On the multisummability of formal solutions of partial differential equations[*]

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Abstract

We study Gevrey properties and multisummability of power series in more than two variables that are formal solutions of a Cauchy problem for linear partial differential equations with constant coefficients. We extend earlier results in an article of Werner Balser, who has considered linear partial differential equations in two variables. Some new problems appear which will be investigated in this paper.

Key words: Partial differential equations, power series solutions, multisummability. 2000 MSC: 35C10, 35C20.

Introduction

In this article, we study formal power series solutions of a Cauchy problem for homogeneous linear partial differential equations in several complex variables t, $z_1, \ldots, z_p$ with constant coefficients. More especially, we will deal with formal power series solutions constructed from a root of the so-called characteristic equation of a given partial differential equation (Proposition 2). These power series correspond to *particular normalized formal power series* in the terminology of the paper [1]. In the latter paper, Werner Balser has shown that the general homogeneous Cauchy problem in two variables t, z satisfies the property that all its particular normalized power series solutions are k-summable in the variable t, for well chosen initial conditions. The picture is however different in the case involving more than two variables as we will show through the following example

$$(\partial_t - \partial_{z_1}^2 - \partial_{z_2}^3)u(t, z_1, z_2) = 0,$$

for a well chosen initial condition $u(0, z_1, z_2)$. In particular, we will show that the latter equation has a particular normalized formal power series which is $(1, 1/2)$-summable and so not k-summable for any k (Example 3).

This example gives rise to the question of sufficient conditions on a given partial differential equation for the existence of k-summable particular normalized power series. This question will be investigated in the third section of this paper (Theorem 2). In the first two sections, we study the existence and the Gevrey properties of power series solutions of linear partial

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differential equations with constant coefficients under certain assumptions on the characteristic equation of the partial differential equation (Theorem 1). In the last section, we give a way to construct linear partial differential equations with constant coefficients having power series solutions which are multisummable in the variable t (Proposition 8). In this section, the example of the multidimensional heat equation

$$(\partial_t - \sum_{i=1}^{p} \partial_{z_i}^2)u = 0,$$

is also considered for particular initial condition $u(0, z_1, \ldots, z_p)$.

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1 Formal power series solutions

We consider a linear partial differential equation with constant coefficients

$$P(\partial_t, \partial_{z_1}, \ldots, \partial_{z_p})u(t, z_1, \ldots, z_p) = 0, \tag{1}$$

where $P \in \mathbb{C}[\partial_t, \partial_{z_1}, \ldots, \partial_{z_p}]$ is a non trivial polynomial with coefficients in $\mathbb{C}$. For notational convenience, we write P in the form

$$P(\partial_t, \partial_{z_1}, \ldots, \partial_p) = \partial_t{}^{\kappa} p(\partial_{z_1}, \ldots, \partial_{z_p}) - \sum_{\nu=1}^{\kappa} \partial_t{}^{\kappa-\nu} p_\nu(\partial_{z_1}, \ldots, \partial_{z_p}),$$

Let

$$\hat{u}(t, z_1, \ldots, z_p) = \sum_{j=0}^{\infty} \frac{t^j}{j!} u_j(z_1, \ldots, z_p) \tag{2}$$

be a formal power series in t, where $u_j(z_1, \ldots, z_p)$, $0 \le j \le \infty$ are holomorphic functions in a polydisc $D_{\rho_1} \times \cdots \times D_{\rho_p}$ (where $D_\rho = \{z \in \mathbb{C} : |z| < \rho\}$). The following proposition gives necessary and sufficient conditions for $\hat{u}$ to be a solution of (1).

Proposition 1 *A formal power series $\hat{u}(t, z_1, \ldots, z_p)$ is a formal solution of* (1) *if and only if the coefficients $u_j(z_1, \ldots, z_p)$ satisfy*

$$p(\partial_{z_1}, \ldots, \partial_{z_p}) u_j(z_1, \ldots, z_p) = \sum_{\nu=1}^{\kappa} p_\nu(\partial_{z_1}, \ldots, \partial_{z_p}) u_{j-\nu}(z_1, \ldots, z_p) \tag{3}$$

for all $(z_1, \ldots, z_p) \in D_{\rho_1} \times \cdots \times D_{\rho_p}$ and $j \ge \kappa$. In particular, the functions u_j, $0 \le j \le \kappa - 1$, can be chosen arbitrarily.

For the proof, insert the power series (2) into (1) and compare coefficients.

In what follows, we will consider particular formal series solutions constructed from roots of the so-called characteristic equation of the partial differential equation (1). We consider the following equation, which will be called the *characteristic equation* of (1)

$$P(\lambda, w_1, \ldots, w_p) = 0. \tag{C}$$

We make an assumption concerning the existence of a root of (C).

Assumption A There exists a function $\lambda(w_1,\ldots,w_p)$ satisfying the following properties i), ii) and iii).

i) There exist $q_1,\ldots,q_p$, non negative integers and $R_0>0$ such that the function $\lambda(w_1^{q_1},\ldots,w_p^{q_p})$ is holomorphic in the variables $w_1,\ldots,w_p$ for $|w_j|\geq R_0^{1/q_j}$, $1\leq j\leq p$.

ii) There exist $C>0$ and non negative integers $m_j>0$, $1\leq j\leq p$, such that for all $w_1,\ldots,w_p$ with $|w_j|\geq R_0$, $1\leq j\leq p$, we have

$$|\lambda(w_1,\ldots,w_p)|\leq C\prod_{k=1}^{p}|w_k|^{m_k}.$$

iii) The function λ is a root of (C), that is $P(\lambda(w_1,\ldots,w_p),w_1,\ldots,w_p)=0$ for $|w_j|\geq R_0$, $1\leq j\leq p$.

Let $\Phi(z_1,\ldots,z_p)=\sum_{n_1,\ldots,n_p=0}^{\infty}\Phi_{n_1,\ldots,n_p}z_1^{n_1}\cdots z_p^{n_p}$ be an analytic function in the polydisc $D_{\rho_1}\times\cdots\times D_{\rho_p}$. We consider, for $R_1,\ldots,R_p\geq R_0$, the series (u_j),

$$u_j(z_1,\ldots,z_p)=\sum_{n_1,\ldots,n_p=0}^{\infty}\Phi_{n_1,\ldots,n_p}\frac{\Pi_{k=1}^{p}n_k!}{(2\pi i)^p\Pi_{k=1}^{p}q_k}\oint_{qC_R}\frac{\lambda^j(w_1,\ldots,w_p)}{\Pi_{k=1}^{p}w_k^{n_k+1}}\Pi_{k=1}^{p}e^{w_kz_k}dw_1\cdots dw_p,$$

where $qC_R=q_1C_{R_1}\times\cdots\times q_pC_{R_p}$ and the symbol $q_jC_{R_j}$ means that we integrate q_j times around the positive oriented circle C_{R_j} of radius R_j centered at 0.

Proposition 2 *Under the assumption* **A**, *the series* (u_j) *converges absolutely and locally uniformly in* $(z_1,\ldots,z_p)\in D_{\rho_1}\times\cdots\times D_{\rho_p}$. *Hence, the function* $u_j(z_1,\ldots,z_p)$ *is holomorphic in* $D_{\rho_1}\times\cdots\times D_{\rho_p}$. *The formal series*

$$\hat{u}(t,z_1,\ldots,z_p)=\sum_{j=0}^{\infty}\frac{t^j}{j!}u_j(z_1,\ldots,z_p)$$

is a formal solution of (1), such that $\hat{u}(0,z_1,\ldots,z_p)=\Phi(z_1,\ldots,z_p)$.

Proof Let $j\geq 0$, $n_j\geq 0$, $1\leq j\leq p$ be non negative integers. In the following, we consider the integral

$$A_{j,n_1,\ldots,n_p}=\oint_{qC_R}\frac{\lambda^j(w_1,\ldots,w_p)}{\Pi_{k=1}^{p}w_k^{n_k+1}}\Pi_{k=1}^{p}e^{w_kz_k}dw_1\cdots dw_p.$$

Using the change of variable $u_k^{q_k}=w_k$, $1\leq k\leq p$, the integral $A_{j,n_1,\ldots,n_p}$ can be written

$$A_{j,n_1,\ldots,n_p}=\oint_{C_{R^{1/q}}}\frac{\lambda^j(u_1^{q_1},u_2^{q_2},\ldots,u_p^{q_p})}{\Pi_{k=1}^{p}u_k^{q_kn_k+1}}\Pi_{k=1}^{p}e^{u_k^{q_k}z_k}q_kdu_1\cdots du_p$$

where $C_{R^{1/q}}=C_{R_1^{1/q_1}}\times\cdots\times C_{R_p^{1/q_p}}$. On the other hand, *the integral* $A_{j,n_1,\ldots,n_p}$ *does not depend on* $R_k\geq R_0$, $1\leq k\leq p$ because the torus $C_{R_1^{1/q_1}}\times\cdots\times C_{R_p^{1/q_p}}$ is homotopic to the torus $C_{{R'}_1^{1/q_1}}\times\cdots\times C_{{R'}_p^{1/q_p}}$ in the set $\mathbb{C}^p\setminus\Theta$, where Θ is the singular set of the analytic function $\lambda(u_1^{q_1},\ldots,u_p^{q_p})$, for $R_k\geq R_0$ and $R'_k\geq R_0$, $1\leq k\leq p$.

