STERNBERG THEOREMS
FOR RANDOM DYNAMICAL SYSTEMS

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Abstract. In this paper, we prove the smooth conjugacy theorems of Sternberg type for random dynamical systems based on their Lyapunov exponents. We also present the stable and unstable manifold theorems of Pesin type which are used in the construction of conjugacy.

1. **Introduction**

The study of reducing a nonlinear deterministic dynamical system to the simplest possible form (called a normal form) goes back to Poincaré and Birkhoff. For a local diffeomorphism $\phi(x)$ with a fixed point $x = 0$, i.e. $\phi(0) = 0$, the classical conjugacy problem is to find a transformation $y = h(x)$ (a local diffeomorphism) such that $h \circ \phi \circ h^{-1}$ has the simplest possible form.

Poincaré [35] first studied the reducibility of an analytic diffeomorphism $\phi(x) = Ax + O(|x|^2)$ in $\mathbb{C}^d$ to its linear part A. He proved that if the eigenvalues $\lambda_1, \cdots, \lambda_d$ of the linear part belong to the so-called Poincaré domain, i.e.

$$\min\{\max_{1\le i\le d} |\lambda_i|, \max_{1\le i\le d} |\lambda_i|^{-1}\} < 1$$

and satisfy the nonresonant conditions, i.e.

$$\lambda_j - \lambda_1^{m_1} \cdots \lambda_d^{m_d} \neq 0, \quad \text{for all },\ 1 \le j \le d,\ n = (m_1, \cdots, m_d) \in \mathbb{Z}_+^d,\ m_1 + \cdots + m_d \ge 2,$$

then $\phi(x)$ is analytically conjugated to its linear part, that is, there is an analytic differomorphism $h(x) = x + O(|x|^2)$ such that

$$h \circ \phi \circ h^{-1} = A.$$

The complement of the Poincaré domain is the so-called Siegel domain. Siegel [42] gave the small divisor conditions for the eigenvalues in the Siegel domain

$$|\lambda_j - \lambda_1^{m_1} \cdots \lambda_d^{m_d}| \ge \frac{C}{|m|^\mu}, \quad 1 \le j \le d,\ m \in \mathbb{Z}_+^d,\ |m| = m_1 + \cdots + m_d \ge 2,$$

under which the nonlinear diffeomorhism is analytically conjugated to its linear part. Brjuno [8] gave a weaker condition than Siegel's condition and proved the result also holds. Siegel's theorem was also proved by Arnold [3], Moser [29], and Zehnder [50]. A simple proof of Poincaré's theorem was given by Meyer [28].

For diffeomorphisms of the circle, there are rich results on smooth linearization, see the works of Arnold[4], Herman [16], Yoccoz[48], Katznelson and Ornstein [19], Sinai and Khanin [43], and

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their references therin. Linearization results for diffeomorphisms of $(\mathbb{C}, 0)$ may be found in the celebrated paper by Yoccoz[49]. For more recent work, we refer to the article by Pérez-Marco [33] and its reference therein.

The study of reducibility of a general local deterministic differentiable dynamical system to a linear normal formal was initiated by Sternberg [44, 45]. He proved that for a local diffeomorphism $\phi(x)$ with a fixed point $x = 0$ and for each integer $k > 0$, there is an integer $N = N(k)$ such that if ϕ is C^N and the eigenvalues of its linearization $A = D\phi(0)$ satisfy the nonresonant conditions up to order N, then there is a C^k local diffeomorphism $h(x) = x + O(|x|^2)$ such that ϕ is conjugate to A, i.e. $h \circ \phi \circ h^{-1} = A$. If ϕ is C^∞ and the nonresonant conditions hold for all orders, then h is C^∞. There is an extensive literature on the problem of finding better conditions for higher regularity. We do not attempt to give an exhaustive list of references. Some classical results may be found in Nagumo and Isé [31] , Chen [10], Hartman [14, 15], and Nelson [21]. For more delicate conditions such that the nonlinear system admits a C^k-smooth linearization we refer to Sell [39, 40] and Beleskii [6, 7]. A result which preserves the geometric structure of the original equation was recently obtained by Banyaga, de la Llave, and Wayne [5]. A C^0 linearization (A Hartman-Grobman Theorem) associated with the local structural stability may be found in Grobman[13], Hartman [14], Pugh [36], and Palmer[32].

In this paper, we study the conjugacy problem for random dynamical systems and extend the Sternberg theorems in deterministic cases to random dynamical systems and settle an open problem raised by L. Arnold. Our results are based on Lyapunov exponents. We also present a stable and unstable manifold theorem of Pesin type for random dynamical systems which play a key role in the construction of conjugacy.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\theta^n)_{n\in\mathbb{Z}}$ be a measurable $\mathbb{P}$-measure preserving dynamical system on Ω. A random dynamical systems (or a cocycle) on the space $\mathbb{R}^d$ over the dynamical system θ^n is a measurable map

$$\phi : \mathbb{Z} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \quad (n, \omega, x) \mapsto \phi(n, \omega, x),$$

such the map $\phi(n, \omega) := \phi(n, \omega, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ forms a cocycle over θ^n:

$$\phi(0, \omega) = Id, \qquad \text{for all } \ \omega \in \Omega,$$

$$\phi(n + m, \omega) = \phi(n, \theta^m \omega) \circ \phi(m, \omega), \qquad \text{for all } \ m, n \in \mathbb{Z}, \quad \omega \in \Omega.$$

A typical example is the solution operator for a stochastic differential equation:

$$dx_t = f_0(x_t)dt + \sum_{k=1}^{d} f_k(x_t) \circ dB_t^k$$

where $x \in \mathbb{R}^d$, $f_k, 0 \le k \le d$, are smooth vector fields, and $B_t = (B_t^1, \cdots, B_t^d)$ is the standard d-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and dB_t^k is the Stratonovich differential. Here, $(\Omega, \mathcal{F}, \mathbb{P})$ is the classic Wiener space, i.e. $\Omega = \{\omega \ : \ \omega(\cdot) \in C(\mathbb{R}, \mathbb{R}^d), \omega(0) = 0\}$ endowed with the open compact topology and $\mathbb{P}$ is the Wiener measure. Define a measurable dynamical system θ^t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by $(\theta^t\omega)(\cdot) = \omega(t + \cdot) - \omega(t)$ for $t > 0$. It is well-known that $\mathbb{P}$ is invariant and ergodic under θ^t.

When $\phi(n, \omega, \cdot)$ is differentiable for each n and ω, ϕ is called a differentiable random dynamical system. We write the time-one map $\phi(1, \omega, x)$ as $\varphi(\omega, x) := \phi(1, \omega, x)$. Then $\varphi(\omega, \cdot)$ is the so-called random diffeomorphism that generates the random dynamical system $\phi(n,\omega,x)$

$$\phi(n,\omega,\cdot)=\begin{cases}\varphi(\theta^{n-1}\omega,\cdot)\circ\cdots\circ\varphi(\omega,\cdot), & n>0\\ I, & n=0\\ \big(\varphi(\theta^{n}\omega,\cdot)\big)^{-1}\circ\cdots\circ\big(\varphi(\theta^{-1}\omega,\cdot)\big)^{-1}, & n<0.\end{cases}$$

We consider C^N, $1<N\le\infty$, random diffeomorphisms $\varphi(\omega,x)$ in $\mathbb{R}^d$ with a fixed point[1] $x=0$, i.e. $\varphi(\omega,0)=0$ for all $\omega\in\Omega$. We assume that $\varphi(\omega,x)$ is a locally tempered C^N random diffeomorphism, that is, there is a tempered ball $U(\omega)=B_{\rho(\omega)}(0)=\{x\mid |x|<\rho(\omega)\}$, where $\rho(\omega)$ is a random variable tempered from below (i.e. $\lim_{n\to\pm\infty}\frac{1}{n}\log^{-}\rho(\theta^n\omega)=0,\ \mathbb{P}-a.s.$), such that each

$$\sup_{x\in U(\omega)}\|D_x^i\varphi(\omega,x)\|=C_i(\omega),\quad 0\le i\le N$$

is tempered from above (i.e. $\lim_{n\to\pm\infty}\frac{1}{n}\log^{+}C_i(\theta^n\omega)=0,\ \mathbb{P}-a.s.$). The size of a tempered ball may decrease as ω varies, but these changes along each orbit $\theta^n\omega$ are at a subexponential rate. The upper bound of $C(\omega)$ may grow to infinity as ω varies. But along each orbit $\theta^n\omega$, it may increase only at a subexponential rate. This nonuniform behavior is one of the intrinsic features of random dynamical systems.

Two local tempered random diffeomporhisms φ and ψ with fixed point $x=0$ are C^k locally conjugate for $1\le k\le\infty$ if there exists an C^k random diffeomorphism $h(\omega,x)$ defined in a tempered ball $V(\omega)$ with $h(\omega,0)=0$ such that

$$h(\theta\omega,\varphi(\omega,x))=\psi(\omega,h(\omega,x))\quad\text{for}\quad x\in V(\omega),\quad a.\ s.\quad \omega\in\Omega$$

This conjugacy relationship implies that h carries orbits of φ to orbits of ψ, when the orbits stay in the corresponding domains.

We write $\varphi(\omega,x)$ as

$$\varphi(\omega,x)=A(\omega)x+F(\omega,x),$$

where $A(\omega)=D\varphi(\omega,0)\in Gl(d,\mathbb{R})$, $F(\omega,0)=0$ and $DF(\omega,0)=0$. We assume that the conditions of the Multiplicative Ergodic Theorem hold, that is,

$$\log^{+}||A(\cdot)||\in L^1(\Omega,\mathcal{F},\mathbb{P})\quad\text{and}\quad\log^{+}||A^{-1}(\cdot)||\in L^1(\Omega,\mathcal{F},\mathbb{P}).$$

Then, by the celebrated Oseledets' Multiplicative Ergodic Theorem (see Theorem 2.4), there exists an invariant set $\tilde{\Omega}\subset\Omega$ of full measure such that for each $\omega\in\tilde{\Omega}$, $\Phi(n,\omega)$ (the linear random dynamical system generated by $A(\omega)$) has $p(\omega)$ Lyapunov exponents:

$$\lambda_1(\omega)>\cdots>\lambda_{p(\omega)}(\omega).$$

Furthermore, $p(\omega)$, $\lambda_i(\omega)$ are θ^n invariant functions, i.e., $p(\theta\omega)=p(\omega)$ and $\lambda_i(\theta\omega)=\lambda_i(\omega)$. When θ^n is ergodic, $p(\omega)$ and $\lambda_i(\omega)$ are independent of ω.

A fixed point of φ is said to be hyperbolic if the linearized system $D\varphi$ at the fixed point has no zero Lyapunov exponents.

Our main results may be summarized as

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[1] A large class of random dynamical systems may be converted into this case. See [1], page 310

Theorem. *Assume that φ is a C^∞ locally tempered random diffeomorphism with a hyperbolic fixed point $x = 0$ and its linearization $A(\omega)$ satisfies the conditions of the Multiplicative Ergodic Theorem. Then,*

- (i) *Conjugacy in Jet Class: For any integer $k > 0$, there exists a θ-invariant $\mathcal{F}$-measurable integer function $N_0(\omega) = N_0(\alpha(\omega), \beta(\omega), k)$ given by (58) such that any C^∞ locally tempered random diffeomorphism ψ with $\varphi - \psi = O(|x|^{N_0})$ is C^k locally conjugate to φ, where $\alpha(\omega) = \max\{|\lambda_i(\omega)| \mid 1 \le i \le p(\omega)\}$ and $\beta(\omega) = \min\{|\lambda_i(\omega)| \mid 1 \le i \le p(\omega)\}$. Any C^∞ locally tempered diffeomorphism ψ with $\varphi - \psi = C^\infty$ flat is C^∞ locally conjugate to φ.*
- (ii) *Sternberg Type: For each integer $k > 0$, if the Lyapunov exponents satisfy the nonresonant conditions up to order $N_0(\omega)$,*

$$\lambda_i(\omega) \neq (\tau, \lambda(\omega)), \quad \text{for all } 1 \le i \le p(\omega), \quad 2 \le |m| \le N_0(\omega).$$

  where $(\tau, \lambda) = \sum_{j=1}^{p(\omega)} \tau_j \lambda_i$, $(\tau_1, \cdots, \tau_{p(\omega)}) \in \mathbb{N}^{p(\omega)}$, $\lambda(\omega) = (\lambda_1(\omega), \cdots, \lambda_{p(\omega)}(\omega))$, and $|m| = \sum_{j=1}^{p(\omega)} m_j$, then φ is C^k locally conjugate to its linear part. If φ is a C^∞ locally tempered random diffeomorphism and the Lyapunov exponents satisfy the nonresonant conditions for all orders, then φ is C^∞ locally conjugate to its linear part.
- (iii) *Chen Type: If two locally tempered C^∞ random diffeomorphisms with hyperbolic fixed point 0 are formally tempered conjugate, then they are C^∞ locally conjugate.*
- (iv) *If all Lyapunov exponents have the same sign, then φ is C^∞ locally conjugate to a random polynormial.*
- (v) *C^k Normal Form: If φ has a diagonal linear part (this is the case when the Lyapunov exponents are distinct), then, for any $k > 0$, φ is C^k locally conjugate to a random polynomial of degree $N_0(\omega)$ which consists of resonant monomials only. Moreover, if the fixed point 0 is stable or unstable,i.e., all Lyapunov exponents have the same sign, then φ is C^∞ locally conjugate to a polynomial which consists of resonant monomials only.*

Remark: The nonresonant conditions here are in terms of Lyapunov exponents, which reflects the limit behavior of the system. That the Lyapunov exponents satisfy a nonresonant condition does not imply that the eigenvalues of the linearized system $A(\omega)$ satisfy a nonresonant condition and vice versa. The linearized system is nonuniformly hyperbolic in the sense of Pesin for deterministic systems (see Lemma 2.6). When the dynamical system θ^n is ergodic, the Liapunov exponents are constant, the nonresonant conditions hold uniformly in ω and $N_0(\alpha(\omega), \beta(\omega), k)$ is independent of $\omega \in \Omega$. For C^k conjugacy in (i) and (ii), both φ and ψ may be required only to be C^{N_0}. When θ^n is not ergodic, qualities such as the dimensions of the stable and unstable Oseledets subspaces may vary as ω changes. The degrees of random polynomials in (iv) and (v) may change as ω varies but do not change along each orbit $\theta^n\omega$. Ω may be decomposed into a union of countably many disjoint θ-invariant measurable sets $\tilde{\Omega}_i$ such that on each $\tilde{\Omega}_i$, $N_0(\omega)$ is constant. In Section 4, we will see that the construction of conjugacy is carried out independently over each invariant set Ω_i. Patching them together gives the conjugacy over the whole Ω. To have a C^k conjugacy in (i) and (ii), we actually need only φ and ψ to be C^N locally tempered over Ω_i. Ω may also be divided into a union of nested invaraint sets on which the regularity of the congugacy may vary. To see this, we assume that the random dynamical system is C^∞ and set for each $m \ge 2$

$$\tilde{\Omega}_m = \{\omega \mid \lambda_i(\omega) \neq (\tau, \lambda(\omega)), \quad 1 \le i \le p(\omega), \quad 2 \le |\tau| \le m\}.$$

Clearly, each $\tilde{\Omega}_m$ is invariant for θ and $\{\tilde{\Omega}_m\}_{m \ge 2}$ is a non-increasing sequence. If $\tilde{\Omega}_\infty \neq \emptyset$, then h is C^∞ for $\omega \in \Omega_\infty$. The smoothness of h may vary from set to set.

The proof of this theorem involves a stable manifold and unstable manifold theorem of Pesin type for the random dynamical systems which are nonuniformly hyperbolic systems. The stable manifold theorem for nonuniformly hyperbolic diffeomorphisms was first established by Pesin in his celebrated paper [34]. The Pesin's theorem was later extended to Hilbert space by Ruelle [37] for not only deterministic dynamical systems but also random dynamical systems. The stable and unstable manifold theorems we present in this paper are very close to Ruelle's results. Most of the properties of stable and unstable manifolds we prove here can be found in Ruelle's theorems. However, for our purpose, we take a weaker assumption on the nonlinearity than Ruelle's. Furthermore, we need to establish the measurability of the stable and unstable manifold with respect to ω in the probability space and to obtain the tempered upper bounds for all derivatives of the stable and unstable manifolds, which are used in establishing the smooth conjugacy. We also prove that if the random dynamical system is C^N, where $1 < N \in \mathbb{N} \cup \{\infty\}$, then the stable and unstable manifolds are C^N submanifolds. In order to construct the random conjugacy, we also establish the existence of the stable foliation and the unstable foliation for the random dynamical system. Our results are also related to Wanner [47] and Arnold [1].

Random dynamical systems arise in the modeling of many phenomena in physics, biology, climatology, economics, etc. when uncertainties or random influences, called noises, are taken into account. These random effects are not only introduced to compensate for the defects in some deterministic models, but also are often rather intrinsic phenomena. The need for studying random dynamical systems was pointed out by Ulam and von Neumann [46] in 1945. It has flourished since 1980's due to the discovery that stochastic ordinary differential equations generate random dynamical systems through the efforts of Harris, Elworthy, Baxendale, Bismut, Ikeda, Kunita, Watanabe, and others. There are many works on this subject for both independent and identically distributed random transformations and stationary random dynamical systems and for which we refer to [1], [20], [21, 22], [38], [26], and [27] and the references therein. Here we simply limit ourselves to the conjugacy problem. For random dynamical systems, Wanner [47] proved a Hartman-Grobman theorem, also see [1] and [12] for topological conjugacy. A structural stability theorem for deterministic dynamical systems under random perturbations was obtained by P-D. Liu [25]. In [2], also see [1], Arnold and Xu gave a theorem on formal linearization of random diffeomorphisms. In [23], we established theorems of Poincaré type for random dynamical systems. A theorem of Siegel type for random dynamical systems was presented in [24].

In Section 2, we introduce basic concepts on random dynamical systems, the Multiplicative Ergodic Theorem, and some basic lemmas; In Section 3, we establish the stable manifold and unstable manifold theorem of Pesin type for random dynamical systems. A theorem on the stable foliation and unstable foliation will also be given. Finally, a theorem of Sternberg type and the related results will be given in Section 4.

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2. **Random Dynamical Systems**

In this section, we review some of the basic concepts and results on random dynamical systems, which are taken from [1]. We first introduce the basic concepts on random dynamical systems.

Then, we state the multiplicative ergodic theorem and related results for random dynamical systems.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$. $\mathbb{T}$ is endowed with its Borel σ-algebra $\mathcal{B}(\mathbb{T})$.

Definition 2.1. *A family $(\theta^t)_{t\in\mathbb{T}}$ of mappings from Ω into itself is called a metric dynamical system if*

1. $(\omega, t) \to \theta^t\omega$ *is* $\mathcal{F} \otimes \mathcal{B}(\mathbb{T})$ *measurable;*
2. $\theta^0 = id_\Omega$, *the identity on* Ω , $\theta^{t+s} = \theta^t \circ \theta^s$ *for all* $t, s \in \mathbb{T}$*;*
3. θ^t *preserves the probability measure* $\mathbb{P}$*.*

Definition 2.2. *A map*

$$\phi : \mathbb{T} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

is called a random dynamical system (or a cocycle) on the space $\mathbb{R}^d$ *over a metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, \theta^t)$ *if*

1. ϕ *is* $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$*-measurable;*
2. *The mappings* $\phi(t, \omega) := \phi(t, \omega, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ *form a cocycle over* θ^t*:*

$$\phi(0, \omega) = Id, \quad \textit{for all } \omega \in \Omega,$$

$$\phi(t + s, \omega) = \phi(t, \theta^s\omega) \circ \phi(s, \omega), \quad \textit{for all } t, s \in \mathbb{T}, \quad \omega \in \Omega.$$

In this paper, we consider time-discrete random dynamical systems. We assume that 0 is a fixed point of $\phi(n, \omega, \cdot)$, i. e., $\phi(n, \omega, 0) = 0$ for any $n \in \mathbb{Z}$ and $\omega \in \Omega$. Then, we rewrite $\phi(n, \omega, x)$ as

$$\phi(n, \omega, x) = \Phi(n, \omega)x + f(n, \omega, x),$$

where $\Phi(n, \omega) \in Gl(d, \mathbb{R})$, $f(n, \omega, 0) = 0$, and $Df(n, \omega, 0) = 0$. Here, Df is the derivative of f with respect to x.

