangle = \overline{\langle u, v\rangle}$.

If $\mathscr{G}, \mathscr{H}, \mathscr{K}$ are Banach spaces such that $\mathscr{G} \subset \mathscr{H}$ continuously and densely and $\mathscr{H} \subset \mathscr{K}$ continuously then we we have a natural continuous embedding $\mathcal{B}(\mathscr{H}) \hookrightarrow \mathcal{B}(\mathscr{G}, \mathscr{K})$ that will be used without comment later on.

A *Friedrichs couple* $(\mathscr{G}, \mathscr{H})$ is a pair of Hilbert spaces $\mathscr{G}, \mathscr{H}$ together with a continuous dense embedding $\mathscr{G} \subset \mathscr{H}$. The *Gelfand triplet* associated to it is obtained by identifying $\mathscr{H} = \mathscr{H}^*$ with the help of the Riesz isomorphism and then taking the adjoint of the inclusion map $\mathscr{G} \to \mathscr{H}$. Thus we get $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ with continuous and dense embeddings. Now if $u \in \mathscr{G}$ and $v \in \mathscr{H} \subset \mathscr{G}^*$ then $\langle u, v\rangle$ is the scalar product in $\mathscr{H}$ of u and v and also the action of the functional v on u. As noted above, we have $\mathcal{B}(\mathscr{H}) \subset \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$.

If X is a locally compact topological space then $B(X)$ is the C^*-algebra of bounded Borel complex functions on X, with norm $\sup_{x\in X} |\varphi(x)|$, and $B_0(X)$ is the subalgebra consisting of functions which tend to zero at infinity. Then $C(X)$, $C_{\mathrm{b}}(X)$, $C_0(X)$ and $C_{\mathrm{c}}(X)$ are the spaces of complex functions on X which are continuous, continuous and bounded, continuous and convergent to zero at infinity, and continuous with compact support respectively. The characteristic function of a subset $S \subset X$ is denoted χ_S .

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2 Banach modules and decay preserving operators

2.1 Banach modules

We use the terminology of [FD] but with some abbreviations, e.g. a *morphism* is a linear multiplicative map between two algebras, and a $*$-*morphism* is a morphism between two $*$-algebras which commutes with the involutions. We recall that an *approximate unit* in a Banach algebra $\mathcal{M}$ is a net $\{J_\alpha\}$ in $\mathcal{M}$ such that $\|J_\alpha\| \le C$ for some constant C and all α and $\lim_\alpha \|J_\alpha M - M\| = \lim_\alpha \|M J_\alpha - M\| = 0$ for all $M \in \mathcal{M}$. An approximate unit exists if and only if there is a number C such that for each $\varepsilon > 0$ and for each finite set $\mathcal{F} \subset \mathcal{M}$ there is $J \in \mathcal{M}$ with $\|J\| \le C$ and $\|JM - M\| \le \varepsilon$, $\|MJ - M\| \le \varepsilon$ for all $M \in \mathcal{F}$. It is well known that any C^*-algebra has an approximate unit. If $\mathscr{H}$ is a Banach space, we shall say that a Banach subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathscr{H})$ is *non-degenerate* if the linear subspace of $\mathscr{H}$ generated by the elements Mu, with $M \in \mathcal{M}$ and $u \in \mathscr{H}$, is dense in $\mathscr{H}$.

In view of its importance in our paper, we state below the Cohen-Hewitt factorization theorem [FD, Ch. V–9.2].

Theorem 2.1 *Let* $\mathcal{C}$ *be a Banach algebra with an approximate unit, let* $\mathscr{E}$ *be a Banach space, and let* $Q : \mathcal{C} \to \mathcal{B}(\mathscr{E})$ *be a continuous morphism. Denote* $\mathscr{E}_0$ *the closed linear subspace of* $\mathscr{E}$ *generated by the elements of the form* $Q(\varphi)v$ *with* $\varphi \in \mathcal{C}$ *and* $v \in \mathscr{E}$*. Then for each* $u \in \mathscr{E}_0$ *there are* $\varphi \in \mathcal{C}$ *and* $v \in \mathscr{E}$ *such that* $u = Q(\varphi)v$*.*

Now we introduce the framework in which we shall work.

Definition 2.2 *A* Banach module *is a couple* $(\mathscr{H}, \mathcal{M})$ *consisting of a Banach space* $\mathscr{H}$ *and a non-degenerate Banach subalgebra* $\mathcal{M}$ *of* $\mathcal{B}(\mathscr{H})$ *which has an approximate unit. If* $\mathscr{H}$ *is a Hilbert space and* $\mathcal{M}$ *is a* C^**-algebra of operators on* $\mathscr{H}$*, we say that* $\mathscr{H}$ *is a* Hilbert module.

We shall adopt the usual *abus de language* and say that $\mathscr{H}$ is a Banach module (over $\mathcal{M}$). The distinguished subalgebra $\mathcal{M}$ will be called *multiplier algebra of* $\mathscr{H}$ and, when required by the clarity of the presentation, we shall denote it $\mathcal{M}(\mathscr{H})$. We are only interested in the case when $\mathcal{M}$ does not have a unit: the operators from $\mathcal{M}$ are the prototype of decay improving (or vanishing at infinity) operators, and the identity operator cannot have such a property. Note that it is implicit in Definition 2.2 that *if* $\mathscr{H}$ *is a Hilbert module then its adjoint space* $\mathscr{H}^*$ *is identified with* $\mathscr{H}$ with the help of the Riesz isomorphism.

If $\{J_\alpha\}$ is an approximate unit of $\mathcal{M}$, then the density in $\mathscr{H}$ of the linear subspace generated by the elements Mu is equivalent to

$$\lim_\alpha \|J_\alpha u - u\| = 0 \quad \text{for all} \quad u \in \mathscr{H}. \tag{2.4}$$

But much more is true:

$$u \in \mathscr{H} \Rightarrow u = Mv \text{ for some } M \in \mathcal{M} \text{ and } v \in \mathscr{H}. \tag{2.5}$$

This follows from the Cohen-Hewitt theorem, see Theorem 2.1. By using (2.4) we could avoid any reference to this result in our later arguments; this would make them more elementary but less simple. From Theorem 2.1 we also get:

Lemma 2.3 *Assume that* $\mathcal{A}$ *is a Banach algebra with approximate unit and that a morphism* $\Phi : \mathcal{A} \to \mathcal{M}(\mathscr{H})$ *with dense image is given. Then each* $u \in \mathscr{H}$ *can be written as* $u = Av$ *where* $A \in \Phi(\mathcal{A})$ *and* $v \in \mathscr{H}$*.*

Example 2.4 The simplest example of Banach module is the following. Let X be a locally compact non-compact topological space and let $\mathscr{H}$ be a Banach space. We say that $\mathscr{H}$ is a *Banach X-module* if a continuous morphism $Q : C_0(X) \to \mathcal{B}(\mathscr{H})$ has been given such that the linear subspace generated by the vectors of the form $Q(\varphi)u$, with $\varphi \in C_0(X)$ and $u \in \mathscr{H}$, is dense in $\mathscr{H}$. If $\mathscr{H}$ is a Hilbert space and Q is a $*$-morphism, we say that $\mathscr{H}$ is a *Hilbert X-module*. We shall use the notation $\varphi(Q) \equiv Q(\varphi)$. The Banach module structure on $\mathscr{H}$ is defined by the closure $\mathcal{M}$ in $\mathcal{B}(\mathscr{H})$ of the set of operators of the form $\varphi(Q)$ with $\varphi \in C_0(X)$. In the case of a Hilbert X-module the closure is not needed and we get a Hilbert module structure (because a $*$-morphism between two C^*-algebras is continuous and its range is a C^*-algebra). Banach X-modules appear naturally in differential geometry as spaces of sections of vector fiber bundles over a manifold X, and this is the point of interest for us.

Remark 2.5 In the case of a Banach X-module, Lemma 2.3 gives: each $u \in \mathscr{H}$ can be written as $u = \psi(Q)v$ with $\psi \in C_0(X)$ and $v \in \mathscr{H}$. In particular, we deduce that *the morphism Q has an extension, also denoted Q, to a unital continuous morphism of $C_\mathrm{b}(X)$ into $\mathcal{B}(\mathscr{H})$* which is uniquely determined by the following strong continuity property: if $\{\varphi_n\}$ is a bounded sequence in $C_\mathrm{b}(X)$ such that $\varphi_n \to \varphi$ locally uniformly, then $\varphi_n(Q) \to \varphi(Q)$ strongly on $\mathscr{H}$. Indeed, we can define $\varphi(Q)u = (\varphi\psi)(Q)v$ for each $\varphi \in C_\mathrm{b}(X)$; then if e_α is an approximate unit for $C_0(X)$ with $\|e_\alpha\| \leq 1$ we get $\varphi(Q)u = \lim(\varphi e_\alpha)(Q)u$ hence the definition is independent of the factorization of u and $\|\varphi(Q)\| \leq \|Q\| \sup|\varphi|$.

Remark 2.6 If $\mathscr{H}$ is a Hilbert X-module one can extend the morphism even further: *Q canonically extends to a $*$-morphism $\varphi \mapsto \varphi(Q)$ of $B(X)$ into $\mathcal{B}(\mathscr{H})$ such that*[2]*: if $\{\varphi_n\}$ is a bounded sequence in $B(X)$ and $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in X$, then* $\text{s-lim}_n\, \varphi_n(Q) = \varphi(Q)$. This follows from standard integration theory see [Be, Lo]. In particular, a separable Hilbert X-module is essentially a direct integral of Hilbert spaces over X, see [Di, II.6.2], but we shall not need this fact.

The class of X-modules is more general than it appears at first sight. Indeed, if $\mathcal{C}$ is an abelian C^*-algebra then one has a canonical identification $\mathcal{C} \equiv C_0(\mathcal{X})$ where $\mathcal{X}$ is the spectrum of $\mathcal{C}$. However, the space $\mathcal{X}$ is in general rather complicated so it is not really useful to take it into account. In particular, this happens in the following class of examples of interest in applications (see Section 7).

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[2] If X is second countable then this property determines uniquely the extension. In general, uniqueness is assured by the property: if $U \subset X$ is open then $\chi_U(Q) = \sup_\varphi \varphi(Q)$, where φ runs over the set of continuous functions with compact support such that $0 \leq \varphi \leq \chi_U$.

Example 2.7 Let X be a set and $\mathcal{F}$ a filter on X. Let us choose a C^*-algebra $\mathcal{C}$ of bounded complex functions on X (with the sup norm) and then let $\mathcal{C}_0$ be the set of $\varphi \in \mathcal{C}$ such that $\lim_{\mathcal{F}} \varphi = 0$. Then $\mathcal{C}_0$ is a C^*-algebra and its spectrum $\mathscr{X}$ contains X but is much larger than X in general.

Let us say that a Banach module structure defined by a Banach algebra $\mathcal{N}$ on $\mathscr{H}$ is *finer* than that defined by $\mathcal{M}$ if $\mathcal{M} \subset \mathcal{N}$. In the next example we show that, by using the same idea as in Example 2.7, one can define on each X-module new Banach module structures finer than the initial one. In Section 7 we shall consider the question of the stability of the essential spectrum in situations of this type, when the perturbation vanishes at infinity in a weak sense.

Example 2.8 Let $\mathscr{H}$ be a Hilbert X-module over a locally compact non-compact topological space X and let $\mathcal{F}$ be a filter on X finer than the Fréchet filter. We extend the morphism Q to all of $B(X)$ as explained in Remark 2.6 and observe that we get a finer Hilbert module structure on $\mathscr{H}$ by taking $\{\varphi(Q) \mid \lim_{\mathcal{F}} \varphi = 0\}$ as multiplier algebra. One can proceed similarly in the case of a Banach X-module, it suffices to replace $B(X)$ by $C_{\mathrm{b}}(X)$.

We now give an example of a non-topological nature.

Example 2.9 Let (X, μ) be measure space with $\mu(X) = \infty$. We define the class of functions which "vanish at infinity" as follows. Let us say that a set $F \subset X$ is of *cofinite measure* if its complement F^{c} is of finite (exterior) measure. The family of sets of cofinite measure is clearly a filter $\mathcal{F}_\mu$. If φ is a function on X then $\lim_{\mathcal{F}_\mu} \varphi = 0$ means that for each $\varepsilon > 0$ the set where $|\varphi(x)| \geq \varepsilon$ is of finite measure. We denote $B_\mu(X)$ the C^*-subalgebra of $L^\infty(X)$ consisting of functions such that $\lim_{\mathcal{F}_\mu} \varphi = 0$. Let $\mathcal{N}_\mu$ be the set of Borel subsets of finite measure of X. Then $\{\chi_N\}_{N \in \mathcal{N}_\mu}$ is an approximate unit of $B_\mu(X)$ because for each $\varphi \in B_\mu(X)$ and each $\varepsilon > 0$ we have $N = \{x \mid |\varphi(x)| \geq \varepsilon\} \in \mathcal{N}_\mu$ and ess-sup $|\varphi - \chi_N \varphi| \leq \varepsilon$. Now it is clear that $L^2(X)$ and, more generally, any direct integral of Hilbert spaces over X, has a natural Hilbert module structure with $B_\mu(X)$ as multiplier algebra.

If $\mathscr{H}$ is a Banach module and the Banach space $\mathscr{H}$ is reflexive we say that $\mathscr{H}$ is a *reflexive Banach module*. In this case the adjoint Banach space $\mathscr{H}^*$ *is equipped with a canonical Banach module structure*, its multiplier algebra being $\mathcal{M}(\mathscr{H}^*) := \{A^* \mid A \in \mathcal{M}(\mathscr{H})\}$. This is a closed subalgebra of $\mathcal{B}(\mathscr{H}^*)$ which clearly has an approximate unit and the linear subspace generated by the elements of the form $A^* v$, with $A \in \mathcal{M}(\mathscr{H})$ and $v \in \mathscr{H}^*$, is weak*-dense, hence dense, in $\mathscr{H}^*$. Indeed, if $u \in \mathscr{H}$ and $\langle u, A^* v\rangle = 0$ for all such A, v then $Au = 0$ for all $A \in \mathcal{M}(\mathscr{H})$ hence $u = 0$ because of (2.4).

Example 2.10 For each real number s let $\mathscr{H}^s := \mathscr{H}^s(\mathbb{R}^n)$ be the Hilbert space of distributions u on $\mathbb{R}^n$ such that $\|u\|_s^2 := \int(1+|k|^2)^s|\widehat{u}(k)|^2dk < \infty$, where $\widehat{u}$ is the Fourier transform of u. This is the usual Sobolev space of order s on $\mathbb{R}^n$. The algebra $\mathscr{S}$ of Schwartz test functions on $\mathbb{R}^n$ is naturally embedded in $\mathcal{B}(\mathscr{H}^s)$, a function $\varphi \in \mathscr{S}$ being identified with the operator of multiplication by φ on $\mathscr{H}^s$. If we denote by $\mathcal{M}^s$ the closure of $\mathscr{S}$ in $\mathcal{B}(\mathscr{H}^s)$, then clearly $(\mathscr{H}^s, \mathcal{M}^s)$ is a Banach module and this Banach module is a Hilbert module if and only if $s = 0$. The module adjoint to $(\mathscr{H}^s, \mathcal{M}^s)$ is identified with $(\mathscr{H}^{-s}, \mathcal{M}^{-s})$. Note that $\mathcal{M}^s$ can be realized as a subalgebra of $\mathcal{M}^0 = C_0(\mathbb{R}^n)$, namely $\mathcal{M}^s$ is the completion of $\mathscr{S}$ for the norm $\|\varphi\|_{\mathcal{M}^s} := \sup_{\|u\|_s=1}\|\varphi u\|_s$, and then we have $\mathcal{M}^s = \mathcal{M}^{-s}$ isometrically and $\mathcal{M}^s \subset \mathcal{M}^t$ if $s \geq t \geq 0$ (by interpolation).

Definition 2.11 *A couple $(\mathscr{G}, \mathscr{H})$ consisting of a Hilbert module $\mathscr{H}$ and a Hilbert space $\mathscr{G}$ such that $\mathscr{G} \subset \mathscr{H}$ continuously and densely will be called a* Friedrichs module. *If $\mathscr{H}$ is a Hilbert X-module over a locally compact space X, we say that $(\mathscr{G}, \mathscr{H})$ is a* Friedrichs X-module. *If $\mathcal{M}(\mathscr{H}) \subset \mathcal{K}(\mathscr{G}, \mathscr{H})$, we say that $(\mathscr{G}, \mathscr{H})$ is a* compact Friedrichs module.

In the situation of this definition we always identify $\mathscr{H}$ with its adjoint space, which gives us a Gelfand triplet $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$. If $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs module then each operator M from $\mathcal{M}(\mathscr{H})$ extends to a compact operator $M : \mathscr{H} \to \mathscr{G}^*$ (this is the adjoint of the compact operator $M^* : \mathscr{G} \to \mathscr{H}$). Thus we shall have $\mathcal{M}(\mathscr{H}) \subset \mathcal{K}(\mathscr{G}, \mathscr{H}) \cap \mathcal{K}(\mathscr{H}, \mathscr{G}^*)$.

Example 2.12 With the notations of Example 2.10, if we set $\mathscr{H} = \mathscr{H}^0$ and take $s > 0$, then $(\mathscr{H}^s, \mathscr{H})$ is a compact Friedrichs module and the associated Gelfand triplet is $\mathscr{H}^s \subset \mathscr{H} \subset \mathscr{H}^{-s}$. Indeed, if $\varphi \in C_0(\mathbb{R}^n)$ then the operator of multiplication by φ is a compact operator $\mathscr{H}^s \to \mathscr{H}$.

2.2 Decay improving operators

Let $\mathscr{H}$ and $\mathscr{K}$ be Banach spaces. If $\mathscr{K}$ is a Banach module then we shall denote by $\mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$ the norm closed linear subspace generated by the operators MT, with $T \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $M \in \mathcal{M}(\mathscr{K})$. We say that an operator in $\mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$ is *decay improving*, or *left vanishes at infinity* (with respect to $\mathcal{M}(\mathscr{K})$, if this is not obvious from the context). If J_α is an approximate unit for $\mathcal{M}(\mathscr{K})$, then for an operator $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ we have:

$$\begin{aligned} S \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{K}) \quad &\Leftrightarrow \quad \lim_\alpha \|J_\alpha S - S\| = 0 \qquad (2.6) \\ &\Leftrightarrow \quad S = MT \text{ for some } M \in \mathcal{M}(\mathscr{K}) \text{ and } T \in \mathcal{B}(\mathscr{H}, \mathscr{K}). \end{aligned}$$

The second equivalence follows from the Cohen-Hewitt theorem (Theorem 2.1).

If $\mathscr{H}$ is a Banach module then one can similarly define $\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$ as the norm closed linear subspace generated by the operators TM with $T\in\mathcal{B}(\mathscr{H},\mathscr{K})$ and $M\in\mathcal{M}(\mathscr{H})$. We say that the elements of $\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$ *right vanish at infinity*. If both $\mathscr{H}$ and $\mathscr{K}$ are Banach modules then both spaces $\mathcal{B}_0^l(\mathscr{H},\mathscr{K})$ and $\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$ make sense and we set $\mathcal{B}_0(\mathscr{H},\mathscr{K})=\mathcal{B}_0^l(\mathscr{H},\mathscr{K})\cap\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$.

Some simple properties of these spaces are described below.

Proposition 2.13 *If $\mathscr{K}$ is a reflexive Banach module and $S\in\mathcal{B}_0^l(\mathscr{H},\mathscr{K})$ then S^* belongs to $\mathcal{B}_0^r(\mathscr{K}^*,\mathscr{H}^*)$.*

Proof: We have $S=MT$ with $M\in\mathcal{M}(\mathscr{K})$ and $T\in\mathcal{B}(\mathscr{H},\mathscr{K})$ by (2.6), which implies $S^*=T^*M^*$ and we have $M^*\in\mathcal{M}(\mathscr{K}^*)$ by definition. ■

Proposition 2.14 *If $\mathscr{H}$ is a Hilbert module then $\mathcal{B}_0(\mathscr{H})$ is a C^*-algebra and an operator $S\in\mathcal{B}(\mathscr{H})$ belongs to it if and only if one can write $S=MTN$ with $M,N\in\mathcal{M}(\mathscr{H})$ and $T\in\mathcal{B}(\mathscr{H})$.*

Proof: $\mathcal{B}_0(\mathscr{H})$ is clearly a C^*-algebra, so if $S\in\mathcal{B}_0(\mathscr{H})$ then $S=S_1S_2$ for some operators $S_1,S_2\in\mathcal{B}_0(\mathscr{H})$. Thus $S_1=MT_1$ and $S_2=T_2N$ for some $M,N\in\mathcal{M}(\mathscr{H})$ and $T_1,T_2\in\mathcal{B}(\mathscr{H})$, hence $S=MT_1T_2N$. ■

Proposition 2.15 *If $\mathscr{K}$ is a Banach module then $\mathcal{K}(\mathscr{H},\mathscr{K})\subset\mathcal{B}_0^l(\mathscr{H},\mathscr{K})$. If $\mathscr{H}$ is a reflexive Banach module, then $\mathcal{K}(\mathscr{H},\mathscr{K})\subset\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$.*

Proof: If $\{J_\alpha\}$ is an approximate unit for $\mathcal{M}(\mathscr{K})$ then s-$\lim_\alpha J_\alpha u=u$ uniformly in u if u belongs to a compact subset of $\mathscr{K}$. Hence if $S\in\mathcal{K}(\mathscr{H},\mathscr{K})$ then $\lim_\alpha\|J_\alpha S-S\|=0$ and thus $S\in\mathcal{B}_0^l(\mathscr{H},\mathscr{K})$ by (2.6). To prove the second part of the proposition, observe that if $S\in\mathcal{K}(\mathscr{H},\mathscr{K})$ then $S^*\in\mathcal{K}(\mathscr{K}^*,\mathscr{H}^*)$, hence $S^*\in\mathcal{B}_0^l(\mathscr{K}^*,\mathscr{H}^*)$ by what we just proved, so $S^{**}\in\mathcal{B}_0^r(\mathscr{H},\mathscr{K}^{**})$ by Proposition 2.13. So $\lim_\alpha\|S^{**}J_\alpha-S^{**}\|=0$ if $\{J_\alpha\}$ is an approximate unit for $\mathcal{M}(\mathscr{H})$. But clearly$S=S^{**}$, hence $S\in\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$. ■

Proposition 2.16 *Let $\mathscr{H}$ be a Banach module and $\mathscr{G}$ a Banach space continuously embedded in $\mathscr{H}$ and such that $\mathcal{M}(\mathscr{H})\subset\mathcal{K}(\mathscr{G},\mathscr{H})$. If $R\in\mathcal{B}_0^l(\mathscr{H})$ and $R\mathscr{H}\subset\mathscr{G}$, then $R\in\mathcal{K}(\mathscr{H})$.*

Proof: According to (2.6) we have $R=\lim_\alpha J_\alpha R$, the limit being taken in norm. But $R\in\mathcal{B}(\mathscr{H},\mathscr{G})$ by the closed graph theorem and $J_\alpha\in\mathcal{K}(\mathscr{G},\mathscr{H})$ by hypothesis, so that $J_\alpha R\in\mathcal{K}(\mathscr{H})$. ■

If $(\mathscr{G},\mathscr{H})$ is a Friedrichs module then $\mathcal{B}(\mathscr{H}) \subset \mathcal{B}(\mathscr{G},\mathscr{G}^*)$ hence we can define

$$\mathcal{B}^l_{00}(\mathscr{G},\mathscr{G}^*) = \text{ norm closure of } \mathcal{B}^l_0(\mathscr{H}) \text{ in } \mathcal{B}(\mathscr{G},\mathscr{G}^*). \tag{2.7}$$

The spaces $\mathcal{B}^r_{00}(\mathscr{G},\mathscr{G}^*)$ and $\mathcal{B}_{00}(\mathscr{G},\mathscr{G}^*)$ are similarly defined and we have

$$\mathcal{K}(\mathscr{G},\mathscr{G}^*) \subset \mathcal{B}_{00}(\mathscr{G},\mathscr{G}^*) \tag{2.8}$$

because $\mathcal{K}(\mathscr{H})$ is a dense subset of $\mathcal{K}(\mathscr{G},\mathscr{G}^*)$ and $\mathcal{K}(\mathscr{H}) \subset \mathcal{B}_0(\mathscr{H})$.