Due to the assumption **A** ii), there exist a constant $k_j > 0$, such that for n_k sufficiently large, $1 \leq k \leq p$ and for $R_k = R_{n_k} = n_k/|z_k|$, $1 \leq k \leq p$,

$$|\Pi_{k=1}^p n_k! A_{j,n_1,\ldots,n_p}| \leq k_j \Pi_{k=1}^p (n_k/|z_k|)^{m_k j} \Pi_{k=1}^p |z_k|^{n_k} \Pi_{k=1}^p n_k! \frac{\Pi_{k=1}^p e^{n_k}}{\Pi_{k=1}^p n_k^{n_k}}$$

Due to the Stirling formula $n! \sim n^n e^{-n}(2\pi n)^{1/2}$ for $n \to \infty$, There exists constants $\tilde{k}_j > 0$, $\tilde{m}_{kj} > 0$, $1 \leq k \leq p$ such that for n_k sufficiently large, $1 \leq k \leq p$,

$$(n!A_j) \qquad |\Pi_{k=1}^p n_k! A_{j,n_1,\ldots,n_p}| \leq \tilde{k}_j \Pi_{k=1}^p (n_k/|z_k|)^{\tilde{m}_{kj}} \Pi_{k=1}^p |z_k|^{n_k}.$$

On the other hand, due to the multidimensional Cauchy formula (see [5]), there exists $M > 0$ and $c > 0$ such that, for all n_k, $1 \leq k \leq p$,

$$(\Phi) \qquad |\Phi_{n_1,\ldots,n_p}| \leq M \Pi_{k=1}^p c^{n_k}.$$

From the estimates $(n!A_j)$ and (Φ), we deduce that (u_j) converges absolutely and locally uniformly in $(z_1, \ldots, z_p) \in D_{\rho_1} \times \cdots \times D_{\rho_p}$.

For $j \geq \kappa$, consider the sequence $\lambda^j(w_1, \ldots, w_p)$. Due to the assumption **A** iii), $\lambda^j(w_1, \ldots, w_p)$ satisfies

$$(\lambda^j) \qquad p(w_1, \ldots, w_p)\lambda^j(w_1, \ldots, w_p) = \sum_{\nu=1}^{\kappa} p_\nu(w_1, \ldots, w_p)\lambda^{j-\nu}(w_1, \ldots, w_p).$$

From this latter equation, we deduce that $u_j(z_1, \ldots, z_p)$ satisfy the relation (3) (the derivations under the signs $\sum$ and $\oint$ are allowed because of uniformly convergence of the series). Hence, the formal series

$$\hat{u}(t, z_1, \ldots, z_p) = \sum_{j=0}^{\infty} \frac{t^j}{j!} u_j(z_1, \ldots, z_p)$$

is a formal solution of (1). Due to the residues theorem, we have for $1 \leq k \leq p$,

$$\frac{n_k!}{(2\pi i)q_k} \oint_{q_k C_{R_k}} \frac{e^{w_k z_k}}{w_k^{n_k+1}} dw_k = z_k^{n_k},$$

from which we deduce that $\hat{u}(0, z_1, \ldots, z_p) = \Phi(z_1, \ldots, z_p)$. □

2 Gevrey estimates

In the following theorem, we give Gevrey estimates for the coefficients of the formal powers series (2) solutions of the given partial differential equation (1).

Theorem 1 *Under the assumption* **A**, *for all* $\rho_k > 0$ *sufficiently small,* $1 \leq k \leq p$, *there exist* $C, K > 0$ *such that*

$$|u_j(z_1, \ldots, z_p)| \leq CK^j \prod_{k=1}^{p} \Gamma(1 + m_k j),$$

for all $j \geq 0$, $|z_k| \leq \rho_k$, $1 \leq k \leq p$.

Proof Due to the assumption **A** ii), there exists $C > 0$ such that

$$|\lambda^j(w_1, \ldots, w_p)| \leq C^{pj} \Pi_{k=1}^{p} |w_k|^{m_k j}$$

for $|w_k| \geq R_0$, $1 \leq k \leq p$. There exists $\tilde{k}$ such that for $|z_k| \leq \rho_k$ and $R_{n_k} = n_k/\rho_k$, $1 \leq k \leq p$,

$$|\frac{\Pi_{k=1}^{p} n_k!}{(2\pi i)^p \Pi_{k=1}^{p} q_k} A_{j,n_1,\ldots,n_p}| \leq \frac{\Pi_{k=1}^{p} n_k!}{(2\pi)^p \Pi_{k=1}^{p} q_k} \tilde{k} C^{pj} \frac{\Pi_{k=1}^{p} (R_{n_k})^{m_k j}}{\Pi_{k=1}^{p} R_{n_k}^{n_k}} \Pi_{k=1}^{p} e^{n_k}.$$

Due to the Fubini theorem, for all $x > 0$,

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{x^{pj}}{\Pi_{k=1}^{p} \Gamma(1 + m_k j)} |u_j(z_1, \ldots, z_p)| \\ &\leq \tilde{k} \sum_{n_1,\ldots,n_p=0}^{\infty} \frac{|\Phi_{n_1,\ldots,n_p}| \Pi_{k=1}^{p} n_k! e^{n_k}}{\Pi_{k=1}^{p} R_{n_k}^{n_k}} \times \sum_{j=0}^{\infty} (\Pi_{k=1}^{p} \frac{(xC)^j (R_{n_k})^{m_k j}}{\Gamma(1 + m_k j)}). \end{aligned}$$

Due to the estimates of the Mittag Leffler functions (see [3] p. 233), there exist $\tilde{k}_k$, $1 \leq k \leq p$, such that

$$\begin{aligned} \sum_{j=0}^{\infty} (\prod_{k=1}^{p} \frac{(xC)^j (R_{n_k})^{m_k j}}{\Gamma(1 + m_k j)}) &\leq \prod_{k=1}^{p} \sum_{j=0}^{\infty} \frac{(xC)^j (R_{n_k})^{m_k j}}{\Gamma(1 + m_k j)} \\ &\leq \prod_{k=1}^{p} (\tilde{k}_k e^{(xc)^{1/m_k} R_{n_k}}). \end{aligned}$$

We deduce that there exists $K > 0$ such that

$$\sum_{j=0}^{\infty} \frac{x^{pj}}{\Pi_{k=1}^{p} \Gamma(1 + m_k j)} |u_j(z_1, \ldots, z_p)| \leq K \prod_{k=1}^{p} \sum_{n_k=0}^{\infty} \frac{c^{n_k} n_k! e^{n_k} e^{(xc)^{1/m_k} R_{n_k}}}{R_{n_k}^{n_k}}$$

Using the Stirling formula, one can show that the right handside of the latter estimate converges for $|x|$ sufficiently small. Hence, the function

$$\psi(x) = \sum_{j=0}^{\infty} \frac{x^{pj}}{\Pi_{k=1}^{p} \Gamma(1 + m_k j)} u_j(z_1, \ldots, z_p)$$

is holomorphic for x such that $|x|$ is small. From the Cauchy formula, we deduce that there exist $C, K > 0$ such that

$$|u_j(z_1, \ldots, z_p)| \leq C K^j \prod_{k=1}^{p} \Gamma(1 + m_k j),$$

for all $j \geq 0$, $|z_k| \leq \rho_k$, $1 \leq k \leq p$. □

3 Summability of formal solutions

All the results of this section can be easily extended to the case of several complex variables t, $z_1, \ldots, z_n$, but for sake of simplicity, we consider now partial differential equations with constant coefficients

$$P(\partial_t, \partial_{z_1}, \partial_{z_2}) u(t, z_1, z_2) = 0, \tag{1}$$

where $P \in \mathbb{C}[\partial_t, \partial_{z_1}, \partial_{z_2}]$ is a non trivial polynomial in three variables with coefficients in $\mathbb{C}$. We assume moreover that P can be written in the form

$$P(\partial_t, \partial_{z_1}, \partial_{z_2}) = {\partial_t}^{\kappa} p(\partial_{z_1}, \partial_{z_2}) - \sum_{\nu=1}^{\kappa} {\partial_t}^{\kappa-\nu} p_\nu(\partial_{z_1}, \partial_{z_2}),$$

where

- $p(w_1, w_2) \in \mathbb{C}[w_1, w_2]$ is a polynomial for which there exists R_0 such that for all $|w_1| \geq R_0$, $|w_2| \geq R_0$, we have $p(w_1, w_2) \neq 0$,
- κ is an integer, p_ν, $1 \leq \nu \leq \kappa$, are polynomials in two variables and p_κ is not identically equal to zero.

We consider the characteristic equation of (1),

$$(C) \qquad P(\lambda, w_1, w_2) = p(w_1, w_2)\lambda^{\kappa} - \sum_{\nu=1}^{\kappa} p_\nu(w_1, w_2)\lambda^{\kappa-\nu} = 0,$$

with $|w_1| \geq R_0$ and $|w_2| \geq R_0$. In the following, we put $\tilde{w}_j = 1/w_j$, $j = 1, 2$.

For m_1, m_2 sufficiently large integers, we multiply the equation (C) by the rational function

$$N(\tilde{w}_1, \tilde{w}_2) = (\tilde{w}_1^{m_1} \tilde{w}_2^{m_2})^{\kappa} / p(1/\tilde{w}_1, 1/\tilde{w}_2),$$

and we get the following equation

$$(\tilde{w}_1^{m_1} \tilde{w}_2^{m_2})^{\kappa} \lambda^{\kappa} - \sum_{\nu=1}^{\kappa} (\tilde{w}_1^{m_1} \tilde{w}_2^{m_2})^{\kappa-1} \frac{\tilde{p}_\nu(\tilde{w}_1, \tilde{w}_2)}{p(1/\tilde{w}_1, 1/\tilde{w}_2)} \lambda^{\kappa-\nu} = 0,$$

where $\tilde{p}_\nu$, $1 \leq \nu \leq \kappa$, are polynomials in the two variables $\tilde{w}_1$ and $\tilde{w}_2$.