We write the time-one map $\phi(1, \omega, x)$ as

$$\phi(\omega, x) := \phi(1, \omega, x) = A(\omega)x + F(\omega, x)$$

where $A(\omega) = \Phi(1, \omega) \in Gl(d, \mathbb{R})$, and $F(\omega, x) = f(1, \omega, x)$. Note that $F(\omega, 0) = 0$ and $DF(\omega, 0) = 0$. Let $\{x_n\}$ be an orbit of $\phi(n, \omega, x)$ with initial point x_0, i. e., $x_n = \phi(n, \omega, x_0)$. Then, $\{x_n\}$ satisfies the equation

$$x_{n+1} = A(\theta^n\omega)x_n + F(\theta^n\omega, x_n).$$

Conversely, a solution of the above equation is also an orbit of $\phi(n, \omega, x_0)$.

The next concept is of fundamental importance in the study of random dynamical systems.

Definition 2.3. *(i) A random variable* $R : \Omega \to (0, \infty)$ *is called tempered with respect to a metric dynamical system* θ^n *if*

$$\lim_{t\to\pm\infty} \frac{1}{n} \log R(\theta^n\omega) = 0 \quad \mathbb{P} - a.s.$$

(ii) $R : \Omega \to [0, \infty)$ *is called tempered from above if*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log^+ R(\theta^n\omega) = 0 \quad \mathbb{P} - a.s.$$

(iii) $R : \Omega \to (0, \infty)$ *is called tempered from below if* $1/R$ *is tempered from above.*

The following theorem is the Multiplicative Ergodic Theorem from [1], pages 134 and 153.

Theorem 2.4. *(Multiplicative Ergodic Theorem) Let* Φ *be a linear random dynamical system with two-side time over the metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, \theta^n)$. *Let*

$$\Phi(n,\omega) = \begin{cases} A(\theta^{n-1}\omega)\cdots A(\omega), & n > 0 \\ I, & n = 0 \\ A^{-1}(\theta^n\omega)\cdots A^{-1}(\theta^{-1}), & n < 0 \end{cases}$$

be generated by $A : \Omega \to Gl(d, \mathbb{R})$ *and assume*

$$\log^+ ||A(\cdot)|| \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \quad \textit{and} \quad \log^+ ||A^{-1}(\cdot)|| \in L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Then there exists an invariant subset $\tilde{\Omega} \subset \Omega$ *of full measure such that for each* $\omega \in \tilde{\Omega}$

(i) *The limit* $\lim_{n\to\infty} \left(\Phi(n,\omega)^*\Phi(n,\omega)\right)^{1/2n} =: \Psi(\omega) > 0$ *exists;*

(ii) *Let* $e^{\lambda_{p(\omega)}(\omega)} < \cdots < e^{\lambda_1(\omega)}$ *be the different eigenvalues of* $\Psi(\omega)$ *and let* $U_{p(\omega)}(\omega), \cdots, U_1(\omega)$ *be the corresponding eigenspaces with multiplicities* $d_i(\omega) := dim\, U_i(\omega)$. *Then*

$$\begin{aligned} p(\theta\omega) &= p(\omega) \\ \lambda_i(\theta\omega) &= \lambda_i(\omega) \quad \textit{for all } i \in \{1, \cdots, p(\omega)\}, \\ d_i(\theta\omega) &= d_i(\omega) \quad \textit{for all } i \in \{1, \cdots, p(\omega)\}, \\ \mathbb{R}^d &= U_1(\omega) \oplus \cdots \oplus U_{p(\omega)}. \end{aligned}$$

(iii) *If* $(\Omega, \mathcal{F}, \mathbb{P}, \theta^n)$ *is ergodic, then the functions* $p(\omega)$, $\lambda_i(\omega)$, *and* $d_i(\omega)$ *are constant on* $\tilde{\Omega}$;

(iv) *For each* $\omega \in \tilde{\Omega}$, *there exists a splitting*

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_{p(\omega)}$$

of $\mathbb{R}^d$ *into random subspaces* $E_i(\omega)$ *with dimension* $d_i(\omega)$. *Moreover, if* $\Pi_i(\omega) : \mathbb{R}^d \to E_i(\omega)$ *denotes the corresponding projection onto* $E_i(\omega)$, *then*

$$A(\omega)\Pi_i(\omega) = \Pi_i(\theta\omega)A(\omega),$$

equivalently

$$A(\omega)E_i(\omega) = E_i(\theta\omega).$$

(v) *We have*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log ||\Phi(n,\omega)x|| = \lambda_i(\omega) \Longleftrightarrow x \in E_i(\omega) \setminus \{0\}.$$

(vi) *Measurability: The functions* $\omega \to p(\omega) \in \{1, \cdots, d\}$, $\omega \to \lambda_i(\omega) \in \mathbb{R}$, $\omega \to d_i(\omega) \in \{1, \cdots, d\}$, $\omega \to E_i(\omega)$, *and* $\omega \to \Pi_i(\omega)$ *are measurable.*

Here $\lambda_i(\omega)$ are the so-called Lyapunov exponents and $E_i(\omega)$ are the so-called Oseledets spaces. For the remainder of this paper, we will denote $\tilde{\Omega}$ by Ω and assume these statements are true for all $\omega \in \Omega$.

Definition 2.5. *A fixed point* $x = 0$ *of random dynamical system* $\phi(n, \omega, x)$ *is called to be hyperbolic if* $D\phi(n, \omega, 0) = \Phi(n, \omega)$ *has no zero Lyapunov exponents.*

For the remainder of this paper, we assume

Hypothesis A:

(i) $\phi(n, \omega, x)$ *is a* C^N *random dynamical system for* $2 \le N \le \infty$ *and* $\Phi(n, \omega) = D\phi(n, \omega, 0)$ *satisfies the conditions of the Multiplicative Ergodic Theorem;*

(ii) $x = 0$ *is a hyperbolic fixed point of* ϕ.

Thus, $\Phi(n, \omega)$ has no zero Lyapunov exponents. We may divide the all Lyapunov exponents into two group based their signs. Let

$$\sigma_u(\omega) := \{\lambda_i(\omega) > 0\} \quad \text{and} \quad \sigma_s(\omega) := \{\lambda_i(\omega) < 0\}$$

and denote

$$E_u(\omega) := \bigoplus_{\lambda_i(\omega)\in\sigma_u(\omega)} E_i(\omega) \quad \text{and} \quad E_s(\omega) := \bigoplus_{\lambda_i(\omega)\in\sigma_s(\omega)} E_i(\omega).$$

Then

$$\mathbb{R}^d = E_u(\omega) \oplus E_s(\omega).$$

We call $E_u(\omega)$ the unstable Oseledets subspace and E_s the stable Oseledets subspace. Let $\Pi_u(\omega)$ and $\Pi_s(\omega)$ be the corresponding projections and let $d_u(\omega)$ and $d_s(\omega)$ denote the dimensions of $E_u(\omega)$ and $E_s(\omega)$, respectively. From the Multiplicative Ergodic Theorem 2.4, both $d_u(\omega)$ and $d_s(\omega)$ are measurable functions from Ω to $\{1, 2, \cdots, d\}$. Thus, Ω can be decomposed into a union of l disjoint θ-invariant measurable sets:

$$\Omega = \bigcup_{i=1}^{l} \Omega_i,$$

where on each Ω_i, $d_u(\omega)$ and $d_s(\omega) = d - d_u(\omega)$ are constant. In the next section, we will build the stable manifold and unstable manifolds over Ω_i. We may patch them together to get dimension-varying stable and unstable manifolds over the whole Ω. In the case of ergodic metric dynamical system, $d_u(\omega)$ and $d_s(\omega)$ are constant over Ω.

The next lemma is a consequence of the Multiplicative Ergodic Theorem 2.4 and its proof follows the same lines as Theorem 4.3.4 in [1].

Lemma 2.6. *There are random variables* $\alpha, \beta : \Omega \to (0, \infty)$ *and a tempered random variable* $K(\omega) : \Omega \to [1, \infty)$ *such that* $\alpha(\theta\omega) = \alpha(\omega)$, $\beta(\theta\omega) = \beta(\omega)$ *for all* $\omega \in \Omega$, *which are constant when* θ^n *is ergodic, and*

$$||\Phi(n, \omega)\Pi_s(\omega)|| \le K(\omega)e^{-\beta(\omega)n} \quad \text{for } n \ge 0 \tag{1}$$

$$||\Phi(n, \omega)\Pi_u(\omega)|| \le K(\omega)e^{\beta(\omega)n} \quad \text{for } n \le 0 \tag{2}$$

$$||\Phi(n, \omega)|| \le K(\omega)e^{\alpha(\omega)|n|} \quad \text{for } n \in \mathbb{Z}. \tag{3}$$

Here $\alpha(\omega)$ is chosen to be larger than the maximum of the absolute values of all Lyapunov exponents and $\beta(\omega)$ is chosen to be smaller than the absolute values of all Lyapunov exponents. For example, one may choose

$$\alpha(\omega) = 2\max\{|\lambda_i(\omega)||i = 1, \cdots, p(\omega)\}$$

and

$$\beta(\omega) = \frac{1}{2}\min\{|\lambda_i(\omega)||i = 1, \cdots, p(\omega)\}.$$

$K(\omega)$ can be chosen such that $e^{-\frac{1}{5}\beta(\omega)|n|}K(\omega) \le K(\theta^n\omega) \le e^{\frac{1}{5}\beta(\omega)|n|}K(\omega)$ which follows from the next lemma. As ω varies, $\beta(\omega)$ may be arbitrarily small and $K(\omega)$ may be arbitrarily large. However, along each orbit $\theta^n\omega$, $\beta(\omega)$ is a constant and $K(\omega)$ can increase only at a subexponential rate, which together with conditions (1) and (2) imply that the linear system $\Phi(n, \omega)$ is nonuniformly hyperbolic in the sense of Pesin.

Lemma 2.7. *Let* $f : \Omega \to (0, \infty)$ *be tempered and* $\gamma : \Omega \to (0, \infty)$ *be a* θ*-invariant random variable. Then, there is a tempered random variable* $R(\omega)$ *such that*

(i) $\frac{1}{R(\omega)} \leq f(\omega) \leq R(\omega)$
(ii) $e^{-\gamma(\omega)|n|}R(\omega) \leq R(\theta^n\omega) \leq e^{\gamma(\omega)|n|}R(\omega)$

This lemma is a consequence of Proposition 4.3.3 ([1], page 187). In fact, letting for $k \in \mathbb{N}$

$$\tilde{\Omega}_k = \Big\{\omega \in \Omega \;\Big|\; \frac{1}{k} \leq \gamma(\omega) < \frac{1}{k-1}\Big\}.$$

then, by Proposition 4.3.3, on each $\tilde{\Omega}_k$ there is a $1/k$-slowly varying random variable $R_{1/k}(\omega)$, i.e., $e^{-\frac{1}{k}|n|}R_{1/k}(\omega) \leq R_{1/k}(\theta^n\omega) \leq e^{\frac{1}{k}|n|}R_{1/k}(\omega)$, such that

$$\frac{1}{R_{1/k}(\omega)} \leq f(\omega) \leq R_{1/k}(\omega).$$

Clearly, $e^{-\gamma(\omega)|n|}R_{1/k}(\omega) \leq R_{1/k}(\theta^n\omega) \leq e^{\gamma(\omega)|n|}R_{1/k}(\omega)$ since $1/k \leq \gamma(\omega)$ and $\gamma(\theta\omega) = \gamma(\omega)$. Define a random variable $R : \Omega \to (0,\infty)$ as $R(\omega) = R_{1/k}(\omega)$ for $\omega \in \tilde{\Omega}_k$. Then, $R(\omega)$ satisfies Lemma (2.7).

The invariant splitting of $\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_{p(\omega)(\omega)}$ gives a random coordinate system with axes $E_i(\omega)$. Following Corollary 4.3.12 in [1], one may choose a new coordinate system such that the corresponding Oseledets spaces are deterministic.

Lemma 2.8. *There is a measurable map*

$$P : \Omega \to Gl(d,\mathbb{R})$$

such that

(1) *$\Phi(n,\omega)$ is conjugate to a block diagonal random dynamical system $\Psi(n,\omega)$ by $P(\omega)$:*

$$P(\theta^n\omega)\Phi(n,\omega)P^{-1}(\omega) = \Psi(n,\omega) = diag(\Psi_1(n,\omega),\cdots,\Psi_p(n,\omega)),$$

where $\Psi_i(n,\omega)$ are cocycles of size d_i.

(2) *The random transformation P preserves the Lyapunov spectrum $\{(\lambda_i(\omega), d_i(\omega)) \mid 1 \leq i \leq p(\omega)\}$ and the corresponding Oseledets spaces are given by*

$$\tilde{E}_i(\omega) = \{0\} \times \cdots \times \{0\} \times \mathbb{R}^{d_i(\omega)} \times \{0\} \times \cdots \times \{0\} \subset \mathbb{R}^d$$

(3) *Both $||P(\omega)||$ and $||P^{-1}(\omega)||$ are tempered.*

Thus, the unstable and stable Oseledets subspaces for Ψ are given by

$$\tilde{E}_u(\omega) = \bigoplus_{\lambda_i(\omega)\in\sigma_u(\omega)} \tilde{E}_i(\omega) \quad \text{and} \quad \tilde{E}_s(\omega) = \bigoplus_{\lambda_i(\omega)\in\sigma_s(\omega)} \tilde{E}_i(\omega)$$

and

$$\mathbb{R}^d = \tilde{E}_u(\omega) \oplus \tilde{E}_s(\omega).$$

is an orthogonal decomposition. We again use $\Pi_u(\omega)$ and $\Pi_s(\omega)$ denote the corresponding projections.

Lemma 2.8 together with Lemma 2.6 gives

Proposition 2.9. *There are random variables α, $\beta : \Omega \to (0,\infty)$ and a tempered random variable $K(\omega) : \Omega \to [1,\infty)$ such that*

(i) *$\alpha(\theta\omega) = \alpha(\omega)$, $\beta(\theta\omega) = \beta(\omega)$ for all $\omega \in \Omega$, which are constant when θ^n is ergodic, and $K(\theta^n\omega) \leq K(\omega)e^{\frac{1}{5}\beta(\omega)|n|}$;*

(ii) *For any* $\omega \in \Omega$,

$$||\Psi(n,\omega)\Pi_s(\omega)|| \le K(\omega)e^{-\beta(\omega)n} \quad \textit{for } n \ge 0 \tag{4}$$
$$||\Psi(n,\omega)\Pi_u(\omega)|| \le K(\omega)e^{\beta(\omega)n} \quad \textit{for } n \le 0$$

(iii) *For each* $\omega \in \Omega$,

$$||\Psi(n,\omega)|| \le K(\omega)e^{\alpha(\omega)|n|}, \quad \textit{for all } \ n \in \mathbb{Z}.$$

The random linear transformation $P(\omega)$ transforms the random dynamical system $\phi(n,\omega,x)$ to

$$\psi(n,\omega,x) = P(\theta^n\omega)\phi(n,\omega,P^{-1}(\omega)x) = \Psi(n,\omega)x + \tilde{f}(n,\omega,x),$$

where $\tilde{f}(n,\omega,x) = P(\theta^n\omega)f(n,\omega,x)P^{-1}(\omega)$.

3. Stable and Unstable Manifolds

Lemma 2.6 implies that the random dynamical systems we study are nonuniformly hyperbolic since $K(\omega)$ may grow to infinity at a subexponential rate and $\beta(\omega)$ may decrease to zero as ω varies. Thus, the size of stable and unstable manifolds may decrease as ω varies even along an orbit. However, these vary at a subexponential rate. Our proof of the existence of stable and unstable manifolds and the existence of stable and unstable foliations are based on the Lyapunov-Perron's approach.

By Proposition 2.8, it suffices to consider a random dynamical system of the block diagonal form

$$\phi(n,\omega,x) = \Phi(n,\omega)x + f(n,\omega,x),$$

where $\Phi(n,\omega)$ is a linear random dynamical system of block diagonal form which can be written as

$$\Phi = \begin{bmatrix} \Phi_u & 0 \\ 0 & \Phi_s \end{bmatrix} \tag{5}$$

and satisfies Proposition 2.9, $f(n,\omega,0) = 0$, and $Df(n,\omega,0) = 0$.

For the nonlinear term $f(1,\omega,x)$ we assume that

Hypothesis B: *There is a ball,* $U(\omega) = B(0,\rho_0(\omega)) = \{x \in \mathbb{R}^d \mid |x| < \rho_0(\omega)\}$, *where* $\rho_0 : \Omega \to (0,\infty)$ *is tempered from below, such that*

$$\sup_{x\in U(\omega)} |D^k f(1,\omega,x)| \le \tilde{B}_k(\omega), \quad \textit{for all } 0 \le k \le N, \quad \omega \in \Omega,$$

where $\tilde{B}_k(\omega)$ *are tempered from above. When* $N = \infty$, *the above holds for all* $k \ge 0$.

We call such $U(\omega)$ a tempered ball and such $\phi(n,\omega,x)$ a local tempered random dynamical system.

In order to construct stable and unstable manifolds, we use the standard cut-off function to modify the nonlinearity $f(1,\omega,x)$.

Let $\sigma(s)$ be a C^∞ function from $(-\infty,\infty)$ to $[0,1]$ with

$$\sigma(s) = 1 \ \text{ for } \ |s| \le 1, \quad \sigma(s) = 0 \ \text{ for } \ |s| \ge 2,$$

$$\sup_{s\in\mathbb{R}} |\sigma'(s)| \le 2.$$

Let $\rho : \Omega \to (0,\infty)$ be a tempered random variable such that $2\rho(\omega) < \rho_0(\omega)$. We consider a modification of $f(1,\omega,x)$. Let

$$F(\omega,x) = \sigma_{\rho(\omega)}(|x|)f(1,\omega,x), \quad \text{where} \quad \sigma_{\rho(\omega)}(|x|) = \sigma\left(\frac{|x|}{\rho(\omega)}\right)$$

with a trivial extension on the outside of $U(\omega)$. An elementary calculation gives

Lemma 3.1. *(i) $F(\omega,x) = f(1,\omega,x)$, for $|x| \le \rho(\omega)$;*

(ii) $||DF(\omega,x)|| \le 10\tilde{B}_2(\omega)\rho(\omega)$ for all $\omega \in \Omega$ and $x \in \mathbb{R}^d$;

(iii) $\sup_{x\in\mathbb{R}^d} ||D^kF(\omega,x)|| \le B_k(\omega)$ for $2 \le k \le N$ and all $\omega \in \Omega$. Here $B_k(\omega)$ are random variables tempered from above.

We choose $\rho(\omega)$ such that

$$\rho(\omega) \le \min\Big\{\frac{\beta(\omega)}{60K(\theta\omega)K(\omega)\tilde{B}_2(\omega)}, \quad \frac{1}{20K(\theta\omega)e^{\alpha(\omega)}\tilde{B}_2(\omega)}\Big\}. \tag{6}$$

The choice of the numbers here is not optimal, but is for the convenience. Clearly, $\rho(\omega)$ is tempered from below. By the choice of $\rho(\omega)$, we have

$$||DF(\omega,x)|| \le \min\Big\{\frac{\beta(\omega)}{6K(\theta\omega)K(\omega)}, \quad \frac{1}{2K(\theta\omega)e^{\alpha(\omega)}}\Big\}. \tag{7}$$

We use $B_1(\omega)$ to denote the right hand side of the above inequality. The condition

$$||DF(\omega,x)|| \le \frac{1}{2K(\theta\omega)e^{\alpha(\omega)}}$$

implies that $\psi(\omega)x := \Phi(1,\omega)x + F(\omega,x)$ is a C^N diffeomorphism on $\mathbb{R}^d$. $\psi(\omega)$ generates a C^N random dynamical system

$$\tilde{\phi}(n,\omega,x) = \begin{cases} \psi(\theta^{n-1}\omega)\circ\cdots\circ\psi(\omega), & n>0 \\ I, & n=0 \\ \big(\psi(\theta^n\omega)\big)^{-1}\circ\cdots\circ\big(\psi(\theta^{-1}\omega)\big)^{-1}, & n<0. \end{cases}$$

We note that for each sequence $x_n \in B(0,\rho(\theta^n\omega))$, $x_n = \phi(n,\omega,x_0)$ if and only if x_n satisfies $x_n = \tilde{\phi}(n,\omega,x_0)$. Equivalently,

$$x_{n+1} = \Phi(1,\theta^n\omega)x_n + F(\theta^n\omega,x_n). \tag{8}$$

Thus, for the remainder of this section, we consider equation (8) and use $\phi(n,\omega,x)$ instead of $\tilde{\phi}(n,\omega,x)$ to denote the modified random dynamical system.