Example 2.17 In the context of Example 2.12 we may consider the two classes of operators $\mathcal{B}^l_{00}(\mathscr{H}^s,\mathscr{H}^{-s})$ and $\mathcal{B}^l_0(\mathscr{H}^s,\mathscr{H}^{-s})$. The first space is included in the second one and the inclusion is strict, for example $\mathcal{B}^l_{00}(\mathscr{H}^s,\mathscr{H}^{-s})$ does not contain operators of order $2s$, while $\mathcal{B}^l_0(\mathscr{H}^s,\mathscr{H}^{-s})$ contains such operators.

2.3 Decay preserving operators

Definition 2.18 *Let $\mathscr{H}$, $\mathscr{K}$ be Banach modules and let $S \in \mathcal{B}(\mathscr{H},\mathscr{K})$. We say that S is* left decay preserving *if for each $M \in \mathcal{M}(\mathscr{H})$ we have $SM \in \mathcal{B}^l_0(\mathscr{H},\mathscr{K})$. We say that S is* right decay preserving *if for each $M \in \mathcal{M}(\mathscr{K})$ we have $MS \in \mathcal{B}^r_0(\mathscr{H},\mathscr{K})$. If S is left and right decay preserving, we say that S is* decay preserving.

We denote $\mathcal{B}^l_q(\mathscr{H},\mathscr{K})$, $\mathcal{B}^r_q(\mathscr{H},\mathscr{K})$ and $\mathcal{B}_q(\mathscr{H},\mathscr{K})$ these classes of operators (the index q comes from *quasilocal*, a terminology which is sometimes more convenient than "decay preserving"). These are closed subspaces of $\mathcal{B}(\mathscr{H},\mathscr{G})$. The next result is obvious; a similar assertion holds in the right decay preserving case.

Proposition 2.19 *Let $\{J_\alpha\}$ be an approximate unit for $\mathcal{M}(\mathscr{H})$ and let S be an operator in $\mathcal{B}(\mathscr{H},\mathscr{K})$. Then S is left decay preserving if and only if one of the following conditions is satisfied:*
(1) *$SJ_\alpha \in \mathcal{B}^l_0(\mathscr{H},\mathscr{K})$ for all α.*
(2) *for each $M \in \mathcal{M}(\mathscr{H})$ there are $T \in \mathcal{B}(\mathscr{H},\mathscr{K})$ and $N \in \mathcal{M}(\mathscr{K})$ such that $SM = NT$.*

The next proposition, which says that the decay preserving property is stable under the usual algebraic operations, is an immediate consequence of Proposition 2.19. There is, of course, a similar statement with "left" and "right" interchanged. We denote by $\mathscr{G},\mathscr{H}$ and $\mathscr{K}$ Banach modules.

Proposition 2.20 (1) *$S \in \mathcal{B}^l_q(\mathscr{H},\mathscr{K})$ and $T \in \mathcal{B}^l_q(\mathscr{G},\mathscr{H}) \Rightarrow ST \in \mathcal{B}^l_q(\mathscr{G},\mathscr{K})$.*
(2) *If $\mathscr{H},\mathscr{K}$ are reflexive and $S \in \mathcal{B}^l_q(\mathscr{H},\mathscr{K})$, then $S^* \in \mathcal{B}^r_q(\mathscr{K}^*,\mathscr{H}^*)$.*
(3) *If $\mathscr{H}$ is a Hilbert module then $\mathcal{B}_q(\mathscr{H})$ is a unital C^*-subalgebra of $\mathcal{B}(\mathscr{H})$.*

Obviously $\mathcal{B}_0^l(\mathscr{H},\mathscr{K}) \subset \mathcal{B}_q^l(\mathscr{H},\mathscr{K})$ and $\mathcal{B}_0^r(\mathscr{H},\mathscr{K}) \subset \mathcal{B}_q^r(\mathscr{H},\mathscr{K})$ but this fact is of no interest. The main results of this paper depend on finding other, more interesting examples of decay preserving operators. We shall give in the rest of this subsection some elementary examples of such operators and in Subsections 2.4 and 7.1 more subtle ones.

From now on in this subsection X will be a locally compact non-compact topological space. The *support* supp $u \subset X$ of an element u of a Banach X-module $\mathscr{H}$ is defined as the smallest closed set such that its complement U has the property $\varphi(Q)u = 0$ if $\varphi \in C_\mathrm{c}(U)$. Clearly, the set $\mathscr{H}_\mathrm{c}$ of elements $u \in \mathscr{H}$ such that supp u is compact is a dense subspace of $\mathscr{H}$.

Let $\mathscr{H},\mathscr{K}$ be Banach X-modules, let $S \in \mathcal{B}(\mathscr{H},\mathscr{K})$, and let $\varphi,\psi \in C(X)$, not necessarily bounded. We say that $\varphi(Q)S\psi(Q)$ is a bounded operator if there is a constant C such that

$$\|\xi(Q)\varphi(Q)S\psi(Q)\eta(Q)\| \le C \sup|\xi| \sup|\eta|$$

for all $\xi,\eta \in C_\mathrm{c}(X)$. The lower bound of the admissible constants C in this estimate is denoted $\|\varphi(Q)S\psi(Q)\|$. If $\mathscr{K}$ is a reflexive Banach X-module, then the product $\varphi(Q)S\psi(Q)$ is well defined as sesquilinear form on the dense subspace $\mathscr{K}_\mathrm{c}^* \times \mathscr{H}_\mathrm{c}$ of $\mathscr{K}^* \times \mathscr{H}$ and the preceding boundedness notion is equivalent to the continuity of this form for the topology induced by $\mathscr{K}^* \times \mathscr{H}$. We similarly define the boundedness of the commutator $[S,\varphi(Q)]$.

Proposition 2.21 *Assume that* $S \in \mathcal{B}(\mathscr{H},\mathscr{K})$ *and let* $\theta : X \to [1,\infty[$ *be a continuous function such that* $\lim_{x\to\infty}\theta(x) = \infty$. *If* $\theta(Q)S\theta^{-1}(Q)$ *is a bounded operator then* S *is left decay preserving. If* $\theta^{-1}(Q)S\theta(Q)$ *is a bounded operator then* S *is right decay preserving.*

Proof: Let $K \subset X$ be compact, let $U \subset X$ be a neighbourhood of infinity in X, and let $\varphi,\psi \in C_\mathrm{b}(X)$ such that supp $\varphi \subset K,$ supp $\psi \subset U$ and $|\varphi| \le 1, |\psi| \le 1$. Then $\theta\varphi$ is a bounded function and $\psi\theta^{-1}$ is bounded and can be made as small as we wish by choosing U conveniently. Thus given $\varepsilon > 0$ we have

$$\|\psi(Q)S\varphi(Q)\| \le \|\psi\theta^{-1}\| \cdot \|\theta(Q)S\theta^{-1}(Q)\| \cdot \|\theta\varphi\| \le \varepsilon$$

if U is a suficiently small neighbourhood of infinity. Then the result follows fom Proposition 2.19(1) and relation (2.6). ■

The boundedness of $\theta(Q)S\theta^{-1}(Q)$ can checked by estimating the commutator $[S,\theta(Q)]$; we give an example for the case of metric spaces. Note that on metric spaces one has a natural class of regular functions, namely the Lipschitz functions, for example the functions which give the distance to subsets: $\rho_K(x) = \inf_{y\in K}\rho(x,y)$ for $K \subset X$.

We say that a locally compact metric space (X, ρ) is *proper* if the metric ρ has the property $\lim_{y \to \infty} \rho(x, y) = \infty$ for some (hence for all) points $x \in X$. Equivalently, if X is not compact but the closed balls are compact.

Corollary 2.22 *Let (X, ρ) be a proper locally compact metric space. If S belongs to $\mathcal{B}(\mathscr{H}, \mathscr{K})$ and if $[S, \theta(Q)]$ is bounded for each positive Lipschitz function θ, then S is decay preserving.*

Proof: Indeed, by taking $\theta = 1 + \rho_K$ and by using the notations of the proof of Proposition 2.21, we easily get the following estimate: there is $C < \infty$ depending only on K such that

$$\|\varphi(Q) S \psi(Q)\| \leq C(1 + \rho(K, U))^{-1},$$

where $\rho(K, U)$ is the distance from K to U. Since S^* has the same properties as S, this proves that S is decay preserving. Note that the boundedness of $[S, \rho_x(Q)]$ for some $x \in X$ suffices in this argument. ■

2.4 X-modules over locally compact groups

A locally compact abelian group X being, in particular a locally compact space, we can consider X-modules in the sense of Example 2.4. However, the group structure of X allows us to associate to it more interesting classes of Banach modules.

We always assume X non-compact and we denote additively the group operation. For example, X could be $\mathbb{R}^n, \mathbb{Z}^n$, or a finite dimensional vector space over a local field, e.g. over the field of p-adic numbers. Let X^* be the abelian locally compact group dual to X.

Definition 2.23 *A* Banach X-module *over the group X is a Banach space $\mathscr{H}$ equipped with a strongly continuous representation $\{V_k\}$ of X^* on $\mathscr{H}$.*

Note that we shall use the same notation V_k for the representations of X^* in different spaces $\mathscr{H}$ whenever this does not lead to ambiguities.

Such *a Banach X-module has a canonical structure of Banach module* that we now define. We choose Haar measures dx and dk on X and X^* normalized by the following condition: if the Fourier transform of a function φ on X is given by $(\mathcal{F}\varphi)(k) \equiv \widehat{\varphi}(k) = \int_X \overline{k(x)} \varphi(x) dx$ then $\varphi(x) = \int_{X^*} k(x) \widehat{\varphi}(k) dk$. Recall that $X^{**} = X$. Let $C^{(a)}(X) := \mathcal{F} L^1_{\mathrm{c}}(X^*)$ be the set of Fourier transforms of integrable functions with compact support on X^*. It is easy to see that $C^{(a)}(X)$ is a

$*$-algebra for the usual algebraic operations; more precisely, it is a dense subalgebra of $C_0(X)$ stable under conjugation. For $\varphi \in C^{(a)}(X)$ we set

$$\varphi(Q) = \int_{X^*} V_k \widehat{\varphi}(k) dk. \tag{2.9}$$

This definition is determined by the formal requirement $k(Q) = V_k$. Then

$$\mathcal{M} := \text{ norm closure of } \{\varphi(Q) \mid \varphi \in C^{(a)}(X)\} \text{ in } \mathcal{B}(\mathscr{H}) \tag{2.10}$$

is a Banach subalgebra of $\mathcal{B}(\mathscr{H})$. By using the next lemma we see that the couple $(\mathscr{H}, \mathcal{M})$ satisfies the conditions of Definition 2.2, which gives us the canonical Banach module structure on $\mathscr{H}$.

Lemma 2.24 *The algebra $\mathcal{M}$ has an approximate unit consisting of elements of the form $e_\alpha(Q)$ with $e_\alpha \in C^{(a)}(X)$.*

Proof: Let us fix a compact neighborhood K of the identity in X^*. The set of compact neighborhoods of the identity α such that $\alpha \subset K$ is ordered by $\alpha_1 \geq \alpha_2 \Leftrightarrow \alpha_1 \subset \alpha_2$. For each such α define e_α by $\widehat{e}_\alpha = \chi_\alpha / |\alpha|$, where $|\alpha|$ is the Haar measure of α. Then $\|e_\alpha(Q)\| \leq \sup_{k \in K} \|V_k\| < \infty$, from which it is easy to infer that $\lim_\alpha \|e_\alpha(Q)\varphi(Q) - \varphi(Q)\| = 0$ for all $\varphi \in C^{(a)}(X)$. ■

Example 2.25 Let $X = \mathbb{R}^n$ with the additive group structure and let $\mathscr{H}$ be the Sobolev space $\mathscr{H}^s(X)$ for some real number s. We identify as usual X^* with X by setting $k(x) = \exp(i\langle x, k\rangle)$, where $\langle x, k\rangle$ is the scalar product in X. Then we get a Banach X-module structure on $\mathscr{H}$ by setting $(V_k u)(x) = \exp(i\langle x, k\rangle)u(x)$, where $\langle x, k\rangle$ is the scalar product. Note that $V_k \mathscr{H}^s \subset \mathscr{H}^s$ and $\|V_k\| \leq C(1 + |k|)^s$. It is easy to see that the Banach module structure associated to this X-module structure coincides with that defined in Example 2.10.

Remark 2.26 Algebras $\mathcal{A}$ as in Lemma 2.3 can be easily constructed in this context. Indeed, if ω is a sub-multiplicative function on X^*, i.e. a Borel map $X^* \to [1, \infty[$ satisfying $\omega(k'k'') \leq \omega(k')\omega(k'')$ (hence is locally bounded), let $C^{(\omega)}(X)$ be the set of functions φ whose Fourier transform $\widehat{\varphi}$ satisfies

$$\|\varphi\|_{C^{(\omega)}} := \int_{X^*} |\widehat{\varphi}(k)|\omega(k) dk < \infty. \tag{2.11}$$

Then $C^{(\omega)}(X)$ is a subalgebra of $C_0(X)$ and is a Banach algebra for the norm (2.11). Moreover, $C^{(a)}(X) \subset C^{(\omega)}(X)$ densely and the net $\{e_\alpha\}$ defined in the proof of Lemma 2.24 is an approximate unit of $C^{(\omega)}(X)$. If $\|V_k\|_{\mathcal{B}(\mathscr{H})} \leq c\omega(k)$

for some number $c > 0$ then $\varphi(Q)$ is well defined for each $\varphi \in C^{(\omega)}(X)$ by the relation (2.9) and $\Phi(\varphi) = \varphi(Q)$ is a continuous morphism with dense range of $C^{(\omega)}(X)$ into $\mathcal{M}(\mathscr{H})$. We could take $\omega(k) = \sup(1, \|V_k\|_{\mathcal{B}(\mathscr{H})})$ but if a second Banach X-module $\mathscr{K}$ is given then it is more convenient to take $\omega(k) = \sup\{1, \|V_k\|_{\mathcal{B}(\mathscr{H})}, \|V_k\|_{\mathcal{B}(\mathscr{K})}\}$.

The adjoint of a reflexive Banach X-module has a natural structure of Banach X-module. Indeed, it is known and easy to prove that a weakly continuous representation is strongly continuous. Thus we can equip the adjoint space $\mathscr{H}^*$ with the Banach X-module structure defined by the representation $k \mapsto (V_{\bar{k}})^*$, where $\bar{k}$ is the complex conjugate of k (i.e. its inverse in X^*).

The group X is, in particular, a locally compact topological space, hence the notion of Banach X-module in the sense of Example 2.4 makes sense. But this is in fact a particular case of that of Banach X-module in the sense of Definition 2.23. Indeed, according to Remark 2.5, we get a strongly continuous representation of X^* on $\mathscr{H}$ by setting $V_k = k(Q)$. In the case of Hilbert X-modules we have a more precise fact.

Lemma 2.27 *Let $\mathscr{H}$ be a Hilbert space. Then giving a Hilbert X-module structure on $\mathscr{H}$ is equivalent with giving on $\mathscr{H}$ a Banach X-module structure over the group X such that the representation $\{V_k\}_{k\in X^*}$ is unitary. The relation between the two structures is determined by the condition $V_k = k(Q)$.*

Proof: If $\mathscr{H}$ is a Hilbert X-module then we can define $V_k = k(Q) \in \mathcal{B}(\mathscr{H})$ and check that $\{V_k\}_{k\in X^*}$ is a strongly continuous unitary representation of X^* on $\mathscr{H}$ with the help of Remark 2.6. Reciprocally, it is well known that such a representation allows one to equip $\mathscr{H}$ with a Hilbert X-module structure. The main point is that the estimate $\|\varphi(Q)\| \le \sup|\varphi|$ holds, see [Lo]. ■

Banach X-modules over the group X which are not Hilbert X-modules often appear in the following context (cf. Example 2.25 in the case $s > 0$).

Definition 2.28 *A* stable Friedrichs X-module *over the group X is a Friedrichs X-module $(\mathscr{G}, \mathscr{H})$ satisfying $V_k\mathscr{G} \subset \mathscr{G}$ for all $k \in X^*$ and such that if $u \in \mathscr{G}$ and if $K \subset X^*$ is compact then* $\sup_{k\in K} \|V_k u\|_{\mathscr{G}} < \infty$.

Here $V_k = k(Q)$. It is clear that $V_k\mathscr{G} \subset \mathscr{G}$ implies $V_k \in \mathcal{B}(\mathscr{G})$ and that the local boundedness condition implies that the map $k \mapsto V_k \in \mathcal{B}(\mathscr{G})$ is a weakly, hence strongly, continuous representation of X^* on $\mathscr{G}$ (not unitary in general). The local boundedness condition is automatically satisfied if X^* is second countable.

Thus, *if $(\mathscr{G}, \mathscr{H})$ is a stable Friedrichs X-module, then $\mathscr{G}$ is equipped with a canonical Banach X-module structure.* Then, by taking adjoints, we get a natural

Banach X-module structure on $\mathscr{G}^*$ too. Our definitions are such that after the identifications $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ the restriction to $\mathscr{H}$ of the operator V_k acting in $\mathscr{G}^*$ is just the initial V_k. Indeed, we have $V_k^* = V_k^{-1} = V_{\bar{k}}$ in $\mathscr{H}$. Thus there is no ambiguity in using the same notation V_k for the representation of X^* in the three spaces $\mathscr{G}, \mathscr{H}$ and $\mathscr{G}^*$.

Lemma 2.29 *If $\mathscr{K}$ is a Banach space then $\mathcal{B}_0^l(\mathscr{K}, \mathscr{G}) \subset \mathcal{B}_0^l(\mathscr{K}, \mathscr{H})$, and if $\mathscr{K}$ is a Banach module then $\mathcal{B}_q^l(\mathscr{K}, \mathscr{G}) \subset \mathcal{B}_q^l(\mathscr{K}, \mathscr{H})$.*

Proof: If $S \in \mathcal{B}_0^l(\mathscr{K}, \mathscr{G})$ then $S = \varphi(Q)T$ for some $\varphi \in C^{(\omega)}(X)$ with $\omega(k) = \sup(1, \|V_k\|_{\mathcal{B}(\mathscr{G})})$ and some $T \in \mathcal{B}(\mathscr{K}, \mathscr{G})$ (see Remark 2.26). But clearly such a $\varphi(Q)$ belongs to the multiplier algebra of $\mathscr{H}$ and $T \in \mathcal{B}(\mathscr{K}, \mathscr{H})$. ■

We show now that, in the case of Banach X-modules over groups, the decay preserving property is related to regularity in the sense of the next definition.

Definition 2.30 *Let $\mathscr{H}$ and $\mathscr{K}$ be Banach X-modules. We say that a continuous operator $S : \mathscr{H} \to \mathscr{K}$ is of class $C^{\mathrm{u}}(Q)$, and we write $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$, if the map $k \mapsto V_k^{-1} S V_k \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ is norm continuous.*

Note that norm continuity at the origin implies norm continuity everywhere. The class of regular operators is stable under algebraic operations:

Proposition 2.31 *Let $\mathscr{G}, \mathscr{H}, \mathscr{K}$ be Banach X-modules.*
(i) If $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ and $T \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{H})$ then $ST \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{K})$.
(ii) If $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ is bijective, then $S^{-1} \in C^{\mathrm{u}}(Q; \mathscr{K}, \mathscr{H})$.
(iii) If $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ and $\mathscr{H}, \mathscr{G}$ are reflexive, then $S^* \in C^{\mathrm{u}}(Q; \mathscr{K}^*, \mathscr{H}^*)$.