Making the change of variable $\lambda = \tilde{\lambda}/\tilde{w}_1^{m_1} \tilde{w}_2^{m_2}$, we get the equation

$$(\tilde{C}) \qquad \tilde{P}(\tilde{\lambda}, \tilde{w}_1, \tilde{w}_2) = \tilde{\lambda}^{\kappa} - \sum_{\nu=1}^{\kappa} (\tilde{w}_1^{m_1} \tilde{w}_2^{m_2})^{\nu-1} \frac{\tilde{p}_\nu(\tilde{w}_1, \tilde{w}_2)}{p(1/\tilde{w}_1, 1/\tilde{w}_2)} \tilde{\lambda}^{\kappa-\nu} = 0.$$

for $(\tilde{w}_1, \tilde{w}_2) \in D_{1/R_0} \times D_{1/R_0}$. In the following, we use the notation $\dot{D}_\rho = \{z \in \mathbb{C} : 0 < |z| < \rho\}$.

We denote by $D_{\tilde{P}}$ the discriminant of the polynomial $\tilde{P}$ in the variable $\tilde{\lambda}$. In this paper, we assume that

$$(Cd1) \qquad D_{\tilde{P}}(\tilde{w}_1, \tilde{w}_2) \neq 0,$$

for all $(\tilde{w}_1, \tilde{w}_2) \in \dot{D}_{1/R_0} \times \dot{D}_{1/R_0}$.

Due to the result of J. Cel (see [4]), there exist two positive integers q_1, q_2 and a holomorphic function H on $D_{(1/R_0)^{1/q_1}} \times D_{(1/R_0)^{1/q_2}}$ such that

$$\tilde{P}(H((\tilde{w}_1)^{1/q_1}, (\tilde{w}_2)^{1/q_2}), \tilde{w}_1, \tilde{w}_2) = 0.$$

Remark. In fact, the result of J. Cel shows that there are κ such holomorphic functions H satisfying the latter equation.

Proposition 3 *Under the conditions* $(Cd1)$*, there exists a function* λ *of the form*

$$\lambda(w_1, w_2) = w_1^{m_1} w_2^{m_2} H(w_1^{-1/q_1}, w_2^{-1/q_2})$$

where H *is a holomorphic function on* $D_{(1/R_0)^{1/q_1}} \times D_{(1/R_0)^{1/q_2}}$ *such that*

$$P(\lambda(w_1, w_2), w_1, w_2) = 0$$

for $|w_1| \geq R_0$*,* $|w_2| \geq R_0$*. In particular, the function* λ *satisfies the assumption* **A** *of the section 1. So that the formal series* (2) *constructed in the proposition 2 with the help of* λ *is a formal solution of* (1) *and satisfies the Gevrey estimates of the theorem 1. In fact, due to the upper mentioned remark, one can construct* κ *such functions* λ *and therefore* κ *formal series* (2) *which are formal solutions of* (1) *and satisfy the Gevrey estimates of the theorem 1.*

In order to study the summability properties of the formal power series obtained in the previous sections, we will make an assumption on the holomorphic function H obtained in the proposition 3. From now on, we assume that the function H can be factorized in the form

$$(Cd2) \qquad H(Z_1, Z_2) = Z_1^{\alpha_1} Z_2^{\alpha_2}(\lambda_0 + Z_1 Z_2 \tilde{H}(Z_1, Z_2))$$

where $\tilde{H}$ is a holomorphic function on $D_{(1/R_0)^{1/q_1}} \times D_{(1/R_0)^{1/q_2}}$, $\lambda_0 \neq 0$ and α_j, $j = 1, 2$ are positive integers such that

$$m_1 - \alpha_1/q_1 + m_2 - \alpha_2/q_2 > 1.$$

Geometrically, the null set $\{H = 0\}$ is in particular an analytic hypersurface with normal crossing. From the condition $(Cd2)$, we deduce that the function λ defined in proposition 3 can be written

$$(Cd21) \qquad \lambda(w_1, w_2) = w_1^{l_1/q_1} w_2^{l_2/q_2}(\lambda_0 + w_1^{-1/q_1} w_2^{-1/q_1} \tilde{H}(w_1^{-1/q_1}, w_2^{-1/q_2}))$$

where $\tilde{H}$ is a holomorphic function on $D_{(1/R_0)^{1/q_1}} \times D_{(1/R_0)^{1/q_2}}$ and

$$1 < \frac{l_1}{q_1} + \frac{l_2}{q_2} \quad , \quad \lambda_0 \neq 0.$$

In the following, for $j \geq 0$, we put $l_j^1 = jl_1/q_1$ and $l_j^2 = jl_2/q_2$. We may expand

$$(exp\lambda^j) \qquad \begin{array}{rcl} \lambda^j(w_1, w_2) & = & w_1^{l_j^1} w_2^{l_j^2}(\lambda_0^j + \sum_{\mu_1,\mu_2=1} \tilde{c}_{j\mu_1\mu_2} w_1^{-\mu_1/q_1} w_2^{-\mu_2/q_2}) \\ & = & w_1^{l_j^1} w_2^{l_j^2} \sum_{\mu_1,\mu_2=0} c_{j\mu_1\mu_2} w_1^{-\mu_1/q_1} w_2^{-\mu_2/q_2} \end{array}$$

for $|w_1| \geq R_0$ and $|w_2| \geq R_0$.

Example 1 Let the characteristic equation of (1) be of the form

$$(E1) \qquad P(\lambda, w_1, w_2) = p(w_1 w_2)\lambda^\kappa - \sum_{\nu=1}^{\kappa} p_\nu(w_1 w_2)\lambda^{\kappa-\nu},$$

where p, $p_\nu \in \mathbb{C}[X]$, $1 \leq \nu \leq \kappa$ and p_κ is a not identically equal to zero. From a classical result (see [1]), there exists an algebraic function $\tau(z)$ with expansion of the form

$$\tau(z) = z^{l/q}(\lambda_0 + z^{-1/q}\tilde{H}(z^{-1/q})),$$

where $\tilde{H}$ is holomorphic on a disc $D_{(1/R_0)^{1/q}}$, $R_0 > 0$, which satisfies the equation

$$p(z)\tau^\kappa - \sum_{\nu=1}^{\kappa} p_\nu(z)\tau^{\kappa-\nu} = 0.$$

We require moreover that $l/q > 1/2$. If we put $z = w_1 w_2$, the function $\lambda(w_1, w_2) = \tau(w_1 w_2)$ satisfies $(Cd21)$ and the equation $P(\lambda(w_1, w_2), w_1, w_2) = 0$ for $|w_1| \geq R_0$ and $|w_2| \geq R_0$.

3.1 Integral representations

In the following, we put for $j \geq 0$,

$$g_j(z_1, z_2) = \sum_{\mu_1,\mu_2=0}^{\infty} c_{j\mu_1\mu_2} \frac{z_1^{\mu_1/q_1} z_2^{\mu_2/q_2}}{\Gamma(1+(\mu_1/q_1))\Gamma(1+(\mu_2/q_2))}$$

for all $(z_1, z_2) \in D_{\rho_1} \times D_{\rho_2}$. The following proposition gives an integral representation of u_j in terms of Φ and g_j.

Proposition 4 *For all $j \geq 0$, $(z_1, z_2) \in D_{\rho_1} \times D_{\rho_2}$,*

$$u_j(z_1, z_2) = \frac{\Gamma(1+l_j^1)\Gamma(1+l_j^2)}{(2\pi i)^2 q_1 q_2} \partial_{z_1}\partial_{z_2} \oint_{q_1 C_{\rho_1}} \oint_{q_2 C_{\rho_2}} \Phi * g_j(w_1, w_2) \frac{dw_1}{(w_1 - z_1)^{(1+l_j^1)}} \frac{dw_2}{(w_2 - z_2)^{(1+l_j^2)}}$$

where $\Phi * g_j(w_1, w_2) = \int_0^{w_2} \int_0^{w_1} \Phi(w_1 - u_1, w_2 - u_2) g_j(u_1, u_2) du_1 du_2$.

Proof Using the expansion $(exp\lambda^j)$ and the residues theorem, we may expand $u_j(z_1, z_2)$ in the form

$$(u_j^{\sum}) \qquad u_j(z_1, z_2) = \sum_{\mathcal{A}_j} c_{j\mu_1\mu_2} \Phi_{n_1 n_2} \frac{n_1! n_2! z_1^{(\mu_1/q_1)+n_1-l_j^1} z_2^{(\mu_2/q_2)+n_2-l_j^2}}{\Gamma(1+(\mu_1/q_1)+n_1-l_j^1)\Gamma(1+(\mu_2/q_2)+n_2-l_j^2)}$$

for all $(z_1, z_2) \in D_{\rho_1} \times D_{\rho_2}$, where $\mathcal{A}_j = \{n_1, n_2 \geq 0, \mu_1, \mu_2 \geq 0 : (\mu_1/q_1) + n_1 - l_j^1 \in \mathbb{N}_0, (\mu_2/q_2) + n_2 - l_j^2 \in \mathbb{N}_0\}$.

