Acoustic scattering by an impedance cylinder immersed halfway between two half spaces

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Abstract. Scattering by an impedance cylinder immersed halfway between two half spaces of different acoustic properties has been studied. Solutions are obtained from an application of a discrete index of Hankel function transform. Expressions for the fields in both half spaces excited by various types of sources are given. Additionally analysis and quantification of transverse leakage are presented.

I. INTRODUCTION

To investigate the features of various media by means of acoustic radiation it is necessary to know the field scattered by inhomogeneties of these media. This problem can be solved by using as a basis a rigorous solution of a basic structure. One of the basic structures is the one considered in this article. The problem under consideration has also acquired practical relevance in fields such as the study of surfaces contaminated and the detection of defects. Additionally the solution of canonical problems such as the one under consideration is important in the sense of scattering and diffraction theories. The aim of this article is to present solutions, in terms of a discrete index of Hankel function transform, to the problem of the scattering of acoustic waves by a circular impedance cylinder immersed halfway between two half spaces of different acoustic properties. Configurations of cylinder and two half spaces have been dealt with before in the literature (Ref. 1 using Fourier/Bessel expansions on the angular variable and Ref. 2,3 using boundary integral equation methods) but the formulation presented in this article, to the best of the author's knowledge, has not been addressed before.

II. FORMULATION

We consider the problem of scattering of harmonic acoustic waves by an infinite cylinder of circular cross-section imbedded halfway between two semi infinite media of different acoustic properties . Let r, ϕ, z be cylindrical coordinates with the axis of the cylinder along the z axis on the interface between the two half spaces. We take the surface of the cylinder as $r = a$. Medium one occupies the half space $R_1 = \{0 < r < \infty,\ 0 \leq \phi \leq \pi,\ -\infty < z < \infty\}$ while medium two occupies the half space $R_2 = \{0 < r < \infty,\ 0 \geq \phi \geq -\pi,\ -\infty < z < \infty\}$.

The time dependence is $e^{-i\omega t}$, which factor will be suppressed throughout. k_1, ρ_1, and λ_1 are respectively the wave number, density, and in-

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compressibility in R_1. k_2, ρ_2, and λ_2 are corresponding quantities in R_2. An impedance boundary condition is imposed on the surface of the cylinder.

The acoustic field is describable by the pressure, P, and velocity, $\mathbf{V}$, obeying the Euler field equations[4]

$$\frac{-i\omega}{\lambda}P + \nabla.\mathbf{V} = -s, \tag{1.a}$$

$$\nabla P - i\omega\rho\mathbf{V} = -\mathbf{f}. \tag{1.b}$$

The excitation terms s and $\mathbf{f}$ represent the scalar particle source and the impressed vector force densities, respectively and ∇ is the spatial gradient operator.

In what follows s is assumed to be zero; $\rho_{1,2}$, $\lambda_{1,2}$ are assumed constant and acoustic field excitation is provided by an impressed line source located at (r_0, ϕ_0) in medium one. It should be pointed out that field expressions for other forms of excitation are recoverable once the ones for line source excitation are obtained. This will be detailed later in the article.

By using the symmetry of the problem structure with respect to the planes $\varphi = \pm\pi/2$, we split up the problem into two independent subproblems. The boundary conditions on the symmetry plane correspond to either a hard or a soft wall. So without loss of generality, we confine our attention to the case of a soft wall only.

Under the above conditions the acoustic pressure in medium one, $P_1(r, \phi)$, satisfies the inhomogeneous Helmholtz equation

$$(\nabla_t^2 + k_1^2)P_1(r, \phi) = -\frac{\delta(r - r_0)\ \delta(\phi - \phi_0)}{r_0}, \tag{2.a}$$

where $k_1 = \frac{\omega}{c_1}$, $c_1 = \sqrt{\frac{\lambda_1}{\rho_1}}$is the acoustic speed in R_1, while the acoustic pressure in R_2, $P_2(r, \phi)$, satisfies the homogeneous Helmholtz equation

$$(\nabla_t^2 + k_2^2)P_2(r, \phi) = 0, \tag{2.b}$$

where $k_2 = \frac{\omega}{c_2}$, $c_2 = \sqrt{\frac{\lambda_2}{\rho_2}}$is the acoustic speed in R_2.