Proof: We prove only (ii). If we set $S_k = V_k^{-1} S V_k$ then $V_k^{-1} S^{-1} V_k = S_k^{-1}$, hence

$$\|V_k^{-1} S^{-1} V_k - S^{-1}\| = \|S_k^{-1} - S^{-1}\| = \|S_k^{-1}(S - S_k) S^{-1}\| \leq C \|S - S_k\|$$

if k is in a compact set, and this tends to zero as $k \to 0$. ■

Proposition 2.32 *If $T \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ then T is decay preserving.*

Proof: We show that $\varphi(Q)T \in \mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$ if $\varphi \in C^{(a)}(X)$. A similar argument gives $T\varphi(Q) \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$. Set $T_k = V_k T V_k^{-1}$, then

$$\varphi(Q)T = \int_{X^*} \widehat{\varphi}(k) V_k T dk = \int_{X^*} T_k \widehat{\varphi}(k) V_k dk.$$

Since $k \mapsto T_k$ is norm continuous on the compact support of $\widehat{\varphi}$, for each $\varepsilon > 0$ we can construct, with the help of a partition of unity, functions $\theta_i \in C_{\mathrm{c}}(X^*)$ and operators $S_i \in \mathcal{B}(\mathscr{H}, \mathscr{K})$, such that $\|T_k - \sum_{i=1}^n \theta_i(k) S_i\| < \varepsilon$ if $\widehat{\varphi}(k) \neq 0$. Thus

$$\|\varphi(Q)T - \sum_{i=1}^{n} \int_{X^*} \theta_i(k) S_i \widehat{\varphi}(k) V_k dk\| \leq \varepsilon \sum_{i=1}^{n} \int_{X^*} |\widehat{\varphi}(k)| \|V_k\|_{\mathcal{B}(\mathscr{H})} dk.$$

Now, since $\mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$ is a norm closed subspace, it suffices to show that the operator $\int_{X^*} \theta_i(k) S_i \widehat{\varphi}(k) V_k dk$ belongs to $\mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$ for each i. But if ψ_i is the inverse Fourier transform of $\theta_i \widehat{\varphi}$ then this is $S_i \psi_i(Q)$ and $\psi_i \in C^{(a)}(X)$. ■

Let $\mathscr{H}, \mathscr{K}$ be Hilbert X-modules over the group X. We say that an operator $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ is of *finite range*[3] if there is a compact neighborhood Λ of the origin such that for any compact sets $H, K \subset X$ with $(H - K) \cap \Lambda = \emptyset$ we have $\chi_H(Q) S \chi_K(Q) = 0$. From Remark 2.6 we get that this is equivalent to $S\chi_K(Q) = \chi_{K+\Lambda}(Q) S \chi_K(Q)$ for any Borel set K. A finite range operator is clearly decay preserving. Moreover, the set of finite range operators is stable under sums and products, and the adjoint of such an operator is also of finite range.

Proposition 2.33 *Each operator of class $C^{\mathrm{u}}(Q)$ is a norm limit of a sequence of finite range operators.*

Proof: We fix a Haar measure dk on X^* and if $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $\theta \in L^1(X^*)$ we define

$$S_\theta = \int_{X^*} V_k^* S V_k \theta(k) dk. \tag{2.12}$$

The integral is well defined because $k \mapsto V_k^* S V_k \in \mathcal{B}(\mathscr{H})$ is a bounded strongly continuous map. In order to explain the main idea of the proof we shall make a formal computation involving the spectral measure $E(A) = \chi_A(Q)$, see Remark 2.6 and Lemma 2.27 (we shall use the same notation for the spectral measures in $\mathscr{H}$ and $\mathscr{K}$). We have for $k \in X^*$ and $\varphi(Q) \in B(X)$

$$\varphi(Q) V_k^* = \varphi(Q) k(Q)^* = (\varphi \overline{k})(Q) = \int \varphi(x) \overline{k}(x) E(dx).$$

Note also that for $x, y \in X$ we have $\overline{k}(x) k(y) = k(-x) k(y) = k(y - x)$. Let $\widehat{\theta}(x) = \int \overline{k(x)} \theta(k) dk$ be the Fourier transform of θ. Then if $\varphi, \psi \in B(X)$:

$$\begin{aligned} \varphi(Q) S_\theta \psi(Q) &= \int_{X^*} \theta(k) dk \int_X \int_X \varphi(x) \overline{k}(x) k(y) \psi(y) E(dx) S E(dy) \\ &= \int_X \int_X \widehat{\theta}(x - y) \varphi(x) \psi(y) E(dx) S E(dy). \end{aligned} \tag{2.13}$$

---

[3] If X is an euclidean space and $\mathscr{H} = \mathscr{K} = L^2(X)$, the next condition means that there is $r < \infty$ such that the distribution kernel of S satisfies $S(x, y) = 0$ for $|x - y| > r$.

We can rigorously justify this computation and give a meaning to the last integral by taking into account that $E(A)SE(B)$ induces a finitely additive measure on the algebra generated by rectangles $A \times B$ in $X \times X$ (note that $\widehat{\theta} \in C_0(X)$). If S is Hilbert-Schmidt then the measure is in fact σ-additive and the result becomes obvious. We shall, however, avoid these questions and we shall directly prove only what we need. Namely, we show the following:

$$(*) \quad \begin{cases} \text{If the support of } \widehat{\theta} \text{ is a compact set } \Lambda \text{ and if supp } \varphi \cap (\Lambda + \text{supp } \psi) = \varnothing \\ \text{then } \varphi(Q)S_\theta\psi(Q) = 0. \end{cases}$$

Observe that if $(*)$ holds for a certain set of operators S then it also holds for the strongly closed linear subspace of $\mathcal{B}(\mathscr{H}, \mathscr{K})$ generated by it. So it suffices to prove $(*)$ for S an operator of rank one $Sf = v\langle u, f\rangle$ with some fixed $u \in \mathscr{H}$ and $v \in \mathscr{K}$. Now the computation giving (2.13) obviously makes sense in the weak topology and gives for $f \in \mathscr{H}$ and $g \in \mathscr{K}$:

$$\langle g, \varphi(Q)S_\theta\psi(Q)f\rangle = \int_X \int_X \widehat{\theta}(x-y)\varphi(x)\psi(y)\langle g, E(dx)u\rangle\langle u, E(dy)f\rangle,$$

hence $(*)$ holds for such S.

Finally, note that if $S \in C^{\mathrm{u}}(Q)$ then S is norm limit of operators of the form S_θ. For this it suffices to take $\theta = |K|^{-1}\chi_K$ where K runs over the set of open relatively compact neighbourhoods of the neutral element of X^*, $|K|$ being the Haar measure of K. Then, by approximating conveniently θ in L^1 norm, one shows that S is norm limit of operators S_θ such that $\widehat{\theta}$ has compact support. ■

Proposition 2.34 *Assume that X is a disjoint union $X = \cup_{a \in A} X_a$ of Borel sets X_a such that: 1) there is a compact set K such that each X_a is a translate of a subset of K, and 2) for each compact neighborhood Λ of the origin, the number of sets $X_b + \Lambda$ which intersects a given X_a is bounded by a constant independent of a. Then a finite range operator is of class $C^{\mathrm{u}}(Q)$.*

Proof: It suffices to prove that a finite range operator S is of class $C^{\mathrm{u}}(Q)$. Let Λ be such that $\chi_H(Q)S\chi_K(Q) = 0$ if H, K are compact sets with $(H-K) \cap \Lambda = \emptyset$. Let χ_a be the characteristic function of X_a and φ_a that of $Y_a = X_a + \Lambda$. We can assume that $A \subset X$ and that $X_a = a + K_a$ for some $K_a \subset \Lambda$. We shall abbreviate $\chi_a = \chi_a(Q)$ and $\varphi_a = \varphi_a(Q)$. We have $\sum_a \chi_a = 1$ strongly on $\mathscr{H}$, cf. Remark 2.6, and $[V_k, S]\chi_a = \varphi_a[V_k, S]\chi_a$ because $V_k = k(Q)$. Thus there is a constant C, depending only on un upper bound for the number of Y_b which intersects a fixed

X_a, such that for $u \in \mathscr{H}$ with compact support:

$$\begin{aligned} \|[V_k, S]u\|^2 &\leq C\sum \|\varphi_a[V_k, S]\chi_a u\|^2 \\ &= C\sum \|\varphi_a[V_k - k(a), S]\chi_a \cdot \chi_a u\|^2 \\ &\leq 2CL_k \sum \|\chi_a u\|^2 = 2CL_k\|u\|^2 \end{aligned}$$

where $L_k = \sup_a \|(V_k - k(a))\varphi_a\|$. But

$$\|(V_k - k(a))\varphi_a\| \leq \sup_{y\in Y_a} |k(y) - k(a)| = \sup_{y\in Y_a} |k(y-a) - 1| \leq \sup_{x\in L} |k(x) - 1|$$

where $L = \Lambda + \Lambda$ is a compact set. Thus $L_k \to 0$ if $k \to 0$ in X^*. ■

If X is an abelian locally compact group then there is enough structure in order to develop a rich pseudo-differential calculus in $L^2(X)$ and Proposition 2.32 shows that many pseudo-differential operators are decay preserving. We give a simple example below. If φ and ψ are bounded Borel functions on X and X^* respectively then, following standard quantum mechanical conventions, we denote by $\varphi(Q)$ the operator of multiplication by φ in $L^2(X)$ and we set $\psi(P) = \mathcal{F}^{-1}M_\psi\mathcal{F}$, where M_ψ is the operator of multiplication by ψ in $L^2(X^*)$. Then one gets more general pseudo-differential operators by considering C^*-algebras generated by products $\varphi(Q)\psi(P)$. We recall that the C^*-algebra generated by such products with φ and ψ bounded Borel and convergent to zero at infinity is $\mathcal{K}(L^2(X))$.

Let $C_{\mathrm{b}}^{\mathrm{u}}(X)$ and $C_{\mathrm{b}}^{\mathrm{u}}(X^*)$ be the algebras of bounded uniformly continuous functions on X and X^* respectively. Below the space $L^2(X)$ is equipped with its natural Hilbert X-module structure.

Proposition 2.35 *The C^*-algebra generated by the operators $\varphi(Q)$ and $\psi(P)$, with $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ and $\psi \in C_{\mathrm{b}}^{\mathrm{u}}(X^*)$, consists of decay preserving operators.*

Proof: By Proposition 2.20, $\mathcal{B}_q(L^2(X))$ is a C^*-algebra, hence it suffices to show that each $\varphi(Q)$ and $\psi(P)$ is decay preserving. For $\varphi(Q)$ the assertion is trivial while for $\psi(P)$ we apply Proposition 2.32. ■

3 Compact perturbations in Banach modules

In this section $(\mathscr{G}, \mathscr{H})$ will always be a *compact Friedrichs module*, see Definition 2.11. As usual, we associate to it a Gelfand triplet $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ and we set $\|\cdot\| = \|\cdot\|_{\mathscr{H}}$. We are interested in criteria which ensure that an operator B is a compact perturbation of an operator A, both operators being unbounded operators in $\mathscr{H}$ obtained as restrictions of some bounded operators $\mathscr{G} \to \mathscr{G}^*$. More precisely, the following is a general assumption (suggested by the statement of Theorem 2.1 in [OS]) which will always be fulfilled:

$$(AB) \begin{cases} A, B \text{ are closed densely defined operators in } \mathscr{H} \text{ with } \rho(A) \cap \rho(B) \neq \emptyset \\ \text{and having the following properties: } \mathcal{D}(A) \subset \mathscr{G} \text{ densely, } \mathcal{D}(A^*) \subset \mathscr{G}, \\ \mathcal{D}(B) \subset \mathscr{G} \text{ and } A, B \text{ extend to continuous operators } \widetilde{A}, \widetilde{B} \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*). \end{cases}$$

Example 3.1 One can construct interesting classes of operators with the properties required in (AB) as follows. Let $\mathscr{G}_a$, $\mathscr{G}_b$ be Hilbert spaces such that $\mathscr{G} \subset \mathscr{G}_a \subset \mathscr{H}$ and $\mathscr{G} \subset \mathscr{G}_b \subset \mathscr{H}$ continuously and densely. Thus we have two scales

$$\mathscr{G} \subset \mathscr{G}_a \subset \mathscr{H} \subset \mathscr{G}_a^* \subset \mathscr{G}^*,$$
$$\mathscr{G} \subset \mathscr{G}_b \subset \mathscr{H} \subset \mathscr{G}_b^* \subset \mathscr{G}^*.$$

Then let $A_0 \in \mathcal{B}(\mathscr{G}_a, \mathscr{G}_a^*)$ and $B_0 \in \mathcal{B}(\mathscr{G}_b, \mathscr{G}_b^*)$ such that $A_0 - z : \mathscr{G}_a \to \mathscr{G}_a^*$ and $B_0 - z : \mathscr{G}_b \to \mathscr{G}_b^*$ are bijective for some number z. According to Lemma A.1 we can associate to A_0, B_0 closed densely defined operators $A = \widehat{A_0}, B = \widehat{B_0}$ in $\mathscr{H}$, such that the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense subspaces of $\mathscr{G}_a$ and the domains $\mathcal{D}(B)$ and $\mathcal{D}(B^*)$ are dense subspaces of $\mathscr{G}_b$. If we also have $\mathcal{D}(A) \subset \mathscr{G}$ densely, $\mathcal{D}(A^*) \subset \mathscr{G}$ and $\mathcal{D}(B) \subset \mathscr{G}$, then all the conditions of the assumption (AB) are fulfilled with $\widetilde{A} = A_0|\mathscr{G}$ and $\widetilde{B} = B_0|\mathscr{G}$.

The rôle of the assumption (AB) is to allow us to give a rigorous meaning to the formal relation, where $z \in \rho(A) \cap \rho(B)$,

$$(A - z)^{-1} - (B - z)^{-1} = (A - z)^{-1}(B - A)(B - z)^{-1}. \tag{3.14}$$

Recall that $z \in \rho(A)$ if and only if $\bar{z} \in \rho(A^*)$ and then $(A^* - \bar{z})^{-1} = (A - z)^{-1*}$. Thus we have $(A - z)^{-1*}\mathscr{H} \subset \mathscr{G}$ by the assumption (AB) and this allows one to deduce that $(A - z)^{-1}$ extends to a unique continuous operator $\mathscr{G}^* \to \mathscr{H}$, that we shall denote for the moment by R_z. From $R_z(A - z)u = u$ for $u \in \mathcal{D}(A)$ we get, by density of $\mathcal{D}(A)$ in $\mathscr{G}$ and continuity, $R_z(\widetilde{A} - z)u = u$ for $u \in \mathscr{G}$, in particular

$$(B - z)^{-1} = R_z(\widetilde{A} - z)(B - z)^{-1}.$$

On the other hand, the identity

$$(A - z)^{-1} = (A - z)^{-1}(B - z)(B - z)^{-1} = R_z(\widetilde{B} - z)(B - z)^{-1}$$

is trivial. Subtracting the last two relations we get

$$(A - z)^{-1} - (B - z)^{-1} = R_z(\widetilde{B} - \widetilde{A})(B - z)^{-1}$$

Since R_z is uniquely determined as extension of $(A-z)^{-1}$ to a continuous map $\mathscr{G}^* \to \mathscr{H}$, we shall keep the notation $(A-z)^{-1}$ for it. With this convention, the rigorous version of (3.14) that we shall use is:

$$(A-z)^{-1} - (B-z)^{-1} = (A-z)^{-1}(\widetilde{B} - \widetilde{A})(B-z)^{-1}. \tag{3.15}$$

Theorem 3.2 *Let A, B satisfy assumption (AB) and let us assume that there are a Banach module $\mathscr{K}$ and operators $S \in \mathcal{B}(\mathscr{K}, \mathscr{G}^*)$ and $T \in \mathcal{B}_0^l(\mathscr{G}, \mathscr{K})$ such that $\widetilde{B} - \widetilde{A} = ST$ and $(A-z)^{-1}S \in \mathcal{B}_q^l(\mathscr{K}, \mathscr{H})$ for some $z \in \rho(A) \cap \rho(B)$. Then the operator B is a compact perturbation of the operator A and $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A)$.*

Proof: It suffices to show that $R \equiv (A-z)^{-1} - (B-z)^{-1} \in \mathcal{B}_0^l(\mathscr{H})$, because the domains of A and B are included in $\mathscr{G}$, hence $R\mathscr{H} \subset \mathscr{G}$, which finishes the proof because of Proposition 2.16. Now due to (3.15) and to the factorization assumption, we can write R as a product $R = [(A-z)^{-1}S][T(B-z)^{-1}]$ where the first factor is in $\mathcal{B}_q^l(\mathscr{K}, \mathscr{H})$ and the second in $\mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$, so the product is in $\mathcal{B}_0^l(\mathscr{H})$. ■

Remarks 3.3 (1) We could have stated the assumptions of Theorem 3.2 in an apparently more general form, namely $B - A = \sum_{k=1}^{n} S_k T_k$ with operators $S_k \in \mathcal{B}(\mathscr{K}_k, \mathscr{G}^*)$ and $T_k \in \mathcal{B}(\mathscr{G}, \mathscr{K}_k)$. But we are reduced to the stated version of the assumption by considering the Hilbert module $\mathscr{K} = \oplus \mathscr{K}_k$ and $S = \oplus S_k, T = \oplus T_k$.
(2) If $V \in \mathcal{K}(\mathscr{G}, \mathscr{G}^*)$ and if $\mathscr{K}$ is an infinite dimensional module, then there are operators $S \in \mathcal{B}(\mathscr{K}, \mathscr{G}^*)$ and $T \in \mathcal{K}(\mathscr{G}, \mathscr{K})$ such that $V = ST$ (the proof is an easy exercise). This and the preceding remark show that compact contributions to $\widetilde{B} - \widetilde{A}$ are trivially covered by the factorization assumption.

If A is self-adjoint then the conditions on A in assumption (AB) are satisfied if $\mathcal{D}(A) \subset \mathscr{G} \subset \mathcal{D}(|A|^{1/2})$ densely (see the Appendix). Moreover, if A is semibounded, then this condition is also necessary. In particular, we have:

Corollary 3.4 *Let A, B be self-adjoint operators on $\mathscr{H}$ such that*

$$\mathcal{D}(A) \subset \mathscr{G} \subset \mathcal{D}(|A|^{1/2}) \text{ and } \mathcal{D}(B) \subset \mathscr{G} \subset \mathcal{D}(|B|^{1/2}) \text{ densely.}$$

Let $\widetilde{A}, \widetilde{B}$ be the unique extensions of A, B to operators in $\mathscr{B}(\mathscr{G}, \mathscr{G}^*)$. Assume that there is a Hilbert module $\mathscr{K}$ and that $\widetilde{B} - \widetilde{A} = S^*T$ for some $S \in \mathcal{B}(\mathscr{G}, \mathscr{K})$ and $T \in \mathcal{B}_0^l(\mathscr{G}, \mathscr{K})$ such that $S(A-z)^{-1} \in \mathcal{B}_q^r(\mathscr{H}, \mathscr{K})$ for some $z \in \rho(A) \cap \rho(B)$. Then B is a compact perturbation of A and $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A)$.

The results which follow are either corollaries of Theorem 3.2 or are versions of the theorem based on essentially the same proof. We shall use the terminology and the facts established in the Appendix.

The next theorem is convenient for applications to differential operators in divergence form. Note that Lemma A.1 is implicitly used: the operators D^*aD and D^*bD considered below belong to $\mathcal{B}(\mathscr{G},\mathscr{G}^*)$ and we denote by Δ_a and Δ_b the operators on $\mathscr{H}$ associated to them. The notation $\mathcal{B}^l_{00}(\mathscr{E},\mathscr{E}^*)$ is introduced in (2.7).

Theorem 3.5 *Let $(\mathscr{E},\mathscr{K})$ be an arbitrary Friedrichs module and let $D\in\mathcal{B}(\mathscr{G},\mathscr{E})$, $a,b\in\mathcal{B}(\mathscr{E},\mathscr{E}^*)$ and $z\in\mathbb{C}$ such that:*
(1) *The operators D^*aD-z and D^*bD-z are bijective maps $\mathscr{G}\to\mathscr{G}^*$,*
(2) $a-b\in\mathcal{B}^l_{00}(\mathscr{E},\mathscr{E}^*)$,
(3) $D(\Delta_a^*-\bar{z})^{-1}\in\mathcal{B}^r_q(\mathscr{H},\mathscr{K})$.
Then Δ_b is a compact perturbation of Δ_a.

Proof: We give a proof independent of Theorem 3.2, although we could apply this theorem. Clearly Δ_a-z and Δ_b-z extend to bijections $\mathscr{G}\to\mathscr{G}^*$ and the identity

$$R:=(\Delta_a-z)^{-1}-(\Delta_b-z)^{-1}=(\Delta_a-z)^{-1}D^*(b-a)D(\Delta_b-z)^{-1}$$

holds in $\mathcal{B}(\mathscr{G}^*,\mathscr{G})$, hence in $\mathcal{B}(\mathscr{H})$. Since the domains of Δ_a and Δ_b are included in $\mathscr{G}$, we have $R\mathscr{H}\subset\mathscr{G}$. Thus, according to Proposition 2.16, it suffices to show that $R\in\mathcal{B}^l_0(\mathscr{H})$. Since the space $\mathcal{B}^l_0(\mathscr{H})$ is norm closed and since by hypothesis we can approach $b-a$ in norm in $\mathcal{B}(\mathscr{E},\mathscr{E}^*)$ by operators in $\mathcal{B}^l_0(\mathscr{K})$, it suffices to show that

$$(D(\Delta_a^*-\bar{z})^{-1})^*cD(\Delta_b-z)^{-1}\in\mathcal{B}^l_0(\mathscr{H})$$

if $c\in\mathcal{B}^l_0(\mathscr{K})$. But this is clear because $cD(\Delta_b-z)^{-1}\in\mathcal{B}^l_0(\mathscr{H},\mathscr{K})$ and $(D(\Delta_a^*-\bar{z})^{-1})^*\in\mathcal{B}^l_q(\mathscr{K},\mathscr{H})$ by Proposition 2.20. ■

By (2.8) we have $\mathcal{K}(\mathscr{E},\mathscr{E}^*)\subset\mathcal{B}^l_{00}(\mathscr{E},\mathscr{E}^*)$, but the case $a-b\in\mathcal{K}(\mathscr{E},\mathscr{E}^*)$ is trivial from the point of view of this paper. Although the space $\mathcal{B}^l_{00}(\mathscr{E},\mathscr{E}^*)$ is much larger than $\mathcal{K}(\mathscr{E},\mathscr{E}^*)$, it is not satisfactory in some applications, cf. Example 2.17 and Remark 4.2. However, we can allow still more general perturbations and obtain more explicit results if we impose more structure on the modules.

We now describe such improvements for the case of X-modules, where X is a non-compact locally compact abelian group We shall need the following fact.

