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RENORMALIZATION OF ISOENERGETICALLY DEGENERATE HAMILTONIAN FLOWS AND ASSOCIATED BIFURCATIONS OF INVARIANT TORI

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Abstract. The paper presents a study of a renormalization group transformation acting on an appropriate space of Hamiltonian functions in two angle and two action variables. In particular, we study the existence of real invariant tori, on which the flow is conjugate to a rotation with a rotation number equal to a quadratic irrational (ω-tori). We demonstrate that the stable manifold of the renormalization operator at the "simple" fixed point contains isoenergetically degenerate Hamiltonians possessing shearless ω-tori. We also show that one-parameter families of Hamiltonians transverse to the stable manifold undergo a bifurcation: for a certain range of the parameter values the members of these families posses two distinct ω-tori, the members of such families lying on the stable manifold posses one shearless ω-torus, while the members corresponding to other parameter values do not posses any.

1. **Introduction.** Invariant tori appear in many physical systems — models of the atmosphere, toroidal plasma devices, fluid dynamics, celestial mechanics and others ([1]–[12]), where they play an important role as barriers to chaotic dynamics.

Mathematically, the issue of existence of such invariant tori is a subject of the KAM theory. It plays a crucial role in explaining the transition from orderly to chaotic behavior in physical systems described by maps or differential equations. Orbits of such systems may behave in a "reasonable" manner if the phase space of the system contains surfaces (tori), which are invariant under motion. The KAM theory is concerned with persistence and break-up of these invariant tori under perturbations. A crucial assumption in most KAM proofs is the *isoenergetic non-degeneracy* condition: a requirement that the orbits on tori wind around these invariant surfaces at a monotonically changing rate as one passes from one torus to another.

In this paper we will present a renormalization group transformation study of the existence of invariant tori for a general class of one parameter-families of two-degree of freedom Hamiltonian flows for which the isoenergetic degeneracy condition is violated for at least one value of the parameter. The invariant tori of such flows, on which this condition is violated, will be called *shearless*.

It has been observed that the transition to chaos and the break-up of the invariant tori exhibit *universality,* i.e. certain characteristics of such break-up are identical for a wide range of systems which possibly share only few common features (e.g. such as being degenerate or non-degenerate). Universality greatly simplifies the analysis of physical phenomena; indeed, if some complicated physical system is known to be

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in a certain *universality class,* many of its properties can be understood by studying a simpler system from the same class.

Another striking feature of many systems nearing a break-up is *self-similarity.* Self-similarity is probably best illustrated by the behavior of KAM tori as they approach a break-up under a perturbation. Such tori begin to exhibit repetitive features (shape, regularity and such) on consecutively smaller scales *ad infinitum.*

As it was first shown by K. Wilson and L. P. Kadanoff in a problem from condensed matter theory (see [13], [14] and, a recent review, [15]) and by M. J. Feigenbaum in [16] and [17], and later by S. Ostlund et al. in [18], D. F. Escande and F. Doveil in [19], S. J. Shenker and L. P. Kadanoff in [20], in several problems from dynamical systems, the best-suited technique for the study of universality and self-similarity is *renormalization.* Since then, the renormalization group transformation has been successfully used to prove the existence of invariant circles/tori both for systems nearing criticality and those close to integrable. The present paper is concerned with the second case: we will prove existence results for Hamiltonian flows close to a certain degenerate integrable flow (for renormalization of degenerate flows at criticality we refer the reader to our numerical work [21], while references [22]-[24] present a numerical study of universality and self-similarity in families of area-preserving non-twist maps at criticality).

We will start by mentioning an important result about invariant circles proved in reference [25] for a class of discrete dynamical systems relevant to this work – non-twist area-preserving maps of a cylinder, i.e. maps of the form $(x',y')=f(x,y)$ for which $\partial_y x'=0$ somewhere in the domain. Such maps serve as discrete analogues of isoenergetically degenerate Hamiltonian flows.