∇_t^2 stands for the transverse (with respect to z) Laplacian in cylindrical coordinates (r, ϕ, z)

$$\nabla_t^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}.$$

The velocity fields are derived from the acoustic pressures as

$$\mathbf{V}_{1,2}(r,\phi) = \frac{-i}{\omega\rho_{1,2}}[\mathbf{r}_0\frac{\partial}{\partial r} + \boldsymbol{\phi}_0\frac{1}{r}\frac{\partial}{\partial\phi}]P_{1,2}(r,\phi). \tag{3}$$

Pressure and normal velocity are continuous across the interface between the two media:

$$P_1(r,\phi=0) = P_2(r,\phi=0), \tag{4.a}$$

$$V_{1\phi}(r,\phi=0) = V_{2\phi}(r,\phi=0). \tag{4.b}$$

Eq. (4.b) reduces on account of Eq. (3) to

$$\frac{1}{\rho_1}\frac{\partial}{\partial\phi}P_1(r,\phi=0) = \frac{1}{\rho_2}\frac{\partial}{\partial\phi}P_2(r,\phi=0) \tag{4.b'}$$

and radiation condition is to be satisfied as $r \to \infty$.

An impedance boundary condition is imposed on the surface of the cylinder requiring

$$\mathbf{n}.\mathbf{V}_{1,2}(r=a,\phi) = Y_s P_{1,2}(r=a,\phi) \tag{4.c}$$

where $\mathbf{n}$ is the outward normal to the surface andY_s is a constant surface admittance.

Eq. (4.c) reduces on account of Eq. (3) to

$$\frac{\partial P_{1,2}(r=a,\phi)}{\partial r} = -ik_{1,2}C_{1,2}P_{1,2}(r=a,\phi) \tag{4.c'}$$

where

$$C_{1,2=}Y_s/Y_{1,2},\ \mathrm{Re}\,C_{1,2} \geq 0. \tag{5}$$

with $Y_{1,2} = \frac{1}{\rho_{1,2}c_{1,2}}$ as the acoustic admittance of $R_{1,2}$ and the condition on the real part of $C_{1,2}$ is a passivity requirement.

We propose to solve the boundary value problem described in Eqs. (2.a), (2.b), and (4.a), (4.b') and (4.c') by means of a discrete index of Hankel function transform.

The transform pair is given by[4]

$$P_{1,2}(r,\phi) = \sum_p p_{1,2}(\nu_{1,2p},\phi)\Phi_p(k_{1,2}r), \tag{6.a}$$

$$p_{1,2}(\nu_{1,2p},\phi) = \int_a^\infty \frac{1}{r}P_{1,2}(r,\phi)\Phi_p(k_{1,2}r)dr, \tag{6.b}$$

with the completeness and orthonormality relation

$$r'\delta(r-r') = \sum_p \Phi_{1,2p}(k_{1,2}r)\Phi_{1,2p}(k_{1,2}r'),\ a \le (r,r') < \infty \tag{6.c}$$

where

$$\Phi_{1,2p}(k_{1,2}r) = \{\frac{-\pi i\ \nu_{1,2p} b(\nu_{1,2p})}{[\frac{d}{d\nu}d(\nu)]_{\nu_{1,2p}}}\}^{1/2} H^{(1)}_{\nu_{1,2p}}(k_{1,2}r), \tag{7.a}$$

$$b(\nu_{1,2p}) = J'_{\nu_{1,2p}}(k_{1,2}a) + iC_{1,2}J_{\nu_{1,2p}}(k_{1,2}a), \tag{7.b}$$

$$d(\nu_{1,2p}) = H^{'(1)}_{\nu_{1,2p}}(k_{1,2}a) + iC_{1,2}H^{(1)}_{\nu_{1,2p}}(k_{1,2}a). \tag{7.c}$$

$J_\nu(z)$ is Bessel function; $H^{(1)}_\nu(z)$ is Hankel function of type one with the prime denoting the derivative with respect to the argument;

$$d(\nu_{1,2p}) = 0,\ \mathrm{Re}\,\nu_{1,2p} > 0,\ \mathrm{Im}\,\nu_{1,2p} > 0. \tag{7.d}$$

Such an index transform has been used before[4] to analyze diffraction by an impedance cylinder in an infinite medium.

Applying the index transform in Eq. (6.a) to Eqs. (2.a) and (2.b) we obtain the ordinary deferential equations

$$(\frac{d^2}{d\phi^2} + \nu^2_{1p})p_1(\nu_{1p},\phi) = -\Phi_{1p}(k_1 r_0)\delta(\phi-\phi_0), \tag{8.a}$$

$$(\frac{d^2}{d\phi^2} + \nu^2_{2p})p_2(\nu_{2p},\phi) = 0. \tag{8.b}$$

We represent $p_1(\nu_{1p},\phi)$ as

$$p_1(\nu_{1p},\phi) = p^{(0)}_1(\nu_{1p},\phi) + p^{(1)}_1(\nu_{1p},\phi) \tag{9}$$