We first recall that a multifunction $M = \{M(\omega)\}_{\omega\in\Omega}$ of nonempty closed sets $M(\omega)$, $\omega\in\Omega$, contained in $\mathbb{R}^d$ is called a *random set* if

$$\omega \to \inf_{y\in M(\omega)} ||x-y||$$

is a random variable for any $x\in\mathbb{R}^d$.

Definition 3.2. *A random set $M(\omega)$ is called an invariant set for a random dynamical system $\phi(t,\omega,x)$ if we have*

$$\phi(t,\omega,M(\omega)) \subset M(\theta^t\omega) \quad \textit{for} \quad t\ge 0.$$

We define for the modified random dynamical system ϕ the stable set

$$W^s(\omega) = \{x_0 \in \mathbb{R}^d \mid x_n = \phi(n, \omega, x_0) \to 0, \text{ as } n \to +\infty\}$$

and the unstable set

$$W^u(\omega) = \{x_0 \in \mathbb{R}^d \mid x_n = \phi(n, \omega, x_0) \to 0, \text{ as } n \to -\infty\}.$$

Clearly, both $W^s(\omega)$ and $W^u(\omega)$ are invariant for the random dynamical system $\phi(n, \omega, x)$.

Let $\gamma : \Omega \to (0, \infty)$ be a random variable with $\gamma(\theta\omega) = \gamma(\omega)$. We define the following Banach spaces

$$C^-_{\gamma(\omega)} = \big\{\mathbf{x} = \{x_n\}_{n\le 0} \mid x_n \in \mathbb{R}^d, \ \sup_{n\le 0} |x_n| e^{-\gamma(\omega)n} < \infty\big\}$$

with the norm

$$|\mathbf{x}|_{C^-_{\gamma(\omega)}} = \sup_{n\le 0} |x_n| e^{-\gamma(\omega)n}$$

and

$$C^+_{\gamma(\omega)} = \big\{\mathbf{x} = \{x_n\}_{n\ge 0} \mid x_n \in \mathbb{R}^d, \ \sup_{n\ge 0} |x_n| e^{\gamma(\omega)n} < \infty\big\}$$

with the norm

$$|\mathbf{x}|_{C^-_{\gamma(\omega)}} = \sup_{n\ge 0} |x_n| e^{\gamma(\omega)n}.$$

As we mentioned in Section 2, Ω may be decomposed into a union of l disjoint θ-invariant measurable sets:

$$\Omega = \bigcup_i^l \Omega_i,$$

where on each Ω_i, $d_u(\omega)$ and $d_s(\omega) = d - d_u(\omega)$ are constant. Thus, on Ω_i $\mathbb{R}^d$ has an invariant splitting $\mathbb{R}^d = E_u \oplus E_s$ with the unstable and stable Oseledets spaces of independent of ω. For each $x \in \mathbb{R}^d$, we may write it as

$$x = x^u + x^s, \quad \text{for some} \quad x^u \in E_u, \ x^s \in E_s.$$

Our construction of the stable and unstable manifolds and the smooth conjugacy will be carried out over Ω_i. Patching them together gives the conjugacy over Ω. When θ^n is ergodic, $d_u(\omega)$ and $d_s(\omega) = d - d_u(\omega)$ are constant over Ω. For simplicity, we will use Ω to denote Ω_i.

Theorem 3.3. *(Unstable Manifold Theorem) Assume that Hypotheses A and B hold. Then, the unstable set $W^u(\omega)$ is a C^N manifold given by*

$$W^u(\omega) = \big\{\eta + h^u(\eta, \omega) \mid \eta \in E_u\big\}$$

where

$$h^u : E_u \times \Omega \to E_s$$

satisfies

(i) *$h^u(\eta, \omega)$ is measurable in (η, ω) and is C^N in η with*

$$\mathit{Lip}\ h^u(\cdot, \omega) < 1, \quad h^u(0, \omega) = 0, \quad Dh^u(0, \omega) = 0.$$

(ii) *$|D^i h^u(\eta, \omega)| \le K_i(\omega)$ for each $0 \le i \le N$ when $N < \infty$ and for all $i \ge 0$ when $N = \infty$, where $K_i(\omega)$ are random variables tempered from above.*

(iii) *$W^u_{loc}(\omega) = \{x \in W^u(\omega) \mid x \in B(0, \rho(\omega))\}$ is a local unstable manifold for the original system.*

Moreover, for each $x_0 \in W^u(\omega)$*, the orbit* $\mathbf{x} = \{x_n\}_{n\le 0}$ *of* ϕ *with the initial point* x_0 *satisfies*

$$\mathbf{x} \in C^-_{\beta/5},$$

which means that $x_n \to 0$ *exponentially as* $n \to -\infty$*.*

The proof of this theorem is based on the following lemma and proposition.

Let $\{x_n\}_{n\le 0}$ be an orbit of $\phi(n,\omega,x)$. Then for each $k < n$, using equation (8) and induction, $\{x_n\}_{n\le 0}$ satisfies the discrete variation of constant formula

$$x_n = \Phi(n-k,\theta^k\omega)x_k + \sum_{i=k}^{n-1} \Phi(n-1-i,\theta^{i+1}\omega)F(\theta^i\omega, x_i). \tag{9}$$

Lemma 3.4. *Let* $\gamma(\omega)$ *be a random variable such that* $0 \le \gamma(\omega) \le \beta(\omega)/5$ *and* $\gamma(\theta\omega) = \gamma(\omega)$*. Then* $\mathbf{x} = \{x_n\}_{n\le 0} \in C^-_{\gamma(\omega)}$ *is an orbit of* $\phi(n,\omega,x)$ *if and only if* $\{x_n\}_{n\le 0} \in C^-_{\gamma(\omega)}$ *and satisfies the following equations*

$$\begin{aligned} x_n &= \Phi_u(n,\omega)x_0^u - \sum_{i=n}^{-1} \Phi_u(n-1-i,\theta^{i+1}\omega)F_u(\theta^i\omega,x_i) \\ &\quad + \sum_{i=-\infty}^{n-1} \Phi_s(n-1-i,\theta^{i+1}\omega)F_s(\theta^i\omega,x_i), \quad \textit{for } n \le -1; \\ x_0 &= x_0^u + \sum_{i=-\infty}^{-1} \Phi_s(-1-i,\theta^{i+1}\omega)F_s(\theta^i\omega,x_i), \end{aligned} \tag{10}$$

where $\Phi_u = \Phi|_{E_u}$*,* $\Phi_s = \Phi|_{E_s}$*,* $F_u = \Pi_u F$*, and* $F_s = \Pi_s F$*.*

Proof. Consider $\mathbf{x} = \{x_n\}_{n\le 0} \in C^-_{\gamma(\omega)}$. Suppose that $\{x_n\}_{n\le 0}$ is an orbit of $\phi(n,\omega,x)$. Then, by equation (9), we have

$$x_n = \Phi(n-k,\theta^k\omega)x_k + \sum_{i=k}^{n-1} \Phi(n-1-i,\theta^{i+1}\omega)F(\theta^i\omega, x_i).$$

Letting $n = 0$ and using the fact $\Phi(-k,\theta^k\omega) = \Phi^{-1}(k,\omega)$, we have for $k \le -1$.

$$x_k = \Phi(k,\omega)x_0 - \sum_{i=k}^{-1} \Phi(k-1-i,\theta^{i+1}\omega)F(\theta^i\omega, x_i).$$

Switching index k back to n gives

$$x_n = \Phi(n,\omega)x_0 - \sum_{i=n}^{-1} \Phi(n-1-i,\theta^{i+1}\omega)F(\theta^i\omega, x_i).$$

Its unstable component satisfies

$$x_n^u = \Phi(n,\omega)x_0^u - \sum_{i=n}^{-1} \Phi_u(n-1-i,\theta^{i+1}\omega)F_u(\theta^i\omega, x_i). \tag{11}$$

Applying the stable projection Π_s to equation (9) gives

$$x_n^s = \Phi_s(n-k,\theta^k\omega)x_k^s + \sum_{i=k}^{n-1} \Phi_s(n-1-i,\theta^{i+1}\omega)F_s(\theta^i\omega, x_i). \tag{12}$$

Since $\{x_k\}_{k\le 0} \in C^-_{\gamma(\omega)}$, using Proposition 2.9, we have

$$|\Phi_s(n-k, \theta^k\omega)x_k^s| \le K(\theta^k\omega)e^{-\beta(\omega)(n-k)}e^{\gamma(\omega)k}|\mathbf{x}|_{C^-_\gamma} \to 0, \quad \text{as } k \to -\infty,$$

where the fact that $K(\omega)$ is tempered is used. Thus, letting $k \to -\infty$ in equation (12), we obtain

$$x_n^s = \sum_{i=-\infty}^{n-1} \Phi_s(n-1-i, \theta^{i+1}\omega)F_s(\theta^i\omega, x_i). \tag{13}$$

Adding equation (11) to equation (13), we get equation (10). The converse follows from a straightforward computation. This completes the proof. $\square$

Proposition 3.5. *Assume that Hypotheses A and B hold. Then for each random variable* $\gamma(\omega) \in [0, \beta(\omega)/5]$ *with* $\gamma(\theta\omega) = \gamma(\omega)$, *equation (10) has a unique solution* $\mathbf{x}(\eta,\omega) = \{x_n(\eta,\omega)\} \in C^-_{\gamma(\omega)}$ *with* $x_0^u = \eta$ *for each* $\eta \in E_u$. *Furthermore,*

(i) *The solution* $\mathbf{x}(\eta,\omega)$ *is independent of* $\gamma(\omega) \in [0, \beta(\omega)/5]$*;*
(ii) $\mathbf{x}(\eta,\omega)$ *is* $\mathcal{B}(E_u)\otimes\mathcal{F}_i$ *measurable;*
(iii) *For each* $\omega \in \Omega$, $\mathbf{x}(\cdot,\omega)$ *is* C^N *from* E_u *to* $C^-_{\beta(\omega)/(5+2N)}$, $\mathbf{x}(0,\omega) = 0$, *and* $D\mathbf{x}(0,\omega) = 0$*;*
(iv) *There are random variables tempered from above* $K_i(\omega)$, $1 \le i \le N$ *when* $N < \infty$ *and* $i \ge 1$ *when* $N = \infty$, *such that*
$$||D^i\mathbf{x}(\eta,\omega)|| \le K_i(\omega).$$

Proof. We prove this proposition in three steps.

Step 1. *The Existence of a Lipschitz Continuous Solution.*

Let $\gamma(\omega)$ be a random variable such that $0 \le \gamma(\omega) \le \beta(\omega)/5$ and $\gamma(\theta\omega) = \gamma(\omega)$. Let $\eta \in E_u$. For each $\mathbf{x} = \{x_n\}_{n\le 0} \in C^-_{\gamma(\omega)}$ with $x_0^u = \eta$, we define a map $\mathbf{y} = \{y_n\}_{n\le 0} = \mathcal{J}^u(\mathbf{x},\eta,\omega)$ by

$$\begin{aligned} y_n &= \Phi_u(n,\omega)\eta - \sum_{i=n}^{-1} \Phi_u(n-1-i, \theta^{i+1}\omega)F_u(\theta^i\omega, x_i) \\ &\quad + \sum_{i=-\infty}^{n-1} \Phi_s(n-1-i, \theta^{i+1}\omega)F_s(\theta^i\omega, x_i), \quad \text{for } n \le -1; \\ y_0 &= \eta + \sum_{i=-\infty}^{-1} \Phi_s(-1-i, \theta^{i+1}\omega)F_s(\theta^i\omega, x_i). \end{aligned}$$

We first show that $\mathcal{J}^u(\cdot,\eta,\omega)$ maps $C^-_{\gamma(\omega)}$ into $C^-_{\gamma(\omega)}$. For each $\mathbf{x} \in C^-_{\gamma(\omega)}$, using (7) and Proposition 2.9, we have for $n \le -1$

$$e^{-\gamma(\omega)n}|y_n| \le K(\omega)e^{(\beta(\omega)-\gamma(\omega))n}|\eta| + \frac{1}{6}\Big(\frac{\beta(\omega)}{\beta(\omega)-\gamma(\omega)} + \frac{\beta(\omega)}{\beta(\omega)+\gamma(\omega)}\Big)|\mathbf{x}|_{C^-_{\gamma(\omega)}}$$

and

$$|y_0| \le |\eta| + \frac{1}{6}\frac{\beta(\omega)}{\beta(\omega)+\gamma(\omega)}|\mathbf{x}|_{C^-_{\gamma(\omega)}}$$

which implies that

$$|\mathcal{J}^u(\mathbf{x},\eta,\omega)|_{C^-_{\gamma(\omega)}} \le K(\omega)|\eta| + \frac{1}{6}\Big(\frac{\beta(\omega)}{\beta(\omega)-\gamma(\omega)} + \frac{\beta(\omega)}{\beta(\omega)+\gamma(\omega)}\Big)|\mathbf{x}|_{C^-_{\gamma(\omega)}}.$$

Next, we show that $\mathcal{J}^u(\mathbf{x},\eta,\omega)$ is a uniform contraction in $\mathbf{x}$ with respect to η. For each $\mathbf{x}$, $\bar{\mathbf{x}} \in C^-_{\gamma(\omega)}$, we have

$$\begin{aligned}
&|\mathcal{J}^u(\mathbf{x},\eta,\omega)-\mathcal{J}(\tilde{\mathbf{x}},\eta,\omega)|_{C^-_{\gamma(\omega)}}\\
&\le \frac{1}{6}\big(\frac{\beta(\omega)}{\beta(\omega)-\gamma(\omega)}+\frac{\beta(\omega)}{\beta(\omega)+\gamma(\omega)}\big)|\mathbf{x}-\tilde{\mathbf{x}}|_{C^-_{\gamma(\omega)}}\\
&\le \frac{9}{24}|\mathbf{x}-\tilde{\mathbf{x}}|_{C^-_{\gamma(\omega)}}
\end{aligned} \tag{14}$$

Hence, $\mathcal{J}^u(\mathbf{x},\eta,\omega)$ is a contraction in $\mathbf{x}$ uniformly in (η,ω). By the contraction principle, for each $\eta \in E_u$, $\mathcal{J}^u(\cdot,\eta,\omega)$ has a unique fixed point $\mathbf{x}(\eta,\omega) \in C^-_{\gamma(\omega)}$, which satisfies equation (10). Clearly, $\mathbf{x}(0,\omega)=0$. Since $C^-_{\gamma_2(\omega)} \subset C^-_{\gamma_1(\omega)}$ for any $0 \le \gamma_1(\omega) \le \gamma_2(\omega)$, a fixed point in $C^-_{\gamma_2(\omega)}$ must be in $C^-_{\gamma_1(\omega)}$. By the uniqueness, $\mathbf{x}(\eta,\omega)$ is independent of $\gamma(\omega) \in [0,\beta(\gamma)/5]$. Furthermore, we have for $\eta,\eta_0 \in E^u$ that

$$|\mathbf{x}(\eta,\omega)-\mathbf{x}(\eta_0,\omega)|_{C^-_{\beta(\omega)/5}} \le \frac{24}{15}K(\omega)|\eta-\eta_0|. \tag{15}$$

Since $\mathbf{x}(\eta,\omega)$ is the ω-wise limit of iteration of contraction mapping $\mathcal{J}^u$ starting at 0 and J^u maps a $\mathcal{F}$-measurable function to a measurable function, $\mathbf{x}(\eta,\cdot)$ is $\mathcal{F}$-measurable. On the other hand, since $\mathbf{x}(\cdot,\omega)$ is Lipschitz continuous, by Castaing and Valadier [9], Lemma III.14, $\mathbf{x}(\eta,\omega)$ is measurable with respect to (η,ω).

Step 2. $\mathbf{x}(\cdot,\omega)$ *is* C^1.

In order to show that $\mathbf{x}(\eta,\omega)$ is C^N, we show by induction that $\mathbf{x}(\cdot,\omega)$ is C^i for any $1 \le i \le N$ from E_u to $C^-_{\beta(\omega)/(5+2i)}$. In this step, we show that $\mathbf{x}(\cdot,\omega)$ is C^1 from E_u to $C^-_{\beta(\omega)/7}$. We first show that $\mathbf{x}(\cdot,\omega)$ is differentiable from E_u to $C^-_{\beta(\omega)/6}$. Then, we prove that $D\mathbf{x}(\cdot,\omega): E_u \to L(E_u, C^-_{\beta(\omega)/7})$ is continuous, where $L(E_u, C^-_{\beta(\omega)/7})$ is the space of all bounded linear operators from E_u to $C^-_{\beta(\omega)/7}$.