We may expand

$$\Phi(w_1 - u_1, w_2 - u_2) g_j(u_1, u_2) = \sum_{m_1=0}^{\infty} \sum_{n_1+\mu_1=m_1} (w_1 - u_1)^{n_1} u_1^{\mu_1/q_1} \frac{1}{\Gamma(1+\mu_1/q_1)} \Phi_{n_1} g_{j,\mu_1}(w_2, u_2)$$

where

$$\Phi_{n_1} g_{j,\mu_1}(w_2, u_2) = (\sum_{n_2=0}^{\infty} \Phi_{n_1 n_2} (w_2 - u_2)^{n_2})(\sum_{\mu_2=0}^{\infty} c_{j\mu_1\mu_2} \frac{u_2^{\mu_2/q_2}}{\Gamma(1+\mu_2/q_2)}).$$

On the other hand, it is well known that (see [3] p. 228),

$$\int_0^1 (1-u)^{n_1} u^{\mu_1/q_1} du = \frac{\Gamma(1+n_1)\Gamma(1+(\mu_1/q_1))}{\Gamma(2+n_1+(\mu_1/q_1))}.$$

Making the change of variable $uw_1 = u_1$, we get

$$\int_0^{w_1} (w_1 - u_1)^{n_1} u_1^{\mu_1/q_1} du_1 = w_1^{n_1+(\mu_1/q_1)+1} \frac{\Gamma(1+n_1)\Gamma(1+(\mu_1/q_1))}{\Gamma(2+n_1+(\mu_1/q_1))}.$$

We deduce that

$$(\Phi * g_j)_1 \qquad \int_0^{w_1} \Phi(w_1 - u_1, w_2 - u_2) g_j(u_1, u_2) du_1 = \sum_{m_1=0}^{\infty} \sum_{n_1+\mu_1=m_1} w_1^{n_1+(\mu_1/q_1)+1} \frac{\Gamma(1+n_1)}{\Gamma(2+n_1+(\mu_1/q_1))} \Phi_{n_1} g_{j,\mu_1}(w_2, u_2).$$

Using the same calculations, one can also show that

$$(\Phi * g_j)_2 \qquad \int_0^{w_2} \Phi_{n_1} g_{j,\mu_1}(w_2, u_2) du_2 = \sum_{m_2=0}^{\infty} \sum_{n_2+\mu_2=m_2} w_2^{n_2+(\mu_2/q_2)+1} \frac{\Gamma(1+n_2)}{\Gamma(2+n_2+(\mu_2/q_2))} \Phi_{n_1 n_2} c_{j\mu_1\mu_2}.$$

On the other hand, by the change of variable $v_k^{q_k} = w_k$, $k = 1, 2$, we get

$$\frac{1}{2\pi i q_k} \oint_{q_k C_{\rho_k}} \frac{w^{n_k+(\mu_k/q_k)+1}}{(w_k - z_k)^{2+l_j^k}} dw_k = \frac{1}{2\pi i} \oint_{C_{\rho_k^{1/q_k}}} \frac{v_k^{q_k(n_k - l_j^k)+\mu_k-1}}{(1 - (z_k/v_k^{q_k}))^{2+l_j^k}} dv_k.$$

From the expansion with respect to v_k,

$$(1 + (z_k/v_k^{q_k}))^{-2-l_j^k} = 1 + \sum_{n=1}^{\infty} \frac{(-2 - l_j^k) \cdots (-2 - l_j^k - (n-1))}{n!} (-z_k/v_k^{q_k})^n$$

and the residues theorem, we deduce that

$$(Q(z_k)) \qquad \frac{1}{2\pi i q_k} \oint_{q_k C_{\rho_k}} \frac{w^{n_k+(\mu_k/q_k)+1}}{(w_k - z_k)^{2+l_j^k}} dw_k = \frac{\Gamma(2 + n_k + (\mu_k/q_k))}{\Gamma(2 + l_j^k)\Gamma(1 + (\mu_k/q_k) + n_k - l_j^k)} z_k^{(\mu_k/q_k)+n_k-l_j^k},$$

when $(\mu_k/q_k) + n_k - l_j^k \in \mathbb{N}_0$. The latter integral is equal to zero if $(\mu_k/q_k) + n_k - l_j^k \notin \mathbb{N}_0$.

From the formula $(\Phi * g_j)_1$, $(\Phi * g_j)_2$ and $(Q(z_k))$, $k = 1, 2$, we deduce that

$$\frac{\Gamma(2 + l_j^1)\Gamma(2 + l_j^2)}{(2\pi i)^2 q_1 q_2} \oint_{q_1 C_{\rho_1}} \oint_{q_2 C_{\rho_2}} \Phi * g_j(w_1, w_2) \frac{dw_1}{(w_1 - z_1)^{(2+l_j^1)}} \frac{dw_2}{(w_2 - z_2)^{(2+l_j^2)}} = \sum_{m_1,m_2=0}^{\infty} \sum_{\mathcal{B}_j} c_{j\mu_1\mu_2} \Phi_{n_1 n_2} \frac{n_1! n_2! z_1^{(\mu_1/q_1)+n_1-l_j^1} z_2^{(\mu_2/q_2)+n_2-l_j^2}}{\Gamma(1 + (\mu_1/q_1) + n_1 - l_j^1)\Gamma(1 + (\mu_2/q_2) + n_2 - l_j^2)}$$

where $\mathcal{B}_j = \{n_1, n_2, \mu_1, \mu_2 : n_1 + \mu_1 = m_1, n_2 + \mu_2 = m_2, (\mu_1/q_1) + n_1 - l_j^1 \in \mathbb{N}_0, (\mu_2/q_2) + n_2 - l_j^2 \in \mathbb{N}_0\}$.

Due to the expansion $(u_j^{\sum})$, we see that the right handside of the latter formula is equal to $u_j(z_1, z_2)$ from which we deduce the expected formula. □

3.2 Analytic continuation and estimates of auxiliary series

In the following, we will use the notation (a, b) to denote the open interval (a, b) or any of its translates $(a + 2k\pi, b + 2k\pi)$, $k \in \mathbb{Z}$. We will denote by $G_\delta(\lambda_0)$ any sector of $\mathbb{C}$, centered at 0, which does not intersect $D(\lambda_0^{-1}, \delta)$,

$$G_\delta(\lambda_0) = \{t = |t|e^{i\theta_t} : \theta_t \in (d^1_{\lambda_0,\delta}, d^2_{\lambda_0,\delta})\}.$$

In the following, we put $q(u_1, u_2) = \lambda(u_1, u_2)/(u_1^{l_1/q_1} u_2^{l_2/q_2})$. From the expansion $(Cd21)$, there exist R_1, R_2 sufficiently large such that for all $t \in G_\delta(\lambda_0)$, $|u_1| > R_1$, $|u_2| > R_2$,

$$(F) \qquad \frac{1}{1 - q(u_1, u_2)t} = \frac{1}{1 - \lambda_0 t} + \frac{q'(u_1, u_2)t}{u_1^{1/q_1} u_2^{1/q_2}(1 - q(u_1, u_2)t)(1 - \lambda_0 t)}$$

is well defined, where $q'(u_1, u_2)$ is bounded by a constant depending only on R_1, R_2.

For $t \in G_\delta(\lambda_0)$ and $z_1, z_2 \in \mathbb{C}$, we put

$$k(t, z_1, z_2) = \frac{1}{(2\pi i)^2} \oint_{\gamma(R_1,z_1)} \oint_{\gamma(R_2,z_2)} \frac{e^{u_1 z_1} e^{u_2 z_2}}{u_1 u_2 (1 - q(u_1, u_2)t)} du_1 du_2,$$

where the path $\gamma(R, z)$ is the Hankel path (see [3], p. 228). From the decomposition (F), we deduce that

$$k(t, z_1, z_2) = \frac{1}{1 - \lambda_0 t} + B(t, z_1, z_2)$$

where

$$B(t, z_1, z_2) = \frac{1}{(2\pi i)^2} \oint_{\gamma(R_1,z_1)} \oint_{\gamma(R_2,z_2)} \frac{e^{u_1 z_1} e^{u_2 z_2} q'(u_1, u_2)t}{u_1^{1+(1/q_1)} u_2^{1+(1/q_2)} (1 - q(u_1, u_2)t)(1 - \lambda_0 t)} du_1 du_2.$$

Proposition 5 *There exists $C_3 > 0$, $\rho > 0$ such that*

$$|k(t, z_1, z_2)| \le C_3 e^{R_1|z_1|} e^{R_2|z_2|}$$

for all $t \in G_\delta(\lambda_0) \cup D(0, \rho)$, $(z_1, z_2) \in \mathbb{C}^2$.

Proof In the following, we write

$$\tilde{q}(t, u_1, u_2) = \frac{q'(u_1, u_2)t}{(1 - q(u_1, u_2)t)(1 - \lambda_0 t)}.$$

By construction, there exists $M_1 > 0$, $M_2 > 0$ such that for all $t \in G_\delta(\lambda_0) \cup D(0, \rho)$, for all $(u_1, u_2) \in \gamma(R_1, z_1) \times \gamma(R_2, z_2)$,

$$|\tilde{q}(t, u_1, u_2)| \le M_1 \quad , \quad \frac{1}{|1 - \lambda_0 t|} \le M_2 \le M_2 e^{R_1|z_1|} e^{R_2|z_2|}.$$

We put $z_l = |z_l| e^{i\theta_{z_l}}$, $u_l = |u_l| e^{i\theta_{u_l}}$, $l = 1, 2$. The path $\gamma(R_l, z_l)$ is composed by the union of the three following paths

$$\begin{aligned} L_l^1 &= \{u_l \in \mathbb{C} : \theta_{z_l} + \theta_{u_l} = -\pi \quad , \quad |u_l| \ge R_l\} \\ C_l &= \{u_l \in \mathbb{C} : -\pi - \theta_{z_l} \le \theta_{u_l} \le \pi - \theta_{z_l} \quad , \quad |u_l| = R_l\} \\ L_l^2 &= \{u_l \in \mathbb{C} : \theta_{z_l} + \theta_{u_l} = \pi \quad , \quad |u_l| \ge R_l\}. \end{aligned}$$

The function $B(t, z_1, z_2)$ is written as a sum of integrals over the curves $(L_1^j, L_2^{j'})$, $1 \le j, j' \le 2$, (C_1, C_2), (C_1, L_2^j), (L_1^j, C_2), $j = 1, 2$.

We show the existence and estimates for the integral over (L_1^1, C_2), the other ones can be treated in the same manner,

$$\begin{aligned} \left| \oint_{L_1^1} \oint_{C_2} \frac{e^{u_1 z_1} e^{u_2 z_2} \tilde{q}(t, u_1, u_2)}{u_1^{1+(1/q_1)} u_2^{1+(1/q_2)}} du_1 du_2 \right| &\le \quad 2\pi M_1 \frac{e^{R_2|z_2|}}{R_2^{1/q_2}} \int_{R_1}^{\infty} \frac{e^{-|u_1||z_1|}}{|u_1|^{1+(1/q_1)}} d|u_1| \\ &\le \quad ((2\pi M_1 q_1)/(R_1^{1/q_1} R_2^{1/q_2})) e^{R_1|z_1|} e^{R_2|z_2|}, \end{aligned}$$

from which the estimate of the proposition 5 follows. □

In the following, we consider the series

$$v(t,z_1,z_2)=\sum_{j=0}^{\infty}\frac{u_j(z_1,z_2)t^j}{\Gamma(1+l_j^1)\Gamma(1+l_j^2)}.$$

From the expansion ($exp\lambda^j$), using the same proof as in the theorem 1, one can show the latter series converge for all t, such that $|t|<\rho$ with ρ sufficiently small and $(z_1,z_2)\in D_{\rho_1}\times D_{\rho_2}$.