$p^{(0)}_1(\nu,\phi)$ satisfies the source conditions:
i) $p^{(0)}_1(\nu,\phi)$ is continuous across $\phi = \phi_0$

$$p^{(0)}_1(\nu_{1p},\phi_0-0) = p^{(0)}_1(\nu_{1p},\phi_0+0), \tag{10.a}$$

ii) $\frac{d}{d\phi}p^{(0)}_1(\nu_{1p},\phi)$ is discontinuous across $= \phi_0$ with

$$\frac{d}{d\phi}p^{(0)}_1(\nu,\phi)|_{\phi_0+0} - \frac{d}{d\phi}p^{(0)}_1(\nu,\phi)|_{\phi_0-0} = -\Phi_{1p}(k_1 r_0). \tag{10.b}$$

and both $p^{(0)}_1(\nu_{1p},\phi)$ and $p^{(1)}_1(\nu_{1p},\phi)$ satisfy the soft boundary condition at $\phi = \frac{\pi}{2}$. Hence

$$p_1^{(0)}(\nu_{1p},\phi) = \frac{\sin\nu_{1p}(\phi_> - \frac{\pi}{2})\cos\nu_{1p}\phi_<}{-\nu_{1p}\cos(\frac{\nu_{1p}\pi}{2})}\Phi_{1p}(k_1 r_0) \qquad (11.a)$$

where $\phi_>|\phi_<$ is the lesser|greater of ϕ, ϕ_0, and

$$p_1^{(1)}(\nu_{1p},\phi) = A_1(\nu_{1p})\sin\nu_{1p}(\phi - \frac{\pi}{2}). \qquad (11.b)$$

In R_2, $p_2(\nu_{2p},\phi)$ satisfies the soft boundary condition at $\phi = \frac{-\pi}{2}$. Hence

$$p_2(\nu_{2p},\phi) = A_2(\nu_{2p})\sin\nu_{2p}(\phi + \frac{\pi}{2}). \qquad (12)$$

$A_1(\nu_{1p})$ and $A_2(\nu_{2p})$ are transform spectra to be determined from the boundary conditions on the interface between the R_1 and R_2 as per Eqs. (4.a) and (4.b').

Remark. The impedance boundary condition on the surface of the cylinder, given in Eq. (4.c'), is built in the eigen functions $\Phi_p(k_{1,2}r)$. This is one advantage of using the discrete index of Hankel transform for problems with boundaries along r = constant.

The field in R_1 now reads

$$P_1(r,\phi) = P_1^{(0)}(r,\phi) + P_1^{(1)}(r,\phi) \qquad (13.a)$$

where

$$P_1^{(0)}(r,\phi) = \sum_p \frac{\sin\nu_{1p}(\phi_> - \frac{\pi}{2})\cos\nu_{1p}\phi_<}{-\nu_{1p}\cos(\frac{\nu_{1p}\pi}{2})}\Phi_p(k_1 r_0)\Phi_p(k_1 r), \qquad (13.b)$$

$$P_1^{(1)}(r,\phi) = \sum_p A_1(\nu_{1p})\sin\nu_{1p}(\phi - \frac{\pi}{2})\Phi_p(k_1 r). \qquad (13.c)$$

The field in R_2 is given by

$$P_2(r,\phi) = \sum_p A_2(\nu_{2p})\sin\nu_{2p}(\phi + \frac{\pi}{2})\Phi_p(k_2 r). \qquad (14)$$

Applying the boundary conditions in Eqs. (4.a) and (4.b') we obtain

$$\sum_p \{\frac{\sin\nu_{1p}(\frac{\pi}{2} - \phi_0)}{\nu_{1p}\cos(\frac{\nu_{1p}\pi}{2})}\Phi_p(k_1 r_0) - A_1(\nu_{1p})\sin(\frac{\nu_{1p}\pi}{2})\}\Phi_p(k_1 r) = \sum_p A_2(\nu_{2p})\sin(\frac{\nu_{2p}\pi}{2})\Phi_p(k_2 r), \qquad (15.a)$$

$$\sum_p \nu_{1p}A_1(\nu_{1p})\cos(\frac{\nu_{1p}\pi}{2})\Phi_p(k_1 r) = \tau\sum_p \nu_{2p}A_2(\nu_{2p})\cos(\frac{\nu_{2p}\pi}{2})\Phi_p(k_2 r), \qquad (15.b)$$

with $\tau = \frac{\rho_1}{\rho_2}$.

III. LINEAR SYSTEM DERIVATION

We multiply Eq. (15.a) and (15.b) by $\frac{1}{r}\Phi_q(k_1 r)$ and integrate on r from a to ∞ utilize the orthonormality relation of Eq. (6.c) to reach

$$\frac{\sin\nu_{1q}(\frac{\pi}{2}-\phi_0)}{\nu_{1q}\cos(\frac{\nu_{1p}\pi}{2})}\Phi_q(k_1 r_0) - A_1(\nu_{1q})\sin(\frac{\nu_{1q}\pi}{2}) = \sum_p A_2(\nu_{2p})\sin(\frac{\nu_{2p}\pi}{2})C_{pq}\ \forall q, \tag{16.a}$$

