Random walks on FKG-horizontally oriented lattices

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5th March 2003

Keywords: Markov chains, random environments, oriented graphs, associated random variables, recurrence vs transience, functional limit theorems.

MSC 2000 Classification: 60K37 (primary), 60F17, 60K35.

Abstract:

We study the asymptotic behavior of the simple random walk on oriented versions of $\mathbb{Z}^2$. The considered lattices are not directed on the vertical axis but unidirectional on the horizontal one, with symmetric random orientations which are positively correlated. We prove that the simple random walk is transient and also prove a functional limit theorem in the space $\mathcal{D}([0,\infty[,\mathbb{R}^2)$ of càdlàg functions, with an unconventional normalization.

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1 Introduction

The use of random walks as a tool in mathematical physics is now well established and they have been for example widely used in classical statistical mechanics to study critical phenomena (see [9]). It has been recently observed that analogous methods in quantum statistical mechanics require the study of random walks on oriented lattices, due to the intrinsic non commutative character of the (quantum) world (see e.g. [5, 13]). Although random walks in random and non-random environments have been intensively studied for many years, only a few results on random walks on oriented lattices are known. The recurrence versus transience properties of simple random walks on oriented versions of $\mathbb{Z}^2$ are studied in [4] when the horizontal lines are unidirectional towards a random or deterministic direction. The interesting behavior of this model is that, depending on the orientation, the walk could be either recurrent or transient. In the deterministic "alternate" case, for which the orientations of horizontal lines are alternated, i.e. oriented rightwards at one level and leftwards at the following level, the recurrence of the simple random walk is proved, whereas the transience naturally arises when the orientations are all identical in infinite regions. More surprisingly, it is also proved that the recurrent character of the simple random walk on $\mathbb{Z}^2$ is lost when the orientations are i.i.d. with zero mean.

In this paper, we prove that the transience of the simple random walk still holds when the orientations are symmetric and positively correlated with a summable power law decay of correlations. We also prove a functional limit theorem for this walk with an unconventional normalization due to the random character of the environment of the walk, solving an open question of [4]. Our paper is organized as follows: the description of our model and the results are stated in Section 2. Section 3 is devoted to the proofs while illustrative examples of orientations, coming from statistical mechanics, are given in Section 4.

2 Model and results

2.1 FKG-horizontally oriented lattices

We consider a canonical probability space $(\Omega, \mathcal{B}, \mathbb{P})$ on which all the random variables are defined, and denote $\mathbb{E}$ (resp. Cov) the expectation (resp. covariance) under $\mathbb{P}$. By *orientations*, we mean a stationary symmetric family of $\{-1,+1\}$-valued random variables $(\epsilon_y)_{y\in\mathbb{Z}}$, with the following properties:

1. Associated random variables:
   For any $m \geq 0$, for any finite collection $(\epsilon_0, \dots, \epsilon_m)$, for any coordinatewise nondecreasing functions f, g on $\{-1,+1\}^m$,

$$\mathrm{Cov}\big[f(\epsilon_0,\dots,\epsilon_m); g(\epsilon_0,\dots,\epsilon_m)\big] \geq 0.$$

2. Summable power-law decay of correlations:
   There exists $\alpha > 1$ such that

$$\mathbb{E}\big[\epsilon_0\epsilon_y\big] = \mathcal{O}\big(\mid y \mid^{-\alpha}\big) \text{ when } \mid \mathrm{y} \mid \longrightarrow +\infty.$$

In our set-up, these orientations are natural extensions of Rademacher random variables of [4]. They have the same one-dimensional law ($\mathbb{P}[\epsilon_0 = +1] = \mathbb{P}[\epsilon_0 = -1] = \frac{1}{2}$) but are *not* *necessarily* independent. The notion of associated random variables (see [15]) is very natural in the context of Gibbs measures in statistical mechanics where it is equivalent to the FKG property of the joint distribution ν of the random field $\epsilon = (\epsilon_y)_{y\in\mathbb{Z}}$ ([10]). In such cases, we also say that the orientations are *positively correlated.* Examples of such distributions are ferromagnetic, possibly long range, Ising models, described in Section 4 at the end of this paper.

We use these associated random variables to build our *FKG-horizontally oriented* lattices. These lattices are oriented versions of $\mathbb{Z}^2$: the vertical lines are not oriented and the horizontal ones are unidirectional, the orientation at a level $y \in \mathbb{Z}$ being given by the random variable ϵ_y (say right if the value is $+1$ and left if it is -1). More formally we give the

Definition 2.1 (FKG-horizontally oriented lattices) *Let* $\epsilon = (\epsilon_y)_{y\in\mathbb{Z}}$ *be a sequence of* $\{-1,+1\}$*-valued, associated and symmetric random variables. The* FKG-horizontally oriented lattice $\mathbb{L}^\epsilon = (\mathbb{V}, \mathbb{A}^\epsilon)$ *is the directed graph with vertex set* $\mathbb{V} = \mathbb{Z}^2$ *and edge set* $\mathbb{A}^\epsilon$ *defined by the condition that for* $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{Z}^2$*,* $(u, v) \in \mathbb{A}^\epsilon$ *if and only if*

1. *either* $v_1 = u_1$ *and* $v_2 = u_2 \pm 1$
2. *or* $v_2 = u_2$ *and* $v_1 = u_1 + \epsilon_{u_2}$*.*

2.2 Simple random walk on $\mathbb{L}^\epsilon$

We consider the usual simple random walk $M = (M_n)_{n\in\mathbb{N}}$ on $\mathbb{L}^\epsilon$. It is a Markov chain with transition probabilities defined for all $(u, v) \in \mathbb{A}^\epsilon$ by

$$\mathbb{P}[M_{n+1} = v | M_n = u] = \begin{cases} \frac{1}{3} & \text{if (u, v)} \in \mathbb{A}^\epsilon \\ \\ 0 & \text{otherwise.} \end{cases}$$

Its transience is proved in [4] for almost every orientation when the directions are given by a sequence of i.i.d. random variables $(\epsilon_y)_{y\in\mathbb{Z}}$, i.e. when the law ν of the random field ϵ is a product probability measure. We generalize this result in this positively correlated and possibly non-independent context.

