Mathematical Addendum for "Infrared Catastrophe for Nelson's Model" (mp_arc 03-512)

By

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Abstract

In [6], the author used the kernel $a(k)$ of annihilation operator of bosons to describe and prove several useful formulas such as the pull-through formula on ground states. So, it is somewhat formal in the light of mathematics. Indeed,it is no wonder some mathematicians think the arguments by the symbolical formulas with $a(k)$ are so formal that they are incredulous of the arguments because of the well-known difficulty on the domain of $a(k)$. Thus, in this addendum, we list the difficulty in Remark 1 first. In order to treat it, we restate the formulas by using the smeared annihilation operator $a(f)$, $f \in L^2(\mathbb{R}^2)$, instead of $a(k)$ and present mathematically rigorous proofs of them. To author's best knowledge, the ways to prove them have not appeared in literatures yet. In this addendum, we use the same notations as in [6].

§1. Introduction

We take a Hilbert space of bosons to be the symmetric Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} \left[\otimes_{\mathrm{s}}^{n} L^2(\mathbb{R}^3)\right]$ over $L^2(\mathbb{R}^3)$, where $\otimes_{\mathrm{s}}^{n} L^2(\mathbb{R}^3)$ denotes the n-fold symmetric tensor product of $L^2(\mathbb{R}^3)$, the space of all square-integrable functions, and $\otimes_{\mathrm{s}}^{0} L^2(\mathbb{R}^3) = \mathbb{C}$. The finite particle space $\mathcal{F}_0$ is defined by $\mathcal{F}_0 = \{ \ \Psi = \Psi^{(0)} \oplus \cdots \oplus \Psi^{(n)} \oplus \cdots \in \mathcal{F} \mid \Psi^{(n)} = 0 \text{ for } n \ge \exists n_0 \ \}$. For every $f \in L^2(\mathbb{R}^3)$ and $\Psi = \Psi^{(0)} \oplus \Psi^{(1)} \oplus \cdots \oplus \Psi^{(n)} \oplus \cdots \in \mathcal{F}_0$, the smeared annihilation operator $a(f)$ of bosons is defined by

$$(1.1) \quad (a(f)\Psi)^{(n)}(k_1, \cdots, k_n) = \sqrt{n+1} \int_{\mathbb{R}^3} f(k)^* \Psi^{(n+1)}(k, k_1, \cdots, k_n) d^3k$$

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as $\otimes_{\mathrm{s}}^{n+1}L^2(\mathbb{R}^3) \ni \Psi^{(n+1)} \to (a(f)\Psi)^{(n)} \in \otimes_{\mathrm{s}}^{n}L^2(\mathbb{R}^3)$ for $n = 0, 1, 2, \cdots$, where $f(k)^*$ is the complex conjugate of $f \in L^2(\mathbb{R}^3)$. Then, $a(f)$ is closable for every $f \in L^2(\mathbb{R}^3)$, so we denote its closure by the same symbol. We define the smeared creation operator $a^\dagger(f)$ by the adjoint operator of $a(f)$, i.e., $a^\dagger(f) = a(f)^*$, for every $f \in L^2(\mathbb{R}^3)$. Fix $k \in \mathbb{R}^3$ arbitrarily. Then, the symbolic kernel $a(k)$ of the annihilation operator is given by

$$(a(k)\Psi)^{(n)}(k_1, \cdots, k_n) = \sqrt{n+1}\Psi^{(n+1)}(k, k_1, \cdots, k_n) \tag{1.2}$$

for $n = 0, 1, 2, \cdots$. We note that $a(k)$ is well-defined as an operator for $\Psi \in D_{\mathcal{S}} = \{\Psi = \Psi^{(0)} \oplus \cdots \oplus \Psi^{(n)} \oplus \cdots \in \mathcal{F}_0 \,|\, \Psi^{(n)} \in \mathcal{S}(\mathbb{R}^3), n \in \mathbb{N}\}$, where $\mathcal{S}(\mathbb{R}^3)$ is the set of all functions in the Schwartz class. Namely, for arbitrarily fixed $k \in \mathbb{R}^3$, $\otimes_{\mathrm{s}}^{n+1}\mathcal{S}(\mathbb{R}^3) \ni \Psi^{(n+1)} \to (a(k)\Psi)^{(n)} \in \otimes_{\mathrm{s}}^{n}\mathcal{S}(\mathbb{R}^3)$ by regarding $(a(k)\Psi)^{(n)}$ as an n-variables function because k is fixed in (1.2). See, for example, [1, §2.2] and [2, §8-3]. Because the mathematical meaning of $a(k)$ is given by pointwise k, we have to pay attention to $a(k)$ as an operator in functional analysis:

Remark 1.