For each $\eta \in E_u$, we define a linear operator $\mathcal{G}: C^-_{\beta(\omega)/6} \to C^-_{\beta(\omega)/6}$ by

$$\begin{aligned}
\big(\mathcal{G}\mathbf{v}\big)_n &= -\sum_{i=n}^{-1}\Phi_u(n-1-i,\theta^{i+1}\omega)DF_u(\theta^i\omega,x_i(\eta_0,\omega))v_i\\
&\quad+\sum_{i=-\infty}^{n-1}\Phi_s(n-1-i,\theta^{i+1}\omega)DF_s(\theta^i\omega,x_i(\eta_0,\omega))v_i, \quad \text{for } n \le -1;\\
\big(\mathcal{G}\mathbf{v}\big)_0 &= \sum_{i=-\infty}^{-1}\Phi_s(-1-i,\theta^{i+1}\omega)DF_s(\theta^i\omega,x_i(\eta_0,\omega))v_i,
\end{aligned} \tag{16}$$

where $\mathbf{v}=\{v_n\}_{n\le 0} \in C^-_{\beta(\omega)/6}$ and define a bounded linear operator $\mathcal{S}$ from E_u to $C^-_{\beta(\omega)/6}$ by $\mathcal{S}(\eta)=\{\mathcal{S}_n(\eta)\}_{n\ge 0}$, where

$$\mathcal{S}_n(\eta)=\Phi(n,\omega)\eta.$$

Then, $\mathcal{G}$ is a bounded linear operator from $C^-_{\beta(\omega)/6}$ into itself and its norm is bounded

$$||\mathcal{G}|| \le \frac{12}{35},$$

which implies that $(Id-\mathcal{G})$ has a bounded inverse in $C^-_{\beta(\omega)/6}$. For $\eta,\eta_0\in E_u$, we set

$$\begin{aligned}
I_n = &-\sum_{i=n}^{-1}\Phi_u(n-1-i,\theta^{i+1}\omega)\Big(F_u(\theta^i\omega,x_i(\eta,\omega))-F_u(\theta^i\omega,x_i(\eta_0,\omega))\\
&\qquad\qquad - DF_u(\theta^i\omega,x_i)\big(x_i(\eta,\omega)-x_i(\eta_0,\omega)\big)\Big)\\
&+\sum_{i=-\infty}^{n-1}\Phi_s(n-1-i,\theta^{i+1}\omega)\Big(F_u(\theta^i\omega,x_i(\eta,\omega))-F_u(\theta^i\omega,x_i(\eta_0,\omega))\\
&\qquad - DF_u(\theta^i\omega,x_i(\eta_0,\omega))\big(x_i(\eta,\omega)-x_i(\eta_0,\omega)\big)\Big),\quad \text{for } n\le -1\\
I_0 = &\sum_{i=-\infty}^{-1}\Phi_s(-1-i,\theta^{i+1}\omega)\Big(F_u(\theta^i\omega,x_i(\eta,\omega))-F_u(\theta^i\omega,x_i(\eta_0,\omega))\\
&\qquad\qquad - DF_u(\theta^i\omega,x_i(\eta_0,\omega))\big(x_i(\eta,\omega)-x_i(\eta_0,\omega)\big)\Big).
\end{aligned}$$

We claim that $|I|_{C^-_{\beta(\omega)/6}}=o(|\eta-\eta_0|)$ as $\eta\to\eta_0$. Using this claim, we obtain

$$\begin{aligned}
&\mathbf{x}(\eta,\omega)-\mathbf{x}(\eta_0,\omega)-\mathcal{G}\big(\mathbf{x}(\eta,\omega)-\mathbf{x}(\eta_0,\omega)\big)\\
&=\mathcal{S}(\eta-\eta_0)+I\\
&=\mathcal{S}(\eta-\eta_0)+o(|\eta-\eta_0|),\qquad \text{as } \eta\to\eta_0,
\end{aligned}\tag{17}$$

which yields

$$\mathbf{x}(\eta,\omega)-\mathbf{x}(\eta_0,\omega)=(Id-\mathcal{G})^{-1}\mathcal{S}(\eta-\eta_0)+o(|\eta-\eta_0|).$$

Hence, $\mathbf{x}(\eta,\omega)$ is differentiable in η and its derivative satisfies $D\mathbf{x}(\eta,\omega)\in L(E_u,C^-_{\beta(\omega)/6})$ and

$$\begin{aligned}
&Dx_n(\eta,\omega)\\
&=\Phi_u(n,\omega)-\sum_{i=n}^{-1}\Phi_u(n-1-i,\theta^{i+1}\omega)DF_u(\theta^i\omega,x_i(\eta,\omega))Dx_i(\eta,\omega)\\
&\quad+\sum_{i=-\infty}^{n-1}\Phi_s(n-1-i,\theta^{i+1}\omega)DF_s(\theta^i\omega,x_i(\eta,\omega))Dx_i(\eta,\omega),\quad \text{for}\ \ n\le -1;\\
&Dx_0(\eta,\omega)\\
&=Id+\sum_{i=-\infty}^{-1}\Phi_s(-1-i,\theta^{i+1}\omega)DF_s(\theta^i\omega,x_i(\eta,\omega))Dx_i(\eta,\omega).
\end{aligned}\tag{18}$$

Furthermore,

$$||D\mathbf{x}(\eta,\omega)||_{L(E_u,C^-_{\beta(\omega)/6})}\le\frac{35}{23}K(\omega)$$

which yields that $||D\mathbf{x}(\eta,\omega)||_{L(E_u,C^-_{\beta(\omega)/6})}$ is tempered from above. The fact $||\mathcal{G}||\le\frac{12}{35}$ implies that $D\mathbf{x}(\eta,\omega)$ is the unique solution of (18). Thus, $D\mathbf{x}(0,\omega)=0$ since $DF(\omega,0)=0$ and $\mathbf{x}(0,\omega)=0$.

We now prove that $|I|_{C^-_{\beta(\omega)/6}} = o(|\eta-\eta_0|)$ as $\eta \to \eta_0$. In order to estimate I, we introduce the following summations. Let N^* be a negative integer to be chosen later and let

$$I_1^u = e^{-\frac{\beta}{6}n}\Big\{ -\sum_{i=n}^{N^*} \Phi_u(n-1-i, \theta^{i+1}\omega)\Big(F_u(\theta^i\omega, x_i(\eta,\omega)) - F_u(\theta^i\omega, x_i(\eta_0,\omega)) - DF_u(\theta^i\omega, x_i(\eta_0,\omega))\big(x_i(\eta,\omega) - x_i(\eta_0,\omega)\big)\Big)\Big\}$$

for $n \le N^*$ and $I_1^u = 0$ for $N^* < n \le 0$;

$$I_2^u = e^{-\frac{\beta}{6}n}\Big\{ -\sum_{i=N^*+1}^{-1} \Phi_u(n-1-i, \theta^{i+1}\omega)\Big(F_u(\theta^i\omega, x_i(\eta,\omega)) - F_u(\theta^i\omega, x_i(\eta_0,\omega)) - DF_u(\theta^i\omega, x_i(\eta_0,\omega))\big(x_i(\eta,\omega) - x_i(\eta_0,\omega)\big)\Big)\Big\}.$$

Let $\tilde N$ be another negative number to be chosen later. For $\tilde N < n \le 0$, we set

$$I_1^s = e^{-\frac{\beta}{6}n}\Big\{ \sum_{i=\tilde N+1}^{n-1} \Phi_s(n-1-i, \theta^{i+1}\omega)\Big(F_s(\theta^i\omega, x_i(\eta,\omega)) - F_s(\theta^i\omega, x_i(\eta_0,\omega)) - DF_s(\theta^i\omega, x_i(\eta_0,\omega))\big(x_i(\eta,\omega) - x_i(\eta_0,\omega)\big)\Big)\Big\};$$

$$I_2^s = e^{-\frac{\beta}{6}n}\Big\{ \sum_{i=-\infty}^{\tilde N} \Phi_s(n-1-i, \theta^{i+1}\omega)\Big(F_s(\theta^i\omega, x_i(\eta,\omega)) - F_s(\theta^i\omega, x_i(\eta_0,\omega)) - DF_s(\theta^i\omega, x_i(\eta_0,\omega))\big(x_i(\eta,\omega) - x_i(\eta_0,\omega)\big)\Big)\Big\}.$$

For $n \le \tilde N$, we set

$$I_3^s = e^{-\frac{\beta}{6}n}\Big\{ \sum_{i=-\infty}^{n-1} \Phi_s(n-1-i, \theta^{i+1}\omega)\Big(F_s(\theta^i\omega, x_i(\eta,\omega)) - F_s(\theta^i\omega, x_i(\eta_0,\omega)) - DF_s(\theta^i\omega, x_i(\eta_0,\omega))\big(x_i(\eta,\omega) - x_i(\eta_0,\omega)\big)\Big)\Big\}.$$

It is sufficient to show that for any $\epsilon > 0$, there is a $\delta > 0$ such that if $|\eta-\eta_0| \le \delta$, then $|I|_{C^-_{\beta(\omega)/6}} \le \epsilon|\eta-\eta_0|$. Note that

$$|I|_{C^-_{\beta(\omega)/6}} \le \sup_{n\le 0}|I_1^u| + \sup_{n\le 0}|I_2^u| + \sup_{\tilde N<n\le 0}|I_1^s| + \sup_{\tilde N<n\le 0}|I_2^s| + \sup_{n\le \tilde N}|I_3^s|.$$

For I_1^u, we have

$$\sup_{n\le 0}|I_1^u| \le \frac{5}{12}e^{\frac{\beta}{30}N^*}|\mathbf{x}(\eta,\omega)-\mathbf{x}(\eta_0,\omega)|_{C^-_{\beta/5}} \le \frac{2}{3}K(\omega)e^{\frac{\beta}{30}N^*}|\eta-\eta_0|.$$

Choose a negative integer N^* large enough that

$$\frac{2}{3}K(\omega)e^{\frac{\beta}{30}N^*} \le \frac{1}{4}\epsilon.$$

Then, for such N^* we have that

$$\sup_{n\le 0} I_1^u \le \frac{1}{4}\epsilon.$$

Fix such N^*, for I_2^u we have that for each $n \leq 0$

$$
\begin{aligned}
|I_2^u| \leq \sum_{i=N^*+1}^{-1} & K(\theta^{i+1}\omega)\big(e^{(\alpha(\omega)+\frac{\beta(\omega)}{6})(|N^*|+1+|i|)} + e^{-i\frac{5\beta}{6}}\big) \\
& \Big\{ | \int_0^1 \Big(DF_u(\theta^i\omega, \tau x_i(\eta,\omega) + (1-\tau)x_i(\eta_0,\omega)) \\
& - DF_u(\theta^i\omega, x_i(\eta_0,\omega))\Big) d\,\tau \Big\} |x_i(\eta,\omega) - x_i(\eta_0,\omega)|_{C^-_{\beta/6}}.
\end{aligned}
$$

Since the above summation contains only $-N^* - 1$ terms and DF is continuous in x, using (15), there is $\delta_1 > 0$ such that if $|\eta - \eta_1| \leq \delta_1$, then

$$
|I_2^u| \leq \frac{1}{4}\epsilon|\eta - \eta_1|, \quad \text{for all } \; n \leq 0.
$$

Therefore, if $|\eta - \eta_1| \leq \delta_1$, then

$$
\sup_{n \leq 0} I_1^u + \sup_{n \leq 0} I_2^u \leq \frac{1}{2}\epsilon|\eta - \eta_1|.
$$

Similarly, by choosing $\tilde{N} < 0$ to be sufficiently large, we have

$$
\sup_{\tilde{N} < n \leq 0} I_2^s + \sup_{n \leq \tilde{N}} I_3^s \leq \frac{1}{4}\epsilon|\eta - \eta_1|.
$$

For the fixed $\tilde{N}$, as we did for I_2^u, there exists $\delta_2 > 0$ such that if $|\eta - \eta_1| \leq \delta_2$, then

$$
\sup_{\tilde{N} < n \leq 0} I_1^s \leq \frac{1}{4}\epsilon|\eta - \eta_1|.
$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, we have that if $|\eta - \eta_1| \leq \delta$, then

$$
|I|_{C^-_{\beta(\omega)/6}} \leq \epsilon|\eta - \eta_0|.
$$

Therefore, $|I|_{C^-_{\beta(\omega)/6}} = o(|\eta-\eta_0|)$ as $\eta \to \eta_0$. We now prove that $D\mathbf{x}(\cdot,\omega)$ is continuous from E_u to $L(E_u, C^-_{\beta/7})$. For $\eta, \eta_0 \in E_u$, using (18), we have

$$\begin{aligned}
&Dx_n(\eta,\omega) - Dx_n(\eta_0,\omega)\\
&= -\sum_{i=n}^{-1} \Phi_u(n-1-i, \theta^{i+1}\omega)\Big(DF_u(\theta^i\omega, x_i(\eta,\omega))Dx_i(\eta,\omega)\\
&\qquad\qquad - DF_u(\theta^i\omega, x_i(\eta_0,\omega))Dx_i(\eta_0,\omega)\Big)\\
&\quad + \sum_{i=-\infty}^{n-1} \Phi_s(n-1-i, \theta^{i+1}\omega)\Big(DF_u(\theta^i\omega, x_i(\eta,\omega))Dx_i(\eta,\omega)\\
&\qquad\qquad - DF_u(\theta^i\omega, x_i(\eta_0,\omega))Dx_i(\eta_0,\omega)\Big), \quad \text{for } n \le -1;\\
&Dx_0(\eta,\omega) - Dx_0(\eta,\omega)\\
&= \sum_{i=-\infty}^{-1} \Phi_s(-1-i, \theta^{i+1}\omega)\Big(DF_u(\theta^i\omega, x_i(\eta,\omega))Dx_i(\eta,\omega)\\
&\qquad\qquad - DF_u(\theta^i\omega, x_i(\eta_0,\omega))Dx_i(\eta_0,\omega)\Big).
\end{aligned}$$

Let

$$\begin{aligned}
&\bar I_n =\\
&-\sum_{i=n}^{-1} \Phi_u(n-1-i, \theta^{i+1}\omega)\Big(DF_u(\theta^i\omega, x_i(\eta,\omega)) - DF_u(\theta^i\omega, x_i(\eta_0,\omega)\Big)Dx_i(\eta_0,\omega)\\
&+\sum_{i=-\infty}^{n-1} \Phi_s(n-1-i, \theta^{i+1}\omega)\Big(DF_u(\theta^i\omega, x_i(\eta,\omega)) - DF_u(\theta^i\omega, x_i(\eta_0,\omega)\Big)Dx_i(\eta_0,\omega)\\
&\bar I_0 =\\
&\sum_{i=-\infty}^{-1} \Phi_s(-1-i, \theta^{i+1}\omega)\Big(DF_u(\theta^i\omega, x_i(\eta,\omega)) - DF_u(\theta^i\omega, x_i(\eta_0,\omega)\Big)Dx_i(\eta_0,\omega).
\end{aligned}$$

Using the same argument we used for proving $|I|_{C^-_{\beta(\omega)/6}} = o(|\eta-\eta_0|)$ as $\eta \to \eta_0$, we have that $|\bar I|_{L(E_u, C^-_{\beta/7})} = o(1)$ as $\eta \to \eta_0$. Hence,

$$|D\mathbf{x}(\eta,\omega) - D\mathbf{x}(\eta_0,\omega)|_{L(E_u, C^-_{\beta/7})} \le \frac{144}{95}|\bar I|_{L(E_u, C^-_{\beta/7})} \to 0 \quad \text{as} \quad \eta \to \eta_0$$

Therefore, $D\mathbf{x}(\cdot,\omega)$ is continuous from E_u to $C^-_{\beta/7}$.

Step 3. *$\mathbf{x}(\eta,\omega)$ is C^N.*

Letting $2 \le m \le N$, by the induction hypothesis, we have that $\mathbf{x}(\cdot,\omega)$ is C^j from E_u to $C^-_{\beta/(5+2j)}$ for all $1 \le j \le m-1$, and there are random variables $K_j(\omega)$ tempered from above such that

$$||D^j\mathbf{x}(\eta,\omega)||_{L^j(E_u, C^-_{\beta/(5+2j)})} \le K_j(\omega), \tag{19}$$

$$||D^j\mathbf{x}(\eta,\omega) - D^j\mathbf{x}(\eta_0,\omega)||_{L^j(E_u, C^-_{\beta/(5+2j)})} \leq K_j(\omega)|\eta-\eta_0|. \tag{20}$$

Here $L^j(E_u, C^-_{\beta/(5+2j)})$ is the usual space of bounded j linear forms. In the case there is no confusion, we simply use $||\cdot||$ denote the operator norm. We want to show that $\mathbf{x}(\cdot,\omega)$ is C^m from E_u to $C^-_{\beta/(5+2m)}$ and $D^m\mathbf{x}(\cdot,\omega)$ is Lipschitz continuous from E_u to $L^m(E_u, C^-_{\beta/(5+2m)})$ if $m < N$. Note that $D^{m-1}\mathbf{x}(\cdot,\omega)$ satisfies the following equation

$$\begin{aligned}
&D^{m-1}x_n(\eta,\omega)\\
&= -\sum_{i=n}^{-1}\Phi_u(n-1-i,\theta^{i+1}\omega)DF_u(\theta^i\omega, x_i(\eta,\omega))D^{m-1}x_i(\eta,\omega)\\
&\quad + \sum_{i=-\infty}^{n-1}\Phi_s(n-1-i,\theta^{i+1}\omega)DF_s(\theta^i\omega, x_i(\eta,\omega))D^{m-1}x_i(\eta,\omega)\\
&- \sum_{i=n}^{-1}\Phi_u(n-1-i,\theta^{i+1}\omega)R^u_{m-1,i}(\eta,\omega)\\
&\quad + \sum_{i=-\infty}^{n-1}\Phi_s(n-1-i,\theta^{i+1}\omega)R^s_{m-1,i}(\eta,\omega), \quad \text{for } \ n\leq -1;\\
&D^{m-1}x_0(\eta,\omega)\\
&= \sum_{i=-\infty}^{-1}\Phi_s(-1-i,\theta^{i+1}\omega)DF_s(\theta^i\omega, x_i(\eta,\omega))D^{m-1}x_i(\eta,\omega)\\
&\quad + \sum_{i=-\infty}^{-1}\Phi_s(-1-i,\theta^{i+1}\omega)R^s_{m-1,i}(\eta,\omega)
\end{aligned} \tag{21}$$

where

$$R^\tau_{m-1,i}(\eta,\omega) = \sum_{l=0}^{m-3}\binom{m-2}{l} D^{m-2-l}_\eta\big(D_xF_\tau(\theta^i\omega, x_i(\eta,\omega))\big)D^{l+1}x_i(\eta,\omega),$$

for $\tau = u, s$. We note that after applying the chain rule to

$$D^{m-2-l}_\eta\big(D_xF_\tau(\theta^i\omega, x_i(\eta,\omega))\big),$$

each term in $R^\tau_{m-1,i}(\eta,\omega)$ contains factors: $D^{l_1}_xF_\tau(\theta^i\omega, x_i(\eta,\omega))$ for some $2\leq l_1\leq m-1$ and at least two derivatives $D^{l_2}_\eta x_i(\eta,\omega))$ and $D^{l_3}_\eta x_i(\eta,\omega))$ for some $l_2, l_2 \in \{1,\cdots,m-2\}$. Since $D^l_\eta\mathbf{x}(\eta,\omega) \in C^-_{\beta/(5+2l)}$ for $l = 1,\cdots,m-1$ and F is C^N, $R^u_{m-1,i}(\cdot,\omega) : E_u \to L^{m-1}\big(E_u, E_u\big)$ and $R^s_{m-1,i}(\cdot,\omega) : E_u \to L^{m-1}\big(E_u, E_s\big)$ are C^1. Furthermore, using (19), we have for $\tau = u, s$

$$||D_\eta R^\tau_{m-1,i}(\eta,\omega)|| \leq \tilde{B}_m(\theta^i\omega)\tilde{K}_m(\omega)e^{\frac{2}{5+2(m-1)}\beta(\omega)i}$$

where $\tilde{B}_m(\omega) = \max_{1\leq i\leq m} B_i(\omega)$ and $\tilde{K}_m(\omega)$ is m-th order polynomial of $K_1(\omega),\cdots, K_{m-1}(\omega)$ with positive integer coefficients, which are tempered from above. If $m < N$, we have that for $\tau = u, s$

$$||D_\eta R^\tau_{m-1,i}(\eta,\omega) - D_\eta R^\tau_{m-1,i}(\eta_0,\omega)|| \leq \tilde{B}_{m+1}(\theta^i\omega)\tilde{K}_{m+1}(\omega)e^{\frac{2}{5+2(m-1)}\beta(\omega)i}.$$

Set $J(\eta,\omega)=\{J_n(\eta,\omega)\}_{n\le 0}$ with

$$J_n(\eta,\omega) = -\sum_{i=n}^{-1}\Phi_u(n-1-i,\theta^{i+1}\omega)R^u_{m-1,i}(\eta,\omega) \tag{22}$$

$$+\sum_{i=-\infty}^{n-1}\Phi_s(n-1-i,\theta^{i+1}\omega)R^u_{m-1,i}(\eta,\omega), \quad \text{for } n\le -1, \tag{23}$$

$$J_0(\eta,\omega) = \sum_{i=-\infty}^{-1}\Phi_s(-1-i,\theta^{i+1}\omega)R^s_{m-1,i}(\eta,\omega). \tag{24}$$

Note that for $n\le i\le -1$

$$\begin{aligned}
&||\Phi_u(n-1-i,\theta^{i+1}\omega)D_\eta R^u_{m-1,i}(\eta,\omega)||\\
&\le K(\theta^{i+1})\tilde{B}_{m+1}(\theta^i\omega)\tilde{K}_{m+1}(\omega)e^{\beta(\omega)(n-1-i)}e^{\frac{2\beta(\omega)}{5+2(m-1)}i}\\
&=\Big(K(\theta^{i+1}\omega)\tilde{B}_{m+1}(\theta^i\omega)e^{\frac{\beta(\omega)}{5+2(m-1)}i}\Big)\tilde{K}_{m+1}(\omega)e^{\beta(\omega)(n-1-i)}e^{\frac{\beta(\omega)}{5+2(m-1)}\beta(\omega)i},
\end{aligned} \tag{25}$$

here $\big(K(\theta^{i+1})\tilde{B}_{m+1}(\theta^i\omega)\}e^{\frac{1}{5+2(m-1)}\beta(\omega)i}\big)$ is bounded by a tempered from above random variable $K^*(\omega)$ independent of i. Similarly, for $i\le n-1$

$$\begin{aligned}
&||\Phi_s(n-1-i,\theta^{i+1}\omega)D_\eta R^s_{m-1,i}(\eta,\omega)||\\
&\le K^*(\omega)\tilde{K}_{m+1}(\omega)e^{-\beta(\omega)(n-1-i)}e^{\frac{1}{5+2(m-1)}\beta(\omega)i}.
\end{aligned} \tag{26}$$