Theorem 2.2 *For* ν*-a.e. realization of the orientation* ϵ*, the simple random walk on the FKG-horizontally oriented lattice* $\mathbb{L}^\epsilon$ *is transient.*

We also answer in this general set-up to an open question of [4] and obtain a functional limit theorem with a suitable and unconventional normalization. We consider a Brownian motion $(W_t)_{t\geq 0}$ and denote $(L_t(x))_{t\geq 0}$ its corresponding local time at $x \in \mathbb{R}$. Moreover, we introduce a pair of independent Brownian motions $Z_+(x), Z_-(x), x \geq 0$. We assume these processes to be defined on one probability space and to be independent of each other so that the following process is well-defined for all $t \geq 0$:

$$\Delta_t = \int_0^\infty L_t(x) dZ_+(x) + \int_0^\infty L_t(-x) dZ_-(x). \tag{2.3}$$

This process is a particular example from a family of processes obtained in [12] as functional limits of $\mathbb{Z}$-valued random walks in random sceneries. Moreover, it has a continuous version which is self-similar with index $\frac{3}{4}$ and has stationary increments. We also introduce a real constant $m = \frac{1}{2}$, defined later as the mean of some geometric random variables related to the behavior of the walk in the horizontal direction.

Theorem 2.4 *The following convergence holds:*

$$\Big(\frac{1}{n^{3/4}}M_{[nt]}\Big)_{t\geq 0} \overset{\mathcal{D}}{\Longrightarrow} \frac{m}{(1+m)^{3/4}}(\Delta_t,0)_{t\geq 0} \tag{2.5}$$

where $\overset{\mathcal{D}}{\Longrightarrow}$ *stands for convergence in the space of càdlàg functions* $\mathcal{D}([0,\infty),\mathbb{R}^2)$ *endowed with the Skorohod topology.*

3 Proofs

3.1 Vertical and horizontal embeddings of the simple random walk

The simple random walk M defined on $(\Omega,\mathcal{B},\mathbb{P})$ can be decomposed into a vertical and an horizontal part by restriction to the corresponding axis. The vertical part is a simple random walk $Y=(Y_n)_{n\in\mathbb{N}}$ on the line. The (independent) σ-algebras generated by this vertical walk Y and the orientation ϵ are denoted respectively by

$$\mathcal{F}=\sigma(Y_n,\ n\in\mathbb{N}) \quad\text{and}\quad \mathcal{G}=\sigma(\epsilon_y,\ y\in\mathbb{Z}).$$

We also define for all $n\in\mathbb{N}$ and $y\in\mathbb{Z}$ the *local time* at level y of the walk Y to be

$$\eta_n(y)=\sum_{k=0}^{n}\mathbf{1}_{Y_k=y}.$$

The horizontal embedding is a random walk with $\mathbb{N}$-valued geometric jumps. More formally, a doubly infinite family $(\xi_i^{(y)})_{i\in\mathbb{N}^*,y\in\mathbb{Z}}$ of independent geometric random variables of parameter $p=\frac{1}{3}$ (and mean $m=\frac{1}{2}$) is given and one defines the embedded horizontal random walk $X=(X_n)_{n\in\mathbb{N}}$ by $X_0=0$ and for $n\geq 1$,

$$X_n=\sum_{y\in\mathbb{Z}}\epsilon_y\sum_{i=1}^{\eta_{n-1}(y)}\xi_i^{(y)}$$

with the convention that the last sum is zero when $\eta_{n-1}(y)=0$. Of course, the walk $\mathbb{M}_n$ does not coincide with (X_n,Y_n) but these objects are closely related: define for all $n\in\mathbb{N}$

$$T_n=n+\sum_{y\in\mathbb{Z}}\sum_{i=1}^{\eta_{n-1}(y)}\xi_i^{(y)}$$

to be the instant just after the random walk M has performed its n^{th} vertical move. The following Lemma is proved in [4].

Lemma 3.6 *1.* $M_{T_n}=(X_n,Y_n)$.

2. For a given orientation ϵ*, the transience of* $(M_{T_n})_{n\in\mathbb{N}}$ *implies the transience of* $(M_n)_{n\in\mathbb{N}}$.

3.2 Associated random variables

The extension from the i.i.d. case to our case is made possible by a comparison of the joint characteristic function of associated random variables with the product of the marginal ones, due to Newman *et al.* ([15]).

Lemma 3.7 *Let $\epsilon = (\epsilon_y)_{y\in\mathbb{Z}}$ be a sequence of associated random variables. Then, for all $t \in \mathbb{R}$, $n \in \mathbb{N}$,*

$$\Big|\mathbb{E}\Big[e^{it\sum_{y\in\mathbb{Z}}\epsilon_y\eta_n(y)}\big|\mathcal{F}\Big] - \prod_{y\in\mathbb{Z}}\mathbb{E}\Big[e^{it\epsilon_y\eta_n(y)}\big|\mathcal{F}\Big]\Big| \leq \frac{1}{2}t^2\sum_{x\neq y}\eta_n(x)\eta_n(y)\mathbb{E}\big[\epsilon_x\epsilon_y\big]. \tag{3.8}$$

Proof. It is based on Theorem 1 in [15], which states that for a finite family of p associated r.v.'s $(Z_1,\dots,Z_p)$ and real numbers $(r_1,\dots,r_p)$,

$$\Big|\mathbb{E}\big[e^{i\sum_{k=1}^{p}r_kZ_k}\big] - \prod_{k=1}^{p}\mathbb{E}\big[e^{ir_kZ_k}\big]\Big| \leq \frac{1}{2}\sum_{1\leq j\neq k\leq p}|r_j||r_k|\mathrm{Cov(Z_j,Z_k)}. \tag{3.9}$$

The sum and product of the l.h.s of (3.8) have a finite number of terms because $\eta_n(y) = 0$ for $|y| > n$. It is thus straightforward to derive (3.8) from (3.9) using the $\mathcal{F}$-measurability of the local times $\eta_n(y)$, the associativity of ϵ and its independence with the vertical walk Y. ∎

3.3 Proof of the transience of the simple random walk

The vertical walk Y is known to be recurrent and its asymptotic behavior is rather well controlled. The transience is due to the behavior of the embedded horizontal random walk X and to exploit it we introduce a partition of Ω between typical or untypical paths of Y.