Lemma 3.6 *Let $(\mathscr{G},\mathscr{H})$ and $(\mathscr{E},\mathscr{K})$ be stable Friedrichs X-modules over the group X. Let $D\in\mathcal{B}(\mathscr{G},\mathscr{E})$ and $a\in\mathcal{B}(\mathscr{E},\mathscr{E}^*)$ be operators of class $C^{\mathrm{u}}(Q)$ such that $D^*aD-z:\mathscr{G}\to\mathscr{G}^*$ is bijective for some complex number z and let Δ_a the operator on $\mathscr{H}$ associated to D^*aD. Then the operator $D(\Delta_a-z)^{-1}\in\mathcal{B}(\mathscr{H},\mathscr{E})$ is decay preserving.*

Proof: The lemma is an easy consequence of Propositions 2.31 and 2.32. Indeed, due to Proposition 2.32, it suffices to show that the operator $D(\Delta_a - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{E})$. We shall prove more, namely that $D(D^* aD - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{E})$. Since D is of class $C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{E})$, and due to (i) of Proposition 2.31, it suffices to show that $(D^* aD - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{G})$. But $D^* aD - z$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{G}^*)$ by (i) and (iii) of Proposition 2.31 and is a bijective map $\mathscr{G} \to \mathscr{G}^*$, so the result follows from (ii) of Proposition 2.31. ■

Theorem 3.7 *Let X be an abelian locally compact group and let $(\mathscr{G}, \mathscr{H})$ be a compact stable Friedrichs X-module and $(\mathscr{E}, \mathscr{K})$ a stable Friedrichs X-modules. Assume that $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$ and $a, b \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ are operators of class $C^{\mathrm{u}}(Q)$ such that the operators $D^* aD - z$ and $D^* bD - z$ are bijective maps $\mathscr{G} \to \mathscr{G}^*$ for some complex number z. If $a - b \in \mathcal{B}_0^l(\mathscr{E}, \mathscr{E}^*)$ then Δ_b is a compact perturbation of Δ_a.*

Proof: The proof is a repetition of that of Theorem 3.5. The only difference is that we write directly

$$R = (D(\Delta_a^* - \bar{z})^{-1})^* (b - a) D(\Delta_b - z)^{-1}$$

and observe that $(b - a)D(\Delta_b - z)^{-1} \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{E}^*)$ and that $(D(\Delta_a^* - \bar{z})^{-1})^*$ as an operator $\mathscr{E}^* \to \mathscr{H}$ is decay preserving by (2) of Proposition 2.20 and because the operator $D(\Delta_a^* - \bar{z})^{-1} : \mathscr{H} \to \mathscr{E}$ is decay preserving by Lemma 3.6. ■

We finish with the simplest corollary of Theorem 3.2 which nevertheless covers interesting examples of differential operators of any order.

Theorem 3.8 *Assume that $(\mathscr{G}, \mathscr{H})$ is a compact stable Friedrichs X-module over the group X and that condition (AB) is satisfied. Assume, furthermore, that $\widetilde{A} - z : \mathscr{G} \to \mathscr{G}^*$ is bijective for some $z \in \rho(A) \cap \rho(B)$ and that $\widetilde{A} \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{G}^*)$. If $\widetilde{B} - \widetilde{A} \in \mathcal{B}_0^l(\mathscr{G}, \mathscr{G}^*)$, then B is a compact perturbation of A.*

Proof: We apply Theorem 3.2 with $\mathscr{K} = \mathscr{G}^*$, S the identity operator and $T = \widetilde{B} - \widetilde{A}$. Then $(\widetilde{A} - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{G})$ by (ii) of Proposition 2.31, hence $(\widetilde{A} - z)^{-1} \in \mathcal{B}_q(\mathscr{G}^*, \mathscr{G})$ by Proposition 2.32. But this is stronger than $(\widetilde{A} - z)^{-1} \in \mathcal{B}_q^l(\mathscr{G}^*, \mathscr{H})$, as follows from Lemma 2.29. ■

4 Pseudo-differential operators

4.1 Operators in divergence form on Euclidean spaces

Our first example is in the context of Theorem 3.7. We shall take $X = \mathbb{R}^n$, let E be a finite dimensional Hilbert space, and let $\mathscr{H} = L^2(X; E)$ with the obvious

Hilbert X-module structure. If $s \in \mathbb{R}$ we denote by $\mathscr{H}^s$ the usual Sobolev space of vector valued functions. Then for each $s > 0$ the couple $(\mathscr{H}^s, \mathscr{H})$ is a compact stable Friedrichs X-module, cf. Examples 2.10, 2.12 and 2.25.

Let us describe the objects which appear in Theorem 3.7 in the present context. We fix an integer $m \geq 1$ and take $\mathscr{G} = \mathscr{H}^m$. Let $\mathscr{K} = \bigoplus_{|\alpha| \leq m} \mathscr{H}_\alpha$, where $\mathscr{H}_\alpha \equiv \mathscr{H}$, with the natural direct sum Hilbert X-module structure. Here α are multi-indices $\alpha \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then we define

$$\mathscr{E} = \bigoplus_{|\alpha| \leq m} \mathscr{H}^{m-|\alpha|} = \{(u_\alpha)_{|\alpha| \leq m} \in \mathscr{K} \mid u_\alpha \in \mathscr{H}^{m-|\alpha|}\}$$

equipped with the Hilbert direct sum structure. It is obvious that $(\mathscr{E}, \mathscr{K})$ is a stable Friedrichs X-module (but not compact).

We set $P_k = -i\partial_k$, where ∂_k is the derivative with respect to the k-th variable, and $P^\alpha = P_1^{\alpha_1} \dots P_n^{\alpha_n}$ if $\alpha \in \mathbb{N}^n$. Then for $u \in \mathscr{G}$ let $Du = (P^\alpha u)_{|\alpha| \leq m} \in \mathscr{K}$. Since

$$\|Du\|^2 = \sum_{|\alpha| \leq m} \|P^\alpha u\|^2 = \|u\|^2_{\mathscr{H}^m}$$

we see that $D : \mathscr{G} \to \mathscr{K}$ is a linear isometry. Moreover, we have defined $\mathscr{E}$ such as to have $D\mathscr{G} \subset \mathscr{E}$, hence $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$. We shall prove now that $D \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{E})$ (in fact, much more). We have, with natural notations,

$$V_k^{-1} D V_k = (V_k^{-1} P^\alpha V_k)_{|\alpha| \leq m} = ((P + k)^\alpha)_{|\alpha| \leq m}$$

and this a polynomial in k with coefficients in $\mathcal{B}(\mathscr{G}, \mathscr{E})$, hence the assertion.

We shall identify $\mathscr{H}^* = \mathscr{H}$ and $\mathscr{K}^* = \mathscr{K}$, which implies $\mathscr{G}^* = \mathscr{H}^{-m}$ and

$$\mathscr{E}^* = \oplus_{|\alpha| \leq m} \mathscr{H}^{|\alpha| - m}.$$

The operator $D^* \in \mathcal{B}(\mathscr{E}^*, \mathscr{G}^*)$ acts as follows:

$$D^*(u_\alpha)_{|\alpha| \leq m} = \sum_{|\alpha| \leq m} P^\alpha u_\alpha \in \mathscr{H}^{-m},$$

because $u_\alpha \in \mathscr{H}^{|\alpha| - m}$.

By taking into account the given expressions for $\mathscr{E}$ and $\mathscr{E}^*$ we see that we can identify an operator $a \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ with a matrix of operators $a = (a_{\alpha\beta})_{|\alpha|, |\beta| \leq m}$, where $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha| - m})$ and

$$a(u_\beta)_{|\beta| \leq m} = \big(\sum_{|\beta| \leq m} a_{\alpha\beta} u_\beta \big)_{|\alpha| \leq m}.$$

Then we clearly have

$$D^*aD = \sum_{|\alpha|,|\beta|\leq m} P^\alpha a_{\alpha\beta} P^\beta. \tag{4.16}$$

which is a general version of a differential operator in divergence form. We must, however, emphasize that our $a_{\alpha\beta}$ are not necessarily ($B(E)$ valued) functions, they could be pseudo-differential or more general operators.

In view of the statement of the next theorem, we note that, since the Sobolev spaces are Banach X-modules, the class of regularity $C^{\mathrm{u}}(Q; \mathscr{H}^s, \mathscr{H}^t)$ is well defined for all real s, t. A bounded operator $S : \mathscr{H}^s \to \mathscr{H}^t$ belongs to this class if and only if the map $k \mapsto V_{-k} S V_k \in \mathcal{B}(\mathscr{H}^s, \mathscr{H}^t)$ is norm continuous. In particular, this condition is trivially satisfied if S is the operator of multiplication by a function, because then V_k commutes with S. Since the coefficients $a_{\alpha\beta}$ of the differential expression (4.16) are usually assumed to be functions, this is barely a restriction in the setting of the next theorem. The condition $S \in \mathcal{B}_0^l(\mathscr{H}^s, \mathscr{H}^t)$ is also well defined and it is easily seen that it is equivalent to

$$\lim_{r\to\infty} \|\theta(Q/r)S\|_{\mathscr{H}^s\to\mathscr{H}^t} = 0 \tag{4.17}$$

where θ is a C^∞ function on X equal to zero on a neighborhood of the origin and equal to one on a neighborhood of infinity. Now we can state the following immediate consequence of Theorem 3.7.

Proposition 4.1 *Let $a_{\alpha\beta}$ and $b_{\alpha\beta}$ be operators of class $C^{\mathrm{u}}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$ and such that the operators $D^*aD - z$ and $D^*bD - z$ are bijective maps $\mathscr{H}^m \to \mathscr{H}^{-m}$ for some complex z. Let Δ_a and Δ_b be the operators in $\mathscr{H}$ associated to D^*aD and D^*bD respectively. Assume that*

$$\lim_{r\to\infty} \|\theta(Q/r)(a_{\alpha\beta} - b_{\alpha\beta})\|_{\mathscr{H}^{m-|\beta|}\to\mathscr{H}^{|\alpha|-m}} = 0 \tag{4.18}$$

for each α, β, where θ is a function as above. Then Δ_b is a compact perturbation of Δ_a and the operators Δ_a and Δ_b have the same essential spectrum.

Example: In the simplest case the coefficients $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the principal parts (i.e. $|\alpha| = |\beta| = m$) are functions. Then the conditions become: $a_{\alpha\beta}$ and $b_{\alpha\beta}$ belong to $L^\infty(X)$ and $|a_{\alpha\beta}(x) - b_{\alpha\beta}(x)| \to 0$ as $|x| \to \infty$. Of course, the assumptions on the lowest order coefficients are much more general.

Example: We show here that "highly oscillating potentials" do not modify the essential spectrum. If $m = 1$ then the terms of order one of D^*aD are of the form $S = \sum_{k=1}^n (P_k v_k' + v_k'' P_k)$, where $v_k' \in \mathcal{B}(\mathscr{H}^1, \mathscr{H})$ and $v_k'' \in \mathcal{B}(\mathscr{H}, \mathscr{H}^{-1})$.

Choose $v_k \in \mathcal{B}(\mathscr{H}^1, \mathscr{H})$ symmetric in $\mathscr{H}$ and let $v'_k = iv_k, v''_k = -iv_k$. Then $S = [iP, v] \equiv \operatorname{div} v$, with natural notations, can also be thought as a term of order zero. Now assume that v_k are bounded Borel functions and consider a similar term $T = [iP, w]$ for D^*bD. Then the condition $|v_k(x) - w_k(x)| \to 0$ as $|x| \to \infty$ suffices to ensure the stability of the essential spectrum. However, the difference $S - T$ could be a function which does not tend to zero at infinity in a simple sense, being only "highly oscillating". An explicit example in the case $n = 1$ is the following: a perturbation of the form $\exp(x)(1 + |x|)^{-1} \cos(\exp(x))$ is allowed because it is the derivative of $(1 + |x|)^{-1} \sin(\exp(x))$ plus a function which tends to zero at infinity.

In order to apply Proposition 4.1 we need that $D^*aD - z : \mathscr{H}^m \to \mathscr{H}^{-m}$ be bijective for some $z \in \mathbb{C}$, and similarly for b. A standard way of checking this is to require the following *coercivity condition*:

$$(C) \begin{cases} \text{there are } \mu, \nu > 0 \text{ such that for all } u \in \mathscr{H}^m : \\ \sum_{|\alpha|,|\beta| \le m} \operatorname{Re} \langle P^\alpha u, a_{\alpha\beta} P^\beta u \rangle \ge \mu \|u\|^2_{\mathscr{H}^m} - \nu \|u\|^2_{\mathscr{H}} \end{cases}$$

Example: One often imposes a stronger ellipticity condition that we describe below. Observe that the coefficients of the highest order part of D^*aD defined by $A_0 = \sum_{|\alpha|=|\beta|=m} P^\alpha a_{\alpha\beta} P^\beta$ are operators $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H})$. Then ellipticity means:

$$(Ell) \begin{cases} \text{there is } \mu > 0 \text{ such that if } u_\alpha \in \mathscr{H} \text{ for } |\alpha| = m \text{ then} \\ \sum_{|\alpha|=|\beta|=m} \operatorname{Re} \langle u_\alpha, a_{\alpha\beta} u_\beta \rangle \ge \mu \sum_{|\alpha|=m} \|u_\alpha\|^2_{\mathscr{H}}. \end{cases}$$

But we emphasize that, our conditions on the lower order terms being very general (e.g. the $a_{\alpha\beta}$ could be differential operators, so the terms of formally lower order could be of order $2m$ in fact), we have to supplement the ellipticity condition (Ell) with a condition saying that the rest of the terms $A_1 = \sum_{|\alpha|+|\beta|<2m} P^\alpha a_{\alpha\beta} P^\beta$ is small with respect to A_0. For example, we may require the existence of some $\delta < \mu$ and $\gamma > 0$ such that

$$| \sum_{|\alpha|+|\beta|<2m} \operatorname{Re} \langle P^\alpha u, a_{\alpha\beta} P^\beta u \rangle | \le \delta \|u\|^2_{\mathscr{H}^m} + \gamma \|u\|^2_{\mathscr{H}}. \tag{4.19}$$

This is satisfied if $A_1 \mathscr{H}^m \subset \mathscr{H}^{-m+\theta}$ for some $\theta > 0$, because for each $\varepsilon > 0$ there is $c(\varepsilon) < \infty$ such that $\|u\|_{\mathscr{H}^{m-\theta}} \le \varepsilon \|u\|_{\mathscr{H}^m} + c(\varepsilon) \|u\|_{\mathscr{H}}$.

Remark 4.2 If we use Theorem 3.5 in the context of this section then we get the same conditions on the coefficients $a_{\alpha\beta} - b_{\alpha\beta}$ of the principal part (i.e. such that $|\alpha| = |\beta| = m$) of the operator $a - b$ but those on the lower order coefficients are less general. Indeed, if $s + t > 0$ the space $\mathcal{B}^l_{00}(\mathscr{H}^s, \mathscr{H}^{-t})$ defined as the closure of $\mathcal{B}^l_0(\mathscr{H})$ in $\mathcal{B}(\mathscr{H}^s, \mathscr{H}^{-t})$ does not contain operators of order $s + t$, while $\mathcal{B}^l_0(\mathscr{H}^s, \mathscr{H}^{-t})$ contains such operators.

4.2 A class of hypoelliptic operators on abelian groups

In this subsection X will be a locally compact non-compact non-discrete abelian group. We also fix a finite dimensional complex Hilbert space E and take $\mathscr{H} = L^2(X;E)$ equipped with its natural Hilbert X-module structure. Note that, according to our conventions, the unitary representation of X^* is given by the multiplication operators $V_k = k(Q)$.

Let $w : X^* \to [1,\infty[$ be a continuous function satisfying $w(k) \to \infty$ as $k \to \infty$ and such that $w(k'k) \leq \omega(k')w(k)$ holds for some function ω and all k', k. We shall assume that ω is the smallest function satisfying the preceding estimate. It is clear then that ω is sub-multiplicative in the sense defined in Remark 2.26 (see [Ho, Section 10.1] for this construction).

Then $w(P)$ is a self-adjoint operator on $\mathscr{H}$ with $w(P) \geq 1$ (see the last part of Subsection 2.4 for this notation). We denote $\mathscr{H}^w = \mathcal{D}(w(P))$ and equip it with the Banach X-module structure given by the norm $\|u\|_w = \|w(P)u\|$ and the representation $V_k|\mathscr{H}^w$. Obviously, this space is a generalization of the usual notion of Sobolev spaces.

Lemma 4.3 *$(\mathscr{H}^w, \mathscr{H})$ is a compact stable Friedrichs X-module.*

Proof: If $\varphi \in C_0(X)$ then $\varphi(Q)w(P)^{-1}$ is a compact operator because w^{-1} belongs to $C_0(X)$, hence $\varphi(Q) \in \mathcal{K}(\mathscr{H}^w, \mathscr{H})$. Then observe that $V_k^{-1}w(P)V_k = w(kP)$ and $w(kP) \leq \omega(k)w(P)$. Thus V_k leaves stable $\mathscr{H}^w$ and we have the estimate $\|V_k\|_{\mathcal{B}(\mathscr{H}^w)} \leq \omega(k)$. ■

We call *uniformly hypoelliptic* an operator A on $\mathscr{H}$ such that there are w as above and an operator $\widetilde{A} \in \mathcal{B}(\mathscr{H}^w, \mathscr{H}^{w*})$ such that $\widetilde{A} - z : \mathscr{H}^w \to \mathscr{H}^{w*}$ is bijective for some complex z and such that A is the operator induced by $\widetilde{A}$ in $\mathscr{H}$ (see the Appendix). For example, the constant coefficients case with $E = \mathbb{C}$ corresponds to the choice $A = h(P)$ with $h : X^* \to \mathbb{C}$ a Borel function such that $c'w^2 \leq 1 + |h| \leq c''w^2$ and such that the range of h is not dense in $\mathbb{C}$. We shall justify our terminology in the remark at the end of this subsection.

Theorem 3.8 is quite well adapted to show the stability of the essential spectrum of such operators under perturbations which are small at infinity. We stress that the differential operators covered by these results can be of any order and that in the usual case when the coefficients are complex measurable functions a condition of the type $\widetilde{A} \in C^{\mathrm{u}}(Q; \mathscr{H}^w, \mathscr{H}^{w*})$ is very general, if not automatically satisfied (see the remark at the end of this subsection). Hence the only condition really relevant in this context is $\widetilde{B} - \widetilde{A} \in \mathcal{B}_0^l(\mathscr{H}^w, \mathscr{H}^{w*})$ and the main point is that it allows perturbations of the higher order coefficients even in the non-smooth case.

It is clear that these results can be used to establish the stability of the essential spectrum of pseudo-differential operators on finite dimensional vector spaces over local fields (see [Sa, Ta]) under perturbations of the same order.

We shall give only one explicit example of physical interest, that of Dirac operators. Let $X = \mathbb{R}^n$ and let $\alpha_0 \equiv \beta, \alpha_1, \dots, \alpha_n$ be symmetric operators on E such that $\alpha_j\alpha_k + \alpha_k\alpha_j = \delta_{jk}$. Then the free Dirac operator is $D = \sum_{k=1}^n \alpha_k P_k + m\beta$ for some real number m. The natural compact stable Friedrichs X-module in this context is $(\mathscr{H}^{1/2}, \mathscr{H})$. Note that we use the same notation $\mathscr{H}^s$ for Sobolev spaces of E-valued functions.

Proposition 4.4 *Let V, W be measurable functions on X with values symmetric operators on E and such that the operators of multiplication by V and W define continuous maps $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ and $V - W \in \mathcal{B}_0(\mathscr{H}^{1/2}, \mathscr{H}^{-1/2})$. Assume that $D + V + i$ and $D + W + i$ are bijective maps $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$. Then $D + V$ and $D + W$ induce self-adjoint operators A and B in $\mathscr{H}$, B is a compact perturbation of A, and $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A)$.*

This follows immediately from Theorem 3.8. We stress that the main new feature of this result is that the "unperturbed" operator A is locally as singular as the "perturbed" one B. The assumptions imposed on V, W are quite general, compare with [Ar, AY, Kl, N1, N2].

Remark: In order to clarify the relation between the notion of uniform hypoellipticity introduced above and the original notion of hypoellipticity due to Hörmander, we shall consider the case of differential operators on $\mathbb{R}^n$ (which is identified with its dual group in the standard way). Assume first that h is a polynomial on $\mathbb{R}^n$ and that $A = h(P)$. Then the function defined by $w(k)^4 = \sum_\alpha |h^{(\alpha)}(k)|^2$ satisfies $w(k' + k) \leq (1 + c|k'|)^{m/2} w(k)$, where c is a number and m is the order of h, see [Ho, Example 10.1.3]. Now the "form domain" of the operator $h(P)$ in $L^2(\mathbb{R}^n)$ is the space $\mathscr{G} = \mathcal{D}(|h(P)|^{1/2})$ and this domain is stable under $V_k = \exp i\langle k, Q\rangle$ if and only the function w satisfies $w^2 \leq c(1 + |h|)$, see Lemma 7.6.7 in [ABG]. On the other hand, Definition 11.1.2 and Theorem 11.1.3 from [Ho] show that A is hypoelliptic if and only if $h^{(\alpha)}(k)/h(k) \to 0$ when $k \to \infty$, for all $\alpha \neq 0$. So in this case we have $c'w^2 \leq 1 + |h| \leq c''w^2$ and the operator $h(P)$ is uniformly hypoelliptic in our sense if $h(\mathbb{R}^n)$ is not dense in $\mathbb{C}$. If $n = 2$ then $h(k) = k_1^4 + k_2^2$ is a simple example of polynomial which satisfies all these conditions but is not elliptic. See [GM, Subsections 2.7-2.10] for the case of matrix valued functions h.

In the variable coefficient case the notion of hypoellipticity defined in [Ho, Definition 13.4.3] is a local one and one may consider different global versions. For instance, [Ho, Theorem 13.4.4] suggests that the notion we introduced above is natural for operators of uniform constant strength. But the uniform constant strength condition is not satisfied by the operators with polynomial coefficients, for example, hence such operators are not uniformly hypoelliptic in our sense in general.

5 Riemannian manifolds

Let $\mathscr{H}, \mathscr{K}$ be two Hilbert spaces identified with their adjoints and d a closed densely defined operator mapping $\mathscr{H}$ into $\mathscr{K}$. Let $\mathscr{G} = \mathcal{D}(\mathrm{d})$ equipped with the graph norm, so $\mathscr{G} \subset \mathscr{H}$ continuously and densely and $\mathrm{d} \in \mathcal{B}(\mathscr{G}, \mathscr{K})$.

Then the quadratic form $\|\mathrm{d}u\|^2_{\mathscr{K}}$ on $\mathscr{H}$ with domain $\mathscr{G}$ is positive densely defined and closed. Let Δ be the positive self-adjoint operator on $\mathscr{H}$ associated to it. In fact $\Delta = \mathrm{d}^*\mathrm{d}$, where the adjoint d^* of d is a closed densely defined operator mapping $\mathscr{K}$ into $\mathscr{H}$.