Let ω_0 be a Diophantine number. Then for every two parameter family $f_{\omega,\epsilon}$ of mappings of $\mathbb{R}\times T^1$, where T^1 is the real torus, such that i) the map $(\omega,\epsilon,p,q)\mapsto f_{\omega,\epsilon}(p,q)$ is analytic in a complex neighborhood of $(\omega_0,0)\times\{0\}\times T^1$ and takes real values when ω, ϵ, p and q are real; ii) $f_{\omega,\epsilon}$ is exact symplectic; iii) $f_{\omega,0}=(p,q+\Gamma(\omega,p))$ with $\Gamma(\omega_0,0)=\omega_0$, $\partial_p\Gamma(\omega_0,0)=0$, $\partial_p^2\Gamma(\omega_0,0)<0$ and $\partial_\omega\Gamma(\omega_0,0)>0$, there exists a $\delta>0$ and a map $\epsilon\mapsto\omega(\epsilon)$, analytic in a complex neighborhood of zero and real-valued for real ϵ, such that

(a) $f_{\omega(\epsilon),\epsilon}$ has exactly one shearless circle in $[-\delta,\delta]\times T^1$ with rotation number ω_0;
(b) if $\omega<\omega(\epsilon)$, $f_{\omega,\epsilon}$ has no invariant circles in $[-\delta,\delta]\times T^1$ with rotation number ω_0;
(c) if $\omega>\omega(\epsilon)$, $f_{\omega,\epsilon}$ has two invariant circles in $[-\delta,\delta]\times T^1$ with rotation number ω_0, which are not shearless.

One of our goals will be to prove a counterpart of this result for the Hamiltonian flows. This will be done using the machinery of the renormalization group transformation and the result will be proved for any one-parameter real family of flows intersecting the stable manifold at the simple fixed point of the renormalization operator (to be defined below) transversally.

As it has been already implied in the beginning of this Introduction, natural candidates for renormalization among continuous dynamical systems are Hamiltonian flows. Recently, most of research in this area has been concentrated around Hamiltonian flows with two or more periodic degrees of freedom $q=(q_1,\ldots,q_d)$ and a Hamiltonian function $H(q,p)$. Of particular interest is the study of invariant tori with a rotation vector $\omega=(1,\omega_2,\ldots,\omega_d)$, for the flow $(\dot q,\dot p)=(\nabla_p H,-\nabla_q H)$. As it was shown in [41], if the components of the rotation vector span an algebraic field of degree d, then there exists an integral $d \times d$ matrix T with the following properties:

(T1) ω is an eigenvector of T corresponding to a simple eigenvalue ϑ of modulus larger than one.
(T2) All other eigenvalues of T are simple, nonzero, and of modulus less than one.
(T3) $\det T = \pm 1$.

Furthermore, as it was shown in [41], the eigenvector ω of such matrix satisfies a Diophantine condition of the from $|\omega \cdot \nu| > c|\nu|^{-\gamma}$ for all $\nu \in \mathbb{Z}^d - \{0\}$, where $\gamma \geq 1$ and c is a real constant.

Approximate renormalization, utilizing such a matrix T, has been studied in [19] and [36]-[40]. In [41], the author introduced a renormalization group transformation $\mathcal{R}$ on a space of Hamiltonian functions, analytic on some subset of $\mathbb{C}^{2d}$, constructed so that

$$\mathcal{R}(H) = H \circ T_\mu \pmod{\mathcal{G}}, \tag{1.1}$$

where $T_\mu(q,p) = (Tq, \mu(T^*)^{-1}p)$, with $\mu \in \mathbb{C}$, and $\mathcal{G}$ is (ideally) a group of transformations that act on the Hamiltonian functions, preserving the rotation numbers of the orbits. The role of the map T in this transformation is simple: if a Hamiltonian H has an orbit with a rotation vector Tw then $\mathcal{R}(H)$ has an orbit with rotation vector cw (the "group" of transformations $\mathcal{G}$ is allowed to include multiplication of the Hamiltonian by a constant complex number c). Obviously, if $w = \omega$ is an eigenvector of T with incommensurate components, $\mathcal{R}(H)$ has an invariant torus with a rotation vector $c\omega$ if H has an invariant torus with the rotation vector ω. We will call an invariant torus on which the flow is conjugate to a rotation with a rotation vector ω, an ω-torus.