In all this proof, for any $i \in \mathbb{N}$, δ_i is a strictly positive real number and we write $d_{n,i} = n^{\frac{1}{2}+\delta_i}$. Define the sets

$$A_n = \big\{\omega\in\Omega;\max_{0\leq k\leq 2n}|Y_k| < n^{\frac{1}{2}+\delta_1}\big\}\ \cap\ \big\{\omega\in\Omega;\max_{y\in\mathbb{Z}}\ \eta_{2n-1}(y) < n^{\frac{1}{2}+\delta_2}\big\}$$

and

$$B_n = \big\{\omega\in A_n;\Big|\sum_{y\in\mathbb{Z}}\epsilon_y\eta_{2n-1}(y)\Big| > n^{\frac{1}{2}+\delta_3}\big\}.$$

By Lemma 3.6, the transience of M will be insured as soon as

$$\sum_{n\in\mathbb{N}}\mathbb{P}[X_{2n} = 0; Y_{2n} = 0]\ <\ \infty \tag{3.10}$$

and to do so we first decompose $\mathbb{P}[X_{2n} = 0; Y_{2n} = 0]$ into

$$\mathbb{P}[X_{2n} = 0; Y_{2n} = 0; A_n^c] + \mathbb{P}[X_{2n} = 0; Y_{2n} = 0; B_n] + \mathbb{P}[X_{2n} = 0; Y_{2n} = 0; A_n\setminus B_n]. \tag{3.11}$$

Some results of the i.i.d. case of [4] still hold and in particular we can prove using standard techniques the following

Lemma 3.12 *For any* $\delta_1, \delta_2 > 0$,

$$\sum_{n \in \mathbb{N}} \mathbb{P}[X_{2n} = 0; Y_{2n} = 0; A_n^c] \ < \ \infty.$$

The second term of (3.11) is also a generic term of convergent series due to the untypical character of the paths in B_n. Again from [4] with standard techniques, we have the

Lemma 3.13 *For any* $\delta_3 > 0$,

$$\sum_{n \in \mathbb{N}} \mathbb{P}[X_{2n} = 0; Y_{2n} = 0; B_n] \ < \ \infty.$$

Now, we denote

$$p_n = \mathbb{P}[X_{2n} = 0; Y_{2n} = 0; A_n \setminus B_n].$$

To prove the theorem, it remains to show that for some $\delta_1, \delta_2, \delta_3 > 0$

$$\sum_{n \in \mathbb{N}} p_n \ < \ \infty. \tag{3.14}$$

Decompose

$$p_n = \mathbb{E}\Big[\mathbf{1}_{Y_{2n}=0} \mathbb{E}\big[\mathbf{1}_{X_{2n}=0} \mathbb{E}\big[\mathbf{1}_{A_n \setminus B_n} \big| \mathcal{F} \vee \mathcal{G}\big] \big| \mathcal{F}\big]\Big]. \tag{3.15}$$

It is well known that for the simple random walk Y, there exists $C > 0$ s.t.

$$\mathbb{P}[Y_{2n} = 0] \sim C \cdot n^{-\frac{1}{2}}, \ n \to +\infty \tag{3.16}$$

and we can prove as in [4] the

Lemma 3.17 *On the set* $A_n \setminus B_n$, *we have,*

$$\mathbb{P}\big[X_{2n} = 0 \big| \mathcal{F} \vee \mathcal{G}\big] = \mathcal{O}\Big(\sqrt{\frac{\ln n}{n}}\Big). \tag{3.18}$$

Hence, the transience of the simple random walk is a direct consequence of the following

Proposition 3.19 *For* $\alpha > 1$, *it is possible to choose* $\delta_1, \delta_2, \delta_3 > 0$ *such that there exists* $\delta > 0$ *and*

$$\mathbb{P}\big[A_n \setminus B_n \big| \mathcal{F}\big] = \mathcal{O}\big(n^{-\delta}\big). \tag{3.20}$$

Proof. We first use an auxiliary centered Gaussian random variable with variance $d_{n,3}^2$, the Anderson's inequality and Plancherel's formula to get

$$\mathbb{P}\big[A_n \setminus B_n \big| \mathcal{F}\big] \le C \cdot n^{\frac{1}{2} + \delta_3} \cdot I_n \tag{3.21}$$

where

$$I_n = \int_{-\pi}^{\pi} \mathbb{E}\big[e^{it \sum_{y \in \mathbb{Z}} \epsilon_y \eta_{2n-1}(y)} \big| \mathcal{F}\big] e^{-t^2 d_{n,3}^2/2} dt.$$

To use that for $t d_{n,3}$ small enough, $e^{-t^2 d_{n,3}^2/2}$ dominates the term under the expectation, we split the integral in two parts. For $b_n = \frac{n^{\delta_2}}{d_{n,3}}$, we write

$$I_n = I_n^1 + I_n^2$$

with

$$I_n^1 = \int_{|t|\le b_n} \mathbb{E}\big[e^{it\sum_{y\in\mathbb{Z}}\epsilon_y\eta_{2n-1}(y)}\big|\mathcal{F}\big]e^{-t^2d_{n,3}^2/2}dt$$
$$I_n^2 = \int_{|t|> b_n} \mathbb{E}\big[e^{it\sum_{y\in\mathbb{Z}}\epsilon_y\eta_{2n-1}(y)}\big|\mathcal{F}\big]e^{-t^2d_{n,3}^2/2}dt.$$