Now let $\lambda \in \mathcal{B}(\mathscr{H})$ and $\Lambda \in \mathcal{B}(\mathscr{K})$ be self-adjoint and such that $\lambda \geq c$ and $\Lambda \geq c$ for some real $c > 0$. Then we can define new Hilbert spaces $\widetilde{\mathscr{H}}$ and $\widetilde{\mathscr{K}}$ as follows:

$$(*) \quad \begin{cases} \widetilde{\mathscr{H}} = \mathscr{H} \text{ as vector space and } \langle u \mid v \rangle_{\widetilde{\mathscr{H}}} = \langle u \mid \lambda v \rangle_{\mathscr{H}}, \\ \widetilde{\mathscr{K}} = \mathscr{K} \text{ as vector space and } \langle u \mid v \rangle_{\widetilde{\mathscr{K}}} = \langle u \mid \Lambda v \rangle_{\mathscr{K}}. \end{cases}$$

Since $\mathscr{H} = \widetilde{\mathscr{H}}$ and $\mathscr{K} = \widetilde{\mathscr{K}}$ as topological vector spaces, the operator $\mathrm{d} : \mathscr{G} \subset \widetilde{\mathscr{H}} \to \widetilde{\mathscr{K}}$ is still a closed densely defined operator, hence the quadratic form $\|\mathrm{d}u\|^2_{\widetilde{\mathscr{K}}}$ on $\widetilde{\mathscr{H}}$ with domain $\mathscr{G}$ is positive, densely defined and closed. We shall denote by $\widetilde{\Delta}$ the positive self-adjoint operator on $\widetilde{\mathscr{H}}$ associated to it.

We can express $\widetilde{\Delta}$ in more explicit terms as follows. Denote by $\widetilde{\mathrm{d}}$ the operator d when viewed as acting from $\widetilde{\mathscr{H}}$ to $\widetilde{\mathscr{K}}$. Then $\widetilde{\Delta} = \widetilde{\mathrm{d}}^*\widetilde{\mathrm{d}}$, where $\widetilde{\mathrm{d}}^* : \mathcal{D}(\widetilde{\mathrm{d}}^*) \subset \widetilde{\mathscr{K}} \to \widetilde{\mathscr{H}}$ is the adjoint of $\widetilde{\mathrm{d}} = \mathrm{d}$ with respect to the new Hilbert space structures (the spaces $\widetilde{\mathscr{H}}, \widetilde{\mathscr{K}}$ being also identified with their adjoints). It is easy to check that $\widetilde{\mathrm{d}}^* = \lambda^{-1}\mathrm{d}^*\Lambda$. Thus $\widetilde{\Delta} = \lambda^{-1}\mathrm{d}^*\Lambda\mathrm{d}$.

Now let (X, ρ) be a proper locally compact metric space (see the definition before Corollary 2.22) and let us assume that $\mathscr{H}$ and $\mathscr{K}$ are Hilbert X-modules.

Definition 5.1 *A closed densely defined map* $\mathrm{d} : \mathcal{D}(\mathrm{d}) \subset \mathscr{H} \to \mathscr{K}$ *is a* first order operator *if there is* $C \in \mathbb{R}$ *such that for each bounded Lipschitz function* φ *on* X *the form* $[\mathrm{d}, \varphi(Q)]$ *is a bounded operator and* $\|[\mathrm{d}, \varphi(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leq C \operatorname{Lip} \varphi$.

Here
$$\operatorname{Lip} \varphi = \inf_{x \neq y} |\varphi(x) - \varphi(y)| \rho(x, y)^{-1}.$$

In more explicit terms, we require

$$|\langle \mathrm{d}^* u, \varphi(Q)v\rangle_{\mathscr{H}} - \langle u, \varphi(Q)\mathrm{d}v\rangle_{\mathscr{K}}| \le C\,\mathrm{Lip}\,\varphi\,\|u\|_{\mathscr{K}}\|v\|_{\mathscr{H}}$$

for all $u \in \mathcal{D}(\mathrm{d}^*)$ and $v \in \mathcal{D}(d)$. Thus $\langle \mathrm{d}^* u, \varphi(Q)v\rangle - \langle u, \varphi(Q)\mathrm{d}v\rangle$ is a sesquilinear form on the dense subspace $\mathcal{D}(\mathrm{d}^*) \times \mathcal{D}(\mathrm{d})$ of $\mathscr{K} \times \mathscr{H}$ which is continuous for the topology induced by $\mathscr{H} \times \mathscr{K}$. Hence there is a unique continuous operator $[\mathrm{d}, \varphi(Q)] : \mathscr{H} \to \mathscr{K}$ such that

$$\langle \mathrm{d}^* u, \varphi(Q)v\rangle_{\mathscr{H}} - \langle u, \varphi(Q)\mathrm{d}v\rangle_{\mathscr{K}} = \langle u, [\mathrm{d}, \varphi(Q)]v\rangle_{\mathscr{K}}$$

for all $u \in \mathcal{D}(\mathrm{d}^*)$, $v \in \mathcal{D}(\mathrm{d})$ and $\|[\mathrm{d}, \varphi(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \le C\,\mathrm{Lip}\,\varphi$.

Lemma 5.2 *The operator* $\mathrm{d}(\Delta+1)^{-1}$ *is decay preserving.*

Proof: We shall prove that $S := \mathrm{d}(\Delta+1)^{-1}$ is a decay preserving operator with the help of Corollary 2.22, more precisely we show that $[S, \varphi(Q)]$ is a bounded operator if φ is a positive Lipschitz function. Let $\varepsilon > 0$ and $\varphi_\epsilon = \varphi(1+\varepsilon\varphi)^{-1}$. Then φ_ε is a bounded function with $|\varphi_\varepsilon| \le \varepsilon^{-1}$ and

$$|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| = \frac{|\varphi(x) - \varphi(y)|}{(1+\varepsilon\varphi(x))(1+\varepsilon\varphi(y))} \le |\varphi(x) - \varphi(y)|$$

hence $\mathrm{Lip}\,\varphi_\varepsilon \le \mathrm{Lip}\,\varphi$. Let $v \in \mathcal{D}(\mathrm{d})$ we have for all $u \in \mathcal{D}(\mathrm{d}^*)$:

$$\begin{aligned} |\langle \mathrm{d}^* u, \varphi_\epsilon(Q)v\rangle_{\mathscr{H}}| &= |\langle u, \varphi_\epsilon(Q)\mathrm{d}v\rangle_{\mathscr{K}} + \langle u, [\mathrm{d}, \varphi_\varepsilon(Q)]v\rangle_{\mathscr{K}}| \\ &\le \|u\|_{\mathscr{K}}(\varepsilon^{-1}\|\mathrm{d}v\|_{\mathscr{K}} + C\,\mathrm{Lip}\,\varphi_\varepsilon\,\|u\|_{\mathscr{H}}). \end{aligned}$$

Hence $\varphi_\varepsilon(Q)v \in \mathcal{D}(\mathrm{d}^{**}) = \mathcal{D}(\mathrm{d})$ because d is closed. Thus $\varphi_\varepsilon(Q)\mathcal{D}(\mathrm{d}) \subset \mathcal{D}(\mathrm{d})$ and by the closed graph theorem we get $\varphi_\varepsilon(Q) \in \mathcal{B}(\mathscr{G})$, where $\mathscr{G}$ is the domain of d equipped with the graph topology. This also implies that $\varphi_\varepsilon(Q)$ extends to an operator in $\mathcal{B}(\mathscr{G}^*)$ (note that $\varphi_\varepsilon(Q)$ is symmetric in $\mathscr{H}$).

Now, if we think of d as a continuous operator $\mathscr{G} \to \mathscr{K}$, then it has an adjoint $\mathrm{d}^* : \mathscr{K} \to \mathscr{G}^*$ which is the unique continuous extension of the operator $\mathrm{d}^* : \mathcal{D}(\mathrm{d}^*) \subset \mathscr{K} \to \mathscr{H} \subset \mathscr{G}^*$. Thus the canonical extension of Δ to an element of $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ is the product of $\mathrm{d} : \mathscr{G} \to \mathscr{K}$ with $\mathrm{d}^* : \mathscr{K} \to \mathscr{G}^*$ (note $\mathcal{D}(\mathrm{d})$ is the form domain of Δ). Then it is trivial to justify that we have in $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$:

$$[\Delta, \varphi_\varepsilon(Q)] = [\mathrm{d}^*, \varphi_\varepsilon(Q)]\mathrm{d} + \mathrm{d}^*[\mathrm{d}, \varphi_\varepsilon(Q)].$$

Here $[\mathrm{d}^*, \varphi_\varepsilon(Q)] = [\varphi_\varepsilon(Q), \mathrm{d}]^* \in \mathcal{B}(\mathscr{K}, \mathscr{H})$. Since $\Delta + 1 : \mathscr{G} \to \mathscr{G}^*$ is a linear homeomorphism, we then have in $\mathcal{B}(\mathscr{G}^*, \mathscr{G})$:

$$\begin{aligned} [\varphi_\varepsilon(Q), (\Delta+1)^{-1}] &= (\Delta+1)^{-1}[\Delta, \varphi_\varepsilon(Q)](\Delta+1)^{-1} \\ &= (\Delta+1)^{-1}[\varphi_\varepsilon(Q), \mathrm{d}]^*\mathrm{d}(\Delta+1)^{-1} \\ &+ (\Delta+1)^{-1}\mathrm{d}^*[\mathrm{d}, \varphi_\varepsilon(Q)](\Delta+1)^{-1}. \end{aligned}$$

Finally, taking once again into account the fact that $\varphi_\varepsilon(Q)$ leaves $\mathscr{G}$ invariant, we have:

$$\begin{aligned}[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}] \;=\; & [\varphi_\varepsilon(Q), \mathrm{d}](\Delta+1)^{-1} \\ & + \mathrm{d}(\Delta+1)^{-1}[\varphi_\varepsilon(Q), \mathrm{d}]^*\mathrm{d}(\Delta+1)^{-1} \\ & + \mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*[\mathrm{d}, \varphi_\varepsilon(Q)](\Delta+1)^{-1}.\end{aligned}$$

Hence:

$$\begin{aligned}\|[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \;\leq\; & \|[\varphi_\varepsilon(Q), \mathrm{d}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H})} \\ + \|\mathrm{d}(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|[\varphi_\varepsilon(Q), \mathrm{d}]^*\|_{\mathcal{B}(\mathscr{K},\mathscr{H})}&\|\mathrm{d}(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \\ + \|\mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{K})}\|[\mathrm{d}, \varphi_\varepsilon(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}&\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H})}.\end{aligned}$$

The most singular factor here is

$$\|\mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{K})} \leq \|\mathrm{d}\|_{\mathcal{B}(\mathscr{G},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})}\|\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{G}^*)}$$

and this is finite. Thus we get for a finite constant C_1:

$$\begin{aligned}\|[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \;\leq\; & C_1\|[\mathrm{d}, \varphi_\varepsilon(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \\ \leq\; & C_1\, C \,\mathrm{Lip}\, \varphi_\varepsilon \leq C_1\, C\, \mathrm{Lip}\, \varphi\end{aligned}$$

Now let $u \in \mathscr{K}_{\mathrm{c}}$ and $v \in \mathscr{H}_{\mathrm{c}}$. We get:

$$\begin{aligned}|\langle \varphi(Q)u, \mathrm{d}(\Delta+1)^{-1}v\rangle - \langle u, \mathrm{d}(\Delta+1)^{-1}\varphi(Q)v\rangle| \;=\; & \\ = \lim_{\varepsilon\to 0} |\langle \varphi_\varepsilon(Q)u, \mathrm{d}(\Delta+1)^{-1}v\rangle - \langle u, \mathrm{d}(\Delta+1)^{-1}\varphi_\varepsilon(Q)v\rangle| & \\ \leq C_1\, C\, \mathrm{Lip}\, \varphi &\end{aligned}$$

Thus $[\varphi(Q), \mathrm{d}(\Delta+1)^{-1}]$ is a bounded operator. ■

Theorem 5.3 *Let (X, ρ) be a proper locally compact metric space. Assume that $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs X-module and that $\mathscr{K}$ is a Hilbert X-module. Let* $\mathrm{d}, \lambda, \Lambda$ *be operators satisfying the following conditions:*
(i) d *is a closed first order operator from $\mathscr{H}$ to $\mathscr{K}$ with $\mathcal{D}(\mathrm{d}) = \mathscr{G}$;*
(ii) λ is a bounded self-adjoint operator on $\mathscr{H}$ with $\inf \lambda > 0$ and such that $\lambda - 1 \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ (e.g. $\lambda - 1 \in \mathcal{B}_0(\mathscr{H})$);
(iii) Λ is a bounded self-adjoint operator on $\mathscr{K}$ with $\inf \Lambda > 0$ and such that $\Lambda - 1 \in \mathcal{B}_0(\mathscr{K})$.
Then the self-adjoint operators Δ and $\widetilde{\Delta}$ have the same essential spectrum.

Proof: In this proof, we shall consider $\widetilde{\Delta}$ as an operator acting on $\mathscr{H}$. Since $\widetilde{\mathscr{H}} = \mathscr{H}$ as topological vector spaces and the notion of spectrum is purely topological, $\widetilde{\Delta}$ is a closed densely defined operator on $\mathscr{H}$ and it has the same spectrum as the self-adjoint $\widetilde{\Delta}$ on $\widetilde{\mathscr{H}}$. Moreover, if we define the essential spectrum $\sigma_{\text{ess}}(A)$ as the set of $z \in \mathbb{C}$ such that either $\ker(A - z)$ is infinite dimensional or the range of $A - z$ is not closed, we see that the essential spectrum is a topological notion, so $\sigma_{\text{ess}}(\widetilde{\Delta})$ is the same, whether we think of $\widetilde{\Delta}$ as operator on $\mathscr{H}$ or on $\widetilde{\mathscr{H}}$. Finally, with this definition of σ_{ess} we have $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$ if $(A - z)^{-1} - (B - z)^{-1}$ is a compact operator for some $z \in \rho(A) \cap \rho(B)$.

Thus it suffices to prove that $(\Delta + 1)^{-1} - (\widetilde{\Delta} + 1)^{-1} \in \mathcal{K}(\mathscr{H})$. Now we observe that

$$\widetilde{\Delta} + 1 = \lambda^{-1} d^* \Lambda d + 1 = \lambda^{-1}(\mathrm{d}^* \Lambda \mathrm{d} + \lambda)$$

and $\Delta_\Lambda = \mathrm{d}^* \Lambda \mathrm{d}$ is the positive self-adjoint operator on $\mathscr{H}$ associated to the closed quadratic form $\|\mathrm{d}u\|^2_{\widetilde{\mathscr{K}}}$ on $\mathscr{H}$ with domain $\mathscr{G}$. Thus $(\widetilde{\Delta} + 1)^{-1} = (\Delta_\Lambda + \lambda)^{-1} \lambda$ and

$$(\widetilde{\Delta} + 1)^{-1} - (\Delta_\Lambda + \lambda)^{-1} = (\Delta_\Lambda + \lambda)^{-1}(\lambda - 1) = [(\lambda - 1)(\Delta_\Lambda + \lambda)^{-1}]^*$$

The range of $(\Delta_\Lambda + \lambda)^{-1}$ is included in the form domain of $\Delta_\Lambda + \lambda$, which is $\mathscr{G}$. The map $(\Delta_\Lambda + \lambda)^{-1} : \mathscr{H} \to \mathscr{G}$ is continuous, by the closed graph theorem, and $\lambda - 1 : \mathscr{G} \to \mathscr{H}$ is compact. Hence $(\widetilde{\Delta} + 1)^{-1} - (\Delta_\Lambda + \lambda)^{-1}$ is compact. Similarly:

$$(\Delta + 1)^{-1} - (\Delta_\Lambda + \lambda)^{-1} = (\mathrm{d}^* \mathrm{d} + 1)^{-1} - (\mathrm{d}^* \Lambda \mathrm{d} + 1)^{-1} \in \mathcal{K}(\mathscr{H})$$

For this we use Theorem 3.5 with: $\mathscr{E} = \mathscr{K}$, $D = \mathrm{d}$, $a = 1$, $b = \Lambda$ and $z = -1$. Since $\mathrm{d}^*\mathrm{d}$ and $\mathrm{d}^* \Lambda d$ are positive self-adjoint operators on $\mathscr{H}$ with the same form domain $\mathscr{G}$, the first condition of Theorem 3.5 is satisfied. Then the second condition holds because $\Lambda - 1 \in \mathcal{B}_0^l(\mathscr{K})$. Thus it remains to observe that the operator $\mathrm{d}(\Delta + 1)^{-1}$ is decay preserving by Lemma 5.2. ■

We shall consider now an application of Theorem 5.3 to concrete Riemannian manifolds. It will be clear from what follows that we could treat Lipschitz manifolds with measurable metrics (see [DP, Hi, Te, We] for example), but the case of C^1 manifolds with locally bounded metrics suffices as an example.

Let X be a *non-compact* differentiable manifold of class C^1 and T^*X be its cotangent manifold, a topological vector fiber bundle over X whose fiber over x will be denoted T_x^*X. If $u : X \to \mathbb{R}$ is differentiable then $\mathrm{d}u(x) \in T_x^*X$ is its differential at the point x and its differential $\mathrm{d}u$ is a section of T^*X. Thus for the moment d is a linear map defined on the space of real $C^1(X)$ functions to the space of sections of T^*X.

We now assume that X is equipped with a measurable locally bounded Riemannian structure. To be precise, each T_x^*X is equipped with a scalar product $\langle\cdot|\cdot\rangle_x$ and the associated norm $\|\cdot\|_x$ satisfies the following condition:

$$(R)\ \begin{cases}\text{if } v \text{ is a continuous section of } T^*X \text{ over a compact set } K \text{ such that}\\ v(x)\neq 0 \text{ for } x\in K, \text{ then } x\mapsto \|v(x)\|_x \text{ is a bounded Borel map on}\\ K \text{ and } \|v(x)\|_x \geq c \text{ for some number } c>0 \text{ and all } x\in K.\end{cases}$$

This structure allows one to construct a metric compatible with the topology on X, the distance between two points being the infimum of the length of the Lipschitz curves connecting the points (see the references above). Since X was assumed to be non-compact, the metric space X is proper in the sense defined in Subsection 2.3 if and only if it is complete.

It will also be convenient to complexify these structures (i.e. replace T_x^*X by $T_x^*X\otimes\mathbb{C}$ and extend the scalar product as usual) and to keep the same notations for the complexified objects.

Now let μ be a positive measure on X such that:

$$(M)\ \begin{cases}\mu \text{ is absolutely continuous and its density is locally bounded}\\ \text{and locally bounded from below by strictly positive constants.}\end{cases}$$

We shall take $\mathscr{H}=L^2(X,\mu)$ and $\mathscr{K}$ equal to the completion of the space of continuous sections of T^*X with compact support under the natural norm

$$\|v\|_{\mathscr{K}}^2=\int_X \|v(x)\|_x^2 d\mu(x).$$

In fact, $\mathscr{K}$ is the space of (suitably defined) square integrable sections of T^*X.

The operator of exterior differentiation d induces a linear map $C_\mathrm{c}^1(X)\to\mathscr{K}$ which is easily seen to be closable as operator from $\mathscr{H}$ to $\mathscr{K}$ (this is a purely local problem and the hypotheses we put on the metric and the measure allow us to reduce ourselves to the Euclidean case). We shall keep the notation d for its closure and we note that its domain $\mathscr{G}$ is the natural first order Sobolev space $\mathscr{H}^1$ defined in this context as the closure of $C_\mathrm{c}^1(X)$ under the norm

$$\|u\|_{\mathscr{H}^1}^2=\int_X \Big(|u(x)|^2+\|\mathrm{d}u(x)\|_x^2\Big)d\mu(x).$$

Note that the self-adjoint operator $\Delta=\mathrm{d}^*\mathrm{d}$ is a slightly generalized form of the Laplace operator associated to the Riemannian structure of X because μ is not necessarily the Riemannian volume element (but we could choose it so).

We shall now consider perturbations of this structure. We assume that the perturbation preserves the local structure, although Theorem 5.3 allows us to go much further.

Theorem 5.4 *Let X be a non-compact manifold of class C^1 equipped with a Riemannian structure and a measure satisfying the conditions (R) and (M) and such that X is complete for the associated metric. Let λ be a bounded Borel function on X such that $\lambda(x) \geq c$ for some number $c > 0$ and $\lim_{x\to\infty} \lambda(x) = 1$. Assume that a new Riemann structure verifying (R) is given on X such that the associated norms $\| \cdot \|'_x$ verify $\alpha(x)\| \cdot \|_x \leq \| \cdot \|'_x \leq \beta(x)\| \cdot \|_x$ for some functions α, β such that $\lim_{x\to\infty} \alpha(x) = \lim_{x\to\infty} \beta(x) = 1$. Let Δ be as above and Δ' be the analog operator associated to the second Riemann structure and to the measure $\mu' = \lambda\mu$. Then $\sigma_{\text{ess}}(\Delta) = \sigma_{\text{ess}}(\Delta')$.*

Proof: We check that the assumptions of Theorem 5.3 are satisfied. We noted above that X is a proper metric space for the metric associated to the initial Riemann structure. The spaces $\mathscr{H}, \mathscr{K}$ have obvious X-module structures and for each $\varphi \in C_c(X)$ the operator $\varphi(Q) : \mathscr{H}^1 \to \mathscr{H}$ is compact. Indeed, by using partitions of unity, we may assume that the support of φ is contained in the domain of a local chart and then we are reduced to a known fact in the Euclidean case. Thus $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs X-module. To see that d is a first order operator we observe that if φ is Lipschitz then $[\mathrm{d}, \varphi]$ is the operator of multiplication by the differential $\mathrm{d}\varphi$ of φ and the estimate ess-sup $\|\mathrm{d}\varphi(x)\|_x \leq$ Lip φ is easy to obtain. The conditions on λ in Theorem 5.3 are trivially verified. So it remains to consider the operator Λ. For each $x \in X$ there is a unique operator $\Lambda_0(x)$ on $T^*_x X$ such that $\langle u|v\rangle'_x = \langle u|\Lambda_0(x)v\rangle_x$ for all $u, v \in T^*_x X$ and we have $\alpha(x)^2 \leq \Lambda_0(x) \leq \beta(x)^2$ by hypothesis. Here the inequalities must be interpreted with respect to the initial scalar product on $T^*_x X$. Thus the operator Λ on $\mathscr{K}$ is just the operator of multiplication by the function $x \mapsto \lambda(x)\Lambda(x)$ and the condition (iii) of Theorem 5.3 is clearly satisfied. ■

We note that if μ is the measure canonically associated to the initial Riemann structure then we can choose λ such that μ' be the measure associated to the second Riemann structure. In particular, if we have two locally L^∞ Riemannian metrics on a non-compact C^1 manifold, if the structures are asymptotically equivalent in the sense made precise in Theorem 5.4, and if the manifold is complete for one of the metrics (hence for the other too), then the Laplacians associated to the two metrics have the same essential spectrum. We stress that this is easy to prove (by using local regularity estimates for elliptic equations) if one of the metrics is locally Lipschitz. On the other hand, it is clear that our proof cover situations when X is not of class C^1. In fact, the arguments work without any modification if X is a Lipschitz manifold and a countable atlas has been chosen, because then the tangent space are well defined almost everywhere and the absolute continuity notions that we have used make sense.