The group $\mathcal{G}$ of transformations introduced in [41] includes 1) canonical coordinate changes $H \mapsto H \circ U$, homotopic to identity, 2) scaling of time and energy $H \mapsto \tau H - \epsilon$ with $\tau \neq 0$, 3) scaling of the action $H \mapsto \mu^{-1} H(., \mu.)$ with $\mu \neq 0$. The parameter τ is chosen so that the torus average of $\partial_{\omega \cdot p} \mathcal{R}(H)$ is equal to 1. The parameter ϵ is taken to be equal to the average of the renormalized Hamiltonian over the real torus, the constant part of the Hamiltonian, which is irrelevant to the dynamics of the flow. The scaling parameter μ is a complex number which generally depends on the renormalized Hamiltonian, however sometimes it can be chosen to be a fixed number (see below).

As we pointed out, the matrix T has one expanding eigendirection, namely, the direction ω. This means that the norm of the terms in the Fourier–Taylor expansion of the Hamiltonian whose frequency vectors are approximately aligned along ω might grow under the map $H \mapsto H \circ T$. These frequencies correspond to "fast" modes, i.e. the modes that are the main contribution to the motion along the orbit on an ω-torus. The idea is to use a canonical transformation to eliminate these terms, namely, the terms whose frequency vectors are in a certain "cone" around the vector ω and whose powers of the action variables are not too large (see the precise definition in Section 3). Another (experimental) indication to the irrelevance of these modes is the fact that the ω-torus is smooth in the direction of ω and exhibits more and more irregularity in the perpendicular direction as the perturbation increases.

The renormalization picture at criticality for generally non-degenerate Hamiltonians has been explored in [42]-[44]. This study focuses on the issue of existence of an invariant ω-torus for an even real-valued Hamiltonian flow in two angle and two action variables. In [42] the authors suggest a particular choice of vector $\omega = (1/\theta, 1)$, where θ is the golden mean $(\sqrt{5}+1)/2$, and matrix T is given by

$T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, and present a numerical demonstration of the existence of a critical fixed point lying on a codimension two stable manifold. Finally, there has been a considerable progress in proving some of these numerical observations at criticality. We refer the reader to [44] which gives a computer assisted proof of the existence of a critical fixed point.

We want to emphasize that, apart from some modifications relevant to our study, the renormalization group transformation used here will be essentially the same as the one introduced in [41].

2. **Main Results.** Our overall objective is to present a renormalization picture for Hamiltonian functions in two angle variables $q = (q_1, q_2)$ and two action variables $p = (p_1, p_2)$ close to a certain fixed integrable degenerate Hamiltonian, which we will call a simple fixed point, and prove a series of results about the "quadratically irrational" invariant tori of families of Hamiltonians intersecting the stable manifold of the renormalization operator at the simple fixed point transversally.

Let $\omega = (1/\theta, 1)$ be a vector in $\mathbb{R}^2$ where θ is an algebraic number of degree two (a quadratic irrational). As we already mentioned, there exists a 2×2 matrix T satisfying conditions (T1)-(T3) with two eigenvectors $\omega = (1/\theta, 1)$ and $\Omega = (1, -1/\theta)$ corresponding to two distinct eigenvalues whose moduli are not equal to one (see [41]). Without loss of generality, one can assume that the eigenvalue corresponding to ω (of modulus larger than one) is positive, and $\det T = -1$. The two eigenvalues will be denoted by ϑ and $-1/\vartheta$.

The outlined task involves the study of a renormalization group transformation whose simple fixed point is the Hamiltonian $H^3_+(q,p) = \omega \cdot p + \gamma(\Omega \cdot p)^3$, where γ is some sufficiently small real positive number. The specialty of this Hamiltonian lies with the fact that it is the simplest example of a Hamiltonian for which the isoenergetic degeneracy (twist) condition on rotation numbers of orbits is violated.

Recall, that if γ is a (lifted) orbit of H, i.e. a curve from $\mathbb{R}$ to the domain of H satisfying $\gamma' = (\mathbb{J}\nabla H) \circ \gamma$, then its rotation vector is defined as $\omega(\gamma) = \lim_{t->\infty} \pi_q(\gamma(t) - \gamma(0))/t$, where π_q is the projection on the periodic coordinates, whenever this limit exists. One also defines the rotation number to be $\omega(\gamma)_1/\omega(\gamma)_2$.

With these definitions, it is easy to see that the rotation number of orbits of the Hamiltonian H^3_+ is minimal and equal to θ^{-1} on the invariant ω-torus passing through the point $p = 0$, while the rotation numbers of all other orbits are larger than θ^{-1}.