Estimates (25) and (26) yield that $J_n(\eta,\omega)$ is C^1 for all $n\le 0$. We consider a linear operator $\mathcal{G}$ from $L^{m-1}(E_u,C^-_{\beta(\omega)/(5+2m-1)})$ into itself that is given by (16) with $\mathbf{v}\in L^{m-1}(E_u,C^-_{\beta(\omega)/(5+2m-1)})$ and set $I^m=\{I^m_n\}_{n\le 0}$ with

$$\begin{aligned}
&I^m_n\\
&=-\sum_{i=n}^{-1}\Phi_u(n-1-i,\theta^{i+1}\omega)\Big(F_u(\theta^i\omega,x_i(\eta,\omega))-F_u(\theta^i\omega,x_i(\eta_0,\omega))\Big)D^{m-1}x_i(\eta,\omega)\\
&+\sum_{i=-\infty}^{n-1}\Phi_s(n-1-i,\theta^{i+1}\omega)\Big(F_u(\theta^i\omega,x_i(\eta,\omega))-F_u(\theta^i\omega,x_i(\eta_0,\omega))\Big)D^{m-1}x_i(\eta,\omega)\\
&I^m_0=\sum_{-\infty}^{-1}\Phi_s(-1-i,\theta^{i+1}\omega)\Big(F_u(\theta^i\omega,x_i(\eta,\omega))-F_u(\theta^i\omega,x_i(\eta_0,\omega))\Big)D^{m-1}x_i(\eta,\omega).
\end{aligned}$$

Then, equation (21) is equivalent to

$$\begin{aligned}
&\big(D^{m-1}\mathbf{x}(\eta,\omega)-D^{m-1}\mathbf{x}(\eta,\omega)\big)-\mathcal{G}\big(D^{m-1}\mathbf{x}(\eta,\omega)-D^{m-1}\mathbf{x}(\eta,\omega)\big)\\
&=J(\eta,\omega)-J(\eta_0,\omega)+I^m.
\end{aligned} \tag{27}$$

Since

$$||G||_{L^{m-1}(E_u,C^-_{\beta(\omega)/(5+2(m-1)+1)})}\le\frac{(5+2m-1)^2}{3(5+2m-2)(5+2m)}<\frac{2}{3},$$

the bounded linear operator $(Id-G)$ has a bounded inverse in

$$L^{m-1}(E_u,C^-_{\beta(\omega)/(5+2m-1)}).$$

Thus,

$$\begin{aligned}&\big(D^{m-1}\mathbf{x}(\eta,\omega)-D^{m-1}\mathbf{x}(\eta,\omega)\big)\\ &=\big(Id-\mathcal{G}\big)^{-1}DJ(\eta_0,\omega)(\eta-\eta_0)\\ &+\big(Id-\mathcal{G}\big)^{-1}\big\{\big(J(\eta,\omega)-J(\eta_0,\omega)-DJ(\eta_0,\omega)(\eta-\eta_0)\big)+I^m\big\}.\end{aligned}\tag{28}$$

Using the same argument that we used to show that $|I|_{C^-_{\beta(\omega)/6}}=o(|\eta-\eta_0|)$ as $\eta\to\eta_0$, we have

$$\begin{aligned}&\big|\big|\big(J(\eta,\omega)-J(\eta_0,\omega)-DJ(\eta_0,\omega)(\eta-\eta_0)\big)+I^m\big|\big|_{L^{m-1}(E_u,C^-_{\beta(\omega)/(5+2m-1)})}\\ &=o(|\eta-\eta_0),\qquad \text{as}\quad \eta\to\eta_0,\end{aligned}$$

which implies that $D^m\mathbf{x}(\eta,\omega)$ exists and

$$D^m\mathbf{x}(\eta,\omega)\in L^m(E_u,C^-_{\beta(\omega)/(5+2m-1)}).$$

Furthermore,

$$D^m\mathbf{x}(\cdot,\omega):E_u\to L^m(E_u,C^-_{\beta(\omega)/(5+2m)})$$

is continuous and (19) holds form $j=m$ and (20) hold for $m<N$. This completes the proof.

□

Proof of Theorem 3.3: We define $h^u:E_u\times\Omega\to E_s$ by

$$h^u(\eta,\omega)=x_0^s(\eta,\omega)=\sum_{i=-\infty}^{-1}\Phi_s(-1-i,\theta^{i+1}\omega)F_s(\theta^i\omega,x_i(\eta,\omega)).$$

By Lemma 3.4 and Proposition 3.5 (i), we see that

$$W^u(\omega)=\{x_0(\eta,\omega)\mid\eta\in E_u\}=\{\eta+h^u(\eta,\omega)\mid\eta\in E_u\}.$$

(ii) and (iii) in Proposition 3.5 imply that (i) and (ii) in Theorem 3.3 hold except of the estimate of the Lipschitz constant of h^u. For each $\eta,\eta_0\in E_u$, we have

$$\begin{aligned}&|h^u(\eta,\omega)-h^u(\eta_0,\omega)|\\ &\le\sum_{i=-\infty}^{-1}e^{-\beta(\omega)(-1-i)}K(\theta^{i+1})B_1(\theta^i\omega)\big(\frac{24}{15}K(\theta)e^{\frac{\beta(\omega)}{5}i}|\eta-\eta_0|\big)\\ &\le\frac{24}{15}\sum_{i=-\infty}^{-1}e^{-\beta(\omega)(-1-i)}K(\theta^{i+1})B_1(\theta^i\omega)K(\theta^i\omega)|\eta-\eta_0|\\ &\le\frac{4}{15}|\eta-\eta_0|\end{aligned}$$

here (15) and the fact $K(\omega)\le K(\theta^i\omega)e^{\frac{\beta(\omega)}{5}|i|}$ and (7) are used in the above estimates. (iii) in the theorem is obvious.

Finally, for each $x_0\in W^u(\omega)$, since there exists $\eta\in E_u$ such that $x_0=\eta+h^u(\eta,\omega)$, by Proposition 3.5, the orbit $\mathbf{x}=\{x_n\}_{n\le 0}=\mathbf{x}(\eta,\omega)$. Hence,

$$\mathbf{x}\in C^-_{\beta/5},$$

which means that $x_n\to 0$ exponentially as $n\to-\infty$.

In order to see that $W^u(\omega)$ is a random set we need to show that for each $x \in \mathbb{R}^d$

$$\omega \to \inf_{y \in \mathbb{R}^d} |x - (y^u + h^u(y^u, \omega))| \tag{29}$$

is measurable, see Castaing and Valadier [9], Theorem III.9. Let $\mathbb{Q}$ be the set of all rational points in $\mathbb{R}^d$. Then the right hand side of (29) is equal to

$$\inf_{y \in \mathbb{Q}} |x - (y^u + h^s(y^u, \omega))| \tag{30}$$

which follows immediately by the continuity of $h^u(\cdot, \omega)$. The measurability of an expression under the infimum of (29) follows since $\omega \to h^u(y^u, \omega)$ is measurable for any $y \in \mathbb{R}^d$. This completes the proof of the theorem. □

Theorem 3.6. *(Stable Manifold Theorem) Assume that Hypotheses A and B hold. Then, the stable set $W^s(\omega)$ is a C^N manifold given by*

$$W^s(\omega) = \{\xi + h^s(\xi, \omega) \mid \xi \in E_s\}$$

where

$$h^s : E_s \times \Omega \to E_u$$

satisfies

(i) *$h^s(\xi, \omega)$ is measurable in (ξ, ω) and is C^N in ξ with*

$$Lip\, h^s(\cdot, \omega) < 1, \quad h^s(0, \omega) = 0, \quad Dh^s(0, \omega) = 0.$$

(ii) *$|D^i h^s(\xi, \omega)| \leq K_i(\omega)$ for each $0 \leq i \leq N$ when $N < \infty$ and for all $i \geq 0$ when $N = \infty$, where $K_i(\omega)$ are random variables tempered from above.*

(iii) *$W^s_{loc}(\omega) = \{x \in W^s(\omega) \mid x \in B(0, \rho(\omega))\}$ is a local unstable manifold for the original system.*

Moreover, for each $x_0 \in W^s(\omega)$, the orbit $\mathbf{x} = \{x_n\}_{n \geq 0}$ of ϕ with the initial point x_0 satisfies

$$\mathbf{x} \in C^+_{\beta/5},$$

which means that $x_n \to 0$ exponentially as $n \to \infty$.

There are two ways to prove this theorem. The first approach is considering the inverse of the random dynamical system ϕ and applying the unstable manifold theorem to the inverse. The unstable manifold of the inverse is the stable manifold of ϕ. The key here is to find a new tempered neighborhood of 0 such that all derivatives of the nonlinearity of the inverse under consideration are bounded by random variables tempered from above. Then, use a cut-off function to modify the nonlinearity such that the properties in Lemma 3.1 and (7) hold. The second approach is proving it like the unstable manifold.

Generally, a few modifications are needed to adapt the proofs presented for the unstable manifolds to the case of stable manifold. The most significant differences are the difference equation and the associated function space. We shall outline the proofs and leave the details to the interested reader.

Corresponding to space $C^-_{\gamma(\omega)}$, we consider space $C^+_{\gamma(\omega)}$ and the stable set

$$W^s(\omega) = \{x_0 \in \mathbb{R}^d \mid x_n = \phi(n, \omega, x_0) \to 0, \text{ as } n \to +\infty\}$$

which is invariant. In order to show that $W^s(\omega)$ is given by the graph of a C^N (or Lipschitz) function, one needs to prove for each $0 \leq \gamma(\omega) \leq \beta(\omega)/5$, $\mathbf{x} = \{x_n\}_{n \geq 0} \in C^+_{\gamma(\omega)}$ is an orbit of

$\phi(n,\omega,x)$ if and only if $\{x_n\}_{n\geq 0}\in C^+_{\gamma(\omega)}$ and satisfies the following equations

$$\begin{aligned} x_n &= \Phi_s(n,\omega)x_0^s + \sum_{i=0}^{n-1}\Phi_s(n-1-i,\theta^{i+1}\omega)F_s(\theta^i\omega,x_i)\\ &\quad -\sum_{i=n}^{\infty}\Phi_u(n-1-i,\theta^{i+1}\omega)F_u(\theta^i\omega,x_i), \quad \text{for} \quad n\geq 1, \\ x_0 &= x_0^s - \sum_{i=0}^{\infty}\Phi_u(-1-i,\theta^{i+1}\omega)F_u(\theta^i\omega,x_i). \end{aligned} \tag{31}$$

The next step is to show that for any given $\xi\in E_s$, equation (31) with $x_0^s=\xi$ has a unique solution in $C^+_{\gamma(\omega)}$. To see this, letting $\mathcal{J}^s_n(\mathbf{x},\eta,\omega)$ denote the right hand side of equation (31), one may show that $\mathcal{J}^s(\mathbf{x},\eta,\omega)=\{\mathcal{J}_n(\mathbf{x},\eta,\omega)\}_{n\geq 0}$ is a unform contraction. Hence, by the uniform contraction mapping principle, we have that for each $\xi\in E_s$, the mapping $\mathcal{J}^s(\cdot,\xi,\omega)$ has a unique fixed point $\mathbf{x}(\xi,\omega)\in C^+_{\beta(\omega)/5}$ and $\mathbf{x}(\cdot,\omega)$ is Lipschitz from E_s to $C^+_{\beta(\omega)/5}$. Thus, $\mathbf{x}(\xi,\omega)\in C^+_{\beta(\omega)/5}$ is a solution of equation (31).

Let $h^s(\xi,\omega)=x_0^u(\xi,\omega)$. Then

$$h^s(\xi,\omega) = -\sum_{i=0}^{\infty}\Phi_u(n-1-i,\theta^{i+1}\omega)F(\theta^i\omega,x_i)$$

and $h^s(0,\omega)=0$.

Therefore,

$$W^s(\omega)=\{\xi+h^s(\xi,\omega)|\xi\in E_s\}.$$

In the same fashion as the case for smoothness of stable manifold, one may show that h^s is C^k and all properties in the theorem hold.

Next, we prove the existence of a stable foliation. Let $\mathbf{x}=\{x_n\}_{n\geq 0}$ be a positive orbit of ϕ. Define the following set

$$\mathcal{M}^s(x_0,\omega)=\big\{y_0 \mid \mathbf{y}-\mathbf{x}\in C^+_{\beta/5}\big\},$$

where $\mathbf{y}=\{y_n\}_{n\geq 0}$ is a positive orbit of ϕ.

Theorem 3.7. *Assume that Hypotheses A and B hold. Then there exists a mapping*

$$g^s: E_s\times\mathbb{R}^d\times\Omega\to E_u$$

such that

(i) $g^s(\xi,x_0,\omega)$ *is measurable in* (ξ,x_0,ω) *and is Lipschitz continuous in* ξ *with*

$$Lip_\xi\, g^s<1.$$

(ii) $\mathcal{M}^s(x_0,\omega)$ *is the graph of* φ_s

$$\mathcal{M}^s(x_0,\omega)=\big\{x_0+\xi+\varphi_s(\xi,x_0,\omega)\mid \xi\in E_s\big\}.$$

(iii) $W^u(\omega)\cap\mathcal{M}^s(x_0,\omega)$ *contains a single point* $\bar{x}_0$ *and*

$$|x_n-\bar{x}_n|\leq\frac{24}{15}K(\omega)|x_0^s-\bar{x}_0^s|e^{-\frac{\beta(\omega)}{5}n}.$$

Furthermore, $\mathbb{R}^d=\cup_{x_0\in W^u(\omega)}\mathcal{M}^s(x_0,\omega)$.

Remark: Each leaf $\mathcal{M}^s(x_0,\omega)$ is in fact a C^N submanifolds and is Hölder continuous in x_0. $\mathcal{M}^s(0,\omega)$ is the stable manifold $W^s(\omega)$.

Proof. Let $\mathbf{x}=\{x_n\}_{n\geq 0}$ be an orbit of ϕ and $\tilde{\mathbf{x}}=\{\tilde{x}_n\}_{n\geq 0}$ be a sequence and let $\mathbf{z}=\tilde{\mathbf{x}}-\mathbf{x}$. Then, $\tilde{\mathbf{x}}$ is an orbit of ϕ and $\tilde{\mathbf{x}}-\mathbf{x}\in C^+_{\beta/5}$ if and only if $\mathbf{z}\in C^+_{\beta/5}$ and satisfies

$$\begin{aligned} z_n &= \Phi_s(n,\omega)\xi+\sum_{i=0}^{n-1}\Phi_s(n-1-i,\theta^{i+1}\omega)\big(F_s(\theta^i\omega,z_i+x_i)-F_s(\theta^i\omega,x_i)\big)\\ &\quad -\sum_{i=n}^{\infty}\Phi_u(n-1-i,\theta^{i+1}\omega)\big(F_u(\theta^i\omega,z_i+x_i)-F_u(\theta^i\omega,x_i)\big),\quad \text{for}\quad n\geq 1,\\ z_0 &= \xi-\sum_{i=0}^{\infty}\Phi_u(-1-i,\theta^{i+1}\omega)\big(F_u(\theta^i\omega,z_i+x_i)-F_u(\theta^i\omega,x_i)\big) \end{aligned}\tag{32}$$

with $\xi=\tilde{x}^s_0-x^s_0$. Let $\mathcal{J}^s_n(\mathbf{z},\xi,x_0,\omega)$ denote the right hand side of the above equation. We first note that $\mathcal{J}^s: C^+_{\beta/5}\times E_s\times\mathbb{R}^d\times\Omega\to C^+_{\beta/5}$. Next, for each $\mathbf{z},\bar{\mathbf{z}}\in C^+_{\beta/5}$, we have

$$|\mathcal{J}^s(\mathbf{z},\xi,x_0,\omega)-\mathcal{J}^s(\bar{\mathbf{z}},\xi,x_0,\omega)|_{C^+_{\beta/5}}\leq\frac{9}{24}|\mathbf{z}-\bar{\mathbf{z}}|_{C^+_{\beta/5}}.$$

Thus, $\mathcal{J}^s$ is a contraction in $\mathbf{z}$ uniformly with respect to (ξ,x_0,ω). Therefore, by the uniform contraction principle, $\mathcal{J}^s$ has a unique fixed point $\mathbf{z}(\xi,x_0,\omega)\in C^+_{\beta/5}$ which satisfies equation (32). Furthermore,

$$|\mathbf{z}(\xi,x_0,\omega)-\mathbf{z}(\tilde{\xi},x_0,\omega)|_{C^+_{\beta/5}}\leq\frac{24}{15}K(\omega)|\xi-\tilde{\xi}|.\tag{33}$$

Let

$$\begin{aligned} g^s(\xi,x_0,\omega)&=z^s_0(\xi,x_0,\omega)\\ &=-\sum_{i=0}^{\infty}\Phi_u(-1-i,\theta^{i+1}\omega)\big(F(\theta^i\omega,z_i(\xi,x_0,\omega)+x_i)-F(\theta^i\omega,x_i)\big) \end{aligned}$$

Then,

$$\begin{aligned} &|g^s(\xi,x_0,\omega)-g^s(\tilde{\xi},x_0,\omega)|\\ &\leq\sum_{i=0}^{\infty}e^{-\beta(\omega)(i+1)}K(\theta^{i+1}\omega)B_1(\theta^i\omega)e^{-\frac{1}{5}\beta(\omega)i}|\mathbf{z}(\xi,x_0,\omega)-\mathbf{z}(\tilde{\xi},x_0,\omega)|_{C^+_{\beta/5}}\\ &\leq\frac{24}{15}\sum_{i=0}^{\infty}e^{-\beta(\omega)(i+1)}K(\theta^{i+1}\omega)B_1(\theta^i\omega)K(\theta^i(\omega))|\xi-\tilde{\xi}|\leq\frac{4}{15}|\xi-\tilde{\xi}| \end{aligned}$$

and

$$\mathcal{M}^s(x_0,\omega)=\big\{x_0+\xi+\varphi_s(\xi,x_0,\omega)\mid\xi\in E_s\big\}.$$

Since $\mathrm{Lip}_\eta\ h^u<1$ and $\mathrm{Lip}_\xi\ g^s<1$, $g^s(h^u(\cdot,\omega)+x^s_0,x_0,\omega)$ is a contraction, which implies that $W^u(\omega)\cap\mathcal{M}^s(x_0,\omega)$ contains a single point. Other properties follow immediately. This completes the proof.

$\square$

Similarly, we have a unstable foliation

Theorem 3.8. *Assume that Hypotheses A and B hold. Then there exists a mapping*

$$g^u : E_u \times \mathbb{R}^d \times \Omega \to E_s$$

such that

(i) $g^u(\eta, x_0, \omega)$ *is measurable in* (η, x_0, ω) *and is Lipschitz continuous in* η *with*

$$Lip_\eta\, g^u < 1$$

(ii) $\mathcal{M}^u(x_0, \omega)$ *is the graph of* g^u

$$\mathcal{M}^u(x_0, \omega) = \{x_0 + \eta + g^u(\eta, x_0, \omega) \mid \eta \in E_u\}$$

(iii) $W^s(\omega) \cap \mathcal{M}^u(x_0, \omega)$ *contains a single point* $\bar{x}_0$ *and*

$$|x_n - \bar{x}_n| \le \frac{24}{15} K(\omega) |x_0^u - \bar{x}_0^u| e^{\frac{\beta(\omega)}{5} n}, \quad \text{for } n \le 0.$$

Furthermore, $\mathbb{R}^d = \cup_{x_0 \in W^u(\omega)} \mathcal{M}^s(x_0, \omega)$.