(1) It is well-known that $a(k)^*$ is not densely defined [10, §X.7], indeed $a(k)^*$ is trivial [2, Proposition 8.2], i.e., $D(a(k)^*) = \{0\}$, so that $a(k)$ is *not closable*. Thus, we have to note that *we cannot use the limiting argument for* $a(k)$. Namely, even if Ψ_n and $a(k)\Psi_n$ converge to Ψ and Φ, respectively, in $\mathcal{F}$ as $n \to \infty$, we are required another difficult argument to check whether $\Psi \in D(a(k))$ and $\Phi = a(k)\Psi$ hold or not. Usually, it is impossible to check it under the present theory.

(2) By (1.2), $(a(k)\Psi)^{(n)}$ is defined as the boundary value of $\Psi^{(n+1)}$ on the hypersurface $\{(\ell_1, \cdots, \ell_{n+1}) \in \mathbb{R}^{3(n+1)} \,|\, (\ell_1 - k)^2 = 0\}$. Thus, the problem on determining the domain in which the operator $a(k)$ is well-defined is somewhat similar to that on the definition of the trace operator for boundary value. We have to note that $a(k)$ *cannot be defined for all elements in the finite particle space* $\mathcal{F}_0$. Namely, $\Psi^{(n+1)}(k, k_1, \cdots, k_n)$ cannot be uniquely determined for any fixed $k \in \mathbb{R}^3$ because $\Psi^{(n+1)} \in \otimes_{\mathrm{s}}^{n+1}L^2(\mathbb{R}^3)$ means an equivalence class of the Hilbert space $\otimes_{\mathrm{s}}^{n+1}L^2(\mathbb{R}^3)$. Even if $\Psi^{(n+1)} = \Phi^{(n+1)} \in \otimes_{\mathrm{s}}^{n+1}L^2(\mathbb{R}^3)$ and both have values at $k_1 = k$, once we fix $k \in \mathbb{R}^3$, $\Psi^{(n+1)}(k, \cdot, \cdots, \cdot)$ is not always equal to $\Phi^{(n+1)}(k, \cdot, \cdots, \cdot)$ in $\otimes_{\mathrm{s}}^{n}L^2(\mathbb{R}^3)$. Thus, we cannot designate a function as the representative element for $(a(k)\Psi)^{(n)}(k_1, \cdots, k_n)$ because we can make many candidates, i.e., $\sqrt{n+1}\Psi^{(n+1)}(k, k_1, \cdots, k_n)$, $\sqrt{n+1}\Phi^{(n+1)}(k, k_1, \cdots, k_n)$, for $\Psi^{(n+1)} = \Phi^{(n+1)} \in \otimes_{\mathrm{s}}^{n+1}L^2(\mathbb{R}^3)$.

(3) In order to justify $a(k)$ and $a^\dagger(k) \equiv a(k)^*$ directly and give a non-trivial meaning of $a^\dagger(k)$ in mathematics, there is a way by using white noise calculus

[9].

In order to treat the difficulty of the domain of $a(k)$, by avoiding to use $a(k)$ and by employing $a(f)$, $f \in L^2(\mathbb{R}^2)$ instead, we restate the formulas used in [6] and prove the restated formulas.

As is well known, we have

$$a(f) = \int_{\mathbb{R}^3} a(k) f(k)^* d^3k \qquad \text{on } D_{\mathcal{S}}, \tag{1.3}$$

where the integral is Bochner's. Using the symbolic kernel of the creation operator denoted by $a^\dagger(k)$, we sometimes write $a^\dagger(f)$ as

$$a^\dagger(f) = \int_{\mathbb{R}^3} a^\dagger(k) f(k) d^3k$$

symbolically. However, we have to remember the triviality of $a^\dagger(k)$ in operator theory as remarked in (1) of Remark 1.