$$\nu_{1q}A_1(\nu_{1q})\cos(\frac{\nu_{1q}\pi}{2}) = \tau\sum_p \nu_{2p}A_2(\nu_{2p})\cos(\frac{\nu_{2p}\pi}{2})C_{pq}\ \forall q \tag{16.b}$$

where

$$C_{qp} = \int_a^\infty \frac{1}{r}\Phi_q(k_1 r)\Phi_p(k_2 r)dr. \tag{17}$$

We cast the linear system as

$$\mathbf{D}_0[\frac{1}{\nu_1\cos(\frac{\nu_1\pi}{2})}]\mathbf{S} - \mathbf{D}_1[\sin(\frac{\nu_1\pi}{2})]\mathbf{A}_1 = \mathbf{C}\mathbf{D}_2[\sin(\frac{\nu_2\pi}{2})]\mathbf{A}_2, \tag{18.a}$$

$$\mathbf{D}_3[\nu_1\cos(\frac{\nu_1\pi}{2})]\mathbf{A}_1 = \mathbf{C}\mathbf{D}_4[\tau\nu_2\cos(\frac{\nu_2\pi}{2})]\mathbf{A}_2 \tag{18.b}$$

where S is the vector

$$S = \{\sin\nu_{1q}(\frac{\pi}{2}-\phi_0)\ \Phi_q(k_1 r_0)\}, \tag{19}$$

$\mathbf{D}_i[.], i = 0:4$ are diagonal matrices with diagonal elements $[.]$ and $\mathbf{A}_{1,2}$ are the vectors of spectral amplitudes.

From the above system we derive

$$\mathbf{A}_1 = \mathbf{M}_1^{-1}\mathbf{S}, \tag{20.a}$$

$$\mathbf{A}_2 = \mathbf{M}_2^{-1}\mathbf{S} \tag{20.b}$$

where

$$\mathbf{M}_1 = \mathbf{D}_0^{-1}\mathbf{C}\mathbf{D}_4^{-1}(\mathbf{D}_2\mathbf{C}^{-1}\mathbf{D}_3 + \mathbf{D}_4\mathbf{C}^{-1}\mathbf{D}_1), \tag{20.c}$$

$$\mathbf{M}_2 = \mathbf{D}_3\mathbf{C}\mathbf{D}_2 + \mathbf{D}_1\mathbf{C}\mathbf{D}_4. \tag{20.d}$$

Remark. Had we used Fourier expansion on the angular variable instead of the discrete index of Hankel function transform, the lack of an orthogonality relation for the Hankel functions of integer order would have resulted in the appearance of dense matrices of the forms

$$\int_a^\infty \frac{1}{r} H_q^{(1)}(k_1 r) H_p^{(1)}(k_1 r) dr, \tag{21.a}$$

$$\int_a^{r_0} \frac{1}{r} H_q^{(1)}(k_1 r) J_p(k_1 r) dr \tag{21.b}$$

and

$$\int_{r_0}^\infty \frac{1}{r} H_q^{(1)}(k_1 r) H_p^{(1)}(k_1 r) dr \tag{21.c}$$

in the linear system.

This is another advantage of using the discrete index of Hankel transform for problems with boundaries along $\phi =$ constant.

A. The isovelocity case

Under the isovelocity condition ($k_1 = k_2$) and with $Y_s = 0$ (or ∞) $\nu_{1p} = \nu_{2p}$, C_{qp} turns into the unit matrix and the linear system simplifies to

$$\frac{\sin \nu_{1q}(\frac{\pi}{2} - \phi_0)}{\nu_{1q} \cos(\frac{\nu_{1q}\pi}{2})} \Phi_q(k_1 r_0) - A_1(\nu_{1q}) \sin(\frac{\nu_{1q}\pi}{2}) = A_2(\nu_{1q}) \sin(\frac{\nu_{1q}\pi}{2}), \tag{22.a}$$

$$A_1(\nu_{1q}) = \tau A_2(\nu_{1q}) \tag{22.b}$$

leading to

$$A_2(\nu_{1q}) = \frac{1}{1+\tau} \frac{\sin \nu_{1q}(\frac{\pi}{2} - \phi_0)}{\nu_{1q} \cos(\frac{\nu_{1q}\pi}{2}) \sin(\frac{\nu_{1q}\pi}{2})} \Phi_q(k_1 r_0). \tag{22.c}$$

