On resonance free domains
for semiclassical Schrödinger operators

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Abstract

We give a simple proof of a result of Martinez on resonance free domains for semiclasssical Schrödinger operators

I. Resonances for semiclassical Schrödinger operators

Let $V:\mathbb{R}^n\to\mathbb{R}$ be a smooth potential satisfying the assumption
(H1) V extends holomorphically to

$$\begin{array}{l} D=\{z\in\mathbb{C}^n||\mathrm{Re}z|>R,\ |\mathrm{Im}z|\leq c|\mathrm{Re}z|\},\ \text{and satisfies} \\ |V(z)|\leq C(1+|z|)^{-\rho},\ z\in D, \end{array}$$

for some R, c, $\rho>0$. We consider the semiclassical Schrödinger operator:

$$H=\frac{h^2}{2}D_x^2+V(x),$$

which is selfadjoint on $H^2(\mathbb{R}^n)$. Let $p(x,\xi)=\frac{1}{2}\xi^2+V(x)$ be the symbol of H. We recall that an energy level $\lambda>0$ is *non-trapping* for p if

$$(H2)\ |\exp tH_p(x,\xi)|\to\infty \text{ when } t\to\pm\infty,\ \forall\,(x,\xi)\in p^{-1}(\lambda),$$

where $\exp tH_p$ is the Hamiltonian flow of p.

The following result has been shown by Martinez in [M].

Theorem 1 *Assume hypotheses (H1) and (H2). Then there exists* $\delta>0$ *such that for any* $C>0$*, there exists* $h_0>0$ *such that for* $0<h\leq h_0$ $H(h)$ *has no resonances in* $[\lambda-\delta,\lambda+\delta]+\mathrm{i}[-Ch|\ln h|,0]$.

The purpose of this note is to give a proof of Thm. 1 which uses only elementary pseudodifferential calculus.

Proof of Thm. 1.

Let S^p be the space of symbols $a(h,x,\xi)$ such that $|\partial_x^\alpha\partial_\xi^\beta a|\leq C_{\alpha,\beta}\langle\xi\rangle^{p-|\beta|}$, for all $\alpha,\beta\in\mathbb{R}^n$, uniformly for $0<h\leq 1$.

We first recall Hunziker's method of analytic distortions [Hu]:

let $v : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field such that $v(x) \equiv 0$ in $|x| \leq R+1$, $v(x) = x$ for $|x| \gg 1$. Let U_s for $s \in \mathbb{R}$, $|s| \ll 1$ be the unitary operator:

$$U_s u(x) = \det(\mathbb{1} + s\nabla v(x))^{\frac{1}{2}} u(x + sv(x)),$$

and $\tilde{H}_s := U_s H U_s^{-1}$. Then if $J_s(x) = \mathbb{1} + s\nabla v(x)$ and $|J_s| = \det J_s$, one has

$$\tilde{H}_s = \frac{h^2}{2} |J_s|^{-\frac{1}{2}} (D_x, {}^tJ_s^{-1} |J_s| J_s^{-1} D_x) |J_s|^{-\frac{1}{2}} + V(x + sv(x)).$$

The family $\tilde{H}_s : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an analytic family and one sets for $0 < t \ll 1$:

$$H_t := \tilde{H}_{\mathrm{i}t}.$$

Then $H_t = p_t(x, hD_x, h)$, where $p_t(x, \xi, h)$ is a second order polynomial in ξ with

$$p_t(x, \xi, h) = p(x + \mathrm{i}tv(x), (\mathbb{1} + \mathrm{i}t\nabla v(x))^{-1}\xi) + h^2 r_{1,t}(x, \xi, h), \tag{1}$$

where $r_{1,t} \in S^0$, uniformly in $|t| \ll 1$, $0 < h \leq 1$. H_t is closed with domain $H^2(\mathbb{R}^n)$, $\sigma_{\mathrm{ess}}(H_t) = (1 + \mathrm{i}t)^{-2}\mathbb{R}^+$ and by definition the resonances of H in

$$S_t = \{z \in \mathbb{C} | \mathrm{Re} z > 0,\ -2\arctan t < \mathrm{Arg} z < 0\}$$

are the eigenvalues of H_t in S_t.

We start with an elementary lemma.

Lemma 2 *i) Let H be a selfadjoint operator on a Hilbert space $\mathcal{H}$ and let $B \in \mathcal{B}(\mathcal{H})$. Assume that $[H, B]$ (as a quadratic form on $\mathcal{D}(H)$) is bounded on $\mathcal{H}$. Then e^{tB} preserves $\mathcal{D}(H)$.*

ii) Let H be a closed operator and $B \in \mathcal{B}(\mathcal{H})$ such that $[H, B]$ (as a quadratic form on $\mathcal{D}(H)$) is bounded on $\mathcal{H}$ and e^{tB} preserves $\mathcal{D}(H)$. Then:

$$\mathrm{e}^B H \mathrm{e}^{-B} = \sum_{k=0}^{n} \frac{1}{k!} \mathrm{ad}_B^k H + \frac{1}{n!} \int_0^1 (1-s)^n \mathrm{e}^{sB} \mathrm{ad}_B^{n+1} H \mathrm{e}^{-sB} \mathrm{d}s,$$

as an identity on $\mathcal{D}(H)$.