§2. Restatement of [6, §5]

Let us fix $0 \le \kappa < \Lambda$. We suppose in this section that Nelson's Hamiltonian H_κ^{N} has a ground state ψ_κ. In [6, (39)], we wrote

$$a(k)\psi_\kappa = -q \frac{\chi_\kappa(k)}{\sqrt{2\omega(k)}} \left(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k)\right)^{-1} e^{-ikx} \psi_\kappa \tag{2.1}$$

as well as (mathematical) physicists do. However, since the domain of $a(k)$ is so narrow that $a(k)$ is not closable as remarked in Remark 1, (2.1) itself should *not* be regarded as an operator equality on the ground state. When we use $a(k)$, we have to go through the following steps. Note $H_\kappa^{\mathrm{N}}\psi \in D_{\mathcal{S}}$ for $\psi \in D_{\mathcal{S}}$ and start with the pull-through formula for $a(k)\psi$ in the same way as in [5, Proposition III.4]. Then, we get

$$(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k))^{-1} a(k)(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}})\psi = a(k)\psi + (H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k))^{-1} \lambda_{\kappa,x}\psi$$

first. We know already (1.3) and the facts that $D_{\mathcal{S}}$ is core for $a(f)$ and that $a(f)$ is closed. So, next, we have to check whether we can make operators,

$$T_1 = \int_{\mathbb{R}^3} f(k)^* (H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k))^{-1} a(k)(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}}) d^3k$$

and

$$T_2 = \int_{\mathbb{R}^3} f(k)^* (H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}} + \omega(k))^{-1} \lambda_{\kappa,x} d^3k$$

for every f in $D_{\mathcal{S}}$, as quadratic forms on $D_{\mathcal{S}} \times D_{\mathcal{S}}$ like [10, Theorem X.44]. Then, we have to also show that $D_{\mathcal{S}}$ is a core for T_1 and T_2. Here we note that it is not so easy to treat T_2 in the case $\kappa = 0$ because $\lambda_{0,x}/\omega \notin L^2(\mathbb{R}^3)$ though it is easy in the case $\kappa > 0$ because T_2 is bounded. After these steps, we have to find a sequence $\{\psi_\nu\}_\nu \subset D_{\mathcal{S}}$ such that $\psi_\nu \to \psi_\kappa$, $a(f)\psi_\nu \to a(f)\psi_\kappa$ for $f \in D_{\mathcal{S}}$, and $T_j\psi_\nu \to T_j\psi_\kappa$ for $j = 1, 2$. These steps are nuisances for even the simple model such as Nelson's. Therefore, in this section we give another mathematically rigorous meaning of (2.1) through an operator identity for $a(f)$ and we consider a mathematical relation between the ground states and the function space to which f belongs.

Before we state our desired proposition, we note that $\psi_\kappa \in D(H_0)$ by [6, Propostion 3.2], provided that ψ_κ is a ground state of H_κ^{N}.

We set

$$Y = \bigcup_{n=1}^{3} Y_n, \tag{2.2}$$

where $Y_n = \big\{(k_1, k_2, k_3) \in \mathbb{R}^3 \,|\, k_n = 0\big\}$ for each $n = 1, 2, 3$.

Proposition 2.1. *(***pull-through formula on ground states***)*
(i) For all $f \in C_0^\infty(\mathbb{R}^3 \setminus Y)$*,*

$$\begin{aligned} &a(f)\psi_\kappa \\ &= -q\int_{\mathbb{R}^3} f(k)^* \frac{\chi_\kappa(k)}{\sqrt{2\omega(k)}} \left(H_\kappa^N - E_\kappa^N + \omega(k)\right)^{-1} e^{-ikx}\psi_\kappa d^3k. \end{aligned} \tag{2.3}$$

(ii) If $\psi_\kappa \in D(N_{\mathrm{f}}^{1/2})$*, then (2.3) holds for all* $f \in L^2(\mathbb{R}^3)$*.*

Proof. (i) We set

$$a_t(f) = e^{itH_\kappa^{\mathrm{N}}} a(e^{-i\omega t} f) e^{-itH_\kappa^{\mathrm{N}}}, \qquad t \in \mathbb{R}.$$

Although our $\lambda_{\kappa,x}$ does not satisfy [4, (A.3)], we can show that $a_\pm(f)\psi = s\text{-}\lim_{t\to\pm\infty} a_t(f)\psi$ exists for all $\psi \in D({H_\kappa^{\mathrm{N}}}^{1/2})$ by using the concrete definition of $\lambda_{\kappa,x}$ and the polar coordinates in the same way as in [4, Lemma 4.2] (also see [7, 8]) and it has the following equality

$$a_\pm(f)\psi = a(f)\psi - iq\int_0^{\pm\infty} e^{itH_\kappa^{\mathrm{N}}} \left(\int_{\mathbb{R}^3} f(k)^* e^{it\omega(k)} \lambda_{\kappa,x}(k) d^3k\right) e^{-itH_\kappa^{\mathrm{N}}}\psi dt.$$