Therefore

$$P_1^{(1)}(r,\phi) = \frac{\tau}{1+\tau} \sum_q \frac{\sin \nu_{1q}(\frac{\pi}{2} - \phi_0)}{\nu_{1q} \cos(\frac{\nu_{1q}\pi}{2}) \sin(\frac{\nu_{1q}\pi}{2})} \sin \nu_{1q}(\phi - \frac{\pi}{2}) \Phi_q(k_1 r_0) \Phi_q(k_1 r), \tag{23.a}$$

$$P_2(r,\phi) = \frac{1}{1+\tau} \sum_q \frac{\sin \nu_{1q}(\frac{\pi}{2} - \phi_0)}{\nu_{1q} \cos(\frac{\nu_{1q}\pi}{2}) \sin(\frac{\nu_{1q}\pi}{2})} \sin \nu_{1q}(\phi + \frac{\pi}{2}) \Phi_q(k_1 r_0) \Phi_q(k_1 r). \tag{23.b}$$

From the asymptotic behavior of the Bessel and Hankel functions[4] as $q \to \infty$ the series in Eqs. (13.b), (23.a) and (23.b) converge exponentially with the order of the q$^{\text{th}}$term $O(e^{i\nu_q|\phi-\phi_0|})$, $O(e^{i\nu_q|\phi+\phi_0|})$, $O(e^{i\nu_q|\phi-\phi_0|})$ respectively. The only exceptions are thus when $\phi = \phi_0$ in Eq. (13.b) and when $\phi = \phi_0 = 0$ in Eqs (23.a)- (23.b) where the corresponding series diverges.

For $P_1^{(0)}(r,\phi)$, if one must compute the fields along $\phi = \phi_0$ where the series representation diverges and is not, relying on field continuity, satisfied by interpolating from neighboring points then a representation convergent when $\phi = \phi_0$ is derived by substituting

$$[\frac{\sin\nu_{1p}(\frac{\pi}{2}-\phi_>)\cos\nu_{1p}\phi_<}{\nu_{1p}\cos(\frac{\nu_{1p}\pi}{2})}] = [\frac{1}{2}\{ie^{i\nu_{1p}|\phi-\phi_0|} - \frac{2ie^{i\pi\nu_{1p}}}{1+e^{i\pi\nu_{1p}}}\cos\nu_{1p}(\phi-\phi_0) + \frac{\sin\nu_{1p}(\frac{\pi}{2}-\phi-\phi_0)}{\cos(\frac{\nu_{1p}\pi}{2})}\}]$$

into Eq. (13.b) to obtain

$$P_1^{(0)}(r,\phi) = P_{11}^{(0)}(r,\phi) + P_{12}^{(0)}(r,\phi) + P_{13}^{(0)}(r,\phi) \tag{24}$$

where

$$P_{11}^{(0)}(r,\phi) = \frac{-i}{2}\sum_p e^{i\nu_{1p}|\phi-\phi_0|}\Phi_p(k_1r_0)\Phi_p(k_1r), \tag{25.a}$$

$$P_{12}^{(0)}(r,\phi) = i\sum_p \frac{e^{i\pi\nu_{1p}}}{1+e^{i\pi\nu_{1p}}}\cos\nu_{1p}(\phi-\phi_0)\Phi_p(k_1r_0)\Phi_p(k_1r), \tag{25.b}$$

$$P_{13}^{(0)}(r,\phi) = \frac{-1}{2}\sum_p \frac{\sin\nu_{1p}(\frac{\pi}{2}-\phi-\phi_0)}{\cos(\frac{\nu_{1p}\pi}{2})}\Phi_p(k_1r_0)\Phi_p(k_1r). \tag{25.c}$$

Applying a Watson transform[5] on the series in Eq. (25.a) we obtain

$$\frac{i}{16}\int_\gamma e^{i\nu|\phi-\phi_0|}\{H_\nu^{(2)}(k_1r_<)H_\nu^{(1)}(k_1r_>) - H_\nu^{(1)}(k_1r)H_\nu^{(1)}(k_1r_0)\frac{H_\nu^{(2)}(k_1a)}{H_\nu^{(1)}(k_1a)}\}d\nu \tag{26}$$

where $r_<|r_>$ is the lesser|greater of $r|r_0$ and γ is the contour around the zeros of $d(\nu) = 0$ in the first quadrant of the complex ν plane. After asymptotic evaluation the integrals are recognized as the geometrical optical incident and reflected fields respectively.

The series in Eq. (25.b) converges for all the observation points while that in Eq. (25.c) converges for all observation points except when $\phi = \phi_0 = 0$.

Therefore we conclude that the formulation renders valid field expressions in terms of series sums for arbitrary source-observer configurations with the exception of the situation when both are on the interface between the two media.