To control the integral I_n^2, we write

$$\begin{aligned}|I_n^2| &\le C\int_{|t|>b_n} e^{-t^2d_{n,3}^2/2}dt = \frac{C}{d_{n,3}}\int_{|s|>n^{\delta_2}} e^{-s^2/2}ds\\ &\le 2\frac{C}{d_{n,3}}n^{-\delta_2}e^{-n^{2\delta_2}/2}\end{aligned}$$

to get

$$|I_n^2| = \mathcal{O}\big(e^{-n^{2\delta_2}/2}\big).$$

Let $(\epsilon'_y)_{y\in\mathbb{Z}}$ be a sequence of i.i.d. random variables with marginal distribution $\mathbb{P}[\epsilon'_y = -1] = \mathbb{P}[\epsilon'_y = +1] = \frac{1}{2}$ and denote I_n^0 the integral which corresponds to I_n^1 in this case. Factorization is possible by independence and we write

$$I_n^0 = \int_{|t|\le b_n}\prod_{y\in\mathbb{Z}}\mathbb{E}\big[e^{it\epsilon'_y\eta_{2n-1}(y)}\big|\mathcal{F}\big]e^{-t^2d_{n,3}^2/2}dt = \int_{|t|\le b_n}\prod_{y\in\mathbb{Z}}\cos(\eta_{2n-1}(y)t)e^{-t^2d_{n,3}^2/2}dt$$

and decompose

$$I_n^1 = I_n^0 + (I_n^1 - I_n^0).$$

In order to get a validity of our result for any summable power law decay of correlations, we estimate I_n^0 by the following

Lemma 3.22 *For $\delta_3 > 2\delta_2$,*

$$|I_n^0| = \mathcal{O}\big(n^{-\frac{3}{4}+\frac{\delta_1}{2}}\big).$$

Proof. We first use Hölder's inequality to get

$$|I_n^0| \le \prod_y\Big[\Big(\int_{|t|\le b_n}|\cos(\eta_{2n-1}(y)t)|^{\frac{2n}{\eta_{2n-1}(y)}}dt\Big)^{\frac{\eta_{2n-1}(y)}{2n}}\Big].$$

Denote for all $y\in\mathbb{Z}, n\in\mathbb{N}$, $p_{n,y} = \frac{\eta_{2n-1}(y)}{2n}$, $C_n = \{y : \eta_{2n-1}(y)\neq 0\}$ and, for $y\in C_n$

$$J_{n,y} = \int_{|t|\le b_n}|\cos(\eta_{2n-1}(y)t)|^{1/p_{n,y}}dt = \frac{1}{\eta_{2n-1}(y)}\int_{|v|\le b_n\eta_{2n-1}(y)}|\cos(v)|^{1/p_{n,y}}dv,$$

to get $|I_n^0| \le \prod_y J_{n,y}^{p_{n,y}}$. Now, using the fact that we work on A_n, we choose $\delta_3 > 2\delta_2$ in order to have $b_n\eta_{2n-1}(y)\to 0$ uniformly in y when n goes to infinity. Using the equivalence around zero between the cosinus and the exponential, one has

$$\begin{aligned}|I_n^0| &\le \prod_y\Big(\sqrt{\frac{2\pi}{2n\eta_{2n-1}(y)}}\Big)^{p_{n,y}}\\ &= (2\pi)^{\frac{1}{2}\sum_y p_{n,y}}\exp\Big(-\frac{1}{2}\sum_y p_{n,y}\log(2n\eta_{2n-1}(y))\Big).\end{aligned}$$

The vector $\mathbf{p} = (p_{n,y})_{y\in C_n}$ defines a probability measure on C_n and we have

$$-\frac{1}{2}\sum_y p_{n,y}\log(2n\eta_{2n-1}(y)) = -\log 2n - \frac{1}{2}\sum_y p_{n,y}\log p_{n,y} = -\log 2n + \frac{1}{2}H(\mathbf{p})$$

where $H(\cdot)$ is the entropy of the probability vector $\mathbf{p}$, always bounded by $\log \mathrm{card} C_n$. We thus have on the set A_n,

$$|I_n^0| \leq \sqrt{2\pi}\exp(-\log 2n + \frac{1}{2}\log(2d_{n,1})) = \frac{\sqrt{\pi d_{n,1}}}{n} = \mathcal{O}(n^{-\frac{3}{4}+\frac{\delta_1}{2}}).$$

■

From Lemma 3.7, it is possible to compare I_n^1 to I_n^0 and control their difference.

Lemma 3.23 *For $\delta_3 > 2\delta_2$ and $\beta = 3\delta_3 + \alpha - 1 - 4\delta_2$,*

$$|I_n^1 - I_n^0| = \mathcal{O}\big(n^{-\beta}\big). \tag{3.24}$$

Proof. We have

$$\Big|I_n^1 - I_n^0\Big| \leq J_n := \int_{|t|\leq b_n}\Big|\mathbb{E}\big[e^{it\sum_{y\in\mathbb{Z}}\epsilon_y\eta_{2n-1}(y)}\big|\mathcal{F}\big] - \prod_{y\in\mathbb{Z}}\mathbb{E}\big[e^{it\epsilon_y\eta_{2n-1}(y)}\big|\mathcal{F}\big]\Big|e^{-t^2d_{n,3}^2/2}dt$$

and by Lemma 3.7 and the $\mathcal{F}$-measurability of the η's:

$$\begin{aligned}J_n &\leq \frac{1}{2}\sum_{x\neq y}\eta_{2n-1}(x)\eta_{2n-1}(y)\mathbb{E}[\epsilon_x\epsilon_y]\int_{|t|\leq b_n}t^2dt.\\ &\leq \frac{b_n^3}{6}\sum_{x\neq y}\eta_{2n-1}(x)\eta_{2n-1}(y)\mathbb{E}[\epsilon_x\epsilon_y].\end{aligned}$$

Using the positivity of the correlations and the fact that we only work on A_n, we rewrite:

$$J_n \leq \frac{n^{3\delta_2}}{6n^{\frac{3}{2}+3\delta_3}}\, n^{\frac{1}{2}+\delta_2}\sum_{y=-2n}^{2n}\eta_{2n-1}(y)\sum_{x=-2n,x\neq y}^{2n}\mathbb{E}[\epsilon_x\epsilon_y].$$