It is a well-known fact that KAM proofs of existence of invariant tori are significantly more difficult for isoenergetically degenerate Hamiltonians (see e.g. [25]). We will avoid these difficulties by using the renormalization group transformation, a non-perturbative technique, to obtain such existence results in the vicinity of the fixed point H^3_+, in particular, we will carry out a construction of the invariant ω-tori for Hamiltonians on the codimension one stable manifold of the renormalization operator at H^3_+.

The Hamiltonians considered here will have a common domain $\mathcal{Q} \times \mathcal{P}$, where $\mathcal{Q}$ is an open neighborhood of the real torus $T^2(\mathbb{R})$ in $\mathbb{C}^2$, and $\mathcal{P}$ is an open neighborhood of zero in $\mathbb{C}^2$.

An invariant ω-torus for such a Hamiltonian H is a differentiable map $\Gamma_H : T^2(\mathbb{R}) \mapsto \mathcal{Q} \times \mathcal{P}$ satisfying

$$(\mathbb{J}\nabla H) \circ \Gamma_H = c\ \omega \cdot \nabla \Gamma_H \tag{2.1}$$

for some (possibly complex) number c. Here, $\mathbb{J} = \left[\begin{smallmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{smallmatrix}\right]$. A torus map Γ_H is said to be real if it takes real values when restricted to real arguments.

We will also consider one-parameter families $\alpha \mapsto F(\alpha)$ of Hamiltonians, where α is restricted to a ball $\mathcal{D}_\delta = \{\alpha \in \mathbb{C} : |\alpha| < \delta\}$ of some fixed radius $\delta > 0$. A family F is said to be real if the map $(\alpha, q, p) \mapsto F(\alpha)(q,p)$ takes real values when restricted to real arguments. We will also consider a special family F^*_+, defined by the equation $F^*_+(\alpha)(q,p) = H^3_+ + \alpha\ \Omega \cdot p$.

Main Theorem. *There exists an open disk $\mathcal{D}_\delta$ in $\mathbb{C}$, an open neighborhood $\mathcal{U}$ of F^* in some space of analytic one-parameter families of Hamiltonians, and an analytic function $F \mapsto \alpha_F$ with values in $\mathcal{D}_\delta$, such that the following holds for every $F \in \mathcal{U}$ and $\alpha \in \mathcal{D}_\delta$.*

- (a) *$F(\alpha)$ possess invariant ω-tori $\Gamma^0_{F(\alpha)}$ and $\Gamma^1_{F(\alpha)}$ which are distinct for $\alpha \neq \alpha_F$ and coincide if $\alpha = \alpha_F$. The maps $(\alpha, F) \mapsto \Gamma^i_{F(\alpha)}$, $i = 0, 1$, are analytic on $\mathcal{D}_\delta \times \mathcal{U} \setminus \{(\alpha_F, F) : F \in \mathcal{U}\}$. Moreover, given a family $F \in \mathcal{U}$ the map $\alpha \mapsto \Gamma^i_{F(\alpha)}$ is continuous at α_F.*
- (b) *For $\alpha = \alpha_F$ the torus $\Gamma^0_{F(\alpha)}$ is shearless (in some exact sense described later).*
- (c) *If F is real and α belongs to the interval $(-\delta, \alpha_F]$, then the tori $\Gamma^i_{F(\alpha)}$ are real.*
- (d) *If F is real and α belongs to the interval (α_F, δ), then the tori $\Gamma^i_{F(\alpha)}$ are not real, and F has no real ω-tori with range in a certain domain $\mathfrak{D} = \mathfrak{D}(\alpha, F) \subset \mathcal{Q} \times \mathcal{P}$ containing the ranges of the maps $\Gamma^i_{F(\alpha)}$, $i = 0, 1$.*

Remark 2.1. Although there are indications that the claim of part (b) of the Theorem is true in its full generality (its counterpart for area-preserving maps has been proved in [25]), in this paper, we present a proof of this claim only for Hamiltonians with a trivial dependence on the variable $\omega \cdot p$. The discussion of the difficulties involved in the proof for general Hamiltonians, together, with the very definition of 'shearlessness' is postponed until Section 8. In the above statement of our main result we have also avoided the use of technical definitions of relevant spaces and domains: the precise formulation of the theorem can be found in Section 9.