In Lemma 2 the multicommutators $\mathrm{ad}_B^k H$ are defined inductively by $\mathrm{ad}_B^0 H = H$, $\mathrm{ad}_B^{k+1} H = [B, ad_B^k H]$.

Proof. Let us first prove *i)*. Clearly we can assume that $t = 1$. Let $\epsilon > 0$. We have

$$\begin{aligned} [H(\mathbb{1} + \mathrm{i}\epsilon H)^{-1}, B] &= -(\mathrm{i}\epsilon)^{-1}[(\mathbb{1} + \mathrm{i}\epsilon H)^{-1}, B] \\ &= (\mathbb{1} + \mathrm{i}\epsilon H)^{-1}[H, B](\mathbb{1} + \mathrm{i}\epsilon H)^{-1} \in O(\epsilon^0). \end{aligned} \tag{2}$$

Let now $u \in \mathcal{H}$ and set $f_\epsilon(t) = H(\mathbb{1} + \mathrm{i}\epsilon H)^{-1}\mathrm{e}^{tB}(H + \mathrm{i})^{-1}u$. Using (2) we see that $f'_\epsilon(t) = Bf_\epsilon(t) + r_\epsilon(t)$, where $\|r_\epsilon(t)\| \leq C\|u\|$ uniformly in $0 < \epsilon \leq 1$, $0 \leq t \leq 1$. Applying then Gronwall's inequality, we obtain that $\|f_\epsilon(1)\| \leq C\|u\|$, uniformly in ϵ, which proves *i)* by letting $\epsilon \to 0$. Part *ii)* is Taylor's formula applied to the C^∞ function $f(t) = \mathrm{e}^{tB}H\mathrm{e}^{-tB}u$ for $u \in \mathcal{D}(H)$. $\square$

Proof of Thm. 1

It is well known (see eg [GM]) that if $\lambda > 0$ is non-trapping, then there exists $\delta, \epsilon_0 > 0$ and a function $m \in C_0^\infty(\mathbb{R}^{2n})$ such that $\{p, x.\xi + m\} \geq \epsilon_0$ on $p^{-1}[\lambda - \delta, \lambda + \delta]$. Now we set

$$r(x,\xi) = m(x,\xi) + \chi \circ p(x,\xi)(x - v(x)).\xi, \tag{3}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ near $[\lambda - \delta, \lambda + \delta]$. Then $r \in C_0^\infty(\mathbb{R}^{2n})$ and if $G_0(x,\xi) = v(x).\xi$, then

$$\{p, G_0 + r\} \geq \epsilon_0 \text{ on } p^{-1}[\lambda - \delta, \lambda + \delta]. \tag{4}$$

Let us now fix $C \gg 1$ and set $B = -Cr(x, hD_x)|\ln h|$, where $r \in C_0^\infty(\mathbb{R}^{2n})$ is defined in (3). Applying Lemma 2 *i)* to $H = D_x^2$, we obtain that e^{tB} preserves $\mathcal{D}(H_t)$. Moreover since $r \in C_0^\infty(\mathbb{R}^{2n})$, $[B, H_t]$ is bounded, hence we can apply Lemma 2 *ii)*. This yields:

$$\mathrm{e}^B H_t \mathrm{e}^{-B} = \sum_{k=0}^{n} \frac{1}{k!} \mathrm{ad}_B^k H_t + \frac{1}{n!} \int_0^1 (1-s)^n \mathrm{e}^{sB} \mathrm{ad}_B^{n+1} H_t \mathrm{e}^{-sB} \mathrm{d}s. \tag{5}$$

We note that by p.d.o. calculus $\mathrm{ad}_B^n H_t \in O((h \ln h)^n)$, and $\mathrm{e}^{sB} \in O(\mathrm{e}^{C_0 C |\ln h|})$, uniformly for $|s| \leq 1$, $0 < h \leq 1$, for $C_0 = \sup_{0<h\leq 1} \|r(x, hD_x)\|$.

Hence picking n large enough in (5), we obtain

$$\mathrm{e}^B H_t \mathrm{e}^{-B} = H_t + [B, H_t] + O(h^{2-\epsilon}),\ \epsilon > 0. \tag{6}$$

Moreover

$$[B, H_t] = -\mathrm{i}Ch|\ln h|\{p_t, r\}(x, hD_x) + O(h^{2-\epsilon}). \tag{7}$$

It follows from (1) that

$$p_t = p - \mathrm{i}t\{p, G_0\} + h^2 s_{0,t} + t^2 s_{2,t},$$

where $s_{i,t} \in S^i$, uniformly for $|t| \ll 1$. This yields

$$\{p_t, r\} = \{p, r\} + t r_{1,t} + h^2 r_{2,t}, \tag{8}$$

where $r_{i,t} \in S^0$, uniformly for $|t| \ll 1$. This implies

$$\begin{aligned} \mathrm{e}^B H_t \mathrm{e}^{-B} = \ & p(x, hD_x) - \mathrm{i}t\{p, G_0\}(x, hD_x) - \mathrm{i}Ch|\ln h|\{p, r\}(x, hD_x) + t^2 s_{2,t}(x, hD_x) \\ & + O(h^{2-\epsilon}) + O(th|\ln h|), \end{aligned} \tag{9}$$

for $s_{2,t} \in S^2$ uniformly in $|t| \ll 1$.