By stationarity of the associated r.v.'s ϵ_y, we have for all $y\in[-2n,2n]$,

$$\sum_{x=-2n,x\neq y}^{2n}\mathbb{E}[\epsilon_x\epsilon_y] = \sum_{x=-2n,x\neq y}^{2n}\mathbb{E}[\epsilon_0\epsilon_{x-y}] = \sum_{z=-2n-y,z\neq 0}^{2n-y}\mathbb{E}[\epsilon_0\epsilon_z] \leq \sum_{z=-4n,z\neq 0}^{4n}\mathbb{E}[\epsilon_0\epsilon_z].$$

Thus, still by stationarity,

$$J_n \leq \frac{n^{3\delta_2}}{6n^{\frac{3}{2}+3\delta_3}}\cdot n^{\frac{1}{2}+\delta_2}\cdot 2n\cdot 2\sum_{z=1}^{4n}\mathbb{E}[\epsilon_0\epsilon_z]$$

In our case of summable power law decay of correlation, we have with $\alpha > 1$

$$\mathbb{E}\big[\epsilon_0\epsilon_y\big] = \mathcal{O}\big(|y|^{-\alpha}\big) \implies \sum_{y=1}^{4n}\mathbb{E}[\epsilon_0\epsilon_y] = \mathcal{O}(n^{1-\alpha})$$

and thus

$$J_n = \mathcal{O}\big(n^{-\beta}\big)$$

with $\beta = 3\delta_3 + \alpha - 1 - 4\delta_2$. ∎

Now, using (3.21), write with the usual notation $d_{n,3} = n^{\frac{1}{2}+\delta_3}$:

$$\mathbb{P}[A_n \setminus B_n | \mathcal{F}] \le C d_{n,3}(|I_n^0| + |I_n^1 - I_n^0| + |I_n^2|).$$

Consider $\delta_3 > 2\delta_2$. By the previous lemmata, we have

$$d_{n,3} \cdot |I_n^0| = \mathcal{O}\big(n^{-\frac{1}{4}+\delta_3+\frac{\delta_1}{2}}\big),\ d_{n,3} \cdot |I_n^2| = \mathcal{O}\big(e^{-n^{2\delta_2}/2}\big)$$

and

$$d_{n,3} \cdot |I_n^1 - I_n^0| = \mathcal{O}\big(n^{\frac{1}{2}+\delta_3-\beta}\big).$$

To find a suitable $\delta > 0$ such that Proposition 3.19 holds, we need the following relations to be verified:

- $\delta_3 < \frac{1}{4} - \frac{\delta_1}{2}$.
- $\frac{1}{2} + \delta_3 - \beta < 0$, or equivalently $\delta_3 > 2\delta_2 + \frac{1}{2}(\frac{3}{2} - \alpha)$

and we still need $\delta_3 > 2\delta_2$. We distinguish two cases:

- $\alpha \in]1, \frac{3}{2}[$: the system reduces to

  $$\begin{cases} \delta_3 > 2\delta_2 + \frac{1}{2}(\frac{3}{2} - \alpha) \\ \delta_3 < \frac{1}{4} - \frac{\delta_1}{2}. \end{cases}$$

  where δ_1 and δ_2 can be taken as small as possible so the existence of $\delta > 0$ in Proposition 3.19 requires that

  $$\frac{1}{2}\big(\frac{3}{2} - \alpha\big) < \frac{1}{4}$$

  i.e. $\alpha > 1$, which is always verified under our hypothesis.

- $\alpha \ge \frac{3}{2}$: the system reduces to

  $$\begin{cases} \delta_3 > 2\delta_2 \\ \delta_3 < \frac{1}{4} - \frac{\delta_1}{2}. \end{cases}$$

  and one only has to choose δ_1 and δ_2 such that $2\delta_2 < \frac{1}{4} - \frac{\delta_1}{2}$ to find a suitable $\delta > 0$.

This proves Proposition 3.19.

∎

Combining Equations (3.15), (3.16), (3.18) and (3.20), we obtain (3.14) and then (3.10). By Borel-Cantelli's Lemma, we get

$$\mathbb{P}[M_{T_n} = (0,0) \text{ i.o.}] = \mathbb{P}[\mathbb{P}[\mathrm{M_{T_n}} = (0,0) \text{ i.o.}|\mathcal{G}]] = 0$$

and thus $(M_{T_n})_{n\in\mathbb{N}}$ is transient for ν-a.e. orientation ϵ. Theorem 2.2 follows from Lemma 3.6.

3.4 Proof of the functional limit theorem.

Proposition 3.25 *The sequence of random processes* $n^{-3/4}(X_{[nt]})_{t\geq 0}$ *weakly converges in the space* $\mathcal{D}([0,\infty[,\mathbb{R})$ *to the process* $(m\Delta_t)_{t\geq 0}$.

Proof. Let us first prove that the finite dimensional distributions of $n^{-3/4}(X_{[nt]})_{t\geq 0}$ converge to those of $(m\Delta_t)_{t\geq 0}$ as $n\to\infty$. We can rewrite for every $n\in\mathbb{N}$,

$$X_n = X_n^{(1)} + X_n^{(2)}$$

where

$$X_n^{(1)} = \sum_{y\in\mathbb{Z}} \epsilon_y \Big(\sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)} - m \Big)$$

and

$$X_n^{(2)} = m \sum_{y\in\mathbb{Z}} \epsilon_y \eta_{n-1}(y).$$

Lemma 3.26 *The sequence of random variables* $n^{-3/4}(X_n^{(1)})_{n\in\mathbb{N}}$ *converges in probability to 0 as* $n\to+\infty$.

Proof. It is enough to prove the convergence to 0 for the L^2-norm.

$$\mathbb{E}\Big[(X_n^{(1)})^2\Big] = \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}} \epsilon_x\epsilon_y \sum_{i=1}^{\eta_{n-1}(x)} \sum_{j=1}^{\eta_{n-1}(y)} \mathbb{E}[(\xi_i^{(x)}-m)(\xi_j^{(y)}-m)|\mathcal{F}\vee\mathcal{G}]\Big]$$

Since by independence of the $\xi_i^{(y)}$'s with both the vertical walk and the orientations,

$$\mathbb{E}[(\xi_i^{(x)}-m)(\xi_j^{(y)}-m)|\mathcal{F}\vee\mathcal{G}] = \mathbb{E}[(\xi_i^{(x)}-m)(\xi_j^{(y)}-m)]m^2\delta_{i,j}\delta_{x,y},$$

we obtain

$$n^{-3/2}\mathbb{E}\Big[(X_n^{(1)})^2\Big] = m^2 n^{-3/2}\sum_{x\in\mathbb{Z}}\eta_{n-1}(x) = m^2 n^{-1/2} = o(1).$$

∎

Lemma 3.27 *The finite dimensional distributions of* $(n^{-3/4}X_{[nt]}^{(2)})_{t\geq 0}$ *converge to those of* $(m\Delta_t)_{t\geq 0}$ *as* $n\to 0$.