Moreover, employing the similar way to [7, Theorem 2], we get $a_\pm(f)\psi_\kappa = 0$. By using these facts and $e^{-itH_\kappa^{\mathrm{N}}}\psi_\kappa = e^{-itE_\kappa^{\mathrm{N}}}\psi_\kappa$, we get

$$a(f)\psi_\kappa = iq\int_0^\infty e^{it(H_\kappa^{\mathrm{N}} - E_\kappa^{\mathrm{N}})} \left(\int_{\mathbb{R}^3} f(k)^* e^{it\omega(k)} \lambda_{\kappa,x}(k) d^3k\right) \psi_\kappa dt. \tag{2.4}$$

Here we note that $e^{it(H_\kappa^{\mathrm{N}}-E_\kappa^{\mathrm{N}})}$ and $\lambda_{\kappa,x}$ are noncommutative because $\lambda_{\kappa,x}$ includes x. This is a different point from the case treated in [4]. So, for $\psi \in D(H_\kappa^{\mathrm{N}})$, by using Fubini's theorem and Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
(2.5)\qquad & \langle \psi\,,\, a(f)\psi_\kappa\rangle_{\mathcal{H}} \\
&= iq\int_0^\infty \langle \psi\,,\, \int_{\mathbb{R}^3} f(k)^* e^{it(H^{\mathrm{N}}-E^{\mathrm{N}}+\omega(k))}\lambda_{\kappa,x}\psi_\kappa\rangle_{\mathcal{H}} dt \\
&= \langle \psi\,,\, iq\int_{\mathbb{R}^3} f(k)^* \int_0^\infty e^{it(H^{\mathrm{N}}-E^{\mathrm{N}}+\omega(k))}\lambda_{\kappa,x}\psi_\kappa dt d^3k\rangle_{\mathcal{H}}.
\end{aligned}$$

Moreover, for every $k \in \mathbb{R}^3 \setminus Y$

$$\begin{aligned}
(2.6)\qquad & \int_0^\infty e^{it(H^{\mathrm{N}}-E^{\mathrm{N}}+\omega(k))}\lambda_{\kappa,x}\psi_\kappa dt \\
&= s\text{-}\lim_{T\to\infty}\, s\text{-}\lim_{\varepsilon\downarrow 0}\int_0^T e^{it(H^{\mathrm{N}}-E^{\mathrm{N}}+\omega(k)+i\varepsilon)}\lambda_{\kappa,x}\psi_\kappa dt \\
&= i\left(H^{\mathrm{N}}-E^{\mathrm{N}}+\omega(k)\right)^{-1}\lambda_{\kappa,x}\psi_\kappa.
\end{aligned}$$

Then, (2.3) follows from (2.5) and (2.6).

(ii) Let ψ_κ be in $D(N_{\mathrm{f}}^{1/2})$ and f be an arbitrary element in $L^2(\mathbb{R}^3)$. Then, since $C_0^\infty(\mathbb{R}^3\setminus Y)$ is dense in $L^2(\mathbb{R}^3)$, we can take a sequence $\{f_\nu\}_{\nu=1}^\infty \subset C_0^\infty(\mathbb{R}^3\setminus Y)$ such that $f_\nu \to f$ in $L^2(\mathbb{R}^3)$ as $\nu\to\infty$. Since $\|a(f_\nu - f)\psi_\kappa\|_{\mathcal{H}} \le \|f_\nu - f\|_{L^2}\|N_{\mathrm{f}}^{1/2}\psi_\kappa\|_{\mathcal{H}}$, we have $s\text{-}\lim_{\nu\to\infty} a(f_\nu)\psi_\kappa = a(f)\psi_\kappa$. By using continuity of inner product, we get for every $\phi\in\mathcal{H}$