The next result allows the functions λ, α, β to vanish at infinity in the weak sense defined by the filter $\mathcal{F}_\mu$ introduced in Example 2.9, but involves an implicit condition on the operator Δ. To simplify the statement we consider only the case when the measures μ and μ' are the Riemannian volume elements. The condition (2) of the next theorem must be interpreted as follows: if $u \in L^2(X) \cap L^p(X)$ then $\mathrm{d}(\Delta+1)^{-1}u$, which is a section of T^*X of finite L^2 norm, must have an L^p norm bounded by $C\|u\|_{L^p}$ for some constant C independent of u.

Theorem 5.5 *Let X be as in Theorem 5.4 and such that: (1) X is of infinite volume; (2) if $F \subset X$ is a Borel set of finite measure then $\chi_F(Q) : \mathscr{H}^1 \to \mathscr{H}$ is compact; (3) the operator $\mathrm{d}(\Delta+1)^{-1}$ induces a bounded operator in L^p for some $p > 2$. Let us consider a new Riemann structure verifying (R) on X such that the associated norms $\|\cdot\|'_x$ verify $\alpha(x)\|\cdot\|_x \leq \|\cdot\|'_x \leq \beta(x)\|\cdot\|_x$ for some Borel functions α, β bounded and bounded from below by a strictly positive constant. Assume that for each $\varepsilon > 0$ the set where one of the inequalities $|\alpha(x)-1| > \varepsilon$ or $|\beta(x)-1| > \varepsilon$ holds is of finite measure. Then $\sigma_{\mathrm{ess}}(\Delta) = \sigma_{\mathrm{ess}}(\Delta')$.*

Proof: Note first that the Riemannian volume elements corresponding to the two metrics are related by the formula $\mu' = \lambda\mu$ where λ is a bounded Borel function, bounded from below by a strictly positive number, such that for for each $\varepsilon > 0$ the set where $|\lambda(x)-1| > \varepsilon$ is of finite measure. Let $\Lambda(x)$ be as in the proof of Theorem 5.4. Clearly that there is a number $C > 0$ such that $C^{-1} \leq \Lambda(x) \leq C$ for all x and that for each $\varepsilon > 0$ the set where $\|\Lambda(x)-1\| > \varepsilon$ is of finite measure (the inequalities and the norm are computed on T^*_xX, which is equipped with the initial scalar product)

Now we proceed as in the proof of Theorem 5.3 but this time we equip $\mathscr{H}$ and $\mathscr{K}$ with the Hilbert module structures described in Example 2.9. To avoid confusions, we denote $\mathcal{B}_\mu(\mathscr{H})$ and $\mathcal{B}_\mu(\mathscr{K})$ the space of decay improving operators relatively to these new module structures. Our assumption (2) implies that $(\mathscr{H}^1, \mathscr{H})$ is a compact Friedrichs module. Moreover, the operator $\lambda(Q) - 1 : \mathscr{H}^1 \to \mathscr{H}$ is compact. Then, as in the proof of Theorem 5.3, we see that it suffices to prove that

$$(\mathrm{d}^*\mathrm{d}+1)^{-1} - (\mathrm{d}^*\Lambda(Q)\mathrm{d}+1)^{-1} \in \mathcal{K}(\mathscr{H}).$$

Clearly $\Lambda(Q) - 1 \in \mathcal{B}_0(\mathscr{K})$. Now we use Theorem 3.5 exactly as in the proof of Theorem 5.3 and we see that the only condition which remains to be checked is (3) of Theorem 3.5, i.e. in our case $\mathrm{d}(\Delta+1)^{-1} \in \mathcal{B}^r_q(\mathscr{H}, \mathscr{K})$, where the decay preserving property is relatively to the algebra $B_\mu(X)$. By Proposition 2.19 this is equivalent to $S \equiv (\mathrm{d}(\Delta+1)^{-1})^* \in \mathcal{B}^l_q(\mathscr{K}, \mathscr{H})$. To get this, it suffices to apply Theorem 7.1. ■

We shall now describe, without going into details, an abstract framework for the study of the Laplace operator acting on forms. Let $\mathscr{H}$ be a Hilbert space and d a closed densely defined operator in $\mathscr{H}$ such that $\mathrm{d}^2 = 0$. For example, $\mathscr{H}$ could be the space of square integrable differential forms over a Lipschitz manifold and d the operator of exterior differentiation. We denote $\delta = \mathrm{d}^*$ and we assume that $\mathscr{G} := \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\delta)$ is dense in $\mathscr{H}$ (which is a rather strong condition in the context of this paper, e.g. in the preceding example it is a differentiability condition on the metric). Then let $D = \mathrm{d} + \delta$ with domain $\mathscr{G}$, observe that $\|Du\|^2 = \|\mathrm{d}u\|^2 + \|\delta u\|^2$ so D is a closed symmetric operator, assume that D is self-adjoint, and define $\Delta = D^2 = \mathrm{d}\delta + \delta\mathrm{d}$ (form sum). Then

$$(\Delta + 1)^{-1} = (D + i)^{-1}(D - i)^{-1}. \tag{5.20}$$

Now let $a \in \mathcal{B}(\mathscr{H})$ with $a \geq \varepsilon > 0$ and such that $a^{\pm 1}\mathscr{G} \subset \mathscr{G}$ and let $\mathscr{H}_a$ be the Hilbert space which is equal to $\mathscr{H}$ as vector space but is equipped with the new the scalar product $\langle u, v\rangle_a = \langle u, av\rangle$. Denote d_a the operator d viewed as operator acting in $\mathscr{H}_a$ with adjoint $\delta_a = a^{-1}\delta a$. We can define as above operators D_a (with domain $\mathscr{G}_a = \mathscr{G}$) and $\Delta_a = D_a^2$ which are self-adjoint in $\mathscr{H}_a$ and satisfy a relation similar to (5.20). Then Δ_a is a compact perturbation of Δ if the operators $(D_a \pm i)^{-1} - (D \pm i)^{-1}$ are compact and this last condition is equivalent to the compactness of the operator $D_a - D : \mathscr{G} \to \mathscr{G}^*$. And this holds if $(\mathscr{G}, \mathscr{H})$ is a compact Hilbert X-module over a metric space X and $a - 1 \in \mathcal{B}_0(\mathscr{H})$.

6 On Maurey's factorization theorem

This section is preparatory for the next one: we shall prove here a version of the following factorization theorem of Bernard Maurey (see Theorems 2 and 8 in [Ma]).

Theorem 6.1 *Let $1 < p < 2$ and let T be an arbitrary continuous linear map from a Hilbert space $\mathscr{H}$ into L^p. Then there is $R \in \mathcal{B}(\mathscr{H}, L^2)$ and there is a function $g \in L^q$, where $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$, such that $T = g(Q)R$.*

We have stated only the particular case we need of the theorem (the result extends easily to larger classes of Banach spaces $\mathscr{H}$). The L^p spaces refer to an arbitrary positive measure space (X, μ). Before going on to our main purpose, we shall state an easy consequence of this theorem which will be needed later on. Let $\{\mathscr{K}(x)\}_{x \in X}$ be a measurable family of Hilbert spaces (see [Di, Ch. II]) such that the dimension of $\mathscr{K}(x)$ is $\leq N$ for some finite N. Let $\mathscr{K} = \int_X^{\oplus} \mathscr{K}(x) d\mu(x)$ be the corresponding direct integral and for each $p \geq 1$ let $\mathscr{K}_p$ be the space of (μ-equivalence classes) of measurable vector fields v such that $\int_X \|v(x)\|^p_{\mathscr{K}(x)} d\mu(x) < \infty$. Thus $\mathscr{K}_p$ is naturally a Banach space and $\mathscr{K}_2 = \mathscr{K}$.

Corollary 6.2 *Let $\mathscr{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathscr{H}, \mathscr{K}_p)$ with $1 < p < 2$. Then there is $R \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and there is a function $g \in L^q$, where $q = 2p/(2-p)$, such that $T = g(Q)R$.*

Proof: For each $n = 1, \dots, N$ let X_n be the set of x such that the dimension of $\mathscr{K}(x)$ is equal to n. Then X is the disjoint union of the measurable sets X_n. For each x there is n such that $x \in X_n$ and we can choose a unitary map $j(x) : \mathscr{K}(x) \to \mathbb{C}^n$ such that $\{j_x\}$ be a measurable family of operators. Let J be the operator acting on vector fields according to the rule $(Jv)(x) = j(x)v(x)$, let Π_n be the operator of multiplication by χ_{X_n}, and let $T_n \equiv \Pi_n JT \in \mathcal{B}(\mathscr{H}, L^p(X_n; \mathbb{C}^n))$. We can write $T_n = (T_n^k)_{1 \le k \le n}$ with $T_n^k \in \mathcal{B}(\mathscr{H}, L^p(X_n))$ and Maurey's theorem gives us a factorization $T_n^k = g_n^k(Q)S_n^k$ with $S_n^k \in \mathcal{B}(\mathscr{H}, L^2(X_n))$ and $g_n^k \in L^q(X_n)$, and clearly we may assume $g_n^k \ge 0$. Let $g_n = \sup_k g_n^k \in L^q(X_n)$ and $S_n \in \mathcal{B}(\mathscr{H}, L^2(X_n; \mathbb{C}^n))$ be the operator with components $(g_n^k g_n^{-1})(Q)R_n^k$. Then $T_n = g_n(Q)S_n$ and if we define $R_n = J^{-1}S_n$ we get

$$g_n(Q)R_n = J^{-1}g_n(Q)S_n = J^{-1}T_n = \Pi_n T.$$

Thus, if we define $g = \sum_n \chi_{X_n} g_n$ and $R = \sum \Pi_n R_n$, we get $T = g(Q)R$. ■

Our purpose in the rest of this section is to extend Theorem 6.1 (in the case $X = \mathbb{R}^n$) to more general classes of spaces of measurable functions, which do not seem to be covered by the results existing in the literature, cf. [Kr]. Our proof follows closely that of Maurey. We first recall Ky Fan's Lemma, see [DJT, 9.10].

Proposition 6.3 *Let $\mathcal{K}$ be a compact convex subset of a Hausdorff topological vector space and let $\mathscr{F}$ be a convex set of functions $F : \mathcal{K} \to]-\infty, +\infty]$ such that each $F \in \mathscr{F}$ is convex and lower semicontinuous. If for each $F \in \mathscr{F}$ there is $g \in \mathcal{K}$ such that $F(g) \le 0$, then there is $g \in \mathcal{K}$ such that $F(g) \le 0$ for all $F \in \mathscr{F}$.*

We need a second general fact that we state below. Let (X, μ) be a σ-finite positive measure space and let $L^0(X)$ be the space of μ-equivalence classes of complex valued measurable functions on X with the topology of convergence in measure. Let $\mathscr{L}$ be a Banach space with $\mathscr{L} \subset L^0(X)$ linearly and continuously and such that if $f \in L^0(X)$, $g \in \mathscr{L}$ and $|f| \le |g|$ (μ-a.e.) then $f \in \mathscr{L}$ and $\|f\|_{\mathscr{L}} \le \|g\|_{\mathscr{L}}$. The next result is a rather straightforward consequence of Khinchin's inequality [DJT, 1.10] (see also [Pi, Section 8]).

Proposition 6.4 *There is a number C, independent of $\mathscr{L}$, such that for any Hilbert space $\mathscr{H}$ and any $T \in \mathcal{B}(\mathscr{H}, \mathscr{L})$ the following inequality holds*

$$\|(\textstyle\sum_j |Tu_j|^2)^{1/2}\|_{\mathscr{L}} \le C\|T\|_{\mathcal{B}(\mathscr{H},\mathscr{L})} (\textstyle\sum_j \|u_j\|^2)^{1/2} \tag{6.21}$$

for all finite families $\{u_j\}$ of vectors in $\mathscr{H}$.

From now on we work in a setting adapted to our needs in Section 7, although it is clear that we could treat by the same methods a general abstract situation. Let $X = \mathbb{R}^n$ equipped with the Lebesgue measure, denote $Z = \mathbb{Z}^n$, and for each $a \in Z$ let $K_a = a + K$, where $K =]-1/2, 1/2]^n$, so that K_a is a unit cube centered at a and we have $X = \bigcup_{a \in Z} K_a$ disjoint union. Let χ_a be the characteristic function of K_a and if $f : X \to \mathbb{C}$ let $f_a = f|K_a$. We fix a number $1 < p < 2$ and a family $\{\lambda_a\}_{a \in Z}$ of strictly positive numbers $\lambda_a > 0$ and we define $\mathscr{L} \equiv \ell^2_\lambda(L^p)$ as the Banach space of all (equivalence classes) of complex functions f on X such that

$$\|f\|_{\mathscr{L}} := \Big(\sum_{a \in Z} \|\lambda_a \chi_a f\|^2_{L^p} \Big)^{1/2} < \infty. \tag{6.22}$$

Here $L^p = L^p(X)$ but note that, by identifying $\chi_a f \equiv f_a$, we can also interpret $\mathscr{L}$ as a conveniently normed direct sum of the spaces $L^p(K_a)$, see [DJT, page XIV]. If $\lambda_a = 1$ for all a we set $\ell^2_\lambda(L^p) = \ell^2(L^p)$. Observe that $\ell^2(L^2) = L^2(X)$.

Let q be given by $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$, so that $1 < p < 2 < q < \infty$. We also need the space $\mathscr{M} \equiv \ell^\infty_\lambda(L^q)$ defined by the condition

$$\|g\|_{\mathscr{M}} := \sup_{a \in Z} \|\lambda_a \chi_a g\|_{L^q} < \infty. \tag{6.23}$$

The definitions are chosen such that $\|gu\|_{\mathscr{L}} \leq \|g\|_{\mathscr{M}} \|u\|_{L^2}$ where $L^2 = L^2(X)$. As explained in [DJT, page XV], the space $\mathscr{M}$ is naturally identified with the dual space of the Banach space $\mathscr{M}_* \equiv \ell^1_{\lambda^{-1}}(L^{q'})$, where $\frac{1}{q} + \frac{1}{q'} = 1$, defined by the norm

$$\|h\|_{\mathscr{M}_*} := \sum_{a \in Z} \|\lambda_a^{-1} \chi_a h\|_{L^{q'}}.$$

Below, when we speak about w^*-topology on $\mathscr{M}$ we mean the $\sigma(\mathscr{M}, \mathscr{M}_*)$-topology. Clearly

$$\mathscr{M}_1^+ = \{ g \in \mathscr{M} \mid g \geq 0, \|g\|_{\mathscr{M}} \leq 1 \}$$

is a convex compact subset of $\mathscr{M}$ for the w^*-topology.

Lemma 6.5 *For each $f \in \mathscr{L}$ there is $g \in \mathscr{M}_1^+$ such that $\|f\|_{\mathscr{L}} = \|g^{-1} f\|_{L^2}$.*

Proof: We can assume $f \geq 0$. Since $1 = \frac{p}{2} + \frac{p}{q}$, we have:

$$\|f_a\|_{L^p} = \|f_a\|^{p/2}_{L^p} \|f_a\|^{p/q}_{L^p} = \|f_a^{p/2}\|_{L^2} \|f_a^{p/q}\|_{L^q} = \|f_a^{-p/q} f\|_{L^2} \|f_a^{p/q}\|_{L^q}$$

with the usual convention $0/0 = 0$. Now we define g_a on K_a as follows. If $f_a = 0$ then we take any $g_a \geq 0$ satisfying $\lambda_a \|g_a\|_{L^q} = 1$. If $f_a \neq 0$ let

$$g_a = \lambda_a^{-1} \big(f_a / \|f_a\|_{L^p} \big)^{p/q} = \lambda_a^{-1} \|f_a^{p/q}\|^{-1}_{L^q} f_a^{p/q}.$$

Thus we have $\lambda_a \|g_a\|_{L^q} = 1$ for all a, in particular $\|g\|_{\mathscr{M}} = 1$. By the preceding computations we also have $\|f_a\|_{L^p} = \|g_a^{-1} f_a\|_{L^2} \|g_a\|_{L^q}$ and so

$$\|f\|_{\mathscr{L}}^2 = \sum \lambda_a^2 \|f_a\|_{L^p}^2 = \sum \lambda_a^2 \|g_a\|_{L^q}^2 \|g_a^{-1} f_a\|_{L^2}^2 = \sum \|g_a^{-1} f_a\|_{L^2}^2$$

which is just $\|g^{-1} f\|_{L^2}^2$. ■

The main technical result follows.

Proposition 6.6 *Let* $(f^u)_{u \in U}$ *be a family of functions in* $\mathscr{L}$ *such that, for each* $\alpha = (\alpha_u)_{u \in U}$ *with* $\alpha_u \in \mathbb{R}$, $\alpha_u \geq 0$ *and* $\alpha_u \neq 0$ *for at most a finite number of* u, *the function* $f^\alpha := (\sum_u |\alpha_u f^u|^2)^{1/2}$ *satisfies* $\|f^\alpha\|_{\mathscr{L}} \leq \|\alpha\|_{\ell^2(U)}$. *Then there is* $g \in \mathscr{M}_1^+$ *such that* $\|g^{-1} f^u\|_{L^2} \leq 1$ *for all* $u \in U$.

Proof: For each α as in the statement of the proposition we define a function $F_\alpha : \mathscr{M}_1^+ \to]-\infty, +\infty]$ as follows:

$$F_\alpha(g) = \|g^{-1} f^\alpha\|_{L^2}^2 - \|\alpha\|_{\ell^2(U)}^2 = \sum_u \alpha_u^2 \big(\|g^{-1} f^u\|_{L^2}^2 - 1\big).$$

Our purpose is to apply Proposition 6.3 with $\mathscr{K} = \mathscr{M}_1^+$ equipped with the w^*-topology and $\mathscr{F}$ equal to the set of all functions F_α defined above. We saw before that $\mathscr{K}$ is a convex compact set. From the second representation of F_α given above it follows that $\mathscr{F}$ is a convex set. Each F_α is a convex function because $\|g^{-1} f^\alpha\|_{L^2}^2 = \int g^{-2} (f^\alpha)^2 \mathrm{d}x$ and the map $t \mapsto t^{-2}$ is convex on $[0, \infty[$. We shall prove in a moment that F_α is lower semicontinuous. From Lemma 6.5 it follows that there is $g_\alpha \in \mathscr{K}$ such that $\|f^\alpha\|_{\mathscr{L}} = \|g_\alpha^{-1} f^\alpha\|_{L^2}$. Then by our assumptions we have

$$F_\alpha(g_\alpha) = \|f^\alpha\|_{\mathscr{L}}^2 - \|\alpha\|_{\ell^2(U)}^2 \leq 0.$$

From Ky Fan's Lemma it follows that one can choose $g \in \mathscr{K}$ such that $F_\alpha(g) \leq 0$ for all α, which finishes the proof of the proposition.

It remains to show the lower semicontinuity of F_α. For this it suffices to prove that $g \mapsto \|g^{-1} f\|_{L^2}^2 \in [0, \infty]$ is lower semicontinuous on $\mathscr{K}$ if $f \in \mathscr{L}, f \geq 0$. But

$$\|g^{-1} f\|_{L^2}^2 = \sum_a \int_{K_a} g_a^{-2} f_a^2 \mathrm{d}x$$

and the set of lower semicontinuous functions $\mathscr{K} \to [0, \infty]$ is stable under sums and upper bounds of arbitrary families. Hence it suffices to prove that each map $g \mapsto \int_{K_a} g_a^{-2} f_a^2 \mathrm{d}x$ is lower semicontinuous. This map can be written as a composition $\phi \circ J_a$ where $J_a : \mathscr{M} \to L^q(K_a)$ is the restriction map $J_a g = g_a$ and

$\phi : L^q(K_a) \to [0,\infty]$ is defined by $\phi(\theta) = \int_{K_a} \theta^{-2} f_a^2 \mathrm{d}x$. The map J_a is continuous if we equip $L^q(K_a)$ with the weak topology and $\mathscr{M}$ with the w^*-topology because it is the adjoint of the norm continuous map $L^{q'}(K_a) \to \mathscr{M}_*$ which sends u into the function equal to u on K_a and 0 elsewhere. Thus it suffices to show that ϕ is lower semicontinuous on the positive part of $L^q(K_a)$ equipped with the weak topology and for this we can use exactly the same argument as Maurey. We must prove that the set $\{\theta \in L^q(K_a) \mid \theta \geq 0, \phi(\theta) \leq r\}$ is weakly closed for each real r. Since ϕ is convex, this set is convex, so it suffices to show that it is norm closed. But this is clear by the Fatou Lemma. ■

Theorem 6.7 *Let $\mathscr{H}$ be a Hilbert space and $T : \mathscr{H} \to \mathscr{L}$ a linear continuous map. Then there exist a linear continuous map $R : \mathscr{H} \to L^2(X)$ and a positive function $g \in \mathscr{M}$ such that $T = g(Q)R$.*

Proof: Let U be the unit ball of $\mathscr{H}$ and for each $u \in U$ let $f^u = Tu$. From Proposition 6.4 we get

$$\|f^\alpha\|_{\mathscr{L}} = \|(\textstyle\sum_u |T(\alpha_u u)|^2)^{1/2}\|_{\mathscr{L}} \leq A(\textstyle\sum_u \|\alpha_u u\|^2)^{1/2} \leq A(\textstyle\sum_u |\alpha_u|^2)^{1/2}$$

where $A = C\|T\|_{\mathcal{B}(\mathscr{H},\mathscr{L})}$. Since there is no loss of generality in assuming $A \leq 1$, we see that the assumptions of Proposition 6.6 are satisfied. So there is $g \in \mathscr{M}_1^+$ such that $\|g^{-1}Tu\|_{L^2(X)} \leq 1$ for all $u \in U$. Thus it suffices to define R by the rule $Ru = g^{-1}Tu$ for all $u \in \mathscr{H}$. ■

7 Weakly vanishing perturbations

7.1 Weak decay preserving operators

The purpose of this section is to reconsider the examples treated in Section 4 and to prove some stability results for perturbations which decay in a generalized sense, as described in Examples 2.7–2.9. This will be done in the next subsection, here we shall point out some classes of weak decay preserving operators.