IV. EXTENSIONS TO OTHER SOURCE TYPES

A. Extension to point source excitation

Staying with the isovelociy case and in order to recover the field expressions for excitation by a point source located at (r_0, ϕ_0, z_0) k_1 in Eqs. (13.b), (23.a) and (23.b) is replaced by χ_1

where

$$\chi_1 = k_1\sqrt{1-\beta^2} \tag{27}$$

and the operation

$$\frac{k_1}{2\pi}\int_{-\infty}^{\infty} d\beta \exp[ik_1\beta(z-z_0)]$$

is performed subsequently rendering

$$P_1^{(0)}(r,\phi,z) = \frac{k_1}{2\pi}\int_{-\infty}^{\infty} d\beta \exp[ik_1\beta(z-z_0)]P_1^{(0)}(r,\phi,\beta), \tag{28.a}$$

$$P_1^{(1)}(r,\phi,z) = \frac{k_1}{2\pi}\int_{-\infty}^{\infty} d\beta \exp[ik_1\beta(z-z_0)]P_1^{(1)}(r,\phi,\beta), \tag{28.b}$$

$$P_2(r,\phi,z) = \frac{k_1}{2\pi}\int_{-\infty}^{\infty} d\beta \exp[ik_1\beta(z-z_0)]P_2(r,\phi,\beta) \tag{28.c}$$

where $P_1^{(0)}(r,\phi,\beta)$, $P_1^{(0)}(r,\phi,\beta)$ and $P_1^{(0)}(r,\phi,\beta)$ given in Eqs. (13.b), (23.a) and (23.b) with k_1 replaced by χ_1.

Since the function χ_1 in Eq. (27) has two branches (depending on picking the + or the $-$ sign of the square root) the complex β plane is two-sheeted. We define

$$\mathrm{Im}\sqrt{1-\beta^2} > 0 \tag{29.a}$$

on the top sheet and

$$\mathrm{Im}\sqrt{1-\beta^2} < 0 \tag{29.b}$$

on the bottom sheet. $\beta = \pm 1$ are branch points with branch cuts along $\mathrm{Im}\sqrt{1-\beta^2} = 0$.

In order to satisfy the radiation condition the integration contour in Eqs. (28.a)-(28.c) is in the top sheet of the complex β plane.

In order to analyze and quantify the leakage (along r or equivalently with respect to z) properties of the structure under consideration we proceed to isolate such behavior. To that end we deform the integration contour to the steepest descent path[4] (SDP) going through the saddle point of the integrand. Poles of the integrand encountered in the process of contour deformation are to be collected.

We show below the details for the field in R_2. Those for R_1 are found similarly. We focus on the case$Y_s \to \infty$ (the case $Y_s = 0$ is treated similarly). Making use of

$$J_\nu(z) = \frac{1}{2}[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)]$$

in Eq. (28.c) we obtain

$$P_2(r,\phi,z) = P_{21}(r,\phi,z) + P_{22}(r,\phi,z) \tag{30.a}$$

where

$$P_{21}(r,\phi,z) = \frac{-ik_1}{2(1+\tau)} \int_{-\infty}^{\infty} d\beta \; e^{ik_1\beta(z-z_0)} \sum_q B(\nu_{1q},\phi,\phi_0) \frac{H_{\nu_{1q}}^{(1)}(\chi_1 a)}{\frac{d}{d\nu_{1q}} H_{\nu_{1q}}^{(1)}(\chi_1 a)} H_{\nu_{1q}}^{(1)}(\chi_1 r_0) H_{\nu_{1q}}^{(1)}(\chi_1 r), \tag{30.b}$$

$$P_{22}(r,\phi,z) = \frac{-ik_1}{2(1+\tau)} \int_{-\infty}^{\infty} d\beta \; e^{ik_1\beta(z-z_0)} \sum_q B(\nu_{1q},\phi,\phi_0) \frac{H_{\nu_{1q}}^{(2)}(\chi_1 a)}{\frac{d}{d\nu_{1q}} H_{\nu_{1q}}^{(1)}(\chi_1 a)} H_{\nu_{1q}}^{(1)}(\chi_1 r_0) H_{\nu_{1q}}^{(1)}(\chi_1 r) \tag{30.c}$$

with

$$B(\nu_{1q},\phi,\phi_0) = \frac{\sin\nu_{1q}(\frac{\pi}{2}-\phi_0)}{2\cos(\frac{\nu_{1q}\pi}{2})\sin(\frac{\nu_{1q}\pi}{2})} \sin\nu_{1q}(\phi+\frac{\pi}{2}). \tag{30.d}$$

We start with Eq. (30.b) and assume that $z > z_0$ (the case $z < z_0$ is treated similarly) . Substituting with the large argument approximation of Hankel functions

$$H_\nu^{(1,2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{\pm i(z-\nu\pi/2-\pi/4)}$$

the saddle point β_s of the resulting phase function of the integrand

$$\Psi(\beta; r, r_0, z, z_0) = k_1(\sqrt{1-\beta^2}(r+r_0) + \beta(z-z_0)) \tag{31.a}$$

is found from

$$\frac{d}{d\beta}\Psi(\beta; r, r_0, z, z_0)|_{\beta=\beta_s} = 0 \tag{31.b}$$

leading to

$$\beta_s = \frac{(z-z_0)}{r+r_0}\frac{1}{\sqrt{1+(\frac{(z-z_0)}{r+r_0})^2}}. \tag{31.c}$$