4. **Smooth Conjugacy**

We first give a general theorem (Theorem 4.2) on smooth conjugacy in a jet class. Then, we present the implications of this theorem including a theorem of Sternberg type, a theorem of Chen type, and a theorem on normal forms. An example of normal forms for two dimensional random dynamical system over an ergodic metric dynamical system is also given.

As we mentioned before, the random dynamical systems are of certain nonuniform features. First, the linearized systems at hyperbolic fixed points are nonuniformly hyperbolic as ω varies no matter the metric dynamical system θ^n is ergodic or not. Second, when θ^n is not ergodic, the quantities such the dimensions of stable and unstable spaces may vary as ω changes. In this section, we will encounter two θ-invariant measurable functions with integer values: the dimension of the stable subspace $d^s(\omega)$ (Note that the dimension of the unstable subspace is $d^u(\omega) = d - d^s(\omega)$) and a conjugacy order function $N_0(\omega) = N_0(\alpha(\omega), \beta(\omega), k)$. When the metric dynamical system θ^n is ergodic, by the Multiplicative Ergodic Theorem, the dimension of stable subspace $d^s(\omega)$ and $N_0(\omega)$ are constant, thus the conjugacy enjoys certain uniform properties for all $\omega \in \Omega$. When the metric dynamical system θ^n is not ergodic, these properties may changes from one part of Ω to another part. Since $d^s(\omega)$ takes finitely many values and $N_0(\omega)$ takes only countably many values, we may decompose Ω into a union of countably many disjoint θ-invariant measurable sets $\tilde{\Omega}_i$, $i \in \mathbb{N}$:

$$\Omega = \cup_{i \ge 1} \tilde{\Omega}_i.$$

On each $\tilde{\Omega}_i$, both $d^s(\omega)$ and $N(\omega)$ are constant. We solve the conjugacy equation (34) on $\tilde{\Omega}_i$ for a solution $h_i(\omega, x)$, where $\omega \in \tilde{\Omega}_i$, $x \in U(\omega)$ = a tempered ball. Patching all $h_i(\omega, x)$ together gives a solution $h(\omega, x)$ of equation (34) over $\tilde{\Omega}$. In fact, define

$$h(\omega, x) = h_i(\omega, x), \quad \omega \in \tilde{\Omega}_i,\ x \in U(\omega).$$

Then, $h(\omega, x)$ is well-defined for all $\omega \in \Omega, x \in U(\omega)$ since Ω is a union of disjoint sets Ω_i and is measurable since Ω is a union of countably many disjoint measurable sets. Furthermore, if h_i is a C^k random diffeomorphism, then h is a C^k random diffeomorphism. Finding $h_i(\omega, x)$ is independence of finding others. The conjugacy equation (34) actually is a θ-pathwise equation. For each fixed $\omega \in \Omega$, $h(\omega, x)$ is determined only by the random dynamical system along the orbit $\theta^n\omega$.

To solve for $h_i(\omega, x)$ (to prove Theorem 4.2), we first use the stable and unstable manifolds obtained in Section 3 as new coordinate planes to transform the random dynamical systems ϕ

and ψ to the equivalent systems $\hat{\phi}$ and $\hat{\psi}$ whose stable and unstable manifolds are flat. Then, we establish the basic estimates on the nonlinearities and on the orbits of the random dynamical systems and their derivatives. The difference of $\hat{\phi}$ and $\hat{\psi}$ is decomposed into two parts, one part is dominated by a power of the stable variable and another is controlled by a power of the unstable variable. These estimates are very crucial to the construction of the transformation. We use a homotopy approach to convert the conjugacy problem to a recurrent linear equation. Based on its recurrence, we construct two formal infinite series solutions of the equation, one is forward in time and another is backward in time. These solutions, in general, may diverge. However, the formal solutions associated with the stable part and the unstable part converge smoothly when the order N is large enough.

Definition 4.1. *Two random dynamical systems $\phi(n,\omega,x)$ and $\psi(n,\omega,x)$ with a fixed point $x=0$ are said to be C^k conjugate locally if there exists a C^k random diffeomorphism $h(\omega,x)$ from a tempered ball $U(\omega)$ to a tempered ball $V(\omega)$ with $h(\omega,0)=0$ such that*

$$h(\theta\omega,\phi(1,\omega)x)=\psi(1,\omega)h(\omega,x) \quad \text{for} \quad x\in U(\omega), \quad a.s.\ \omega\in\Omega. \tag{34}$$

The conjugacy relationship (34) implies that $h(\theta^n\omega,\phi(n,\omega)x)=\psi(n,\omega)h(\omega,x)$ when the orbits stay in the corresponding domains.

The next theorem gives a smooth conjugacy in a jet class.

Theorem 4.2. *Let ϕ be a random dynamical system satisfying Hypothesis A and B. Then*

- (i) *For any integer $k>0$, there exists a θ-invariant measurable integer function $N_0(\omega)=N_0(\alpha(\omega),\beta(\omega),k)$ such that*
  - (i_1) *if $\phi\big|_{\tilde{\Omega}_i}$ (the restriction of ϕ to $\tilde{\Omega}_i$) is C^{N_0} locally tempered, ψ is a random dynamical system in $\mathbb{R}^d$ with $\phi-\psi=O(|x|^{N_0})$ for $\omega\in\tilde{\Omega}_i$ and $\psi\big|_{\tilde{\Omega}_i}$ is C^{N_0} locally tempered, then $\phi\big|_{\tilde{\Omega}_i}$ and $\psi\big|_{\tilde{\Omega}_i}$ are C^k locally conjugate.*
  - (i_2) *if ϕ is a C^∞ locally tempered random dynamical system, then any C^∞ tempered random dynamical system ψ in $\mathbb{R}^d$ with $\phi-\psi=O(|x|^{N_0})$ is C^k locally conjugate to ϕ.*
- (ii) *Any C^∞ tempered random dynamical system ψ with $\phi-\psi=C^\infty$ flat is C^∞ locally conjugate to ϕ.*

The proof of this theorem is based on the following lemmas and proposition. As we did in section 3, since $d^s:\Omega\to\{1,\cdots,d\}$ is a measurable function, Ω may be decomposed into a union of $l>0$ disjoint θ-invariant measurable sets:

$$\Omega=\cup_{i=1}^l\Omega_i.$$

On each Ω_i, $d^s(\omega)$ is constant. In the following, we will restrict our discussions to Ω_i. For simplicity, we use Ω to denote the Ω_i. We first applying the unstable manifold theorem (Theorem 3.3) and the stable manifold theorem (Theorem 3.6) to ϕ. Then, ϕ has a C^N (resp. C^∞) local unstable manifold

$$W^u_{loc}(\omega)=\{\eta+h^u(\eta,\omega)\mid\eta\in E_u,\ |\eta|\le\rho(\omega)\}$$

and a C^N (resp. C^∞) local stable manifold

$$W^s_{loc}(\omega)=\{\xi+h^s(\xi,\omega)\mid\xi\in E_s,\ |\xi|\le\rho(\omega)\}$$

We may identify E_u and E_s with $\mathbb{R}^{d_u}$ and $\mathbb{R}^{d_s}$, respectively. Next, we use the stable and unstable manifolds as new axes to rewrite ϕ. Consider the coordinate random transformation defined on

$B_{r(\omega)}(0)$

$$\begin{aligned} y &= \eta - h^s(\xi, \omega), \\ z &= \xi - h^u(\eta, \omega). \end{aligned} \tag{35}$$

It also follows from Theorem 3.3 and 3.6 that this transformation is a C^N (resp. C^∞) random diffeomorphism. By this transformation, the random diffeomorphism $\phi(1, \omega)$ is locally conjugate to a C^N (resp. C^∞) random diffeomorphism

$$\hat{\phi}(1, \omega) \ : \ x = (y, z)^T \mapsto A(\omega)x + F_1(\omega, x)$$

where $A(\omega) = D\phi(1, \omega)$ is written as $diag(A^s(\omega), A^u(\omega))$, $F_1(\omega, x)$ satisfies Hypothesis B, and $F_1(\omega, 0) = 0$, $DF_1(\omega, 0) = 0$. Furthermore,

$$F_1 = \begin{pmatrix} F_{1,s}(\omega, y, z) \\ F_{1,u}(\omega, y, z) \end{pmatrix}, \quad F_{1,s}(\omega, 0, z) = 0, \quad F_{1,u}(\omega, y, 0) = 0,$$

which implies that $y = 0$ is the stable manifold and $z = 0$ is the unstable manifold for the new random dynamical system.

Similarly, $\psi(1, \omega)$ is C^N (resp. C^∞) conjugate to

$$\hat{\psi}(1, \omega) \ : \ x = (y, z)^T \mapsto A(\omega)x + F_2(\omega, x)$$

where $F_2(\omega, x)$ satisfies Hypothesis B, and $F_2(\omega, 0) = 0$, $DF_2(\omega, 0) = 0$. Furthermore,

$$F_2 = \begin{pmatrix} F_{2,s}(\omega, y, z) \\ F_{2,u}(\omega, y, z) \end{pmatrix}, \quad F_{2,s}(\omega, 0, z) = 0, \quad F_{2,u}(\omega, y, 0) = 0.$$

Let

$$R = \begin{pmatrix} R_s(\omega, y, z) \\ R_u(\omega, y, z) \end{pmatrix} = F_1 - F_2 = \begin{pmatrix} F_{1,s}(\omega, y, z) - F_{2,s}(\omega, y, z) \\ F_{1,s}(\omega, y, z) - F_{2,s}(\omega, y, z) \end{pmatrix}.$$

Then $R_s(\omega, 0, z) = 0, R_u(\omega, y, 0) = 0$. When $\phi - \psi = O(|x|^N)$, the order of the difference between the corresponding stable manifolds and order of the difference between unstable manifolds are $O(|x|^N)$, thus $\hat{\phi} - \hat{\psi} = O(|x|^N)$. When $\phi - \psi = C^\infty$ *flat*, $\hat{\phi} - \hat{\psi} = C^\infty$ *flat.*

In the following, we will decompose R into two parts, one part is dominated by a power of y while another is dominated by a power of z.

Lemma 4.3. *The function R can be written as $R = R_1 + R_2$, where $R_i(\omega, x)$ are measurable and $R_i(\omega, \cdot)$ are $C^{[N/2]}$ (resp. C^∞) tempered functions satisfying*

$$\Pi_s R_2(\omega, 0, z) = 0, \quad \Pi_u R_1(\omega, y, 0) = 0,$$

$$|R_1| \le C_0(\omega)|y|^{[N/2]}, \quad |R_2| \le C_0(\omega)|z|^{[N/2]}, \quad x \in U(\omega) \tag{36}$$

if $jet^N_{x=0}R = 0$, *and*

$$|R_1| \le C_m(\omega)|y|^m, \quad |R_2| \le C_m(\omega)|z|^m, \quad \textit{for all integers } m > 0, \quad x \in U(\omega) \tag{37}$$

if $jet^\infty_{x=0}R = 0$, *where* $C_m(\omega), m \ge 0$, *are tempered from above.*

Proof. We first consider the case $\text{jet}^N_{x=0}R = 0$. Consider the Taylor expansion of R of order $[N/2]$ with respect to the variable y:

$$R(\omega, y, z) = \sum_{i=0}^{[N/2]} a_i(\omega, y, z) + R_{[N/2]}(\omega, y, z),$$

where $R_{[N/2]} = O(|y|^{[N/2]})$ and a_i is a homogeneous polynomial of variable y of degree i whose coefficients are the measurable functions of (ω, z) and $C^{[N/2]}$ with respect to the variable z. Since $jet^N_{x=0}R = 0$, we have $\|a_i\| = O(\|z\|^{[N/2]})$. Let $R_1 = R_{[N/2]}$ and $R_2 = \sum_{i=0}^{[N/2]} a_i(\omega, y, z)$. Then (36) holds. Other properties of R_1 and R_2 follows from the properties of F_1 and F_2.

Next, we consider the case $\text{jet}^{\infty}_{x=0}R = 0$. Let $\chi(r) : \mathbb{R}^+ \to [0, 1]$ be a C^∞ cut-off function with the following properties:

$$\chi(r) = \begin{cases} 1, & 0 \le r \le \frac{1}{2}, \\ 0, & r > 1 \end{cases}$$

For $k = (k_1, \cdots, k_{d_s}) \in \mathbb{Z}_+^{d_s}$, we denote

$$k! = k_1!\, k_2! \cdots k_{d_s}!, \quad |k| = \sum_{i=1}^{d_s} k_i.$$

Let

$$\mu_k(\omega) = \sup_{x \in U(\omega), 0 \le i \le 2|k|} \|D^i R(\omega, x)\| + k!.$$

Then μ_k are tempered from above. Let

$$R_2(\omega, x) = \sum_{k \in \mathbb{Z}_+^{d_s}, |k|=0}^{\infty} \frac{y^k}{k!} \frac{\partial^k}{\partial y^k} R(\omega, 0, z) \chi(\mu_k(\omega)|y|) \quad \text{and} \quad R_1 = R - R_2.$$

For each $l = (l_1, \cdots, l_{d_s}) \in \mathbb{Z}_+^{d_s}$ and $m = (m_1, \cdots, m_{d_u}) \in \mathbb{Z}_+^{d_u}$, since

$$\frac{1}{k!} \frac{\partial^l}{\partial y^l} \big(y^k \chi(\mu_k(\omega)|y|)\big) \frac{\partial^m}{\partial z^m} \big(\frac{\partial^k}{\partial y^k} R(\omega, 0, z)\big) = O\Big(\frac{1}{\big(\mu_k(\omega)\big)^{|k|-|l|-1}}\Big) = O\Big(\frac{1}{\big(k!\big)^{|k|-|l|-1}}\Big)$$

we have

$$\frac{\partial^l}{\partial y^l} \frac{\partial^m}{\partial z^m} R_2(\omega, x) = \sum_{k \in \mathbb{Z}_+^{d_s}, |k|=0}^{\infty} \frac{1}{k!} \frac{\partial^l}{\partial y^l} \big(y^k \chi(\mu_{2k}(\omega)|y|)\big) \frac{\partial^m}{\partial z^m} \big(\frac{\partial^k}{\partial y^k} R(\omega, 0, z)\big)$$

that is tempered from above. This together with R being C^∞ flat gives (39) for R_2. Since $R_1 = R - R_2$ and $\frac{\partial^l}{\partial y^l} R_1(\omega, 0) = 0$, R_1 is a C^∞ tempered function and (37) holds for R_1. This completes the proof of this lemma.

□

Next, we use the cut-off procedure introduced at the beginning of Section 3 to modify the functions, F_1, F_2, R_1, and R_2. We set for $i = 1, 2$

$$\tilde{F}_i(\omega, x) = \sigma_{\rho_1(\omega)}(|x|) F_i(\omega, x),$$

$$\tilde{R}_i(\omega, x) = \sigma_{\rho_1(\omega)}(|x|) R_i(\omega, x).$$

We choose a random variable $\rho_1(\omega)$ tempered from below such that $\rho_1(\omega) \le \rho(\omega)$ and satisfies the corresponding estimate (6) for each F_i and R_i. Then, $\tilde{F}_1$ and $\tilde{F}_2$ are C^N (resp. C^∞) tempered function on $\mathbb{R}^d$ and $\tilde{R}_1$ and $\tilde{R}_2$ are $C^{[N/2]}$ (resp. C^∞) tempered function on $\mathbb{R}^d$. Moreover, the following hold:

$$|\tilde{R}_1| \le C_0(\omega)|y|^{[N/2]} \sigma_{\rho_1(\omega)}(|y|), \quad |\tilde{R}_2| \le C_0(\omega)|z|^{[N/2]} \sigma_{\rho_1(\omega)}(|z|), \quad \text{for } x \in \mathbb{R}^d, \tag{38}$$

if $jet^N_{x=0}R = 0$; and

$$|\tilde{R}_1| \le C_m(\omega)|y|^m \sigma_{\rho_1(\omega)}(|y|), \quad |\tilde{R}_2| \le C_m(\omega)|z|^m \sigma_{\rho_1(\omega)}(|z|), \quad \text{for } x \in \mathbb{R}^d, \text{for all } m > 0, \tag{39}$$

if $jet^{\infty}_{x=0}R = 0$.

Let $\tilde{\phi} = A(\omega)x + \tilde{F}_1$ and $\tilde{\psi} = A(\omega)x + \tilde{F}_2$. Then

$$\hat{\phi} = \tilde{\phi}, \quad \hat{\psi} = \tilde{\psi}, \text{ for } |x| < \rho_1(\omega).$$

Let $\phi_\tau(m, \omega, x)$ and $\psi_\tau(m, \omega, x), \tau \in [0, 1]$, be the families of random dynamical systems whose time-one maps have the forms

$$\phi_\tau(1, \omega, x) = \tilde{\phi}(\omega, x) + \tau \tilde{R}_1$$

and

$$\psi_\tau(1, \omega, x) = \tilde{\phi}(\omega, x) + \tilde{R}_1 + \tau \tilde{R}_2$$

respectively.

Lemma 4.4. *For any given integer $k > 0$, there exist a θ-invariant measurable function $M_k(\omega) = M_k(\alpha(\omega), \beta(\omega))$ and a random variable $d_k(\omega)$ tempered from above such that*

$$\max_{1 \le |\alpha| \le k} \sup_{x \in \mathbb{R}^d} \|D^{\alpha}_{(x,\tau)} \phi^{\pm 1}_{\tau}(n, \omega, x)\| \le d_k(\omega) M_k(\omega)^{|n|}, \quad n \in \mathbb{Z}, \tag{40}$$

$$\max_{1 \le |\alpha| \le k} \sup_{x \in \mathbb{R}^d} \|D^{\alpha}_{(x,\tau)} \psi^{\pm 1}_{\tau}(n, \omega, x)\| \le d_k(\omega) M_k(\omega)^{|n|}, \quad n \in \mathbb{Z}. \tag{41}$$

Proof. We only prove (40) since (41) can be proved in the same fashion. We prove this lemma by induction on k. We first consider positive orbits. Denote by $x_n = \phi_\tau(n, \omega, x), x_0 = x$. Since x_n satisfies the constant of variation formula

$$x_n = \Phi(n, \omega)x + \sum_{i=0}^{n-1} \Phi(n - 1 - i, \theta^{i+1}\omega)(\tilde{F}_1(\theta^i\omega, x_i) + \tau \tilde{R}_1(\theta^i\omega, x_i)). \tag{42}$$

Using (42),

$$\|D_x(x_n)\| \le e^{\alpha(\omega)n} K(\omega) + \sum_{i=0}^{n-1} e^{\alpha(\omega)(n-1-i)} K(\theta^{1+i}\omega) \left(||D\tilde{F}_1(\theta^i\omega, \cdot)|| + \tau ||D\tilde{R}_1(\theta^i\omega, \cdot)|| \right) \|D_x(x_i)\|.$$

Using (7), we obtain

$$e^{-\alpha(\omega)n} \|D_x(x_n)\| \le K(\omega) + \frac{\beta(\omega)}{3} \sum_{i=0}^{n-1} e^{-\alpha(\omega)i} \|D_x(x_i)\|.$$

It follows from the Discrete Time Gronwall-Bellman Lemma ([1], page 327) that

$$\|D(x_n)_x\| \le K(\omega)|x_0| e^{\alpha(\omega)n} (1 + \frac{\beta(\omega)}{3})^n.$$

Taking the derivative with respect to τ on both sides of (42), we obtain

$$\begin{aligned} D_\tau(x_n) = & \sum_{i=0}^{n-1} \Phi(n - 1 - i, \theta^{i+1}\omega) \tilde{R}_1(\theta^i\omega, x_i) \\ & + \sum_{i=0}^{n-1} \Phi(n - 1 - i, \theta^{i+1}\omega) \left(D\tilde{F}_1((\theta^i\omega, x_i)) + \tau D(\tilde{R}_1(\theta^i\omega, x_i) \right) D_\tau x_i, \end{aligned} \tag{43}$$

which implies

$$\begin{aligned}\|D_\tau x_n\| \le& \sum_{i=0}^{n-1} e^{\alpha(\omega)(n-1-i)} K(\theta^{i+1}\omega)||\tilde{R}_1(\theta^i\omega,\cdot)|| \\ &+ \sum_{i=0}^{n-1} e^{\alpha(\omega)(n-1-i)} K(\theta^{i+1}\omega)\left(||D\tilde{F}_1(\theta^i\omega,\cdot)|| + \tau||D\tilde{R}_1(\theta^i\omega,\cdot)||\right)\|D_\tau x_i\|.\end{aligned} \tag{44}$$