$$\begin{aligned}
&\left\langle \phi\,,\, \int_{\mathbb{R}^3} f_\nu(k)^* \frac{\chi_\kappa(k)}{\sqrt{2\omega(k)}}\left(H_\kappa^{\mathrm{N}}-E_\kappa^{\mathrm{N}}+\omega(k)\right)^{-1}e^{-ikx}\psi_\kappa d^3k\right\rangle_{\mathcal{H}} \\
&= \left\langle \phi\,,\, \left(f_\nu\,,\,\left(H_\kappa^{\mathrm{N}}-E_\kappa^{\mathrm{N}}+\omega(k)\right)^{-1}\lambda_{\kappa,x}\psi_\kappa\right)_{L^2}\right\rangle_{\mathcal{H}} \\
&\longrightarrow \left\langle \phi\,,\, \left(f\,,\,\left(H_\kappa^{\mathrm{N}}-E_\kappa^{\mathrm{N}}+\omega(k)\right)^{-1}\lambda_{\kappa,x}\psi_\kappa\right)_{L^2}\right\rangle_{\mathcal{H}} \\
&= \left\langle \phi\,,\, \int_{\mathbb{R}^3} f(k)^* \frac{\chi_\kappa(k)}{\sqrt{2\omega(k)}}\left(H_\kappa^{\mathrm{N}}-E_\kappa^{\mathrm{N}}+\omega(k)\right)^{-1}e^{-ikx}\psi_\kappa d^3k\right\rangle_{\mathcal{H}}
\end{aligned}$$

as $\nu\to\infty$. Applying these limits to (2.3) for f_ν, we obtain the second statement of the proposition. □

Remark 2.

(1) It could be interesting to consider what is the maximal class of f such that (2.3) holds for $\psi_\kappa \notin D(N_{\mathrm{f}}^{1/2})$. Considering this problem, $f \in L^2(\mathbb{R}^3)$ needs certain regularity around $k = 0$ because we can use only $\|a(f_\nu - f)\psi_\kappa\|_{\mathcal{H}} \le \|(f_\nu - f)/\sqrt{\omega}\|_{L^2}\|H_{\mathrm{f}}^{1/2}\psi_\kappa\|_{\mathcal{H}}$ instead of $\|a(f_\nu - f)\psi_\kappa\|_{\mathcal{H}} \le \|f_\nu - f\|_{L^2}\|N_{\mathrm{f}}^{1/2}\psi_\kappa\|_{\mathcal{H}}$.

(2) The method presented above to prove the pull-through formula on the ground states is useful to the other models of nonrelativistic QED. We are preparing a paper on it.

From (2.3) we have the following identity.

Proposition 2.2. *Let us fix* $0 \le \kappa < \Lambda$, *and suppose that* H_κ^N *has a ground state* ψ_κ. *Then, for all* $f \in C_0^\infty(\mathbb{R}^3 \setminus Y)$,

$$a(f)\psi_\kappa = \sum_{j=1}^{5} \int_{\mathbb{R}^3} f(k)^* I_j(k)\psi_\kappa d^3k. \tag{2.7}$$

§3. Restatement of [6, §6]

In this section, we assume $\kappa > 0$ and we justify the following symbolic identity

$$\langle \psi_\kappa\,,\, N_{\mathrm{f}}\psi_\kappa\rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \|a(k)\psi_\kappa\|_{\mathcal{H}}^2 d^3k, \tag{3.1}$$

in mathematics. See Proposition 3.2 below.

First we slightly improve [3, Lemma 4.2] to meet our problem. Let $\mathbb{R}^3_{\le\kappa} = \{k \in \mathbb{R}^3 \,|\, |k| \le \kappa\}$ and $\mathbb{R}^3_{>\kappa} = \{k \in \mathbb{R}^3 \,|\, |k| > \kappa\}$. We set $N_{\mathrm{f}}^{\le\kappa} = d\Gamma(\mathbb{1}_{[0,\kappa]})$ and $N_{\mathrm{f}}^{>\kappa} = d\Gamma(\mathbb{1}_{(\kappa,\infty)})$ acting in $\bigoplus_{n=0}^\infty \otimes_{\mathrm{s}}^n L^2(\mathbb{R}^3_{\le\kappa})$ and $\bigoplus_{n=0}^\infty \otimes_{\mathrm{s}}^n L^2(\mathbb{R}^3_{>\kappa})$, respectively.

Before the following proposition, we note that $C_0^\infty(\mathbb{R}^3 \setminus Y)$ is dense in $L^2(\mathbb{R}^3)$.