In the next proposition, we give an integral representation of $v(t,z_1,z_2)$ in terms of the function k.

Proposition 6 *For all t, such that $|t|<\rho$ with ρ sufficiently small and $(z_1,z_2)\in D_{\rho_1}\times D_{\rho_2}$,*

$$v(t,z_1,z_2)=\frac{1}{(2\pi i)^2}\partial_{z_1}\partial_{z_2}\oint_{C_{\rho_1^{1/q_1}}}\oint_{C_{\rho_2^{1/q_2}}}\Phi*k(t,v_1,v_2)\frac{v_1^{q_1-1}v_2^{q_2-1}}{(v_1^{q_1}-z_1)(v_2^{q_2}-z_2)}dv_1dv_2,$$

where

$$\Phi*k(t,v_1,v_2)=\int_0^{v_2^{q_2}}\int_0^{v_1^{q_1}}\Phi(v_1^{q_1}-u_1,v_2^{q_2}-u_2)k(\frac{t}{(v_1^{q_1}-z_1)^{l_1/q_1}(v_2^{q_2}-z_2)^{l_2/q_2}},u_1,u_2)du_1du_2.$$

Proof We first show that

$$(k^{\Sigma})\qquad k(t,z_1,z_2)=\sum_{j=0}^{\infty}g_j(z_1,z_2)t^j$$

for all t, such that $|t|<\rho$ with ρ sufficiently small and $(z_1,z_2)\in D_{\rho_1}\times D_{\rho_2}$. For $|u_1|\geq R_1$, $|u_2|\geq R_2$,

$$\begin{aligned}\frac{1}{u_1u_2(1-q(u_1,u_2)t)} &= \textstyle\sum_{j=0}^{\infty}(\lambda^j(u_1,u_2)/(u_1^{l_j^1+1}u_2^{l_j^2+1}))t^j\\ &= \textstyle\sum_{j=0}^{\infty}(\lambda_0^ju_1^{-1}u_2^{-1}+\sum_{\mu_1,\mu_2=1}^{\infty}\tilde{c}_{j\mu_1\mu_2}u_1^{-(\mu_1/q_1)-1}u_2^{-(\mu_2/q_2)-1})t^j\end{aligned}$$

From the Fubini theorem and the formula (see [3] p. 228), for $\mu_l\geq 1$,

$$\frac{z_l^{\mu_l/q_l}}{\Gamma(1+(\mu_l/q_l))}=\frac{1}{2\pi i}\oint_{\gamma(R_l,z_l)}\frac{e^{u_lz_l}}{u_l^{1+(\mu_l/q_l)}}du_l,$$

we deduce the expansion (k^{Σ}). Finally, from the integral representation of $u_j(z_1,z_2)$ in proposition 4, where we make the change of variable $w_l=v_l^{q_l}$, $l=1,2$, and the latter expansion (k^{Σ}), we deduce the expected integral formula. □

In the following, we will try to continue analytically and give growth estimates of the function $v(t,z_1,z_2)$ with respect to t in a sector $S_\delta\subset\mathbb{C}$. We will give simple sufficient conditions on the function Φ in such a way that the following properties are satisfied. There exists a curve $\mathcal{C}$ in $\mathbb{C}^2$ which is homotopic to the torus $C_{\rho_1^{1/q_1}}\times C_{\rho_2^{1/q_2}}$ such that the function $\Phi*k(t,v_1,v_2)$ is well defined for $t\in S_\delta$, $(v_1,v_2)\in\mathcal{C}$. Moreover, there exist constants $\tilde{C}>0$ and $\tilde{K}>0$, such that $\Phi*k$ has growth estimates

$$|\Phi*k(t,v_1,v_2)|\leq\tilde{C}e^{\tilde{K}|t|^m},$$

for $t \in S_\delta$, $(v_1, v_2) \in \mathcal{C}$, where $m = ((l_1/q_1) + (l_2/q_2) - 1)^{-1}$. Simple calculations then show that $v(t, z_1, z_2)$ has an analytic continuation and satisfies the same growth estimates as $\Phi * k$.

Now, consider a sector $S_\delta = \{t = |t|e^{i\theta_t} : \theta_t \in (D_{1,\delta}, D_{2,\delta})\}$. We denote by $\mathcal{M}$ the subset of $\mathbb{C}^2$,

$$\mathcal{M} = \{(v_1, v_2) \in \mathbb{C}^2 : \arg(v_1^{q_1} - z_1)^{\frac{l_1}{q_1}} + \arg(v_2^{q_2} - z_2)^{\frac{l_2}{q_2}} \in (D_{2,\delta} - d^2_{\lambda_0,\delta}, D_{1,\delta} - d^1_{\lambda_0,\delta})\}.$$

Let $[m_{1,j}, m_{1,j+1}]$, $[m_{2,k}, m_{2,k+1}]$, $1 \le j \le \alpha_1$, $1 \le k \le \alpha_2$ be intervals included in $(-\pi, \pi)$. We denote by $\mathcal{M}_1$, the subset of $\mathbb{C}$,

$$\mathcal{M}_1 = \{v_1 \in \mathbb{C} : \arg(v_1) \in \cup_{j=1}^{\alpha_1} [m_{1,j}, m_{1,j+1}]\},$$

and likewise we denote by $\mathcal{M}_2$, the subset of $\mathbb{C}$,

$$\mathcal{M}_2 = \{v_2 \in \mathbb{C} : \arg(v_2) \in \cup_{k=1}^{\alpha_2} [m_{2,k}, m_{2,k+1}]\}.$$

We assume that the upper mentioned intervals are chosen such that the following inclusion holds,

$$\mathcal{N} = \{(v_1, v_2) \in \mathbb{C}^2 : v_1 \notin \mathcal{M}_1 \ \text{ and } \ v_2 \notin \mathcal{M}_2\} \subset \mathcal{M}.$$

We make now an assumption on the function Φ. The function $\Phi(w_1^{q_1}, w_2^{q_2})$ has an analytical continuation on $\mathcal{M}_1 \times \mathcal{M}_2$ and there exist $C > 0$, $K_1 > 0$, $K_2 > 0$, such that

$$(Cd3) \qquad |\Phi(w_1^{q_1}, w_2^{q_2})| \le C e^{K_1 |w_1^{q_1}|^{k_1}} e^{K_2 |w_2^{q_2}|^{k_2}}$$

for all $(w_1, w_2) \in \mathcal{M}_1 \times \mathcal{M}_2$, where $k_1 > 0$, $k_2 > 0$ satisfy the following assumption.

If $l_1 \neq 0$ and $l_2 \neq 0$, then

$$\sup(\sup(1, k_1)\frac{q_1}{l_1}, \sup(1, k_2)\frac{q_2}{l_2}) \le \frac{1}{(l_1/q_1) + (l_2/q_2) - 1}.$$

Notice that the latter inequality implies that $\sup(q_1/l_1, q_2/l_2) \le ((l_1/q_1) + (l_2/q_2) - 1)^{-1}$ and therefore that $l_1/q_1 \le 1$ and $l_2/q_2 \le 1$.

If $l_2 = 0$ then

$$k_1 = \frac{l_1/q_1}{(l_1/q_1) - 1} \quad , \quad k_2 > 0,$$

If $l_1 = 0$ then

$$k_1 > 0 \quad , \quad k_2 = \frac{l_2/q_2}{(l_2/q_2) - 1}.$$

Proposition 7 *There exists $\epsilon > 0$ such that the function $v(t, z_1, z_2)$ has an analytical continuation on the domain $(S_\delta \cup D(0, \rho)) \times D(0, \epsilon) \times D(0, \epsilon)$, and there exists $\tilde{C} > 0$, $\tilde{K} > 0$, such that for all $t \in S_\delta \cup D(0, \rho)$, $z_1 \in D(0, \epsilon)$, $z_2 \in D(0, \epsilon)$,*

$$(|v|) \qquad |v(t, z_1, z_2)| \le \tilde{C} e^{\tilde{K}|t|^m},$$

where

$$m = \frac{1}{(l_1/q_1) + (l_2/q_2) - 1}.$$

Proof

Case $l_1 \neq 0$ **and** $l_2 \neq 0$.