The SDP is defined by

$$\mathrm{Re}\,\Psi(\beta; r, r_0, z, z_0) = \mathrm{Re}\,\Psi(\beta_s; r, r_0, z, z_0) \text{ with } \mathrm{Im}\,\Psi(\beta; r, r_0, z, z_0) > 0 \tag{32}$$

Since from Eq. (31.c)

$$\beta_s \text{ (the saddle point)} < 1 \text{ (the branch point)} \tag{33}$$

and from Eq. (32) we infer that part of the deformed integration contour (the SDP) runs in the bottom sheet of the first quadrant of the complex β plane.

The integrand in Eq. (30.b) has poles satisfying

$$\sin \nu_{1q}\pi = 0 \tag{34.a}$$

leading to

$$\nu_{1q} = n, n = 1, 2, 3, .., \infty. \tag{34.b}$$

The conditions in Eqs. (34.a)-(34.b), making use of Eqs. (7.c)-(7.d) translate to

$$d(n, \beta) = 0. \tag{34.c}$$

Eq. (34.c) leads to

$$H_n^{(1)}(k_1 a\sqrt{1-\beta_{nm}^2}) = 0, \ (n, m) = 1, 2, 3, ..., \infty \tag{34.d}$$

where β_{nm} defines the location of the poles of the integrand of Eq. (30.b) in the complex β plane.

These poles are located in the bottom sheet of the complex β plane in the four quadrants[6] and thus the residues of the ones in the domain

bounded by the SDP and the coordinate axis in the first quadrant during the process of contour deformation are to be collected.

Hence the leaky (along r) behavior is quantified by the contribution

$$P_{21}^{(l)}(r,\phi,z) = \sum_{n=N_1}^{n=N_2} \sum_{m=1}^{m=M(n)} \hat{B}_{nm} \frac{H_n^{(1)}(\chi_{1_{nm}} a)}{H_n'^{(1)}(\chi_{1_{nm}} a)} H_n^{(1)}(\chi_{1_{nm}} r_0) H_n^{(1)}(\chi_{1_{nm}} r) e^{i\beta_{nm}(z-z_0)} \tag{35.a}$$

with

$$\hat{B}_{nm} = \frac{(-1)^n}{a(1+\tau)} \sin n(\frac{\pi}{2} - \phi_0) \sin n(\phi + \frac{\pi}{2}) \frac{\sqrt{1-\beta_{nm}^2}}{\beta_{nm}}$$

where the sum in Eq. (35.a) is on the residues of the poles encountered. From Ref. 6 it is inferred that

$$\sqrt{1 - (\frac{N_1}{k_1 a})^2} < \beta_s \tag{35.b}$$

and

$$\sqrt{(\frac{N_2}{k_1 a})^2 - 1} < \beta_i \tag{35.c}$$

where $(0, i\beta_i)$ is the intersection point between the SDP and the +ive imaginary axis of the complex β plane and is given by

$$\sqrt{1+\beta_i^2}(r + r_0) = \sqrt{1-\beta_s^2}(r + r_0) + \beta_s(z - z_0) \tag{35.d}$$

and $M(n)$ defines the number of poles collected for each index n through the requirement that

$$\beta_{nm} \in \Omega \tag{35.e}$$

where Ω is the closed domain on the bottom sheet in the first quadrant of the complex β plane bounded by the coordinate axis and the SDP. Eqs. (35.b)-(35.e) reveal that the number of contributing poles to the laky behavior are source-observer configuration dependent.

The analysis of Eq. (30.c) are similar to that performed above with the substitution of $(r + r_0 - 2a)$ for $(r + r_0)$.

The non-leaky part of the field is obtained from the saddle point contribution.

Remark. For the arbitrary refractive index case ($k_1 \neq k_2$) with arbitrary Y_s the field due to a point source excitation is recovered from that of the line source field following the same steps performed above with

(i) $k_{1,2}$ replaced by $\chi_{1,2}$ where

$$\chi_1 = k_2\sqrt{(\frac{k_1}{k_2})^2 - \beta^2}, \tag{36.a}$$

$$\chi_2 = k_2\sqrt{1-\beta^2} \tag{36.b}$$

and the operation

$$\frac{k_2}{2\pi}\int_{-\infty}^{\infty} d\beta \exp[ik_2\beta(z-z_0)]$$

is performed subsequently.