Proposition 3.1. *Let* $\kappa > 0$ *and* $\{f_\nu\}_{\nu=1}^\infty \subset C_0^\infty(\mathbb{R}^3 \setminus Y)$ *be a complete orthonormal system of* $L^2(\mathbb{R}^3_{>\kappa})$ *such that* $\sqrt{\omega \mathbb{1}_{(\kappa,\infty)}} f_\nu \in L^2(\mathbb{R}^3_{>\kappa})$, $\nu \in \mathbb{N}$. *Then, for all* $\Psi \in D((N_{\mathrm{f}}^{>\kappa})^{1/2}) \subset \bigoplus_{n=0}^\infty \otimes_{\mathrm{s}}^n L^2(\mathbb{R}^3_{>\kappa}) \equiv \mathcal{F}_{>\kappa}$,

$$\|(N_{\mathrm{f}}^{>\kappa})^{1/2}\Psi\|_{\mathcal{F}_{>\kappa}}^2 = \sum_{\nu=1}^\infty \|a(f_\nu)\Psi\|_{\mathcal{F}_{>\kappa}}^2. \tag{3.2}$$

Proof. By (1.1), for each $M \in \mathbb{N}$ we get

$$\begin{aligned}\sum_{\ell=1}^{M} \|a(f_\nu)\Psi\|^2_{\mathcal{F}_{>\kappa}} &= \sum_{\ell=1}^{M} \langle a(f_\nu)\Psi \,,\, a(f_\nu)\Psi\rangle_{\mathcal{F}_{>\kappa}} \\ &= \sum_{n=0}^{\infty}(n+1)\int_{\mathbb{R}^3_{>\kappa}} \cdots \int_{\mathbb{R}^3_{>\kappa}} \Psi^{(n)}_M(k_1, \cdots, k_n) d^3k_1 \cdots d^3k_n,\end{aligned}$$

where

$$\begin{aligned}&\Psi^{(n)}_M(k_1, \cdots, k_n) \\ &= \sum_{\nu=1}^{M} \int_{\mathbb{R}^3_{>\kappa}} \int_{\mathbb{R}^3_{>\kappa}} f_\nu(k) f_\nu(k')^* \\ &\qquad\qquad \times \Psi^{(n+1)}(k, k_1, \cdots, k_n)^* \Psi^{(n+1)}(k', k_1, \cdots, k_n) d^3k d^3k'.\end{aligned}$$

By using the Schwarz inequality on $\sum_{\nu=1}^{M}$ and the Bessel inequality for the orthogonal system $\{f_\nu\}_{\nu=1}^{M}$ and noting $1 \le \kappa^{-1}\omega \mathbb{1}_{(\kappa,\infty)}$,

$$\begin{aligned}|\Psi^{(n)}_M(k_1, \cdots, k_n)| \le &\left(\int_{\mathbb{R}^3_{>\kappa}} |\Psi^{(n+1)}(k', k_1, \cdots, k_n)|^2 d^3k'\right)^{1/2} \\ &\times \left(\int_{\mathbb{R}^3_{>\kappa}} |\Psi^{(n+1)}(k, k_1, \cdots, k_n)|^2 d^3k\right)^{1/2}.\end{aligned}$$

It follows from the completeness of $\{f_\nu\}_{nu=1}^{\infty}$ that

$$\lim_{M\to\infty} \Psi^{(n)}_M(k_1, \cdots, k_n) = \int_{\mathbb{R}^3_{>\kappa}} |\Psi^{(n+1)}(k, k_1, \cdots, k_n)|^2 d^3k.$$

It is easy to see that

$$\begin{aligned}&\sum_{n=0}^{\infty} \int_{\mathbb{R}^3_{>\kappa}} \cdots \int_{\mathbb{R}^3_{>\kappa}} (n+1) \left(\int_{\mathbb{R}^3_{>\kappa}} |\Psi^{(n+1)}(k', k_1, \cdots, k_n)|^2 d^3k'\right)^{1/2} \\ &\qquad \times \left(\int_{\mathbb{R}^3_{>\kappa}} |\Psi^{(n+1)}(k, k_1, \cdots, k_n)|^2 d^3k\right)^{1/2} d^3k_1 \cdots d^3k_n \\ &\le \left(\sum_{n=0}^{\infty} \int_{\mathbb{R}^3_{>\kappa}} \cdots \int_{\mathbb{R}^3_{>\kappa}} (n+1)|\Psi^{(n+1)}(k', k_1, \cdots, k_n)|^2 d^3k' d^3k_1 \cdots d^3k_n\right)^{1/2} \\ &\times \left(\sum_{n=0}^{\infty} \int_{\mathbb{R}^3_{>\kappa}} \cdots \int_{\mathbb{R}^3_{>\kappa}} (n+1)|\Psi^{(n+1)}(k, k_1, \cdots, k_n)|^2 d^3k d^3k_1 \cdots d^3k_n\right)^{1/2} \\ &\le \|(N_{\mathrm{f}}^{>\kappa})^{1/2}\Psi\|^2_{\mathcal{F}_{>\kappa}} < \infty.\end{aligned}$$