We consider a curve $\mathcal{C} = C^d_{\rho_1^{1/q_1}} \times C^d_{\rho_2^{1/q_2}}$ where $C^d_{\rho_1^{1/q_1}}$ is homotopic to the circle $C_{\rho_1^{1/q_1}}$ and $C^d_{\rho_2^{1/q_2}}$ is homotopic to the circle $C_{\rho_2^{1/q_2}}$ and satisfies the following properties.

i) For $(v_1, v_2) \in \mathcal{C}$ such that $(v_1, v_2) \in \mathcal{N}$, we have $(v_1, v_2) \in C_{\rho_1^{1/q_1}} \times C_{\rho_2^{1/q_2}}$,
ii) There exists $a > 0$, $b > 0$, $\epsilon > 0$, such that for $(v_1, v_2) \in \mathcal{C}$ with $(v_1, v_2) \in \mathcal{N}^c$ (where $\mathcal{N}^c$ is the complement of $\mathcal{N}$ in $\mathbb{C}^2$), we have that for all $(z_1, z_2) \in D(0, \epsilon) \times D(0, \epsilon)$,

$$0 < \frac{|t|}{|v_1^{q_1} - z_1|^{l_1/q_1} |v_2^{q_2} - z_2|^{l_2/q_2}} < \rho \quad , \quad |v_1^{q_1}| \leq a|t|^{q_1/l_1} \quad , \quad |v_2^{q_2}| \leq b|t|^{q_2/l_2}.$$

From the latter definition, we notice that

· For $(v_1, v_2) \in \mathcal{C}$ with $(v_1, v_2) \in \mathcal{N}$, we get

$$\arg\left(\frac{t}{(v_1^{q_1} - z_1)^{l_1/q_1} (v_2^{q_2} - z_2)^{l_2/q_2}}\right) \in (d^1_{\lambda_0, \delta}, d^2_{\lambda_0, \delta}),$$

for all $t \in S_\delta$. So that if we put $t' = t/(v_1^{q_1} - z_1)^{l_1/q_1}(v_2^{q_2} - z_2)^{l_2/q_2}$, we have $t' \in G_\delta(\lambda_0)$ and $k(t', u_1, u_2)$ is well defined for all $(u_1, u_2) \in \mathbb{C}^2$.

· If we put $t' = t/(v_1^{q_1} - z_1)^{l_1/q_1}(v_2^{q_2} - z_2)^{l_2/q_2}$, for $(v_1, v_2) \in \mathcal{C}$ with $(v_1, v_2) \in \mathcal{N}^c$, we have $|t'| < \rho$ and the function $k(t', u_1, u_2)$ is well defined for all $(u_1, u_2) \in \mathbb{C}^2$.

On the other hand, the value of the integral representation in the proposition 6 does not depend on the deformation of the curve $C_{\rho_1^{1/q_1}} \times C_{\rho_2^{1/q_2}}$ to the curve $\mathcal{C}$, because they are homotopic to each other. Therefore, we deduce an analytic continuation of the function $v(t, z_1, z_2)$ with respect to $t \in S_\delta \cup D(0, \rho)$, $z_1 \in D(0, \epsilon)$, $z_2 \in D(0, \epsilon)$ which will be also denoted by

$$v(t, z_1, z_2) = \frac{1}{(2\pi i)^2} \partial_{z_1} \partial_{z_2} \oint_{\mathcal{C}} \Phi * k(t, v_1, v_2) \frac{v_1^{q_1 - 1} v_2^{q_2 - 1}}{(v_1^{q_1} - z_1)(v_2^{q_2} - z_2)} dv_1 dv_2.$$

The curve $\mathcal{C}$ can be written as the union of two disjointed parts $\mathcal{C}_1 = \{(v_1, v_2) \in \mathcal{C} : (v_1, v_2) \in \mathcal{N}\}$, $\mathcal{C}_2 = \{(v_1, v_2) \in \mathcal{C} : (v_1, v_2) \in \mathcal{N}^c\}$. In the following, we put

$$A_1(t, z_1, z_2) = \frac{1}{(2\pi i)^2} \partial_{z_1} \partial_{z_2} \oint_{\mathcal{C}_1} \Phi * k(t, v_1, v_2) \frac{v_1^{q_1 - 1} v_2^{q_2 - 1}}{(v_1^{q_1} - z_1)(v_2^{q_2} - z_2)} dv_1 dv_2,$$

and

$$A_2(t, z_1, z_2) = \frac{1}{(2\pi i)^2} \partial_{z_1} \partial_{z_2} \oint_{\mathcal{C}_2} \Phi * k(t, v_1, v_2) \frac{v_1^{q_1 - 1} v_2^{q_2 - 1}}{(v_1^{q_1} - z_1)(v_2^{q_2} - z_2)} dv_1 dv_2,$$

so, that $v(t, z_1, z_2) = A_1(t, z_1, z_2) + A_2(t, z_1, z_2)$.

Due to the continuity of the function Φ and the estimates of the proposition 5, there exists $K^1 > 0$ with the following properties. For $t \in S_\delta \cup D(0, \rho)$, $z_1 \in D(0, \epsilon)$, $z_2 \in D(0, \epsilon)$, $(v_1, v_2) \in \mathcal{C}_1$ ($|v_1|^{q_1} = \rho_1$, $|v_2|^{q_2} = \rho_2$, $0 \leq |u_1| \leq \rho_1$ and $0 \leq |u_2| \leq \rho_2$),

$$|\Phi(v_1^{q_1} - u_1, v_2^{q_2} - u_2) \frac{v_1^{q_1 - 1}}{(v_1^{q_1} - z_1)} \frac{v_2^{q_2 - 1}}{(v_2^{q_2} - z_2)}| \leq K^1$$

and

$$|k(\frac{t}{(v_1^{q_1}-z_1)^{l_1/q_1}(v_2^{q_2}-z_2)^{l_2/q_2}},u_1,u_2)| \leq C_3 e^{R_1 u_1} e^{R_2 u_2} \leq K^1.$$

From the latter estimates, we deduce that there exists $\tilde{K}_1 > 0$, $\tilde{C} > 0$ and $\tilde{K} > 0$ such that for all $t \in S_\delta \cup D(0,\rho)$, $z_1 \in D(0,\epsilon)$, $z_2 \in D(0,\epsilon)$,

$$(A_1) \qquad |A_1(t,z_1,z_2)| \leq \tilde{K}_1 \leq \tilde{C} e^{\tilde{K}|t|^m}.$$

From the assumption $(Cd3)$ and the estimates in the proposition 5, there exist $K_1' > 0$, $K_2' > 0$, $\tilde{K}_1' > 0$, $M > 0$ with the following properties. For $t \in S_\delta \cup D(0,\rho)$, $z_1 \in D(0,\epsilon)$, $z_2 \in D(0,\epsilon)$, $(v_1,v_2) \in \mathcal{C}_2$ ($|v_1|^{q_1} \leq a|t|^{q_1/l_1}$, $|v_2|^{q_2} \leq b|t|^{q_2/l_2}$, $u_1 = v_1^{q_1}s_1$, $s_1 \in [0,1]$, $u_2 = v_1^{q_2}s_2$, $s_2 \in [0,1]$), we put

$$A = |\Phi(v_1^{q_1}-u_1, v_2^{q_2}-u_2)||k(\frac{t}{(v_1^{q_1}-z_1)^{l_1/q_1}(v_2^{q_2}-z_2)^{l_2/q_2}},u_1,u_2)|.$$

We get

$$A \leq CC_3 \exp(K_1|v_1^{q_1}-u_1|^{k_1} + R_1|u_1|)\exp(K_2|v_2^{q_2}-u_2|^{k_2} + R_2|u_2|),$$

$$A \leq CC_3 \exp(K_1 a^{k_1}|t|^{k_1 q_1/l_1}(1-s_1)^{k_1} + R_1 a|t|^{q_1/l_1}s_1) \\ \times \exp(K_2 b^{k_2}|t|^{k_2 q_2/l_2}(1-s_2)^{k_2} + R_2 b|t|^{q_2/l_2}s_2).$$

For $|t|$ sufficiently large, we get

$$A \leq CC_3 \exp(K_1'|t|^{\sup(1,k_1)q_1/l_1})\exp(K_2'|t|^{\sup(1,k_2)q_2/l_2}),$$

and therefore

$$A \leq CC_3 \exp(\tilde{K}_1'|t|^{\sup(\sup(1,k_1)\frac{q_1}{l_1},\sup(1,k_2)\frac{q_2}{l_2})}).$$

On the other hand, we have

$$|\frac{v_1^{q_1-1}}{(v_1^{q_1}-z_1)}\frac{v_2^{q_2-1}}{(v_2^{q_2}-z_2)}| \leq M.$$

We deduce that there exist $\tilde{m} > 0$, $\hat{C} > 0$, $\hat{K} > 0$ such that for all $t \in S_\delta \cup D(0,\rho)$, $z_1 \in D(0,\epsilon)$, $z_2 \in D(0,\epsilon)$,

$$|A_2(t,z_1,z_2)| \leq \hat{C}|t|^{\tilde{m}} \exp(\hat{K}|t|^{\sup(\sup(1,k_1)\frac{q_1}{l_1},\sup(1,k_2)\frac{q_2}{l_2})}).$$

From the latter estimate we deduce finally that there exist $\tilde{C}_0 > 0$, $\tilde{C} > 0$ and $\tilde{K}_0 > 0$, $\tilde{K} > 0$ such that for all $t \in S_\delta \cup D(0,\rho)$, $z_1 \in D(0,\epsilon)$, $z_2 \in D(0,\epsilon)$,

$$(A_2) \qquad |A_2(t,z_1,z_2)| \leq \tilde{C}_0 \exp(\tilde{K}_0|t|^{\sup(\sup(1,k_1)\frac{q_1}{l_1},\sup(1,k_2)\frac{q_2}{l_2})}) \leq \tilde{C}\exp(\tilde{K}|t|^m).$$

In conclusion, from the estimates (A_1) and (A_2), we deduce the estimate $(|v|)$.

Case $l_2 = 0$ (the case $l_1 = 0$ can be treated in the same way).