Proof. Let $0\leq t_1\leq t_2\leq\ldots\leq t_k$ and $\theta_1,\theta_2,\ldots,\theta_k\in\mathbb{R}$. By the definition of $X_n^{(2)}$, we have

$$n^{-3/4}\sum_{j=1}^k \theta_j X_{[nt_j]}^{(2)} = mn^{-\frac{3}{4}}\sum_{j=1}^k\theta_j\sum_{y\in\mathbb{Z}}\epsilon_y\eta_{[nt_j]-1}(y).$$

For $\delta_1>0$, we define the event

$$D_n = \{\omega\in\Omega;\ \max_{y\in\mathbb{Z}}\eta_n(y) < n^{\frac{1}{2}+\delta_1}\}.$$

One has

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(in^{-3/4}\sum_{j=1}^{k}\theta_j X^{(2)}_{[nt_j]}\Big)\Big] &= \mathbb{E}\Big[\mathbb{E}[\exp(imn^{-3/4}\sum_{j=1}^{k}\theta_j\sum_{y\in\mathbb{Z}}\epsilon_y\eta_{[nt_j]-1}(y))|\mathcal{F}]\Big]\\
&= \mathbb{E}\Big[\mathbb{E}[\exp(imn^{-3/4}\sum_{j=1}^{k}\theta_j\sum_{y\in\mathbb{Z}}\epsilon_y\eta_{[nt_j]-1}(y))|\mathcal{F}]\mathbf{1}_{D_n}\Big]\\
&+ \mathbb{E}\Big[\mathbb{E}[\exp(imn^{-3/4}\sum_{j=1}^{k}\theta_j\sum_{y\in\mathbb{Z}}\epsilon_y\eta_{[nt_j]-1}(y))|\mathcal{F}]\mathbf{1}_{D_n^c}\Big]\\
&= \Sigma_1(n)+\Sigma_2(n)\ \ \text{(say)}
\end{aligned}$$

Firstly, we can use standard properties of the local time for the simple random walk on the line to estimate the second term:

$$|\Sigma_2(n)|\leq \mathbb{P}(D_n^c)\leq e^{-cn^{\delta_2}}$$

for some c and δ_2 strictly positive.
Secondly, we compare on the particular set D_n (on which uniformly in $y\in\mathbb{Z}$, the local time of the simple random walk is dominated by $n^{3/4}$) the characteristic function of the linear combinations of our process conditionally to the random walk with the marginal characteristic functions, using Lemma 3.7. Therefore we decompose

$$\Sigma_1(n)=\Sigma_{1,1}(n)+\Sigma_{1,2}(n)$$

where

$$\begin{aligned}
\Sigma_{1,1}(n) &= \mathbb{E}\Big[\mathbf{1}_{D_n}\Big\{\mathbb{E}[\exp(imn^{-3/4}\sum_{j=1}^{k}\theta_j\sum_{y\in\mathbb{Z}}\epsilon_y\eta_{[nt_j]-1}(y))|\mathcal{F}]\\
&- \prod_{y\in\mathbb{Z}}\mathbb{E}[\exp(imn^{-3/4}\epsilon_y\sum_{j=1}^{k}\theta_j\eta_{[nt_j]-1}(y))|\mathcal{F}]\Big\}\Big]
\end{aligned}$$

and

$$\Sigma_{1,2}(n)=\mathbb{E}\Big[\mathbf{1}_{D_n}\prod_{y\in\mathbb{Z}}\mathbb{E}[\exp(imn^{-3/4}\epsilon_y\sum_{j=1}^{k}\theta_j\eta_{[nt_j]-1}(y))|\mathcal{F}]\Big].$$

From Proposition 1 in [12], we have that

$$\begin{aligned}
\lim_{n\to\infty}\Sigma_{1,2}(n) &= \lim_{n\to\infty}\mathbb{E}\Big[\exp\Big(-\frac{m^2}{2}n^{-\frac{3}{2}}\sum_{y\in\mathbb{Z}}(\sum_{j=1}^{k}\theta_j\eta_{[nt_j]-1}(y))^2\Big)\Big]\\
&= \mathbb{E}\Big[\exp\Big(-\frac{m^2}{2}\int_{-\infty}^{\infty}(\sum_{j=1}^{k}\theta_j L_{t_j}(x))^2dx\Big)\Big]\ \text{by Lemma 6 in [12]}\\
&= \mathbb{E}\Big[\exp\Big(im\sum_{j=1}^{k}\theta_j\Delta_{t_j}\Big)\Big],\ \text{see Lemma 5 in [12].}
\end{aligned}$$

It remains to prove that $\Sigma_{1,1}(n)$ tends to 0 as n goes to infinity. By Lemma 3.7, we have that

$$|\Sigma_{1,1}(n)| \leq \frac{m^2}{2n^{3/2}} \sum_{x \neq y} \sum_{i,j=1}^{k} \theta_i \theta_j \mathbb{E}\Big[\epsilon_x \epsilon_y \eta_{[nt_i]-1}(x) \eta_{[nt_j]-1}(y) \mathbf{1}_{D_n}\Big]$$

Using the fact that we work on D_n, there exists $C > 0$ such that

$$|\Sigma_{1,1}(n)| \leq C \frac{n^{\frac{3}{2}+\delta_1}}{n^{\frac{3}{2}}} \sum_{z=1}^{[t_k n]} \mathbb{E}[\epsilon_0 \epsilon_z].$$

From the hypothesis on the power-law decay of correlations, there exists $\gamma > 0$ such that

$$\sum_{z=1}^{n} \mathbb{E}[\epsilon_0 \epsilon_z] = \mathcal{O}(n^{-\gamma}).$$

So it is enough to choose $\delta_1 < \gamma$ in order to have $\Sigma_{1,1}(n) = o(1)$ as n goes to infinity. ∎

From Lemma 3.26 and Lemma 3.27, we deduce the convergence of the finite dimensional distributions of $n^{-3/4}(X_{[nt]})_{t \geq 0}$ to those of $(m\Delta_t)_{t \geq 0}$.