Thus, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}&\lim_{M\to\infty}\sum_{\nu=1}^{M}\|a(f_\nu)\Psi\|^2_{\mathcal{F}_{>\kappa}}\\&=\sum_{n=0}^{\infty}(n+1)\int_{\mathbb{R}^3_{>\kappa}}\cdots\int_{\mathbb{R}^3_{>\kappa}}|\Psi^{(n)}(k,k_1,\cdots,k_n)|^2d^3kd^3k_1\cdots d^3k_n\\&=\|(N_{\mathrm{f}}^{>\kappa})^{1/2}\Psi\|^2_{\mathcal{F}_{>\kappa}}.\end{aligned}$$

Namely, desired statement is obtained. □

We note

$$D((H_{\mathrm{f}}^{>\kappa})^s)\subset D((N_{\mathrm{f}}^{>\kappa})^s) \tag{3.3}$$

for $s>0$.

Through the unitary equivalence $H_\kappa^{\mathrm{N}}\cong H_{\mathrm{f}}^{\le\kappa}\otimes I+I\otimes H_{\mathrm{N}}^{>\kappa}$, the ground state ψ_κ of H_κ^{N} is represented by $\Omega_0\otimes\psi_\kappa^{>\kappa}$, where $\psi_\kappa^{>\kappa}$ is a ground state of $H_{\mathrm{N}}^{>\kappa}$ and Ω_0 the Fock vacuum. We note $N_{\mathrm{f}}\cong N_{\mathrm{f}}^{\le\kappa}\otimes I+I\otimes N_{\mathrm{f}}^{>\kappa}$. Since $D(H_{\mathrm{f}}^{>\kappa})\subset D(N_{\mathrm{f}}^{>\kappa})$ by (3.3), $\psi_\kappa^{>\kappa}$ has to be in $D(N_{\mathrm{f}}^{>\kappa})$, i.e., $\psi_\kappa^{>\kappa}\in D(N_{\mathrm{f}}^{>\kappa})$, by [6, Proposition 3.2]. Therefore, by Proposition 3.1 we obtain

Proposition 3.2. *Let $\kappa>0$ and $\{f_\nu\}_{\nu=1}^{\infty}\subset C_0^\infty(\mathbb{R}^3\setminus Y)$ be a complete orthonormal system of $L^2(\mathbb{R}^3)$ such that $\sqrt{\omega}f_\nu\in L^2(\mathbb{R}^3)$, $\nu\in\mathbb{N}$. Then,*

$$\langle\psi_\kappa\,,\,N_{\mathrm{f}}\psi_\kappa\rangle_{\mathcal{H}}=\|N_{\mathrm{f}}^{1/2}\psi_\kappa\|^2_{\mathcal{H}}=\sum_{\nu=1}^{\infty}\|a(f_\nu)\psi_\kappa\|^2_{\mathcal{H}}. \tag{3.4}$$

Following this restatement Proposition 3.2, we have to change the proof of [6, Theorem 3.4] in the following way.

We take a normalized ground state ψ_κ. We set

$$K_\nu=\sum_{j=2}^{5}\int_{\mathbb{R}^3}f_\nu(k)^*I_j(k)\psi_\kappa d^3k\in\mathcal{H}$$

for the complete orthonormal system of $L^2(\mathbb{R}^3)$ in Proposition 3.2.

For each $M\in\mathbb{N}$, by using the Fubini theorem, the Bessel inequality, the completeness of $\{f_\nu\}_{\nu=1}^{\infty}$, and [6, Lemma 5.2], we get

$$\begin{aligned}
&\sum_{\nu=1}^{M}\left\|\int_{\mathbb{R}^3} f_\nu(k)^* I_j(k)\psi_\kappa d^3k\right\|_{\mathcal{H}}^2\\
&=\sum_{\nu=1}^{M}\left\langle\int_{\mathbb{R}^3} f_\nu(k')^* I_j(k')\psi_\kappa d^3k'\,,\int_{\mathbb{R}^3} f_\nu(k)^* I_j(k)\psi_\kappa d^3k\right\rangle_{\mathcal{H}}\\
&=\int_{\mathbb{R}^3}\left\langle\psi_\kappa\,,\left\{\sum_{\nu=1}^{M}\left(\int_{\mathbb{R}^3} f_\nu(k')^* I_j(k')d^3k'\right)f_\nu(k)\right\}^* I_j(k)\psi_\kappa\right\rangle_{\mathcal{H}} d^3k\\
&\le\int_{\mathbb{R}^3}\|I_j(k)\psi_\kappa\|_{\mathcal{H}}^2 d^3k<\infty
\end{aligned}$$