Picking $t = Ch|\ln h|$, we obtain

$$\mathrm{e}^B H_t \mathrm{e}^{-B} = q(h, x, hD_x) + O(h^{2-\epsilon}), \tag{10}$$

for

$$q(h, x, \xi) = p(x,\xi) - \mathrm{i}Ch|\ln h|\{p, G\}(x,\xi) + (h \ln h)^2 s_2(x,\xi),$$

where $G = G_0 + r$ and $s_2 \in S^2$. Let now $z \in [\lambda - \delta/4, \lambda + \delta/4] - \mathrm{i}[-C\epsilon_0 h|\ln h|/2, 0]$, where ϵ_0 and δ are fixed in (4). Then it is easy to see that for $h \ll 1$ $|q(h,x,\xi) - z| \geq ch|\ln h|$. From this degenerate ellipticity it should be easy to conclude, by contructing a parametrix, that for

$h \ll 1$ $(q(h,x,hD_x)-z)^{-1}$ exists and has a norm $O(|h\ln h|^{-1})$. Using (10) this would imply that $\mathrm{e}^B H_t \mathrm{e}^{-B} - z$ and hence $H_t - z$ is invertible. For completeness we give below another argument:

let $z \in \mathbb{C}$ be as above and let us assume that $\mathrm{Ker}(H_t - z) \neq \{0\}$. Since e^B preserves $H^2(\mathbb{R}^n)$, this implies that $\mathrm{Ker}(\mathrm{e}^B H_t \mathrm{e}^{-B} - z) \neq \{0\}$. Let hence $u \in H^2(\mathbb{R}^n)$ with

$$(\mathrm{e}^B H_t \mathrm{e}^{-B} - z)u = 0, \ \|u\| = 1.$$

Let us pick $\chi_0 \in C_0^\infty(\mathbb{R})$, $\chi_+, \chi_- \in C^\infty(\mathbb{R})$ such that $\mathrm{supp}\chi_0 \subset [\lambda - \delta, \lambda + \delta]$, $\chi_0 \equiv 1$ on $[\lambda - \delta/2, \lambda + \delta/2]$, $\mathrm{supp}\chi_+ \subset [\lambda + \delta/2, +\infty[$, $\mathrm{supp}\chi_- \subset] - \infty, \lambda - \delta/2]$ and $\chi_-^2 + \chi_0^2 + \chi_+^2 \equiv 1$. We set then $f_\epsilon(x,\xi) = \chi_\epsilon \circ p(x,\xi)$ and $F_\epsilon = f_\epsilon(x, hD_x)$ for $\epsilon = -, 0, +$.

By p.d.o. calculus, we deduce from (10) that

$$0 = (u, F_\epsilon^2(\mathrm{e}^B H_t \mathrm{e}^{-B} - z)u) = (u, F_\epsilon(q(h,x,hD_x) - z)F_\epsilon u) + O(h)\|u\|^2. \tag{11}$$

Recall that by (4) $\{p, G\} \geq \epsilon_0$ on $\mathrm{supp} f_0$, which implies that for $h \ll 1$ $-\mathrm{Im} q \geq Ch|\ln h|\epsilon_0/2$ on $\mathrm{supp} f_0$. Using also that $\mathrm{Im} z \in [-C\epsilon_0 h|\ln h|/2, 0]$, we obtain from Gärding's inequality:

$$-\mathrm{Im}(u, F_0(q(h,x,hD_x) - z)F_0 u) \geq c_0 h \ln h (u, F_0^2 u) + O(h)\|u\|^2. \tag{12}$$

Similarly since $\mathrm{Re} z \in [\lambda - \delta/4, \lambda + \delta/4]$, we obtain that $\pm(\mathrm{Re}(q - z) \geq \pm\epsilon_1 > 0$ on $\mathrm{supp} f_\pm$, which again by Gärdings inequality gives:

$$\pm\mathrm{Re}(u, F_\pm(q(h,x,hD_x) - z)F_\pm u) \geq \pm c_1(u, F_\pm^2 u) + O(h)\|u\|^2. \tag{13}$$

Now by (11) the left hand sides in (12), (13) are of size $O(h)\|u\|^2$, which yields

$$(u, F_0^2 u) \leq c_0 |\ln h|^{-1} \|u\|^2, \ (u, F_\pm^2 u) \leq c_0 h \|u\|^2. \tag{14}$$

But since $f_-^2 + f_0^2 + f_+^2 \equiv 1$, we have

$$\|u\|^2 = (u, F_-^2 u) + (u, F_0^2 u) + (u, F_+^2 u) + O(h)\|u\|^2,$$

which by (14) yields contradicts the fact that $\|u\| = 1$. This completes the proof of the theorem.

References

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