(ii) The complex β plane now has extra branch points at $\beta_b = \pm\frac{k_1}{k_2}$ with branch cuts along $\mathrm{Im}\sqrt{(\frac{k_1}{k_2})^2 - \beta^2} = 0$ and two extra sheets due to the square root in Eq. (36.a). Therefore the complex β plane has now four sheets defined by

$$Sh_{tt} : \mathrm{Im}\sqrt{1-\beta^2} > 0, \mathrm{Im}\sqrt{(\frac{k_1}{k_2})^2 - \beta^2} > 0 \tag{37.a}$$

$$Sh_{tb} : \mathrm{Im}\sqrt{1-\beta^2} > 0, \mathrm{Im}\sqrt{(\frac{k_1}{k_2})^2 - \beta^2} < 0 \tag{37.b}$$

$$Sh_{bt} : \mathrm{Im}\sqrt{1-\beta^2} < 0, \mathrm{Im}\sqrt{(\frac{k_1}{k_2})^2 - \beta^2} > 0 \tag{37.c}$$

$$Sh_{bb} : \mathrm{Im}\sqrt{1-\beta^2} < 0, \mathrm{Im}\sqrt{(\frac{k_1}{k_2})^2 - \beta^2} < 0 \tag{37.d}$$

(iii) β_{nm} is obtained from

$$d(\nu_{1,2q}, \beta) = 0 \tag{38.a}$$

$$\det(M_2) = 0 \tag{38.b}$$

satisfied simultaneously. The poles β_{nm} are located in the Sh_{bb} sheet. When $\frac{k_1}{k_2} > \beta_s$ ($\frac{k_1}{k_2} < \beta_s$) the poles contributing to the leaky behavior are those in the closed domain on the Sh_{bb} sheet in the first quadrant of the complex β plane bounded by the coordinate axis and the SDP (the SDP of the branch cut integral).

When $\frac{k_1}{k_2} < \beta_s$ a branch point contribution from β_b needs also to be collected during the process of contour deformation and together with the saddle point contribution give the non-leaky part of the field description.

B. Extension to normally incident plane wave

In order to recover the field expressions for a normally (to z) incident plane wave at an angle of ϕ_0 one first lets $r_0 \to \infty$ in the field expressions for the line source excitation making use of the large argument approximation of Hankel function then sets

$$\frac{1}{4}\sqrt{\frac{2}{\pi k_1 r_0}}e^{i(k_1 r_0+\frac{\pi}{4})} = 1 \tag{39}$$

The convergence properties of the resulting series remain the same as discussed before.

C. Extension to obliquely incident plane wave

In order to recover the field expressions for a plane wave incident obliquely along the direction (ϕ_0, θ_0) we make the replacements

$$k_1 \to k_2\sqrt{(\frac{k_1}{k_2})^2 - (\cos\theta_0)^2}, \tag{40.a}$$

$$k_2 \to k_2 \sin\theta_0 \tag{40.b}$$

in the field expressions for the normally incident plane wave and subsequently multiply the field expressions by

$$\exp(ik_2 z\ \cos\theta_0). \tag{40.c}$$

D. Extension to beam excitation

The fields due to 2-D (3-D) beam excitation is derived from that of 2-D (3-D) plane wave excitation by attaching a profile $S(\phi_0)$ ($S(\phi_0, \theta_0)$) to the incident plane wave and making use of the superposition principle.

Analysis and quantification of the leakage phenomena in the 3-D case proceed similar to that performed in connection with point source excitation with $\cos\theta_0$ assuming the role of β.

V. CONCLUSION

The problem of scattering of acoustic waves by a circular cylinder with a surface impedance boundary and buried halfway between two half spaces has been formulated in terms of the index of Hankel function transform. Expressions for the total field as well as the leaky part of the field have been derived. Conditions of validity for each of these field expressions for a given set of problem parameters have been established. It is concluded that the index of Hankel function transform formulation suits naturally expressing the fields in domains bounded by both radial and angular boundaries. It also allows for expressing the acoustic field for arbitrary refractive index and surface impedance and source-observer

configurations. This is apart from the configuration disallowed by the formulation when both source and observer are on the interface between the two half spaces.

REFERENCES

[1]Philip L. Marston, " Scattering of plane evanescent waves by buried cylinders: Modeling the coupling to guided waves and resonances," J. Acoust. Soc. Am. **113**, 2320 (2003).

[2]John A. Fawcett, "Acoustic scattering from cylindrical objects embedded between two half-spaces," J. Acoust. Soc. Am. **100**, 3053 (1996).

[3]John A. Fawcett, "The computation of the scattered pressure field from a cylinder embedded between two half-spaces with different densities," J. Acoust. Soc. of Am., **99**, 2435-2438 (1996).

[4]L.B. Felsen and N. Marcuvitz, *Radiation and scattering of waves* (Prentice-Hall inc., New Jersey, USA 1973).

[5]G.N. Watson, "Diffraction of electric waves by the earth," Proc. Roy. Soc. (London), **A95**, 83-99 (1919).

[6]M. Abramowitz and I. Stegun, *Handbook of mathematical functions.* (Dover Publications, New York, USA, 1970).