since $\kappa > 0$ in this section. Thus, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&\sum_{\nu=1}^{\infty}\left\|\int_{\mathbb{R}^3} f_\nu(k) I_j(k)\psi_\kappa d^3k\right\|_{\mathcal{H}}^2\\
&=\int_{\mathbb{R}^3}\left\langle\psi_\kappa\,,\left\{\sum_{\nu=1}^{\infty}\left(\int_{\mathbb{R}^3} f_\nu(k') I_j(k')d^3k'\right)f_\nu(k)\right\}^* I_j(k)\psi_\kappa\right\rangle_{\mathcal{H}} d^3k\\
&=\int_{\mathbb{R}^3}\|I_j(k)\psi_\kappa\|_{\mathcal{H}}^2 d^3k.
\end{aligned}\tag{3.5}$$

Applying the trigonometrical inequality to (2.7) and using (3.5), we get

$$\begin{aligned}
\langle\psi_\kappa\,,N_{\mathrm{f}}\psi_\kappa\rangle_{\mathcal{H}}&=\sum_{\nu=1}^{\infty}\|a(f_\nu)\psi_\kappa\|_{\mathcal{H}}^2\\
&\le 2\sum_{\nu=1}^{\infty}\left\|\int_{\mathbb{R}^3} f_\nu(k)^* I_1(k)\psi_\kappa d^3k\right\|_{\mathcal{H}}^2+2\sum_{\nu=1}^{\infty}\|K_\nu\|_{\mathcal{H}}^2\\
&\le 2\int_{\mathbb{R}^3}\|I_1(k)\psi_\kappa\|_{\mathcal{H}}^2+8\sum_{j=2}^{5}\int_{\mathbb{R}^3}\|I_j(k)\psi_\kappa\|_{\mathcal{H}}^2.
\end{aligned}\tag{3.6}$$

By [6, (49),(51)] and (3.6), we have

$$\langle\psi_\kappa\,,N_{\mathrm{f}}\psi_\kappa\rangle_{\mathcal{H}}\le 2\left\{\frac{q^2}{4\pi^2}\left(\log\frac{\Lambda}{\kappa}\right)+4M\right\}.\tag{3.7}$$

By (2.7) again, we get

$$\int_{\mathbb{R}^3} f_\nu(k)^* I_1(k)\psi_\kappa d^3k=a(f_\nu)\psi_\kappa-K_\nu.$$

In the same way as above, we get

$$\begin{aligned}(3.8)\quad \int_{\mathbb{R}^3} \|I_1(k)\psi_\kappa\|^2_{\mathcal{H}} d^3k &\le 2\langle \psi_\kappa\, , \, N_{\mathrm{f}}\psi_\kappa\rangle_{\mathcal{H}} + 2\sum_{\nu=1}^{\infty} \|K_\nu\|^2_{\mathcal{H}} \\ &\le 2\langle \psi_\kappa\, , \, N_{\mathrm{f}}\psi_\kappa\rangle_{\mathcal{H}} + 8\sum_{j=2}^{5} \int_{\mathbb{R}^3} \|I_j(k)\psi_\kappa\|^2_{\mathcal{H}} d^3k.\end{aligned}$$

By [6, (49),(51)] and (3.8), we have

$$\frac{q^2}{4\pi^2}\left(\log\frac{\Lambda}{\kappa}\right) \le 2\langle \psi_\kappa\, , \, N_{\mathrm{f}}\psi_\kappa\rangle_{\mathcal{H}} + 8M,$$

which implies

$$(3.9)\qquad \left\{\frac{q^2}{8\pi^2}\left(\log\frac{\Lambda}{\kappa}\right) - 4M\right\} \le \langle \psi_\kappa\, , \, N_{\mathrm{f}}\psi_\kappa\rangle_{\mathcal{H}}.$$

Therefore, considering $\|\psi_\kappa\|^{-1}_{\mathcal{H}}\psi_\kappa$ is normalization for the ground state ψ_κ which is not normalized, [6, Theorem 3.4] follows from (3.7) and (3.9).

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