We first consider the setting of Example 2.9: (X, μ) is a positive measure such that $\mu(X) = \infty$, $\mathcal{F}_\mu$ is the filter of sets of cofinite measure[4], and $B_\mu(X)$ is the algebra of bounded measurable $\mathcal{F}_\mu$-vanishing functions. We recall that any direct integral of Hilbert spaces over X has a canonical Hilbert module structure

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[4] Note that if X is a locally compact space and μ a Radon measure then $\mathcal{F}_\mu$ is finer than the Fréchet filter. Moreover, if X is an abelian locally compact non-compact group then $\mathcal{F}_\mu$ is strictly included in the filter $\mathcal{F}_{\mathrm{w}}$ which will be defined below.

with $B_\mu(X)$ as multiplier algebra. To avoid ambiguities, we shall speak of $\mathcal{F}_\mu$-decay preserving operators when we refer to this algebra. Let $\{\mathscr{H}(x)\}_{x\in X}$ and $\{\mathscr{K}(x)\}_{x\in X}$ be measurable families of Hilbert spaces such with dimensions $\leq N$ for some finite N. We shall use the notations introduced before Corollary 6.2.

Theorem 7.1 *If $S \in \mathcal{B}(\mathscr{H}, \mathscr{K}) \cap \mathcal{B}(\mathscr{H}_p, \mathscr{K}_p)$ for some $p < 2$, then S is left $\mathcal{F}_\mu$-decay preserving.*

Proof: We shall prove that for each measurable set N of finite measure the operator $T = S\chi_N(Q)$ has the property: if $\varepsilon > 0$ then there is a Borel set $F \in \mathcal{F}_\mu$ such that $\|\chi_F(Q)T\| \leq \varepsilon$ (then Proposition 2.19 implies that S is left $\mathcal{F}_\mu$-decay preserving). Since N is of finite measure, $\chi_N(Q)$ is a bounded operator $\mathscr{H} \to \mathscr{H}_p$, hence $T \in \mathcal{B}(\mathscr{H}, \mathscr{K}_p)$. The rest of the proof is a straightforward application of Corollary 6.2. Let $a > 0$ real and let F be the set of points x such that $|g(x)| \leq a$. Since $g \in L^q$ with $q < \infty$, we have $F \in \mathcal{F}_\mu$ and

$$\|\chi_F(Q)T\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} = \|\chi_F(Q)g(Q)R\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leq a\|R\|_{\mathscr{H},\mathscr{K})}.$$

Thus it suffices to choose a such that $a\|R\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} = \varepsilon$. ■

Let X be a locally compact non-compact topological space and let $\mathscr{H}$ be a Hilbert X-module. Then, due to Remark 2.6, the operator $\varphi(Q) \in \mathcal{B}(\mathscr{H})$ is well defined for all $\varphi \in B(X)$. If $\mathcal{F}$ is a filter finer than the Fréchet filter on X then

$$B_{\mathcal{F}}(X) := \{\varphi \in B(X) \mid \lim_{\mathcal{F}} \varphi = 0\} \tag{7.24}$$

is a C^*-algebra and we can consider on $\mathscr{H}$ the Hilbert module structure defined by the multiplier algebra $\mathcal{M}_{\mathcal{F}} := \{\varphi(Q) \mid \varphi \in B_{\mathcal{F}}(X)\}$. We are interested in the corresponding classes of decay improving or decay preserving operators. To be precise, we shall speak in this context of (left or right) *$\mathcal{F}$-vanishing at infinity* or of (left or right) *$\mathcal{F}$-decay preserving* operators. Below and later on we use the notation $N^{\mathrm{c}} = X \setminus N$.

Lemma 7.2 *Let $\mathscr{H}, \mathscr{K}$ be Hilbert X-modules. Then an operator $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ is left $\mathcal{F}$-decay preserving if and only if for each Borel set N with $N^{\mathrm{c}} \in \mathcal{F}$ and for each $\varepsilon > 0$ there is a Borel set $F \in \mathcal{F}$ such that $\|\chi_F(Q)S\chi_N(Q)\| \leq \varepsilon$.*

Proof: We note first that the family of operators χ_N, where N runs over the family of Borel sets with complement in $\mathcal{F}$, is an approximate unit for $B_{\mathcal{F}}(X)$. Indeed, if $\varepsilon > 0$ and $\varphi \in B_{\mathcal{F}}(X)$ then the set $N = \{x \mid |\varphi(x)| > \varepsilon\}$ is Borel, its complement is in $\mathcal{F}$, and $\sup_x |\varphi(x)(1 - \chi_N(x))| \leq \varepsilon$. Thus, according to Proposition 2.19, S is left $\mathcal{F}$-decay preserving if and only if $S\chi_N(Q)$ is left $\mathcal{F}$-vanishing at infinity for each N. Now the result follows from (2.6). ■

The main restriction we have to impose on $\mathcal{F}$ comes from the fact that the Friedrichs couple $(\mathscr{G},\mathscr{H})$ which is involved in our abstract compactness criteria must be such that $\varphi(Q)\in\mathcal{K}(\mathscr{G},\mathscr{H})$ if $\varphi\in B_{\mathcal{F}}(X)$. In some cases this condition is easy to express explicitly (see Lemma 7.4 for a more general fact):

Lemma 7.3 *Let X be an Euclidean space, $\mathscr{H}=L^2(X)$, and let $\mathscr{G}=\mathscr{H}^s$ be a Sobolev space of order $s>0$. If $\varphi\in B(X)$ then $\varphi(Q)\in\mathcal{K}(\mathscr{G},\mathscr{H})$ if and only if*

$$\lim_{a\to\infty}\int_{|x-a|\leq 1}|\varphi(x)|dx=0. \tag{7.25}$$

The importance of such a condition in questions of stability of the essential spectrum has been noticed in [He, LV, OS, We]. That it is a natural condition follows also from the characterizations that we shall give below in a more general context.

Let X be a locally compact non-compact abelian group. We shall say that a function $\varphi\in B(X)$ is *weakly vanishing (at infinity)* if

$$\lim_{a\to\infty}\int_{a+K}|\varphi(x)|dx=0 \text{ for each compact set } K. \tag{7.26}$$

We shall denote by $B_{\mathrm{w}}(X)$ the set of functions φ satisfying (7.26). This is clearly a C^*-algebra. Note that it suffices that the convergence condition in (7.26) be satisfied for only one compact set K with non-empty interior.

Let us now express the condition (7.26) in terms of convergence to zero along a filter. We denote $|K|$ the exterior (Haar) measure of a set $K\subset X$ and we set $K_a=a+K$ if $a\in X$. A subset N is called w-*small (at infinity)* if there is a compact neighborhood K of the origin such that $\lim_{a\to\infty}|N\cap K_a|=0$. The complement of a w-small set will be called w-*large (at infinity)*. The family $\mathcal{F}_{\mathrm{w}}$ of all w-large sets is clearly a filter on X finer than the Fréchet filter.

Observe that a Borel set is w-small if and only if its characteristic function weakly vanishes at infinity. Denote $f*g$ the convolution of two functions on X.

Lemma 7.4 *For a function $\varphi\in B(X)$ the following conditions are equivalent: (1) φ is weakly vanishing; (2) $\theta*|\varphi|\in C_0(X)$ if $\theta\in C_{\mathrm{c}}(X)$; (3) $\lim_{\mathcal{F}_{\mathrm{w}}}\varphi=0$; (4) $\varphi(Q)\psi(P)$ is a compact operator on $L^2(X)$ for all $\psi\in C_0(X)$.*

Proof: The equivalence of (1) and (2) is clear because $\int_{K_a}|\varphi|dx=(\chi_K*|\varphi|)(a)$. Then (3) means that for each $\varepsilon>0$ the Borel set N where $|\varphi(x)|>\varepsilon$ is w-small. Since $\chi_N\leq\varphi/\varepsilon$, the implication (2) $\Rightarrow$ (3) is clear, while the reciprocal implication follows from $\chi_K*|\varphi|\leq\sup|\varphi|\chi_K*\chi_N+\varepsilon|K|$. If (4) holds, let us choose ψ such that its Fourier transform $\widehat{\psi}$ be a positive function in $C_{\mathrm{c}}(X)$ and let

$f \in C_{\mathrm{c}}(X)$ be positive and not zero. Since $\psi(P)f$ is essentially the convolution of $\widehat{\psi}$ with f, there is a compact set K with non-empty interior such that $\psi(P)f \geq c\chi_K$ with a number $c > 0$. Let U_a be the unitary operator of translation by a in $L^2(X)$, then $U_a f \to 0$ weakly when $a \to \infty$, hence $\|\varphi(Q)U_a\psi(P)f\| = \|\varphi(Q)\psi(P)U_a f\| \to 0$. Since $U_a^*\varphi(Q)U_a = \varphi(Q-a)$ we get $\|\varphi(Q-a)\chi_K\| \to 0$, hence (1) holds.

Finally, let us prove that (1) $\Rightarrow$ (4). It suffices to prove that $\varphi(Q)\psi(P)$ is compact if $\widehat{\psi} \in C_{\mathrm{c}}(X)$ and for this it suffices that $\bar{\psi}(P)|\varphi|^2(Q)\psi(P)$ be compact. Since $\xi := |\varphi|^2 \in B_{\mathrm{w}}(X)$ and since $\psi(P)$ is the operator of convolution by a function in $\theta \in C_{\mathrm{c}}(X)$, we are reduced to proving that the integral operator S with kernel $S(x,y) = \int \bar{\theta}(z-x)\xi(z)\theta(z-y)dz$ is compact. If $K = \operatorname{supp}\theta$ and Λ is the compact set $K - K$, then clearly there is a number C such that

$$|S(x,y)| \leq C \int_{K_x} \xi(z)dz\chi_\Lambda(x-y) \equiv \phi(x)\chi_\Lambda(x-y)$$

where $\phi \in C_0(X)$. The last term here is a kernel which defines a compact operator T. Thus $\eta(Q)S$ is a Hilbert-Schmidt operator for each $\eta \in C_{\mathrm{c}}(X)$ and from the preceding estimate we get $\|(S - \eta(Q)S)u\| \leq \|(1-\eta(Q))T|u|\|$ for each $u \in L^2(X)$. Thus $\|S - \eta(Q)S\| \leq \|(1-\eta(Q))T\|$ and the right hand side tends to zero if $\eta \equiv \eta_\alpha$ is an approximate unit for $C_0(X)$. ■

We shall consider now a general class of filters defined in terms of the metric and measure space structure. We consider only the case of an Euclidean space X, the extension to the case of locally compact groups or metric spaces being obvious. We set $B_a(r) = \{x \in X \mid |x-a| < r\}$, $B_a = B_a(1)$ and $B(r) = B_0(r)$. To each function $\nu : X \to]0,\infty[$ such that $\liminf_{a\to\infty}\nu(a) = 0$ we associate a set of subsets of X as follows:

$$\mathscr{N}_\nu = \{N \subset X \mid \limsup_{a\to\infty} \nu(a)^{-1}|N \cap B_a| < \infty\}. \tag{7.27}$$

Clearly $\mathscr{F}_\nu = \{F \subset X \mid F^{\mathrm{c}} \in \mathscr{N}_\nu\}$ is a filter on X finer than the Fréchet filter.

Theorem 7.5 *Let $X = \mathbb{R}^n$ and let $\nu : X \to]0,\infty[$ such that $\liminf_{a\to\infty}\nu(a) = 0$ and $\sup_{|b-a|\leq r}\nu(b)/\nu(a) < \infty$ for each real r. If $S \in \mathcal{B}(L^2(X))$ is of class $C^{\mathrm{u}}(Q)$ and if $S \in \mathcal{B}(L^p(X))$ for some $p < 2$, then S is left $\mathscr{F}_\nu$-decay preserving.*

Proof: We can approximate in norm in $\mathcal{B}(L^2(X))$ the operator S by operators which are in $\mathcal{B}(L^2(X)) \cap \mathcal{B}(L^p(X))$ and have finite range. Indeed, the approximation procedure (2.12) used in the proof of Proposition 2.33 is such that it leaves $\mathcal{B}(L^2(X)) \cap \mathcal{B}(L^p(X))$ invariant (because V_k are isometries in L^p too). Since the set of left $\mathscr{F}_\nu$-decay preserving operators is norm closed in $\mathcal{B}(L^2(X))$, we may assume in the rest of the proof that S is of finite range. According to Lemma 7.2, it suffices to show that, for a given Borel set $N \in \mathscr{N}_\nu$ and for any number $\varepsilon > 0$, there is a Borel set $M \in \mathscr{N}_\nu$ such that $\|\chi_{M^c}(Q) S \chi_N(Q)\| < \varepsilon$.

In the rest of the proof we shall freely use the notations introduced in Section 6 (see also the proof of Proposition 2.33). In particular, q is defined by $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. If $f \in L^2(X)$ we have

$$\|\chi_N f\|_{L^p(K_a)} \leq \|\chi_N\|_{L^q(K_a)} \|f\|_{L^2(K_a)} \leq |N \cap K_a|^{1/q} \|f\|_{L^2(K_a)}.$$

Since $N \in \mathscr{N}_\nu$ we can find a constant c such that $|N \cap K_a| \leq c\nu(a)$ (note that the definition (7.27) does not involve the restriction of ν to bounded sets). Thus, if we take $\lambda_a = \nu(a)^{-1/q}$ for $a \in Z \equiv \mathbb{Z}^n$, we get $\chi_N f \in \mathscr{L}$ with the notations of Section 6. In other terms, we see that we have $\chi_N(Q) \in \mathcal{B}(L^2(X), \mathscr{L})$. Let $T = S\chi_N(Q)$ and let us assume that we also have $S \in \mathcal{B}(\mathscr{L})$. Then $T \in \mathcal{B}(L^2(X), \mathscr{L})$ and we can apply the Maurey type factorization theorem Theorem 6.7, where $\mathscr{H} = L^2(X)$. Thus we can write $T = g(Q)R$ for some $R \in \mathcal{B}(L^2(X))$ and some function $g \in \mathscr{M}$, which means that $G := \sup_{a \in Z} \nu(a)^{-1/q} \|g\|_{L^q(K_a)}$ is a finite number. If $t > 0$ and $M = \{x \mid g(x) > t\}$ then we get for all $a \in Z$:

$$|M \cap K_a| = \|\chi_M\|^q_{L^q(K_a)} \leq \|g/t\|^q_{L^q(K_a)} \leq (G/t)^q \nu(a).$$

Note that the second condition imposed on ν in Theorem 7.5 ca be stated as follows: there is an increasing strictly positive function δ on $[0, \infty[$ such that $\nu(b) \leq \delta(|b-a|)\nu(a)$ for all a, b. Indeed, we may take $\delta(r) = \sup_{|b-a| \leq r} \nu(b)/\nu(a)$. Now let $a \in X$ and let $D(a)$ be the set of $b \in Z$ such that K_b intersects B_a. Clearly $D(a)$ contains at most 2^n points b all of them satisfying $|b - a| \leq \sqrt{n} + 1$. Hence:

$$|M \cap K_a| \leq \sum_{b \in D(a)} |M \cap K_b| \leq 2^n \sup_{b \in D(a)} (G/t)^q \nu(b) \leq 2^n (G/t)^q \delta(\sqrt{n}+1)\nu(a),$$

which proves that M belongs to $\mathscr{N}_\nu$. On the other hand, we have:

$$\|\chi_{M^c}(Q) T\| = \|\chi_{M^c}(Q) g(Q) R\| \leq \|\chi_{M^c} g\|_{L^\infty} \|R\| \leq t\|R\|.$$

To finish the proof of the theorem it suffices to take $t = \varepsilon/\|R\|$.

We still have to prove that $S \in \mathcal{B}(\mathscr{L})$. Since S is of finite range, there is a number r such that $\chi_a(Q)\chi_b(Q) = 0$ if $|a - b| \geq r$. Then for any $f \in \mathscr{L}$:

$$\sum_a \lambda_a^2 \|\chi_a S f\|^2_{L^p} = \sum_a \lambda_a^2 \|\sum_{|b-a|<r} \chi_a S \chi_b f\|^2_{L^p} \leq C \sum_{|b-a|<r} \lambda_a^2 \|\chi_a S \chi_b f\|^2_{L^p}$$

where C is a number depending only on r and n. Since S is bounded in L^p the last term is less than $CC'\sum_{|b-a|<r}\lambda_a^2\|\chi_b f\|_{L^p}^2$ for some constant C'. Finally, from $\nu(b) \leq \delta(|b-a|)\nu(a) \leq \delta(r)\nu(a)$ we get

$$\sum_{|a-b|<r} \lambda_a^2 = \sum_{|a-b|<r} \nu(a)^{-2/q} \leq L(r)\delta(r)^{2/q}\lambda_b^2$$

where $L(r)$ is the maximum number of points from Z inside a ball of radius r. Thus we have $\|S\|_{\mathcal{B}(\mathscr{L})}^2 \leq CC'L(r)\delta(r)^{2/q}$. ■

Theorem 7.6 *Let $X = \mathbb{R}^n$ and let S be a pseudo-differential operator of class S^0. Then S is $\mathcal{F}_{\mathrm{w}}$-decay preserving, i.e. if $\varphi \in B_{\mathrm{w}}(X)$ then $\varphi(Q)S = T_1\psi_1(Q)$ and $S\varphi(Q) = \psi_2(Q)T_2$ for some $\psi_1, \psi_2 \in B_{\mathrm{w}}(X)$ and $T_1, T_2 \in \mathcal{B}(L^2(X))$.*

Proof: Since the adjoint of S is also a pseudo-differential operator of class S^0, it suffices to show that S is left $\mathcal{F}_{\mathrm{w}}$-decay preserving. We have $S \in \mathcal{B}(L^p(X))$ for all $1 < p < \infty$ and S is of class $C^{\mathrm{u}}(Q)$ because the commutators $[Q_j, S]$ are bounded operators for all $1 \leq j \leq n$. Thus we can apply Theorem 7.5 and deduce that for any function ν as in the statement of the theorem, for any $\varepsilon > 0$, and for any $N \in \mathscr{N}_\nu$ there is $M \in \mathscr{N}_\nu$ such that $\|\chi_{M^c}(Q)S\chi_N(Q)\| \leq \varepsilon$. Now let N be a Borel w-small set, i.e. such that $|N \cap B_a| \to 0$ if $a \to \infty$. We shall prove that there is a function ν with the properties required in Theorem 7.5 and with $\lim_{a\to\infty}\nu(a) = 0$ such that $N \in \mathscr{N}_\nu$. This finishes the proof of the corollary because the relation $M \in \mathscr{N}_\nu$ implies now that M is w-small.

We construct ν as follows. The relation $\theta(r) = \sup_{|a|\geq r}|N \cap B_a|$ defines a positive decreasing function on $[0,\infty[$ which tends to zero at infinity and such that $|N \cap B_a| \leq \theta(|a|)$ for all $a \in X$. We set $\xi(t) = \theta(0)$ if $0 \leq t < 1$ and for $k \geq 0$ integer and $2^k \leq t < 2^{k+1}$ we define $\xi(t) = \max\{\xi(2^{k-1})/2, \theta(2^k)\}$. So ξ is a strictly positive decreasing function on $[0,\infty[$ which tends to zero at infinity and such that $\theta \leq \xi$. Moreover, if $2^k \leq s < 2^{k+1}$ and $2^{k+p} \leq t < 2^{k+p+1}$ then

$$\xi(t) = \xi(2^{k+p}) \geq \xi(2^{k+p-1})/2 \geq \ldots \geq 2^{-p}\xi(2^k) = 2^{-p}\xi(s)$$

hence $\xi(s) \geq \xi(t) \geq \frac{s}{2t}\xi(s)$ if $1 \leq s \leq t$. We take $\nu(a) = \xi(|a|)$, so ν is a bounded strictly positive function on X with $\lim_{a\to\infty}\nu(a) = 0$ and $|N \cap B_a| \leq \nu(a)$ for all a. If a, b are points with $|a|, |b| \geq 1$ and $|a-b| \leq r$ then $\nu(b)/\nu(a) \leq 1$ if $|a| \leq |b|$ and if $|a| > |b|$ then

$$\frac{\nu(b)}{\nu(a)} = \frac{\xi(|b|)}{\xi(|a|)} \leq \frac{2|a|}{|b|} \leq 2(1+r).$$

Thus the second condition imposed on ν in Theorem 7.5 is also satisfied. ■

Remark 7.7 We stress that we shall need this theorem for a very simple class of operators, namely $S = \psi(P)$ with $\psi(k) = k^\alpha(\sum_{|\beta|\leq m} k^{2\beta})^{-1/2}$ and $|\alpha| \leq m$.

As a final example, we introduce now classes of vanishing at infinity functions of a more topological nature. Let us fix a *uniformly discrete set* $L \subset X$, i.e. a set such that $\inf |a-b| > 0$ where the infimum is taken over couples of distinct points $a, b \in L$. Let $L_\varepsilon = L + B(\varepsilon)$ be the set of points at distance $< \varepsilon$ from L. We say that a subset $N \subset X$ is *L-thin* if for each $\varepsilon > 0$ there is $r < \infty$ such that $N \setminus B(r) \subset L_\varepsilon$. In other terms, N is L-thin if there is a family $\{\delta_a\}_{a\in L}$ of positive real numbers with $\delta_a \to 0$ as $a \to \infty$ such that $N \subset \bigcup B_a(\delta_a)$. The complement of such a set will be called *L-fat*. We denote $\mathcal{F}_L$ the family of L-fat sets, we note that $\mathcal{F}_L$ is a filter on X contained in $\mathcal{F}_{\rm w}$ and finer than the Fréchet filter, and we denote $B_L(X)$ the set of bounded Borel functions such that $\lim_{\mathcal{F}_L} \varphi = 0$. So $\varphi \in B(X)$ belongs to $B_L(X)$ if and only if the set $\{|\varphi| \geq \lambda\}$ is L-thin for each $\lambda > 0$.