Since $K(\omega)$ is tempered from above and $\tilde{R}_1(\theta^i\omega,\cdot)$ is bounded by a random variable tempered from above, by Lemma 2.7, then

$$K(\theta^{i+1}\omega)||\tilde{R}_1(\theta^i\omega,\cdot)|| \le \tilde{K}_0(\omega)e^i,$$

for some random variable $\tilde{K}_0(\omega)$, which together with (44) implies

$$e^{-\alpha(\omega)n}\|D_\tau x_n\| \le \sum_{i=0}^{n-1}\tilde{K}_0(\omega)e^{-(1+i)\alpha(\omega)+i} + \frac{\beta(\omega)}{3}\sum_{i=0}^{n-1}e^{-\alpha(\omega)i}\|D_\tau x_i\|.$$

It follows from the Discrete Time Gronwall-Bellman Lemma that

$$\begin{aligned}\|D_\tau x_n\| &\le e^{\alpha(\omega)n}\sum_{i=1}^{n}(1+\frac{\beta(\omega)}{6})^{n-i}\tilde{K}_0(\omega)e^{-(1+i)\alpha(\omega)+i} \\ &\le e^{\alpha(\omega)(n-1)}\tilde{K}_0(\omega)(1+\frac{\beta(\omega)}{6}+e^{1-\alpha(\omega)})^n \\ &= \tilde{K}_0(\omega)e^{-\alpha(\omega)}(e^{\alpha(\omega)}+e^{\alpha(\omega)}\frac{\beta(\omega)}{6}+e)^n.\end{aligned}$$

Next, we use induction to show the estimation of derivatives of any order hold for $\phi_\tau(n,\omega,x), n>0$. We assume that it holds for all derivatives of order $\le m$. Taking the derivative of order $m+1$ on the both sides of (42), we obtain

$$D^{m+1}x_n = \sum_{i=0}^{n-1}\Phi(n-1-i,\theta^{i+1}\omega)\left(D\tilde{F}_1(\theta^i\omega,x_i)+\tau D\tilde{R}_1(\theta^i\omega,x_i)\right)D^{m+1}x_i + \sum(\cdot), \tag{45}$$

where $\sum(\cdot)$ consists of the terms with derivatives of order $\le m$. Using the induction assumption, there exist a θ-invariant function $a_m = a_m(\alpha(\omega),\beta(\omega))$ and a random variable $\tilde{K}_m(\omega)$ tempered from above such that

$$\|\sum(\cdot)\| < \tilde{K}_m(\omega)a_m(\omega)^n,$$

which, together with (45), implies

$$\begin{aligned}&||D^{m+1}x_n|| \\ &\le \sum_{i=0}^{n-1}K(\theta^{i+1}\omega)e^{(n-1-i)\alpha(\omega)}\left(||D\tilde{F}_1(\theta^i\omega,\cdot)||+\tau||D\tilde{R}_1(\theta^i\omega,\cdot)||\right)||D^{m+1}x_i|| + \tilde{K}_m(\omega)a_m(\omega)^n\end{aligned}$$

or

$$e^{-\alpha(\omega)n}|D^{m+1}x_n| \le \frac{\beta(\omega)}{3}\sum_{i=0}^{n-1}e^{-i\alpha(\omega)}|D^{m+1}x_i| + \tilde{K}_m(\omega)(a_m(\omega)e^{-\alpha(\omega)})^n.$$

By using the Discrete Time Gronwall-Bellman Lemma again, we have

$$\begin{aligned}
e^{-\alpha(\omega)n}|D^{m+1}x_n| &\le \tilde{K}_m(\omega)\left(1+\frac{\beta(\omega)}{3}\right)^n + \sum_{i=1}^{n}\left(1+\frac{\beta(\omega)}{3}\right)^{n-i}\tilde{K}_m(\omega)(a_m(\omega)e^{-\alpha(\omega)})^i \\
&\le \tilde{K}_m(\omega)\left[\left(1+\frac{\beta(\omega)}{3}\right)^n + \left(1+\frac{\beta(\omega)}{3}+a_m(\omega)e^{-\alpha(\omega)}\right)^n\right] \\
&\le 2\tilde{K}_m(\omega)\left(1+\frac{\beta(\omega)}{3}+a_m(\omega)e^{-\alpha(\omega)}\right)^n .
\end{aligned}$$

Hence,

$$|D^{m+1}x_n| \le 2\tilde{K}_m(\omega)\left(e^{\alpha(\omega)}+\frac{e^{\alpha(\omega)}\beta(\omega)}{3}+a_m(\omega)\right)^n .$$

Letting $d_{m+1} = \max\{d_m(\omega), 2\tilde{K}_m\}$ and $M_{m+1} = \max\left\{M_m, \left(e^{\alpha(\omega)}+\frac{e^{\alpha(\omega)}\beta(\omega)}{3}+a_m(\omega)\right)\right\}$, we have that (40) holds for $k = m+1$ and $n \ge 0$. Since $a_m(\omega) = a_m(\alpha(\omega), \beta(\omega))$, M_{m+1} depends only on $\alpha(\omega)$ and $\beta(\omega)$.

For the negative orbit $x_n, n < 0$, we first choose $\rho_1(\omega)$ smaller such that (7) holds for the nonlinearity for $\phi_\tau(-1, \omega, x)$. Then the negative orbits of ϕ_τ are the positive orbits of ϕ_τ^{-1}. So the above results may be applied and the estimations of the derivatives of $\phi_\tau(n, \omega, x)$ hold for all $n \le 0$.

Finally, we show the estimations for $\phi_\tau^{-1}(n, \omega, x)$. By the cocycle properties,

$$\|D^m\phi_\tau^{-1}(n,\omega,x)\| = \|D^m\phi_\tau(-n,\theta^n\omega,x)\| \le \hat{d}(\theta^n\omega)(\hat{M}_m(\omega))^n < \tilde{d}_m(\omega)(\hat{M}_m(\omega)e)^n,$$

here again Lemma 2.7 is used. This completes the proof.

□

The next lemma states that one may reduce the problem of C^k conjugacy (34) to a problem of solving a linear recurrent functional equation. This idea is based on the so-called homotopy method which was used in [17] and [18] for finitely smooth normal forms of deterministic dynamical systems. Here we extend it further so that it can be applied to random dynamical systems where the hyperbolicity is nonuniform. Our modified approach also allow us to establish theorems of Poincaré type [23].

Lemma 4.5. *Let $\phi_\tau(n, \omega), \tau \in [0, 1], n \in \mathbb{Z}$, be a family of C^1 random dynamical systems in $\mathbb{R}^d$ with fixed point 0 whose time-one mapping has the form*

$$\phi_\tau(1,\omega)x = \varphi_\tau(\omega,x) = \varphi(\omega,x) + \tau R(\omega,x).$$

If there exists a measurable mapping $r : \Omega \times \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d$ such that

(i) *$r(\omega, \cdot, \cdot)$ is C^k and satisfies*

$$|r(\omega,x,\tau)| \le a(\omega)|x|^2, \text{ for } |x| \le 1, \tag{46}$$

where $a(\omega)$ is a tempered function from above;

(ii) *r satisfies the following equation*

$$D\varphi_\tau(\omega,x)r(\omega,x,\tau) - r(\theta\omega,\varphi_\tau(\omega,x),\tau) = -R(\omega,x), \tag{47}$$

then ϕ_0 and ϕ_1 are C^k locally conjugate.

Proof. To prove this lemma, we first introduce in $\mathbb{R}^d \times [0, 1]$ a family of diffeomorphisms:

$$\psi_\omega : (x,\tau) \mapsto (\varphi_\tau(\omega,x),\tau)$$

and a family of vector fields:

$$V_\omega = (r(\omega, x, \tau), 1).$$

Here, ω is regarded as a parameter. Then, from (47) it follows that

$$D\psi_\omega V_\omega = V_{\theta\omega} \circ \psi_\omega. \tag{48}$$

Denote by g^t_ω the local flow generated by V_ω, which is a C^k random diffeomorphism. Then (48) implies that

$$\psi_\omega \circ g^t_\omega = g^t_{\theta\omega} \circ \psi_\omega.$$

Using(46), we have that g^t_ω is well-defined for $x \in B_{\rho(\omega)}, \rho(\omega) = \frac{1}{1+a(\omega)}$, and $t \in [0,1]$. Write $g^1_\omega(x, 0) = (h(\omega, x), 1)$. Then we have

$$\psi_\omega \circ g^1_\omega(x, 0) = \psi_\omega(h(\omega, x), 1) = (\phi_1(\omega, h(\omega, x)), 1) = ((\varphi + R)(\omega, h(\omega, x)), 1)$$

and

$$g^1_{\theta\omega} \circ \psi_\omega(x, 0) = g^1_{\theta\omega}(\phi_0(\omega, x), 0) = g^1_{\theta\omega}(\varphi(\omega, x), 0) = (h(\theta\omega, \phi(\omega, x)), 1).$$

Therefore, $h(\theta\omega, \varphi(\omega, x)) = (\varphi + R)(\omega, h(\omega, x))$. Clearly, $h(\omega, x)$ is a C^k random diffeomorphism. This completes the proof. ☐

Next, we construct formal solutions of linear functional equation (47) in terms of an infinite series. We will prove later that they are convergent for R being R_1 and R_2.

For simplicity, we set $(f_*R)(x) := (Df\, R) \circ f^{-1}(x)$ for a diffeomorphism $f : \mathbb{R}^d \to \mathbb{R}^d$ and a vector field R in $\mathbb{R}^d$. Notice that $f_*(g_*R) = (f \circ g)_*R$. Then, we have two formal solutions to equation (47).

Lemma 4.6. *The functions*

$$r_1(\omega, x, \tau) = -\sum_{m=1}^{\infty} \Big(\big(\phi_\tau^{-1}(m, \omega)\big)_* R\big(\theta^{m-1}\omega, \varphi_\tau^{-1}(\theta^{m-1}\omega, \cdot)\big) \Big)(x) \tag{49}$$

and

$$r_2(\omega, x, \tau) = \sum_{m=0}^{\infty} \Big(\big(\phi_\tau^{-1}(-m, \omega)\big)_* R\big(\theta^{-m-1}\omega, \varphi_\tau^{-1}(\theta^{-m-1}\omega, \cdot)\big) \Big)(x) \tag{50}$$

are formal solutions of (47).

Proof. The proof of this lemma is quite straightforward. Let $\tilde{R}(\omega, x, \tau) = R(\omega, \varphi_\tau^{-1}(\omega, x))$. Then, equation (47) is equivalent to

$$\big((\varphi_\tau)_* r(\omega, \cdot, \tau)\big)(x) = r(\theta\omega, x, \tau) - \tilde{R}(\omega, x, \tau). \tag{51}$$

Note that $\varphi_\tau(\omega, \cdot) = \phi_\tau(1, \omega)$. We calculate

$$\begin{aligned}
(\varphi_\tau)_* r_1 &= -\sum_{m=1}^{\infty} \big((\phi_\tau(1, \omega)\phi_\tau(m, \omega)^{-1})_* \tilde{R}(\theta^{m-1}\omega, \cdot, \tau)\big)(x) \\
&= -\sum_{m=1}^{\infty} \big((\phi_\tau(m-1, \theta\omega)^{-1})_* \tilde{R}(\theta^{m-1}\omega, \cdot, \tau)\big)(x) \\
&= -\tilde{R}(\omega, x, \tau) - \sum_{m=1}^{\infty} \big((\phi_\tau(m, \theta\omega)^{-1})_* \tilde{R}(\theta^{m}\omega, \cdot, \tau)\big)(x) \\
&= -\tilde{R}(\omega, x, \tau) + r_1(\theta\omega, x, \tau).
\end{aligned}$$

Similarly, for the second function we obtain

$$\begin{aligned}(\phi_\tau)_* r_2 &= \sum_{m=0}^{\infty} \big((\phi_\tau(1,\omega)\phi_\tau(-m,\omega)^{-1})_* \tilde{R}(\theta^{-m-1}\omega,\cdot,\tau)\big)(x)\\ &= -\tilde{R}(\omega,x,\tau) + r_2(\theta\omega,x,\tau).\end{aligned}$$

This completes the proof.

□

We are now ready to show the convergence of the formal series solutions. Let

$$\bar{R}_1(\omega,x,\tau) = \tilde{R}_1(\omega,\phi_\tau^{-1}(1,\omega,x)), \quad \bar{R}_2(\omega,x,\tau) = \tilde{R}_2(\omega,\psi_\tau^{-1}(1,\omega,x)),$$

then $\bar{R}_1$ and $\bar{R}_2$ are $C^{[N/2]}$ (resp. C^∞) tempered and satisfy

$$|\bar{R}_1(\omega,x,s)| \le \begin{cases} \bar{C}_0(\omega)|y|^{N/2}\sigma_{\rho_1(\omega)}(|y|), & \text{if } jet^N_{x=0}R = 0\\ \bar{C}_m(\omega)|y|^m\sigma_{\rho_1(\omega)}(|y|) \text{ for all } m\in\mathbb{N}, & \text{if } jet^\infty_{x=0}R = 0\end{cases} \tag{52}$$

$$|\bar{R}_2(\omega,x,s)| \le \begin{cases} \bar{C}_0(\omega)|z|^{N/2}\sigma_{\rho_1(\omega)}(|z|), & \text{if } jet^N_{x=0}R = 0\\ \bar{C}_m(\omega)|z|^m\sigma_{\rho_1(\omega)}(|z|) \text{ for all } m\in\mathbb{N}, & \text{if } jet^\infty_{x=0}R = 0\end{cases} \tag{53}$$

where functions $\bar{C}_m, m \ge 0$, are tempered from above.

The next lemma gives two smooth solutions of (47) with $R = \bar{R}_1$ and $\bar{R}_2$. We will encounter a C^k-conjugacy order function $N_0(\omega) = N_0(\alpha(\omega),\beta(\omega),k)$ depending only on $\alpha(\omega),\beta(\omega)$, and k, which is θ-invariant and has integer values. Recall that we may decompose Ω into a union of countably many disjoint θ-invariant measurable sets: $\Omega = \cup_{i\ge1}\tilde{\Omega}_i$. On $\tilde{\Omega}_i$, $N_0(\omega)$ is a constant.

Lemma 4.7. (i) *For any given integer $k > 0$, there exists a θ-invariant measurable integer function $N_0 = N_0(\alpha(\omega),\beta(\omega),k)$ such that if $N \ge N_0(\omega)$ for $\omega \in \tilde{\Omega}_i$ and $jet^N_{x=0}R = 0$, then the functions $r_1(\cdot,\cdot,\cdot)$ and $r_2(\cdot,\cdot,\cdot)$ defined by*

$$r_1(\omega,x,s) = -\sum_{n=1}^{\infty}\Big(\big(\phi_\tau^{-1}(n,\omega)\big)_*\bar{R}_1(\theta^{n-1}\omega,\cdot,\tau)\Big)(x) \tag{54}$$

and

$$r_2(\omega,x,\tau) = \sum_{n=0}^{\infty}\Big(\big(\psi_\tau^{-1}(-n,\omega)\big)_*\bar{R}_2(\theta^{-n-1}\omega,\cdot,\tau)\Big)(x) \tag{55}$$

are measurable from $\tilde{\Omega}_i \times \mathbb{R}^d \times [0,1]$ to $\mathbb{R}^d$ and are C^k smooth with respect to (x,τ) for each fixed $\omega \in \tilde{\Omega}_i$ and satisfy (46).

(ii) *If $jet^\infty_{x=0}R = 0$, then $r_1(\omega,\cdot,\cdot)$, $r_2(\omega,\cdot,\cdot) : \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$ are C^∞ for all $\omega \in \Omega$.*

Proof. We only prove the lemma for r_1. The proof for r_2 is the same as r_1. We first prove (i). First, by Theorem 3.7, we have

$$|\Pi_s\phi_\tau(n,\omega,x)| \le 2K(\omega)e^{-\beta(\omega)n/5}|y|. \tag{56}$$

The proof for r_1 is based on the following key arguments. First, from (56), the y component of ϕ_τ decays along its orbits at at the least exponential rate $e^{-\beta(\omega)n/5}$. Hence, from (52) the composition $\bar{R}_1 \circ \phi_\tau(n,\omega,x)$ decays not slower than the exponential function $\exp(-N\beta(\omega)n/10)$, where N may be chosen arbitrarily large. On the other hand, from (40) the norm $\|D\phi_\tau^{-1}(n,\omega,x)\|$ is dominated by $d_1(\omega)M_1(\omega)^n$. Hence, by choosing N sufficiently large, the terms in the series (54) goes to zero exponentially, and the series converges. The same arguments work for the derivatives of r_1 up to order k.

Now, we look at how the arguments work exactly. We first show that r_1 is well defined and (46) holds for $N > 10 + \frac{10}{\beta(\omega)} \ln M_1(\omega)$. From Lemma 4.4 and inequality (56) and (52), we have

$$\begin{aligned}
|r_1(\omega, x, \tau)| &\le \sum_{n=1}^{\infty} |\Big(\big(\phi_\tau^{-1}(n,\omega)\big)_* \bar{R}_1(\theta^{n-1}\omega, \cdot, \tau)\Big)(x)| \\
&\le \sum_{n=1}^{\infty} d_1(\omega) M_1(\omega)^n \bar{C}_0(\theta^{n-1}\omega) \|\Pi_s \phi_\tau(n,\omega,x)\|^{N/2} \\
&\le \sum_{n=1}^{\infty} d_1(\omega) M_1(\omega)^n \bar{C}_0(\theta^{n-1}\omega) (2K(\omega) e^{-\beta(\omega)n/5}|y|)^{N/2} \\
&\le \sum_{n=1}^{\infty} d_1(\omega) \bar{C}_0(\theta^{n-1}\omega)(2K(\omega))^{N/2} \Big[M_1(\omega) e^{-\beta(\omega)N/10}\Big]^n |y|^{N/2} \\
&\le \sum_{n=1}^{\infty} d_1(\omega)(2K(\omega))^{N/2} \tilde{C}(\omega) \Big[M_1(\omega) e^{\beta(\omega)-\beta(\omega)N/10}\Big]^n |y|^{N/2} \\
&\le d_1(\omega)(2K(\omega))^{N/2} \tilde{C}(\omega)/(1 - M_1(\omega) e^{\beta(\omega)-\beta(\omega)N/10})|y|^{N/2} \\
&\le a(\omega)|x|^{N/2} < \infty
\end{aligned} \tag{57}$$

where Lemma 2.7 is used to bound $\bar{C}(\theta^{n-1}\omega)$ by $\tilde{C}(\omega)e^{\beta(\omega)n}$ and $a(\omega) = d_1(\omega)(2K(\omega))^{N/2}\tilde{C}(\omega)/(1 - M_1(\omega)e^{\beta(\omega)-\beta(\omega)N/10})$ which is a tempered function from above.