In order to prove the weak convergence of $(n^{-3/4}X_{[nt]})_{t \geq 0}$ to $(m\Delta_t)_{t \geq 0}$ in $\mathcal{D}([0,\infty), \mathbb{R})$, it remains to prove the tightness of the family $(n^{-3/4}X_{[nt]})_{t \geq 0, n \geq 1}$ in $\mathcal{D}([0,\infty), \mathbb{R})$. By Theorem 15.6 from Billingsley ([2]), it is enough to prove that there exists $C > 0$ such that for all $t_1 \leq t \leq t_2 \in [0,T], T < \infty$, for all $n \geq 1$,

$$\mathbb{E}\Big[|X_{[nt_2]} - X_{[nt]}||X_{[nt]} - X_{[nt_1]}|\Big] \leq C|t_2 - t_1|^{\frac{3}{2}}. \tag{3.28}$$

Let us estimate

$$\begin{aligned}
\mathbb{E}\Big[|X_{[nt_2]} - X_{[nt]}|^2\Big] &\leq 2m^2 \sum_{x,y \in \mathbb{Z}} \mathbb{E}\Big[(\eta_{[nt_2]-1}(x) - \eta_{[nt]-1}(x))(\eta_{[nt_2]-1}(y) - \eta_{[nt]-1}(y))\Big]\mathbb{E}[\epsilon_x \epsilon_y] \\
&= 2m^2 \sum_{z=-2n}^{2n} \mathbb{E}[\epsilon_0 \epsilon_z] \sum_{x=-n}^{n} \mathbb{E}\Big[\Big(\sum_{k=[nt]}^{[nt_2]-1} \mathbf{1}_{Y_k = x}\Big)\Big(\sum_{l=[nt]}^{[nt_2]-1} \mathbf{1}_{Y_l = x+z}\Big)\Big] \\
&= 2m^2 \sum_{z=-2n}^{2n} \mathbb{E}[\epsilon_0 \epsilon_z] \sum_{k,l=[nt]}^{[nt_2]-1} \mathbb{P}[Y_k - Y_l = z] \\
&= 2m^2 \Big\{2 \sum_{z=-2n}^{2n} \mathbb{E}[\epsilon_0 \epsilon_z] \sum_{k,l=[nt];k<l}^{[nt_2]-1} \mathbb{P}[Y_{l-k} = z] + [nt_2] - [nt]\Big\}.
\end{aligned}$$

Now, it is well-known that when $(Y_k)_{k \geq 0}$ is a simple random walk on $\mathbb{Z}$, the probabilities of transition from 0 to z satisfy uniformly in $z \in \mathbb{Z}$,

$$\mathbb{P}[Y_n = z] = \mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big)$$

which implies that

$$\begin{aligned}
\sum_{k,l=[nt];k<l}^{[nt_2]-1} \mathbb{P}[Y_{l-k} = z] &= \mathcal{O}\Big(([nt_2] - 1 - [nt])^{3/2}\Big) \\
&= \mathcal{O}\Big(n^{3/2}(t_2 - t_1)^{3/2}\Big).
\end{aligned}$$

Using the hypothesis on the power-law decay of correlations and their positivity,

$$\sum_{z=-\infty}^{\infty} \mathbb{E}[\epsilon_0 \epsilon_z] < \infty.$$

So we deduce that there exists $C > 0$ such that

$$\mathbb{E}[|X_{[nt_2]} - X_{[nt]}|^2] \leq C n^{3/2} |t_2 - t_1|^{3/2}.$$

By Cauchy-Schwarz inequality, we obtain that there exists $C' > 0$ such that

$$\begin{aligned} n^{-3/2}\mathbb{E}[|X_{[nt_2]} - X_{[nt]}||X_{[nt]} - X_{[nt_1]}|] &\leq n^{-3/2}\mathbb{E}[|X_{[nt_2]} - X_{[nt]}|^2]^{1/2}\mathbb{E}[|X_{[nt]} - X_{[nt_1]}|^2]^{1/2} \\ &\leq C'|t_2 - t_1|^{3/2} \end{aligned}$$

so the tightness is proved. ∎

Let us recall that $M_{T_n} = (X_n, Y_n)$ for every $n \geq 1$. The sequence of random processes $n^{-3/4}(Y_{[nt]})_{t\geq 0}$ weakly converges in $\mathcal{D}([0,\infty[,\mathbb{R})$ to 0, thus the sequence of $\mathbb{R}^2-$valued random processes $n^{-3/4}(M_{T_{[nt]}})_{t\geq 0}$ weakly converges in $\mathcal{D}([0,\infty[,\mathbb{R}^2)$ to the process $(m\Delta_t, 0)_{t\geq 0}$. Theorem 2.4 follows from this remark and the next lemma

Lemma 3.29 *The sequence of random variables* $(\frac{T_n}{n})_{n\geq 1}$ *converge in probability to* $1 + m$ *as* $n \to +\infty$.