We consider a curve $\mathcal{C} = C^d_{\rho_1^{1/q_1}} \times C_{\rho_2^{1/q_2}}$ where $C^d_{\rho_1^{1/q_1}}$ is homotopic to the circle $C_{\rho_1^{1/q_1}}$ and satisfies the following properties.

i) For $(v_1,v_2)\in\mathcal{C}$ such that $(v_1,v_2)\in\mathcal{N}$, we have $(v_1,v_2)\in C_{\rho_1^{1/q_1}}\times C_{\rho_2^{1/q_2}}$,
ii) There exists $a>0$, $\epsilon>0$, such that for $(v_1,v_2)\in\mathcal{C}$ with $(v_1,v_2)\in\mathcal{N}^c$ (where $\mathcal{N}^c$ is the complement of $\mathcal{N}$ in $\mathbb{C}^2$), we have that for all $(z_1,z_2)\in D(0,\epsilon)\times D(0,\epsilon)$,

$$0<\frac{|t|}{|v_1^{q_1}-z_1|^{l_1/q_1}}<\rho \quad , \quad |v_1^{q_1}|\le a|t|^{q_1/l_1}.$$

The curve $\mathcal{C}$ can be written as the union of two disjointed parts $\mathcal{C}_1=\{(v_1,v_2)\in\mathcal{C}:(v_1,v_2)\in\mathcal{N}\}$, $\mathcal{C}_2=\{(v_1,v_2)\in\mathcal{C}:(v_1,v_2)\in\mathcal{N}^c\}$. In the following, we put

$$A_1(t,z_1,z_2)=\frac{1}{(2\pi i)^2}\partial_{z_1}\partial_{z_2}\oint_{\mathcal{C}_1}\Phi*k(t,v_1,v_2)\frac{v_1^{q_1-1}v_2^{q_2-1}}{(v_1^{q_1}-z_1)(v_2^{q_2}-z_2)}dv_1dv_2,$$

and

$$A_2(t,z_1,z_2)=\frac{1}{(2\pi i)^2}\partial_{z_1}\partial_{z_2}\oint_{\mathcal{C}_2}\Phi*k(t,v_1,v_2)\frac{v_1^{q_1-1}v_2^{q_2-1}}{(v_1^{q_1}-z_1)(v_2^{q_2}-z_2)}dv_1dv_2,$$

so, that $v(t,z_1,z_2)=A_1(t,z_1,z_2)+A_2(t,z_1,z_2)$.

As previously, one can show that there exist $\tilde{K}_1>0$, $\tilde{C}>0$ and $\tilde{K}>0$, such that for all $t\in S_\delta\cup D(0,\rho)$, $z_1\in D(0,\epsilon)$, $z_2\in D(0,\epsilon)$,

$$(A_1') \qquad |A_1(t,z_1,z_2)|\le\tilde{K}_1\le\tilde{C}e^{\tilde{K}|t|^m}.$$

On the other hand, there exist $K_1'>0$, $K_2'>0$ with the following properties. For $t\in S_\delta\cup D(0,\rho)$, $z_1\in D(0,\epsilon)$, $z_2\in D(0,\epsilon)$, $(v_1,v_2)\in\mathcal{C}_2$ ($|v_1|^{q_1}\le a|t|^{q_1/l_1}$, $|v_2|^{q_2}=\rho_2$, $u_1=v_1^{q_1}s_1$, $s_1\in[0,1]$, $u_2=v_1^{q_2}s_2$, $s_2\in[0,1]$), we put

$$A=|\Phi(v_1^{q_1}-u_1,v_2^{q_2}-u_2)||k(\frac{t}{(v_1^{q_1}-z_1)^{l_1/q_1}},u_1,u_2)|.$$

From the assumption $(Cd3)$ and the estimates in the proposition 5, we get

$$A\le CC_3K_2'\exp(K_1a^{k_1}|t|^{k_1q_1/l_1}(1-s_1)^{k_1}+R_1a|t|^{q_1/l_1}s_1).$$

Due to the fact that $m=k_1q_1/l_1\ge q_1/l_1$, we get for $|t|$ sufficiently large,

$$A\le CC_3K_2'\exp(K_1'|t|^m).$$

As previously, one can show that there exist $\tilde{C}>0$, $\tilde{K}>0$ such that for all $t\in S_\delta\cup D(0,\rho)$, $z_1\in D(0,\epsilon)$, $z_2\in D(0,\epsilon)$,

$$(A_2') \qquad |A_2(t,z_1,z_2)|\le\tilde{C}\exp(\tilde{K}|t|^m).$$

In conclusion, from the estimates (A_1') and (A_2'), we deduce the estimate $(|v|)$. □

3.3 Summability of formal solutions

Definition 1 *A series*

$$\hat{u}(t,z_1,z_2)=\sum_{j=0}^{\infty}\frac{t^j}{j!}u_j(z_1,z_2)$$

where $u_j(z_1,z_2)$, $0 \leq j \leq \infty$ *are holomorphic functions in a polydisc* $D_{\rho_1} \times D_{\rho_2}$, *is said to be* $k-$*summable in the direction* $d \in \mathbb{R}$ *if*

i) *There exist* $r_1 > 0$, $r_2 > 0$, $R(r_1,r_2) > 0$, *such that the formal Borel transform with respect to the variable* t *of order* k,

$$w(t,z_1,z_2) = \sum_{j=0}^{\infty} \frac{u_j(z_1,z_2)t^j}{j!\Gamma(1+j/k)},$$

is absolutely convergent for $|z_1| < r_1$, $|z_2| < r_2$ *and* $|t| < R(r_1,r_2)$,

ii) *There exists* $\delta > 0$ *such that for all* $(z_1,z_2) \in \bar{D}(0,r_1) \times \bar{D}(0,r_2)$, *the series* $w(t,z_1,z_2)$ *can be analytically continued with respect to* t *in a sector* $S_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| < \delta\}$. *Moreover, there exist* $\delta_1 < \delta$, $C > 0$, *and* $K > 0$ *such that*

$$\sup_{|z_1| \leq r_1, |z_2| \leq r_2} |w(t,z_1,z_2)| \leq Ce^{K|t|^k}$$

for all $t \in S_{d,\delta_1}$.

If this is so, the Laplace transform of order k of $w(t,z_1,z_2)$,

$$u(t,z_1,z_2) = t^{-k} \int_0^{\infty(\gamma)} w(t,z_1,z_2) e^{-(\tau/t)^k} d\tau^k,$$

where the integration is made along the ray $\arg(\tau) = \gamma$ with $|d - \gamma| < \delta$, is called the $k-$ sum of the formal series $\hat{u}(t,z_1,z_2)$.

We consider the formal Borel transform

$$w(t,z_1,z_2) = \sum_{j=0}^{\infty} \frac{u_j(z_1,z_2)t^j}{j!\Gamma(1+j/m)},$$

where $u_j(z_1,z_2)$ is defined by the formula (u_j) in the section 1 and $m = ((l_1/q_1) + (l_2/q_2) - 1)^{-1}$. Due to the general theory of moment summability (see [3], p. 107–109), we deduce from the proposition 7 that there exists a direction $d \in \mathbb{R}$ and $\delta > 0$ with $S_{d,\delta} \subset S_\delta$ such that $w(t,z_1,z_2)$ satisfies the properties **i)** and **ii)** of the latter definition. Finally, we get the main result of this section.

Theorem 2 *Let* $m = ((l_1/q_1) + (l_2/q_2) - 1)^{-1}$. *Under the conditions* $(Cd1)$, $(Cd2)$ *and* $(Cd3)$ *there exists a direction* $d \in \mathbb{R}$ *and* $\delta > 0$ *with* $S_{d,\delta} \subset S_\delta$ *such that the formal solution*

$$\hat{u}(t,z_1,z_2) = \sum_{j=0}^{\infty} \frac{t^j}{j!} u_j(z_1,z_2)$$

where u_j *are defined by the formula* (u_j), *is* $m-$*summable in the direction* d.

4 Multisummability of formal solutions

The next result explains how to construct linear partial differential equations with constant coefficients having solutions which are products of power series solutions of given partial differential equations.

Proposition 8 *Let* $P_\alpha \in \mathbb{C}[t, z_1, \ldots, z_d]$, $P_\beta \in \mathbb{C}[t, z_{d+1}, \ldots, z_p]$ *be two polynomials and assume that the assumption* **A** *is satisfied for each of them (see section 1). Let* $\Phi_\alpha(z_1, \ldots, z_d)$ *and* $\Phi_\beta(z_{d+1}, \ldots, z_p)$ *be holomorphic functions on polydiscs* $D_{\rho_1} \times \ldots \times D_{\rho_d}$ *resp.* $D_{\rho_{d+1}} \times \ldots \times D_{\rho_p}$. *Consider formal power series* $\hat{u}_\alpha(t, z_1, \ldots, z_d)$ *and* $\hat{u}_\beta(t, z_{d+1}, \ldots, z_p)$ *constructed as in the proposition 2, satisfying*

$$P_\alpha(\partial_t, \partial_{z_1}, \ldots, \partial_{z_d})\hat{u}_\alpha = 0 \quad , \quad P_\beta(\partial_t, \partial_{z_{d+1}}, \ldots, \partial_{z_p})\hat{u}_\beta = 0,$$

with initial conditions

$$\hat{u}_\alpha(0, z_1, \ldots, z_d) = \Phi_\alpha(z_1, \ldots, z_d) \quad , \quad \hat{u}_\beta(0, z_{d+1}, \ldots, z_p) = \Phi_\beta(z_{d+1}, \ldots, z_p).$$

Then, there exists a polynomial $P_{\alpha\beta}(t, z_1, \ldots, z_p)$ *for which the assumption* **A** *is satisfied and such that* $P_{\alpha\beta}(\partial_t, \partial_{z_1}, \ldots, \partial_{z_p})\hat{u}_\alpha\hat{u}_\beta = 0$. *In particular, if* $\hat{u}_\alpha$ *(resp.* $\hat{u}_\beta$*) is* $(m_1, \ldots, m_{\alpha_1})$*-summable (resp.* $(m'_1, \ldots, m'_{\beta_1})$*-summable) with respect to* t *in the direction* d*, then the formal series* $\hat{u}_{\alpha\beta} = \hat{u}_\alpha\hat{u}_\beta$ *is a solution of the Cauchy problem*

$$P_{\alpha\beta}(\partial_t, \partial_{z_1}, \ldots, \partial_{z_p})\hat{u}_{\alpha\beta} = 0 \quad , \quad \hat{u}_{\alpha\beta}(0, z_1, \ldots, z_p) = \Phi_\alpha\Phi_\beta,$$

which is $(\bar{m}_1, \ldots, \bar{m}_{\gamma_1})$*-summable with respect to* t *in the direction* d*, where* $(\bar{m}_1, \ldots, \bar{m}_{\gamma_1})$ *is the type of multisummability corresponding to the union* $\{m_1, \ldots, m_{\alpha_1}\} \cup \{m'_1, \ldots, m'_{\beta_1}\}$.