Proposition 7.8 *Let $X = \mathbb{R}^n$ and let S be a bounded operator on $L^2(X)$ such that on the region $x \neq y$ its distribution kernel is a function satisfying the estimate $|S(x,y)| \leq c|x-y|^{-m}$ for some $m > n$. Then S is $\mathcal{F}_L$-decay preserving.*

Proof: Let $\theta \in C_{\rm b}(X)$ such that $\theta(x) = 0$ on a neighborhood of the origin and $S_\theta(x,y) = \theta(x-y)S(x,y)$. If $\xi(x) = \theta(x)|x|^{-m}$ then for the operator S_θ of kernel $S_\theta(x,y)$ we have $\|S_\theta u\| \leq c\|\xi * |u|\|$ hence $\|S_\theta\| \leq c\|\xi\|_{L^1}$ By choosing a convenient sequence of functions θ we see that S is the norm limit of a sequence of operators which besides the properties from the statement of the proposition are such that $S(x,y) = 0$ if $|x-y| > R(S)$. Since the set of $\mathcal{F}_L$-decay preserving operators is closed in norm (see Subsection 2.3), we may assume in the rest of the proof that the kernel of S has this property. In fact, in order to simplify the notations and without loss of generality, we shall assume $S(x,y) = 0$ if $|x-y| > 1$.

Let N be an L-thin Borel set and let $\varepsilon > 0$. We shall construct an L-fat Borel set with $F \subset N^{\rm c}$ such that $\|\chi_N(Q)S\chi_F(Q)\| \leq \varepsilon$. Since the adjoint operator S^* has the same properties as S, this suffices to prove that it is decay preserving.

We shall only need two simple estimates. First, if $\rho_x(G)$ is the distance from a Borel set G to a point x, then

$$\int_G \frac{dy}{|x-y|^{2m}} \leq C(m,n)\rho_x(G)^{n-2m}. \tag{7.28}$$

Then, if B_0, B are two balls with the same center and radiuses δ and $\delta + \varepsilon$, then

$$\int_{B_0} \rho_x(B^{\rm c})^{n-2m} dx \leq C(m,n)\varepsilon^{n-2m}\delta^n. \tag{7.29}$$

We shall choose $\varepsilon = \delta^{n/2m}$. Then $\chi_{B_0}(Q)S\chi_{B^{\mathrm{c}}}(Q)$ is an operator with integral kernel and we can estimate its Hilbert-Schmidt norm as follows:

$$\begin{aligned} \|\chi_{B_0}(Q)S\chi_{B^{\mathrm{c}}}(Q)\|_{HS}^2 &= \int_{X\times X} \chi_{B_0}(x)|S(x,y)|^2\chi_{B^{\mathrm{c}}}(y)dxdy \\ &\leq c\int_{B_0} dx \int_{B^{\mathrm{c}}} \frac{dy}{|x-y|^{2m}} \leq C\int_{B_0} \rho_x(B^{\mathrm{c}})^{n-2m}dx \\ &\leq C'\varepsilon^{n-2m}\delta^n = C'\delta^\lambda \end{aligned} \tag{7.30}$$

where $\lambda = n^2/2m > 0$.

We can assume that $N = \bigcup_a B_a(\delta_a)$, where the sequence of numbers δ_a satisfies $\delta_a \to 0$ as $a \to \infty$. Denote $N_a = B_a(\delta_a)$ and $M_a = B_a(\delta_a + \varepsilon_a)$, where we choose $\varepsilon_a = \delta_a^{n/2m}$ as above. Choose r such that the balls N_a are pairwise disjoint and $\delta_a + \varepsilon_a < 1$ if $|a| > r$ and let R such that $\chi_{N_a}(Q)S\chi_{B(R)^{\mathrm{c}}}(Q) = 0$ if $|a| \leq r$. Let $M = \bigcup M_a$ and $F = M^{\mathrm{c}} \setminus B(R)$, so that F is a closed L-fat set. Then for any $u \in L^2(X)$ we have:

$$\|\chi_N(Q)S\chi_F(Q)u\|^2 = \sum_{|a|>r} \|\chi_{N_a}(Q)S\chi_F(Q)u\|^2.$$

Since S is of range 1 we have $\chi_{N_a}(Q)S\chi_{B_a(2)^{\mathrm{c}}}(Q) = 0$ if $\delta_a < 1$. Thus

$$\|\chi_N(Q)S\chi_F(Q)u\|^2 \leq \sum_{|a|>r} \|\chi_{N_a}(Q)S\chi_{F\cap B_a(2)}(Q)\|^2 \, \|\chi_{B_a(2)}(Q)u\|^2$$

The number of $b \in L$ such that $B_b(2)$ meets $B_a(2)$ is a bounded function of a, hence there is a constant C depending only on L such that

$$\|\chi_N(Q)S\chi_F(Q)u\| \leq C \sup_{|a|>r} \|\chi_{N_a}(Q)S\chi_{F\cap B_a(2)}(Q)\| \, \|u\|.$$

We have $F \subset M^{\mathrm{c}} \subset M_a^{\mathrm{c}}$ hence

$$\|\chi_{N_a}(Q)S\chi_{F\cap B_a(2)}(Q)\| \leq \|\chi_{N_a}(Q)S\chi_{M_a^{\mathrm{c}}}(Q)\|_{HS} \leq C'\delta_a^{\lambda/2}$$

because of (7.30). So the norm $\|\chi_N(Q)S\chi_F(Q)\|$ can be made as small as we wish by choosing r large enough. ■

Corollary 7.9 *Let $X = \mathbb{R}^n$, μ the Lebesgue measure, and L a uniformly discrete subset of $\mathbb{R}^n$. Then a pseudo-differential operator of class S^0 on $L^2(X)$ is decay preserving with respect to $\mathcal{F}_\mu$, $\mathcal{F}_{\mathrm{w}}$ and $\mathcal{F}_L$.*

Proof: In the first case we use Theorem 7.1 by taking into account that a pseudo-differential operator of class S^0 belongs to $\mathcal{B}(L^p(X))$ for all $1 < p < \infty$ and that the adjoint of such an operator is also pseudo-differential of class S^0. The second case has already been considered in Theorem 7.6. For the third case, note that the distribution kernel of such an operator verifies the estimates $|S(x,y)| \leq C_k|x-y|^{-n}(1+|x-y|)^{-k}$ for any $k > 0$, see [Ho]. ■

7.2 Applications

We shall use the results of Subsection 7.1 in the framework of Subsection 4.1. We consider on $\mathscr{H}$ the class of "vanishing at infinity" functions corresponding to the C^*-algebra of multipliers $B_{\mathrm{w}}(X)$. The conditions of decay at infinity (4.18) imposed in Proposition 4.1 come from the consideration of $\mathscr{H}$ equipped with the Hilbert module structure defined by the algebra $B_0(X)$. Note that if we equip $\mathscr{H}$ with the Hilbert module structure defined by the algebra $B_{\mathrm{w}}(X)$ the property of compactness of the Friedrichs module $(\mathscr{G}, \mathscr{H})$ remains valid due to Lemma 7.3, and the space $\mathscr{K}$ inherits a natural direct sum Hilbert module structure.

Our purpose is to apply Theorem 3.5 in this setting. The only thing which remains to be checked is the right $\mathcal{F}_{\mathrm{w}}$-decay preserving property of the operator $D(\Delta_a^* - \bar{z})^{-1}$. We shall establish such a result below assuming that the lower order coefficients are also bounded operators, but it is clear that this assumption can be replaced by much more general ones. Note also that in this subsection we are less precise and identify the operators D^*aD and Δ_a, although they act in different spaces.

In the next lemma we consider only the filter $\mathcal{F}_{\mathrm{w}}$. Of course, the result remains true if $\mathcal{F}_{\mathrm{w}}$ is replaced by $\mathcal{F}_\mu$ or $\mathcal{F}_L$.

Lemma 7.10 *Let* $\Delta_a = \sum_{|\alpha|,|\beta| \leq m} P^\alpha a_{\alpha\beta} P^\beta$ *with* $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H})$ $\mathcal{F}_{\mathrm{w}}$*-decay preserving (e.g.* $a_{\alpha\beta} \in B(X)$*) and let as assume that the operator* $\Delta_a : \mathscr{H}^m \to \mathscr{H}^{-m}$ *is coercive, i.e. there are numbers* $\mu, \nu > 0$ *such that for all* $u \in \mathscr{H}^m$*:*

$$\operatorname{Re} \langle u, \Delta_a u \rangle \geq \mu \|u\|^2_{\mathscr{H}^m} - \nu \|u\|^2_{\mathscr{H}}. \tag{7.31}$$

Then $P^\alpha(\Delta_a + z)^{-1}$ *is* $\mathcal{F}_{\mathrm{w}}$*-decay preserving if* $|\alpha| \leq m$ *and if* Re $z > 0$ *is large.*

Proof: We shall denote by Δ the operator Δ_a corresponding to the case when a is the identity matrix, so $\Delta = D^*D = \sum_{|\alpha| \leq m} P^{2\alpha}$ (of course, this is not the Laplace operator). In fact, Δ is the canonical (Riesz) positive isomorphism of $\mathscr{G}$ onto $\mathscr{G}^*$ and (7.31) means Re $\Delta_a \geq \mu\Delta - \nu$. Note that we can include ν in the term of order zero of Δ_a, hence there is no loss of generality if we assume $\nu = 0$. Later computations look simpler if $\mu = 1$ and we can reduce ourselves to this situations by replacing a by a/μ. Thus we may assume that we have the estimate Re $\Delta_a \geq \Delta$. Now let us decompose $\Delta_a = \Delta + D^*(a-1)D \equiv \Delta + V$ and, if θ is a positive number, let us set $A_\theta = \Delta + \theta V$. Then $A_\theta \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ and we have Re $A_\theta \geq \Delta$, so that if Re $z \geq 0$ then $A_\theta + z : \mathscr{G} \to \mathscr{G}^*$ is bijective and $\|(A_\theta + z)^{-1}\|_{\mathcal{B}(\mathscr{G}^*, \mathscr{G})} \leq 1$ (see the Appendix). It follows easily that the function $\theta \mapsto (A_\theta + z)^{-1} \in \mathcal{B}(\mathscr{G}^*, \mathscr{G})$ is real analytic on $]0, \infty[$ which implies that the function $\theta \mapsto P^\alpha (A_\theta + z)^{-1} \in \mathcal{B}(\mathscr{H})$ is real analytic too. The set of $\mathcal{F}_\mathrm{w}$-decay preserving operators is a closed subspace of the Banach space $\mathcal{B}(\mathscr{H})$ and an analytic function which on an open set takes values in a closed subspace remains in that subspace for ever. Thus it suffices to show that the operator $P^\alpha (A_\theta + z)^{-1}$ is $\mathcal{F}_\mathrm{w}$-decay preserving for small values of θ. The operator $P^\alpha (A_\theta + z)^{-1}$ is also a holomorphic function of z in the region Re $z > 0$, so by a similar argument we see that it suffices to consider $z \geq 0$. Below we shall take $z = 0$, the argument in general is identical.

For reasons of simplicity, we change again the notations: we set $b = \theta(a-1)$, we assume $\|b\|_{\mathcal{B}(\mathscr{K})} < 1$, and denote $V = D^* b D$ and $A = \Delta + V$. Let $S = \Delta^{-1/2}$, where Δ is considered as self-adjoint operator on $\mathscr{H}$. Note that S is an isometry of $\mathscr{G}^*$ onto $\mathscr{H}$ and of $\mathscr{H}$ onto $\mathscr{G}$. Then we have:

$$A^{-1} = S(1 + SVS)^{-1} S = \sum_{k \geq 0} (-1)^k S(SVS)^k S$$

the series being norm convergent in $\mathcal{B}(\mathscr{G}^*, \mathscr{G})$. Indeed, $\|DS\|_{\mathcal{B}(\mathscr{H}, \mathscr{K})} = 1$, hence

$$\|S(SVS)^k S\|_{\mathcal{B}(\mathscr{G}^*, \mathscr{G})} = \|(SD^* b DS)^k\|_{\mathcal{B}(\mathscr{H})} \leq \|b\|^k_{\mathcal{B}(\mathscr{K})}$$

and $\|b\|_{\mathcal{B}(\mathscr{K})} < 1$. Thus $P^\alpha A^{-1}$ is a sum of terms $P^\alpha (-1)^k S(SVS)^k S$ which converges in norm, so it suffices that each of them be $\mathcal{F}_\mathrm{w}$-decay preserving. But

$$P^\alpha S(SVS)^k S = (P^\alpha S)(SD^*) b (DS) \dots (SD^*) b (DS) S$$

and each factor in the product is $\mathcal{F}_\mathrm{w}$-decay preserving: for b by assumption, and for $P^\alpha S$, DS and SD^* due to Theorem 7.6. ■

Below we give the simplest consequence of Theorem 3.5. The conditions on the lower order coefficients can be improved without difficulty.

Theorem 7.11 *Let Δ_a be as in Lemma 7.10 and let $b = (b_{\alpha\beta})_{|\alpha|,|\beta| \leq m}$ with $b_{\alpha\beta}$ bounded operators $\mathscr{H}^{m-|\beta|} \to \mathscr{H}^{|\alpha|-m}$ such that Δ_b is coercive. For $|\alpha| + |\beta| = m$ assume that $b_{\alpha\beta} - a_{\alpha\beta}$ is left $\mathcal{F}_\mathrm{w}$-vanishing at infinity (which holds if $b_{\alpha\beta} - a_{\alpha\beta} \in B_\mathrm{w}(X)$). If $|\alpha| + |\beta| < m$ we assume $b_{\alpha\beta} - a_{\alpha\beta} \in \mathcal{K}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$. Then the operator Δ_b is a compact perturbation of Δ_a, in particular $\sigma_\mathrm{ess}(\Delta_a) = \sigma_\mathrm{ess}(\Delta_b)$.*

Proof: We check the conditions of Theorem 3.5. Because of the coercivity assumptions, condition (1) is fulfilled, and (2) is satisfied by Lemma 7.10. The part of condition (3) involving the coefficients such that $|\alpha| + |\beta| = m$) is satisfied by definition, for the lower order coefficients it suffices to use (2.8). ■

Remark 7.12 If $b_{\alpha\beta} \in B(X)$ and $b_{\alpha\beta} - a_{\alpha\beta} \in B_{\mathrm{w}}(X)$ for all α, β, then the compactness conditions on the lower order coefficients are satisfied. Indeed, if $\varphi \in B_{\mathrm{w}}(X)$ then $\varphi(Q) : \mathscr{H}^s \to \mathscr{H}^{-t}$ is compact if $s, t \geq 0$ and one of them is not zero, see Lemma 7.3.

A Appendix

In this Appendix we discuss some elementary abstract facts which are used without comment in the main text.

Let $(\mathscr{G}, \mathscr{H})$ be a Friedrichs couple and $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ the Gelfand triplet associated to it. To an operator $S \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ (which is the same as a continuous sesquilinear form on $\mathscr{G}$) we associate an operator $\widehat{S}$ acting in $\mathscr{H}$ according to the rules: $\mathcal{D}(\widehat{S}) = S^{-1}(\mathscr{H})$, $\widehat{S} = S|\mathcal{D}(\widehat{S})$. Due to the identification $\mathscr{G}^{**} = \mathscr{G}$, the operator S^* is an element of $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$, so $\widehat{S^*}$ makes sense. On the other hand, if $\widehat{S}$ is densely defined in $\mathscr{H}$ then the adjoint $\widehat{S}^*$ of $\widehat{S}$ with respect to $\mathscr{H}$ is also well defined and we clearly have $\widehat{S^*} \subset \widehat{S}^*$.

Lemma A.1 *If $S - z : \mathscr{G} \to \mathscr{G}^*$ is bijective for some $z \in \mathbb{C}$, then $\widehat{S}$ is a closed densely defined operator, we have $\widehat{S}^* = \widehat{S^*}$ and $z \in \rho(\widehat{S})$. Moreover, the domains $\mathcal{D}(\widehat{S})$ and $\mathcal{D}(\widehat{S}^*)$ are dense subspaces of $\mathscr{G}$.*

Proof: Clearly we can assume $z = 0$. From the bijectivity of $S : \mathscr{G} \to \mathscr{G}^*$ and the inverse mapping theorem it follows that S and S^* are homeomorphisms of $\mathscr{G}$ onto $\mathscr{G}^*$. Since $\mathscr{H}$ is dense in $\mathscr{G}^*$, we see that $\mathcal{D}(\widehat{S})$ and $\mathcal{D}(\widehat{S^*})$ are dense in $\mathscr{G}$, hence in $\mathscr{H}$. Since $\widehat{S^*} \subset \widehat{S}^*$, the operator $\widehat{S}^*$ is also densely defined in $\mathscr{H}$. Thus $\widehat{S}$ is densely defined and closable. We now show that it is closed. Consider a sequence of elements $u_n \in \mathcal{D}(\widehat{S})$ such that $u_n \to u$ and $\widehat{S}u_n \to v$ in $\mathscr{H}$. Then $Su_n \to v$ in $\mathscr{G}^*$ hence, S^{-1} being continuous, $u_n \to S^{-1}v$ in $\mathscr{G}$, so in $\mathscr{H}$. Hence $u = S^{-1}v \in \mathcal{D}(\widehat{S})$ and $\widehat{S}u = v$.

We have proved that $\widehat{S}$ is densely defined and closed and clearly $0 \in \rho(\widehat{S})$. Then we also have $0 \in \rho(\widehat{S}^*)$, so $\widehat{S}^* : \mathcal{D}(\widehat{S}^*) \to \mathscr{H}$ is bijective. Since $\widehat{S^*} : \mathcal{D}(\widehat{S^*}) \to \mathscr{H}$ is also bijective and $\widehat{S}^*$ is an extension of $\widehat{S^*}$, we get $\widehat{S^*} = \widehat{S}^*$. ■

A standard example of operator satisfying the condition required above is a *coercive* operator, i.e. such that Re $\langle u, Su \rangle \geq \mu \|u\|^2_{\mathscr{G}} - \nu \|u\|^2_{\mathscr{H}}$ for some strictly positive constants μ, ν and all $u \in \mathscr{G}$. Indeed, replacing S by $S+\nu$, we may assume Re $\langle u, Su\rangle \geq \mu\|u\|^2_{\mathscr{G}}$. Since S^* verifies the same estimate, this clearly gives $\|Su\|_{\mathscr{G}^*} \geq \mu\|u\|_{\mathscr{G}}$ and $\|S^*u\|_{\mathscr{G}^*} \geq \mu\|u\|_{\mathscr{G}}$ for all $u \in \mathscr{G}$. Thus S and S^* are injective operators with closed range, which implies that they are bijective.

If A is a self-adjoint operator on $\mathscr{H}$ then there is a natural Gelfand triplet associated to it, namely $\mathcal{D}(|A|^{1/2}) \subset \mathscr{H} \subset \mathcal{D}(|A|^{1/2})^*$. Then A extends to a continuous operator $A_0 : \mathcal{D}(|A|^{1/2}) \to \mathcal{D}(|A|^{1/2})^*$ which fulfills the conditions of Lemma A.1 and one has $\widetilde{A_0} = A$. In our applications it is interesting to know whether there are other Gelfand triplets $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ with $\mathcal{D}(A) \subset \mathscr{G}$ and such that A extends to a continuous operator $\mathscr{G} \to \mathscr{G}^*$. For not semibounded operators, e.g. for Dirac operators, many other possibilities exist such that $\mathscr{G}$ is not comparable to $\mathcal{D}(|A|^{1/2})$. But if A is semibounded, then the class of spaces $\mathscr{G}$ is rather restricted, as the next lemma shows.

Lemma A.2 *Assume that A is a bounded from below self-adjoint operator on $\mathscr{H}$ and such that $\mathcal{D}(A) \subset \mathscr{G}$ densely. Then A extends to a continuous operator $\widetilde{A} : \mathscr{G} \to \mathscr{G}^*$ if and only if $\mathscr{G} \subset \mathcal{D}(|A|^{1/2})$ and in this case $\widetilde{A} = A_0|_{\mathscr{G}}$.*

Proof: We prove only the nontrivial implication of the lemma. So let us assume that A extends to some $\widetilde{A} \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$. Replacing A by $A+\lambda$ with λ a large enough number, we can assume that $A \geq 1$. For $u \in \mathcal{D}(A)$ we have

$$\|A^{1/2}u\|_{\mathscr{H}} = \sqrt{\langle u, Au\rangle} = \sqrt{\langle u, \widetilde{A}u\rangle} \leq C\|u\|_{\mathscr{G}},$$

where $C^2 = \|\widetilde{A}\|_{\mathscr{G}\to\mathscr{G}^*}$. Since $\mathcal{D}(A)$ is dense in $\mathscr{G}$, it follows that the inclusion map $\mathcal{D}(A) \to \mathcal{D}(A^{1/2})$ extends to a continuous linear map $J : \mathscr{G} \to \mathcal{D}(A^{1/2})$. If $u \in \mathscr{G}$ then there is a sequence $\{u_n\}$ in $\mathcal{D}(A)$ such that $u_n \to u$ in $\mathscr{G}$. Then $J(u_n) \to J(u)$ in $\mathcal{D}(A^{1/2})$. Since $\mathscr{G}$ and $\mathcal{D}(A^{1/2})$ are continuously embedded in $\mathscr{H}$ we shall have $u_n \to u$ in $\mathscr{H}$ and $u_n = J(u_n) \to J(u)$ in $\mathscr{H}$, hence $J(u) = u$ for all $u \in \mathscr{G}$. In other terms, $\mathscr{G} \subset \mathcal{D}(A^{1/2})$. ■

We note that, under the conditions of the lemma, the inclusions $\mathcal{D}(A) \subset \mathscr{G}$ and $\mathscr{G} \subset \mathcal{D}(|A|^{1/2})$ are continuous (by the closed graph theorem), so we have a scale

$$\mathcal{D}(A) \subset \mathscr{G} \subset \mathcal{D}(|A|^{1/2}) \subset \mathscr{H} \subset \mathcal{D}(|A|^{1/2})^* \subset \mathscr{G}^* \subset \mathcal{D}(A)^*$$

with continuous and dense embeddings (because $\mathcal{D}(A)$ is dense in $\mathcal{D}(|A|^{1/2})$).

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