Proof. Let us remark that

$$T_n = n + \sum_{y\in\mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} (\xi_i^{(y)} - m) + m \sum_{y\in\mathbb{Z}} \eta_{n-1}(y).$$

Now,

$$\begin{aligned} \mathbb{E}\Big[\Big(\sum_{y\in\mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} (\xi_i^{(y)} - m)\Big)^2\Big] &= \sum_{x,y\in\mathbb{Z}} \mathbb{E}\Big[\sum_{i=1}^{\eta_{n-1}(x)} \sum_{j=1}^{\eta_{n-1}(y)} (\xi_i^{(x)} - m)(\xi_j^{(y)} - m)\Big] \\ &= \sum_{x,y\in\mathbb{Z}} \mathbb{E}\Big[\sum_{i=1}^{\eta_{n-1}(x)} \sum_{j=1}^{\eta_{n-1}(y)} \mathbb{E}[(\xi_i^{(x)} - m)(\xi_j^{(y)} - m)|\mathcal{F} \vee \mathcal{G}]\Big] \\ &= m^2 \sum_{x,y\in\mathbb{Z}} \mathbb{E}\Big[\sum_{i=1}^{\eta_{n-1}(x)} \sum_{j=1}^{\eta_{n-1}(y)} \delta_{i,j}\delta_{x,y}\Big] \\ &= m^2 \sum_{x\in\mathbb{Z}} \mathbb{E}[\eta_{n-1}(x)] \\ &= m^2 n = \mathcal{O}\Big(n^2\Big). \end{aligned}$$

From this calculation and the fact that $\sum_{x\in\mathbb{Z}} \eta_{n-1}(x) = n$, we deduce the lemma. ∎

4 Examples

Our framework includes this of [4] where i.i.d. orientations are considered but it also includes orientations whose joint distribution is not a product measure. Natural examples of non-product measures are given by *Gibbs measures in statistical mechanics* ([11]). To destroy the independence of the random variables ϵ_y, a family of measurable functions $\Phi = (\Phi_A)_{A \in \mathcal{S}}$ indexed by the set $\mathcal{S}$ of finite subsets of $\mathbb{Z}$, called *potential*, is introduced. For all $\epsilon, \Phi_A(\epsilon)$ represents the interaction between the random variables $(\epsilon_y)_{y \in A}$. A translation-invariant measure ν^β on $\{-1,+1\}$ is a *Gibbs measure at inverse temperature* $\beta > 0$ for the interaction Φ when it is an equilibrium state regard to some variational properties in terms of thermodynamic functions. An equivalent definition characterizes Gibbs measures in terms of continuity of their conditional probabilities w.r.t. the outside of finite sets, or via the well-known *DLR equation.* In some domains of temperature, there could be more than one Gibbs measure for an interaction Φ, and we then say that a *phase transition* holds. In this Gibbsian context, FKG property requires *ferromagnetic pair interactions* ([10]), i.e. a potential Φ such that $\Phi_A = 0$ if card(A) > 2 and $\Phi_{\{i,j\}}(\epsilon) = J(i,j)\epsilon_i\epsilon_j$ with $J(i,j) \geq 0$ for all $i, j \in \mathbb{Z}$. This provides us a wide family of examples suitable to our set-up.

1. Ferromagnetic nearest neighbors Ising model:
   The coupling is translation-invariant, positive for nearest neighbors pairs $\{i,j\} \subset \mathbb{Z}$ and null otherwise:
   $$J(i,j) \ = \begin{cases} J \ \geq \ 0 & \text{if } |\mathrm{i}-\mathrm{j}| = 1 \\ \\ 0 & \text{otherwise.} \end{cases}$$
   It is well known ([11]) that this one-dimensional model does not exhibit a phase transition. By translation invariance of the potential, the unique Gibbs measure is then translation invariant, and equivalently the family ϵ is stationary. Moreover, the decay of correlations of the random field ϵ of law ν is known to be exponential ([6, 7]). The absence of phase transition and the translation invariance property of the potential prove also that the orientations are symmetric and thus fit in our framework.

2. Long range ferromagnetic Ising model:
   It is similarly defined but the coupling $J(i,j)$ is non null for any pair $\{i,j\}$ and has a power law decay: there exists $\alpha > 1$ and $J \geq 0$ such that
   $$J(i,j) = J.|i-j|^{-\alpha}.$$
   Depending on the value of α, there could be a phase transition in some domains of temperature, and in particular two different regimes with summable power law decay of correlations are relevant in our set-up (see [1, 8, 14, 17]).

   (a) $\alpha > 2$.
   There is no phase transition and the Gibbs measure for this potential is translation invariant. The variables are thus symmetric and one could also learn in the literature that α also governs a power law decay of correlations:
   $$\mathbb{E}\big[\epsilon_0\epsilon_y\big] = \mathcal{O}\big(\mid y \mid^{-\alpha}\big) \text{ when } \mid \mathrm{y} \mid \longrightarrow \ +\infty.$$

(b) $\alpha \in]1,2[$.

There exists a critical inverse temperature $\beta_c > 0$ which separates the domain of temperature in two different regimes. In the high temperature regime, there is no phase transition and the picture is as above: the power law decay is the same as this of the interaction, i.e. governed by α. By translation-invariance of the unique phase, this provides us examples with very slow but summable power law decay of correlations for which our theorems hold.

The later is not necessarily true at low (or critical) temperature. Firstly, when a phase transition holds, the random variables are not necessarily centered and secondly, the power law decay of correlations is governed by $\alpha - 1 < 1$, and thus the correlations are not summable.

5 Comments

We have extended the results of [4] to positively correlated orientations and solved one of their open problems. In particular, we have proved that the simple random walk is still transient for ferromagnetic models in absence of phase transition. As the walk can be recurrent for deterministic orientations, it would be interesting to perturb deterministic cases in order to get a full picture of the transience versus recurrence properties and identify a sort of phase transition. As a perturbation of the alternate lattice on which the walk is recurrent, we are studying anti-ferromagnetic systems (i.e. the case of a coupling $J \leq 0$) for which the recurrence might be conserved at low temperature. Similarly, one could consider negatively correlated orientations but this requires finer results on such distributions and a complete theory of negative dependence has not been established yet (see e.g. [16, 3]).

The ingredients used to prove the functional limit theorems still hold for any $m > 0$ and Theorem 2.4 should therefore also be satisfied for more general random walks than the simple random walk on the lattice $\mathbb{L}^{\epsilon}$. This question is currently under considerations.

Acknowledgments: We thank D. Pétritis for having introduced us to this subject and indicated to us the proof of Lemma 3.22. We are also grateful to Aernout van Enter and Frank Redig for many useful comments and references. This work has been partially achieved under the RDSES program of the ESF.

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