Proof Let $\lambda_\alpha(w_1, \ldots, w_d)$ and $\lambda_\beta(w_{d+1}, \ldots, w_p)$ be two functions satisfying the assumptions **A** i), ii) and $P_\alpha(\lambda_\alpha, w_1, \ldots, w_d) = 0$, $P_\beta(\lambda_\beta, w_{d+1}, \ldots, w_p) = 0$. In particular, λ_α and λ_β are algebraic functions over the fields $\mathbb{C}(w_1, \ldots, w_p)$. From a well known result of basic algebra, we deduce that the sum $\lambda_{\alpha\beta} = \lambda_\alpha + \lambda_\beta$ is also an algebraic function over the fields $\mathbb{C}(w_1, \ldots, w_p)$. So that there exists a polynomial $P_{\alpha\beta}(t, w_1, \ldots, w_p)$ such that

$$P_{\alpha\beta}(\lambda_{\alpha\beta}, w_1, \ldots, w_p) = 0.$$

We recall briefly the proof of this result. Consider the vector space V over the fields $K = \mathbb{C}(w_1, \ldots, w_p)$ which is spanned by the functions $\lambda_\alpha^i\lambda_\beta^j$, for i, j non negative integers. Denote by κ_α the degree of P_α in the variable t and by κ_β the degree of P_β in the variable t. From the fact that λ_α and λ_β satisfy polynomial equations of degree lower than κ_α (resp. κ_β), we get that $\dim_K(V) \leq \kappa_\alpha\kappa_\beta$. From this, we deduce that the functions of V, $(\lambda_\alpha + \lambda_\beta)^j$, $0 \leq j \leq \kappa_\alpha\kappa_\beta$ are linearly dependent over K. We conclude that there is a polynomial $P_{\alpha\beta} \in \mathbb{C}[t, w_1, \ldots, w_p]$ of degree lower or equal $\kappa_\alpha\kappa_\beta$ in the variable t such that $P_{\alpha\beta}(\lambda_\alpha + \lambda_\beta, w_1, \ldots, w_p) = 0$.

From the assumption **A** ii), there exist $C_\alpha > 0$ and $C_\beta > 0$ such that the functions λ_α and λ_β satisfy estimates

$$|\lambda_\alpha(w_1, \ldots, w_d)| \leq C_\alpha \prod_{k=1}^{d} |w_k|^{m_k} \quad , \quad |\lambda_\beta(w_{d+1}, \ldots, w_p)| \leq C_\beta \prod_{k=d+1}^{p} |w_k|^{m_k},$$

for $|w_k| \geq R_0$, $1 \leq k \leq p$. We deduce that

$$|\lambda_\alpha(w_1, \ldots, w_d) + \lambda_\beta(w_{d+1}, \ldots, w_p)| \leq (C_\alpha + C_\beta) \prod_{k=1}^{p} |w_k|^{m_k},$$

for $|w_k| \geq R'_0 = \sup(R_0, 1)$, $1 \leq j \leq p$. So that the assumptions **A** i), ii), iii) are satisfied for the polynomial $P_{\alpha\beta}$.

In the following, we keep the same notations as in the section 1. For all non negative integers m, n_k, $1 \leq k \leq p$, we have that

$$(B) \qquad \oint_{qC_R} \frac{(\lambda_\alpha + \lambda_\beta)^m}{\Pi_{k=1}^p w_k^{n_k+1}} \Pi_{k=1}^p e^{w_k z_k} dw_1 \cdots dw_p = \sum_{k+l=m} \frac{m!}{k!l!} A_{k,n_1,\dots,n_d} A_{l,n_{d+1},\dots,n_p},$$

where

$$A_{k,n_1,\dots,n_d} = \oint_{qC_R} \frac{\lambda_\alpha^k(w_1,\dots,w_d)}{\Pi_{k=1}^d w_k^{n_k+1}} \Pi_{k=1}^d e^{w_k z_k} dw_1 \cdots dw_d$$

and

$$A_{l,n_{d+1},\dots,n_p} = \oint_{qC_R} \frac{\lambda_\beta^l(w_{d+1},\dots,w_p)}{\Pi_{k=d+1}^p w_k^{n_k+1}} \Pi_{k=d+1}^p e^{w_k z_k} dw_{d+1} \cdots dw_p.$$

Consider now the series u_k^α of the proposition 2 constructed from the function λ_α and Φ_α, the series u_l^β of the proposition 2 constructed from the function λ_β and Φ_β and the series $u_m^{\alpha\beta}$ of the proposition 2 constructed from the function $\lambda_\alpha + \lambda_\beta$ and $\Phi_\alpha\Phi_\beta$. From the equality (B), we get that

$$\frac{u_m^{\alpha\beta}(z_1,\dots,z_p)}{m!} = \sum_{k+l=m} \frac{u_k^\alpha(z_1,\dots,z_d)}{k!} \frac{u_l^\beta(z_{d+1},\dots,z_p)}{l!}.$$

We conclude that the formal series $\hat{u}_{\alpha\beta}$ of the proposition 2 constructed from the series $(u_m^{\alpha\beta})$ satisfies $\hat{u}_{\alpha\beta} = \hat{u}_\alpha \hat{u}_\beta$. From the proposition 2 it also follows that

$$P_{\alpha\beta}(\partial_t, \partial_{z_1}, \dots, \partial_{z_p})\hat{u}_{\alpha\beta} = 0$$

and $\hat{u}_{\alpha\beta}(0, z_1, \dots, z_p) = \Phi_\alpha \Phi_\beta$. □

Example 2 We consider the multidimensional heat equation

$$(E2) \qquad (\partial_t - \sum_{k=1}^p \partial_{z_k}^2) u(t, z_1, \dots, z_p) = 0,$$

with $u(0, z_1, \dots, z_p) = \Phi_1(z_1) \cdots \Phi_p(z_p)$ where $\Phi_k(z_k)$, $1 \leq k \leq p$ are rational functions. We will show by induction that the equation $(E2)$ has a formal series solution (2), as described in the proposition 2, which is 1-summable with respect to t in all but finitely many directions. Assume that the heat equation

$$P_\alpha(\partial_t, \partial_{z_1}, \dots, \partial_{z_{p-1}})\hat{u}_\alpha = 0,$$

where $P_\alpha(t, z_1, \dots, z_{p-1}) = t - \sum_{k=1}^{p-1} z_k^2$, with $\hat{u}_\alpha(0, z_1, \dots, z_{p-1}) = \Phi_1(z_1) \cdots \Phi_{p-1}(z_{p-1})$, has a formal series $\hat{u}_\alpha$ solution (2) which is 1-summable with respect to t in all but finitely many directions. In this case the root of P_α in the variable t is the function $\lambda_\alpha = \sum_{k=1}^{d-1} z_k^2$. From the papers [2], [6], we know that heat equation

$$P_\beta(\partial_t, \partial_{z_p})\hat{u}_\alpha = 0,$$

where $P_\beta(t, z_p) = t - z_p^2$, with $\hat{u}_\beta(0, z_p) = \Phi_p(z_p)$, has a formal series $\hat{u}_\beta$ solution (2) which is 1-summable with respect to t in all but finitely many directions. The root of P_β in the variable t is the function $\lambda_\beta = z_p^2$. On the other hand, the polynomial $P_{\alpha\beta}(t, z_1, \dots, z_p) = t - \sum_{k=1}^p z_k^2$ satisfy $P_{\alpha\beta}(\lambda_\alpha + \lambda_\beta, z_1, \dots, z_p) = 0$. From the proposition 8, we deduce that the equation $(E2)$ has the formal series $\hat{u}_\alpha \hat{u}_\beta$ as solution, which is 1-summable with respect to t in all but finitely many directions.

Example 3 Consider the partial differential equation

$$(E3) \qquad (\partial_t - \partial_{z_1}^2 - \partial_{z_2}^3)u(t, z_1, z_2) = 0,$$

with $u(0, z_1, z_2) = \Phi_1(z_1)\Phi_2(z_2)$, where Φ_1 and Φ_2 are rational functions. From the papers [2], [6], we know that the equation

$$P_\alpha(\partial_t, \partial_{z_1})\hat{u}_\alpha = 0,$$

where $P_\alpha(t, z_1) = t - z_1^2$, with $\hat{u}_\alpha(0, z_1) = \Phi_1(z_1)$, has a formal series $\hat{u}_\alpha$ solution (2) which is 1-summable with respect to t in all but finitely many directions. In this case the root of P_α in the variable t is the function $\lambda_\alpha = z_1^2$. Likewise, from the papers [2], [7], we know that the equation

$$P_\beta(\partial_t, \partial_{z_2})\hat{u}_\beta = 0,$$

where $P_\beta(t, z_2) = t - z_2^3$, with $\hat{u}_\beta(0, z_2) = \Phi_2(z_2)$, has a formal series $\hat{u}_\beta$ solution (2) which is 1/2-summable with respect to t in all but finitely many directions. In this case the root of P_β in the variable t is the function $\lambda_\beta = z_2^3$. On the other hand, the polynomial $P_{\alpha\beta}(t, z_1, z_2) = t - z_1^2 - z_2^3$ satisfy $P_{\alpha\beta}(\lambda_\alpha + \lambda_\beta, z_1, z_2) = 0$. From the proposition 8, we deduce that the equation $(E3)$ has the formal series $\hat{u}_\alpha\hat{u}_\beta$ as solution, which is $(1, 1/2)$-summable with respect to t in all but finitely many directions, so that the series $\hat{u}_\alpha\hat{u}_\beta$ is not m-summable in any direction d, for any $m \in \mathbb{R}_+$.

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