Next, we show $r_1(\omega, \cdot, \cdot)$ is C^k smooth if N is large enough. It follows from (40) that the derivatives of ϕ_τ^{-1} of order no greater than $k+1$ are bounded by $\max_{1\le i\le k+1}\{d_i(\omega)M_i(\omega)^n\}$. On the other hand, each term of the derivatives $D^k\phi_\tau^{-1}(n,\omega,x)_*\bar{R}_1(\theta^{n-1}\omega, x, \tau)$ has the factor $(D^m\bar{R}_1)\circ \phi_\tau(n,\omega,x)$ for some $m \le k$, which decays faster than $\exp((-N+2k)\beta(\omega)n/10)$. All other factors (the maximum number of them are $k+1$) are bounded by $\max_{1\le i\le k+1}\{d_i(\omega)M_i(\omega)^n\}$. Thus, we choose N_0 to be

$$N_0(\alpha(\omega), \beta(\omega), k) = \Big(10 + 2k + \big[\frac{10}{\beta(\omega)} \ln\big(\max_{1\le i\le k+1} M_i(\alpha(\omega), \beta(\omega))\big)^{k+1}\big]\Big) + 1, \tag{58}$$

where $[a]$ is the usual notation for the integer part of a. Then, for $N \ge N_0(\alpha(\omega), \beta(\omega))$,

$$D^k\phi_\tau^{-1}(n,\omega,x)_*\bar{R}_1(\theta^{n-1}\omega, x, \tau)$$

decays exponentially as $n \to \infty$ and the corresponding series converges. Furthermore, we have

$$\|D^k r_1(\omega, x, s)\| \le a_k(\omega)|x|^{N/2-k},$$

where $a_k(\omega)$ is a random variable tempered from above. Hence, the series (54) converges in C^k. Since $N_0(\alpha(\omega), \beta(\omega), k)$ is constant on $\tilde{\Omega}_i$, for fixed $N \ge N_0$, the series (54) converges in C^k for every $\omega \in \Omega_i$. Clearly, r_1 is measurable from $\tilde{\Omega}_i \times \mathbb{R}^d \times [0,1]$ to $\mathbb{R}^d$. Next, we prove the conclusion (ii). Since $jet^{\infty}_{x=0}R = 0$, applying (i) to each $\tilde{\Omega}_j$, we have that r_1 is C^∞ for each $\omega \in \Omega$ and $r : \Omega \times \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$ is measurable since Ω is the union of countably many disjoint measurable sets $\tilde{\Omega}_i$. This completes the proof of the lemma. □

Summarizing the above discussions gives

Proposition 4.8. (i) *For each $k \in \mathbb{N}$, there exists an integer $N_0 = N_0(\alpha(\omega), \beta(\omega), k)$ such that*

- *(i_1) if $\tilde{\phi}|_{\tilde{\Omega}_i}$ and $\tilde{\psi}|_{\tilde{\Omega}_i}$ are C^{N_0} and (38) holds with $N = N_0$ for $\omega \in \tilde{\Omega}_i$, then $\tilde{\phi}|_{\tilde{\Omega}_i}$ and $(\tilde{\phi} + \tilde{R}_1)|_{\tilde{\Omega}_i}$, $(\tilde{\phi} + \tilde{R}_1)|_{\tilde{\Omega}_i}$ and $(\tilde{\phi} + \tilde{R}_1 + \tilde{R}_2)|_{\tilde{\Omega}_i} = \tilde{\psi}_{\tilde{\Omega}_i}$ are C^k locally conjugate;*
- *(i_2) if ϕ and ψ are C^∞ locally tempered and (38) holds with $N = N_0$ for $\omega \in \Omega$, then $\tilde{\phi}$ and $(\tilde{\phi} + \tilde{R}_1)$, $(\tilde{\phi} + \tilde{R}_1)$ and $(\tilde{\phi} + \tilde{R}_1 + \tilde{R}_2) = \tilde{\psi}$ are C^k locally conjugate;*

(ii) *if ϕ and ψ are C^∞ locally tempered and (39) holds, then $\tilde{\phi}$ and $\tilde{\phi} + \tilde{R}_1$, $\tilde{\phi} + \tilde{R}_1$ and $\tilde{\phi} + \tilde{R}_1 + \tilde{R}_2 = \tilde{\psi}$ are C^∞ locally conjugate.*

This proposition together with Lemma 4.3 and the random tranformation (35) gives Theorem 4.2.

Now, we give the corollaries of Theorem 4.2. We first introduce a concept called formal conjugacy.

Definition 4.9. *Two C^∞ random diffeomorphisms ϕ and ψ with fixed point 0 in $\mathbb{R}^d$ are said to be formally tempered conjugate if there exists a formal random coordinate transformation $h(\omega, x) \sim x + \sum_{n \geq 2} h_n(\omega, x)$ where all coefficients of the homogeneous polynomials h_n of degree n for all $n \geq 2$ are tempered from above such that*

$$jet^\infty_{x=0}\psi(\omega) = jet^\infty_{x=0} h(\theta\omega)^{-1} \circ \phi(\omega) \circ h(\omega). \tag{59}$$

The next theorem is analog to Chen's theorem in deterministic case [10].

Theorem 4.10. *(Chen's theorem for random diffeomorphisms) If two locally tempered C^∞ random diffeomorphisms in $\mathbb{R}^d$ with hyperbolic fixed point $x = 0$ are formally tempered conjugate, then they are C^∞ locally congugated.*

To prove this theorem, we need the following lemma.

Lemma 4.11. *Given a formal random Taylor series $h(\omega, x)$ with tempered coefficients, there exists a C^∞ locally tempered random mapping $\tilde{h}(\omega, x)$ whose formal random Taylor series expansion is equal to h.*

Proof. All notations used here are the same as in the proof of Lemma 4.3. Suppose that

$$h(\omega, x) \sim \sum_{k \in \mathbb{Z}^d_+} \frac{x^k}{k!} h_k(\omega).$$

Let

$$\mu_k(\omega) = (|h_k(\omega)| + 1)k!.$$

Then μ_k are tempered from above. Let

$$\tilde{h}(\omega, x) = \sum_{k \in \mathbb{Z}^d_+}^{\infty} \frac{x^k}{k!} h_k(\omega) \chi(\mu_k |x|).$$

Then $\tilde{h}$ is a C^∞ tempered mapping with the prescribed formal random power series h. □

The proof of Theorem 4.10. Theorem 4.10 is just a corollary of the conclusion (ii) in Theorem 4.2 and Lemma 4.11. Indeed, suppose that two C^∞ diffeomorphisms ϕ and ψ are formally tempered conjugate, i.e., there is a formally tempered random coordinate transformation h such that (59) hold. By Lemma 4.11, there exists a locally tempered C^∞ random diffeomophism $\tilde{h}$ such that the Taylor series expansion of $\tilde{h}$ is equal to h. Let

$$\tilde{\psi}(\omega) = \tilde{h}(\theta\omega)^{-1} \circ \phi(\omega) \circ \tilde{h}(\omega). \tag{60}$$

Then, ψ and $\tilde{\psi}$ have the same Taylor series expansion at $x = 0$. By the conclusion (ii) of Theorem 4.2, they are C^∞ locally conjugate. By (60), ϕ and ψ C^∞ locally conjugate. The proof is completed. □

Theorem 4.12. *(Sternberg Type) Let ϕ be a locally tempered random diffeomorphism in $\mathbb{R}^d$ with a hyperbolic fixed point $x = 0$. Then,*

- (i) *for any $k \in \mathbb{N} \cup \{\infty\}$, there exists a θ-invariant measurable integer function $N_0(\omega) = N_0(\alpha(\omega), \beta(\omega), k)$ given by (58) such that*
  - (i_1) *if $\phi\big|_{\tilde{\Omega}_i}$ is C^{N_0} tempered and the Lyapunov exponents of linear random dynamical system generated by $D\phi(\omega, 0)$ satisfy the nonresonant conditions of orders N_0, i.e.*

    $$\lambda_i \neq (\lambda, \tau), \quad 2 \leq |\tau| < N_0,$$

    then $\phi\big|_{\tilde{\Omega}_i}$ is C^k locally conjugate to its linear part $D\phi(\omega, 0)\big|_{\tilde{\Omega}_i}$.
  - (i_2) *if ϕ is C^∞ locally tempered and the Lyapunov exponents satisfy the nonresonant conditions of orders N_0, then ϕ is C^k locally conjugate to its linear part $D\phi(\omega, 0)$.*
- (ii) *if ϕ is C^∞ locally tempered and the Lyapunov exponents satisfy the nonresonant conditions of any orders, then ϕ is C^∞ locally conjugate to its linear part $D\phi(\omega, 0)$.*

Remark: (ii) in this theorem settles an open problem raised by L. Arnold in [1].

Proof. We first look at the case $k < \infty$ and limit ourselves to $\tilde{\Omega}_i$. Suppose $\psi\big|_{\tilde{\Omega}}$ is C^{N_0} locally tempered. Thus, restricting θ to Ω_i and using Corollary 8.2.10 in [1], page 419, we obtain a unique near-identity polynomial transformation $H_{i,N_0}(\omega, x) = x + \sum_{n=2}^{N_0} h_{i,n}(\omega, x)$ over $\tilde{\Omega}_i$ with tempered coefficients such that ϕ is transformed by H_{i,N_0} to ψ, whose N_0-jet in the Taylor expansion of order N is linear, $\psi_i(\omega, x) = A(\omega)x + O(|x|^{N_0})$. By Theorem 4.2, $\psi\big|_{\tilde{\Omega}}$ is C^k locally conjugate to its linear part $A(\omega)\big|_{\tilde{\Omega}}$. This proves (i_1). If ϕ is C^∞ locally tempered, then the above statements are true for all $\tilde{\Omega}_i$. Since Ω is the union of countably many disjoint θ-invariant measurable sets Ω_i, patching H_{i,N_0} and ψ_i together, respectively, we obtain a near-identity random polynomial $H(\omega, x)$ and a C^∞ random diffeomorphism ψ with the form $\psi(\omega, x) = A(\omega)x + O(|x|^{N_0})$ such that ϕ is C^∞ locally conjugate to ψ through $H(\omega, x)$. By Theorem 4.2, ψ is C^k conjugate to its linear part in some tempered ball. This completes the proof of (i_2).

For the case $k = \infty$, by the assumption, the linear cocycle generated by $A(\omega)$ is nonresonant of any order $n \geq 2$. By Theorem 8.2.11 in [1], page 419, ϕ is formally tempered conjugate to its linear part. Thus, Sternberg's theorem follows from Theorem 4.10. □

Theorem 4.13. *Let ϕ be a C^∞ tempered random diffeomorphism in $\mathbb{R}^d$ with a fixed point $x = 0$. If all Lyapunov exponents of the linear cocycle generated by $A = D\phi(\cdot, 0)$ have the same sign, then ϕ is C^∞ locally conjugate to a polynomial random diffeomorphism.*

Proof. Since all Lyapunov exponents $\lambda_1(\omega), \cdots, \lambda_d(\omega)$ have the same sign,

$$\lambda_i(\omega) \neq (\lambda(\omega), \tau), \text{ for } n = |\tau| > \max|\lambda_i(\omega)| / \min|\lambda_i(\omega)|,$$

which implies A is nonresonant of order n for n large enough. Let

$$N_1(\omega) = [\max|\lambda_i(\omega)|/\min|\lambda_i(\omega)] + 1.$$

Clearly, $N_1(\omega)$ is a θ-invariant measurable function with integer values. Thus we may decompose Ω into a union of countably many disjoint invariant measurable sets

$$\Omega = \cup_i \Omega_i^*$$

such that $N_1(\omega)$ is constant on each Ω_i^*. Restricting θ to Ω_i^* and using Proposition 8.2.9 in [1],page 418, $\phi\big|_{\Omega_i^*}$ is formally tempered conjugate to a polynomial random diffeomorphism. Patching them together gives ϕ is formally tempered conjugate to a polynomial random diffeomorphism. Then the theorem follows from Theorem 4.10. □

Finally, we give the theory of normal forms for random diffeomorphisms with diagonal linear part. We follow Arnold[1]. Let ϕ be a C^∞ random diffeomorphism in $\mathbb{R}^d$ with fixed point $x = 0$, i.e. $\phi(\omega)0 = 0$. The idea of the normal form theory is to find a smooth random coordinate transformation h such that the conjugate random diffeomorphism $\psi(\omega) = h^{-1}(\theta\omega) \circ \phi(\omega) \circ h(\omega)$ has the simplest possible form, which is equivalent to

$$h(\theta\omega, \psi(\omega, x)) = \phi(\omega, h(\omega, x)), \quad a.s. \quad \omega \in \Omega. \tag{61}$$

Such the simplest form is easier to be studied and provides useful information for investigating the original system. Theorem 4.12 means that if the linear cocycle generated by $A(\omega) = D\phi(\omega, 0)$ is nonresonant, ϕ is smoothly conjugate to is linear part $A(\omega)$, i.e. its normal form is a linear form. We now consider the resonant case. We assume that there is a random basis $\mathcal{S}(\omega) = (u_1(\omega), \dots, u_d(\omega))$ of $\mathbb{R}^d$ so that in the $\mathcal{S}(\omega), \mathcal{S}(\theta\omega)$ bases the matrix representation $A(\omega)$ of $D\phi(\omega, 0)$ is diagonal and the change from the standard basis to the random basis $\mathcal{S}(\omega)$ is a Lyapunov cohomology(this is possible if the Lyapunov exponents of $A(\omega)$ are distinct, see[1]). We denote by $H_{n,d}$ the vector space of homogeneous polynomials of degree n in d variables with values in $\mathbb{R}^d$ and let

$$\mathcal{H}_{n,d} := \{f : \Omega \to H_{n,d} \text{ measurable}\}$$

be the vector space of random homogeneous polynomials of degree $n \geq 2$ in d variables with values in $\mathbb{R}^d$. Let $A(\omega) = diag(a_1(\omega), \dots, a_d(\omega))$ and $\lambda(\omega) = (\lambda_1(\omega), \dots, \lambda_d(\omega))$ be the Lyapunov exponents of the linear cocycle generated by $A(\omega)$.

Definition 4.14. *The vector-valued monomial $x^m u_i(\omega)$ is said to be resonant if $\lambda_i(\omega) = (m, \lambda(\omega))$ and $|m| \geq 2$.*

We now expand the three mappings ϕ, h and ψ in equation (61) as formal Taylor series at 0:

$$\phi(\omega, x) \sim A(\omega)x + \sum_{n=2}^{\infty} \phi_n(\omega, x),$$

$$\psi(\omega, x) \sim A(\omega)x + \sum_{n=2}^{\infty} \psi_n(\omega, x),$$

and

$$h(\omega, x) \sim x + \sum_{n=2}^{\infty} h_n(\omega, x),$$

where $\phi_n, \psi_n, h_n \in \mathcal{H}_{n,d}$. Inserting these expansions into equation (61), equating coefficients yields the following result. The terms of order n of the formal power series expansion of ψ are successively given by the terms of order n of ϕ and h and lower order terms. More specifically, $\psi_1(\omega) = \phi_1(\omega) = A(\omega)$ and

$$A(\omega)h_n(\omega, x) - h_n(\theta\omega, A(\omega)x) = \psi_n - \gamma_n, \quad n \geq 2. \tag{62}$$

Here

$$\gamma_n(\omega) = T_n((\phi(\omega) - A(\omega)) \circ S_{n-1}h(\omega) + S_{n-1}h(\theta\omega) \circ S_{n-1}\psi(\omega)),$$

where $T_n(f)$ denotes the terms of order n in the formal Taylor expansion of f and $S_N(f) := \sum_{n=1}^N T_n(f)$ is the N-jet of f.

Let $h^i_m(\omega), \psi^i_m(\omega)$, and $\gamma^i_m(\omega)$ denote the coefficients of $x^m u_i(\omega)$ of functions h_n, ψ_n and γ_n, respectively. For $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$, let $a(\omega)^m = a_1(\omega)^{m_1} \cdots a_d(\omega)^{m_d}$. Then the homological equation (62) gives

$$a_i(\omega)h^i_m(\omega) - a(\omega)^m h^i_m(\theta\omega) = \psi^i_m(\omega) - \gamma^i_m(\omega),$$

which has a tempered solution for $\psi^i_m(\omega) = 0$:

$$h^i_m(\omega) = \begin{cases} -a_i(\omega)^{-1}\gamma^i_m(\omega) - a_i(\omega)^{-1}\sum_{k=1}^{\infty}\prod_{j=1}^{k}(a(\theta^{j-1}\omega)^m a_i(\theta^j\omega)^{-1})\gamma^i_m(\theta^k\omega), \\ \qquad \text{if } \lambda_i(\omega) > (m, \lambda(\omega)) \\ \sum_{k=0}^{\infty}(\prod_{j=0}^{k} a_i(\theta^{-j}\omega)a(\theta^{-j-1}\omega)^{-m})\gamma^i_m(\theta^{-k-1}\omega), \\ \qquad \text{if } \lambda_i(\omega) < (m, \lambda(\omega)) \end{cases}$$

and has a solution $h^i_m(\omega) = 0$ for $\psi^i_m(\omega) = \gamma^i_m(\omega)$ if $\lambda_i(\omega) = (m, \lambda(\omega))$. Thus, together with Lemma 4.11, we have the following

Theorem 4.15. *Any C^∞ tempered random diffeomorphism in $\mathbb{R}^d$ with a fixed point $x = 0$ and a diagonal linear part is formally tempered conjugate to a C^∞ random diffeomorphism whose formal random Taylor series expansion has resonant monomials only.*

As a corollary of Theorem 4.2, 4.15, and 4.13 we have the following

Theorem 4.16. *Let ϕ be a C^∞ random diffeomorphism in $\mathbb{R}^d$ with a hyperbolic fixed point 0 and with a diagonal matrix of linear part. Then, for each $k > 0$, there exists a number $N_0(\omega) = N_0(\alpha(\omega), \beta(\omega), k)$ such that ϕ is C^k locally conjugate to a random polynomial mapping of degree $N_0(\omega)$ which consists of resonant monomials only. Moreover, if the fixed point 0 is stable or unstable,i.e., all Lyapunov exponents have the same sign, then ϕ is C^∞ conjugate to a polynomial mapping which consists of resonant monomials only.*

Example. Let ϕ be a C^∞ tempered random diffeomorphism in $\mathbb{R}^2$ with a hyperbolic fixed point $x = 0$ over an ergodic metric dynamical system $\{\theta^n\omega\}_{n\in\mathbb{Z}}$. Let λ_1, λ_2 with $|\lambda_1| \geq |\lambda_2|$ be the Lyapunov exponents of ϕ. Then we have

(i) When $\lambda_1\lambda_2 > 0$,
  - (i_1) if $\lambda_1/\lambda_2 \notin \mathbb{N} \setminus \{1\}$, then ϕ is C^∞ locally conjugate to its linear part.
  - (i_2) if $\lambda_1/\lambda_2 = n \in \mathbb{N} \setminus \{1\}$, then ϕ is C^∞ locally conjugate to

$$\psi(\omega, x) = \begin{pmatrix} a_1(\omega)x_1 + c(\omega)x_2^n \\ a_2(\omega)x_2 \end{pmatrix}.$$

(ii) When $\lambda_1\lambda_2 < 0$,
  - (ii_1) if $\lambda_1/\lambda_2 \notin \mathbb{Q}$, then ϕ is C^∞ locally conjugate to its linear part.

(ii_2) if $\lambda_1/\lambda_2 = -m/n \in \mathbb{Q}$, then for any $k > 0$, there is a number $N = N(k)$, ϕ is C^k locally conjugate to the following polynomial random diffeomorphism

$$\psi(\omega, x) = \begin{pmatrix} x_1(a_1(\omega) + \sum_{i=1}^N c_i(\omega)(x_1^n x_2^m)^i) \\ x_2(a_2(\omega) + \sum_{i=1}^N d_i(\omega)(x_1^n x_2^m)^i) \end{pmatrix}.$$

Remark. The theorem of Sternberg type holds for continuous time random dynamical systems. For simplicity, consider a C^∞ continuous time random dynamical system $\phi(t, \omega, x)$ with a hyperbolic fixed point $x = 0$, $\phi(t, \omega, 0)$=0. Assume that $D\phi(t, \omega, 0)$ satisfies the conditions of the Multiplicative Ergodic Theorem and their Lyapunov exponents satisfy the resonant conditions for all orders. Then by Theorem 4.12, the random diffeomorphism $\phi(1, \omega, \cdot)$ is C^∞ locally conjugate to its linear part $D\phi(1, \omega, 0)$ through a random diffeomorphism $h(\omega, x)$. Furthermore, one can show that $h(\theta^t\omega, x)$ is continuous in t. Using the standard transformation modified from the deterministic case

$$H(\omega, x) = \int_0^1 D\phi(s, \theta^s\omega, 0)h\big(\theta^s, \phi(s, \omega, x)\big)\, ds,$$

we have

$$H\big(\theta^t\omega, \phi(t, \omega, x)\big) = D\phi(t, \omega, 0)H(\omega, x).$$

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