This theorem parallels the result of [25] about the non-twist maps, mentioned in the Introduction.

3. **Definitions and Preliminaries.**

We will build on the ideas described in the Introduction by giving some rigorous definitions. In our following exposition, we will use the variables $(x, y, u, z) = (\omega' \cdot q, \Omega' \cdot q, \omega \cdot p, \Omega \cdot p)$, where ω' and Ω' are the constant multiples of ω and Ω such that $\omega' \cdot \omega = 1$ and $\Omega' \cdot \Omega = 1$, whenever convenient, together with the standard variables (q, p). The next two definitions describe the space of analytic Hamiltonian functions which will be the objects of renormalization.

Definition 3.1.

- *Given a real positive number ρ, define $\mathcal{D}_\rho = \{z \in \mathbb{C} : |z| < \rho\}$.*
- *Given a pair $\rho = (\rho_1, \rho_2)$ of real positive numbers ρ_1 and ρ_2, define $\mathcal{D}_1(\rho) = \left\{(x, y) \in \mathbb{C}^2 : |\Im(x)| < \rho_1, |\Im(y)| < \rho_2, \right\}$. We shall also use an alternative notation $\mathcal{D}_1(\rho_1, \rho_2)$.*
- *Given a pair $\rho = (\rho_3, \rho_4)$ of real positive numbers ρ_3 and ρ_4, define $\mathcal{D}_2(\rho) = \mathcal{D}_{\rho_3} \times \mathcal{D}_{\rho_4} = \left\{(u, z) \in \mathbb{C}^2 : |u| < \rho_3, |z| < \rho_4, \right\}$. We shall also use an alternative notation $\mathcal{D}_2(\rho_3, \rho_4)$.*

- *Given a quadruple* $\rho=(\rho_1,\rho_2,\rho_3,\rho_4)$ *of real positive numbers* ρ_1, ρ_2, ρ_3 *and* ρ_4, *define* $\mathcal{D}(\rho)=\mathcal{D}_1(\rho_1,\rho_2)\times\mathcal{D}_2(\rho_3,\rho_4)$. *We shall also use an alternative notation* $\mathcal{D}(\rho_1,\rho_2,\rho_3,\rho_4)$. *The numbers in the quadruple* ρ *will be also referred to as "components".*

Definition 3.2.

- *Given a quadruple* ρ *of four real positive numbers, define* $\mathcal{A}(\rho)$ *to be the Banach space of all analytic functions on* $\mathcal{D}(\rho)$, *which extend continuously to the boundary of* $\mathcal{D}(\rho)$, *of the form*
$$H(x,y,u,z)=\sum_{\nu\in\mathbb{Z}^2,k\in\mathbb{N},n\in\mathbb{N}} H_{(\nu,k,n)}e^{i(\nu\cdot\omega x+\nu\cdot\Omega y)}u^k z^n \tag{3.1}$$
for which the norm
$$\|H\|_\rho=\sum_{\nu\in\mathbb{Z}^2,k\in\mathbb{N},n\in\mathbb{N}} \left|H_{(\nu,k,n)}\right| e^{|\nu\cdot\omega|\rho_1+|\nu\cdot\Omega|\rho_2}\rho_3^k\rho_4^n \tag{3.2}$$
is finite.
- *Given a quadruple* ρ *of four real positive numbers, define* $\mathcal{A}'(\rho)$ *to be the vector space of all analytic functions on* $\mathcal{D}(\rho)$ *whose partial derivatives lie in* $\mathcal{A}(\rho)$. *For every* $\phi\in\mathcal{A}'(\rho)$ *define two seminorms:*
$$\|\phi\|'_\rho=\max_{i=1..4}\{\|\partial_i\phi\|_\rho\} \quad \textit{and} \quad |\phi|'_\rho=\sum_{i=1}^{4}\|\partial_i\phi\|_\rho. \tag{3.3}$$
- *Given a quadruple* ρ *of four real positive numbers, define a product space* $\mathcal{A}^4(\rho)=\mathcal{A}(\rho)\times\mathcal{A}(\rho)\times\mathcal{A}(\rho)\times\mathcal{A}(\rho)$ *and, for every* $\phi=(\phi_1,\phi_2,\phi_3,\phi_4)\in\mathcal{A}^4(\rho)$, *a norm* $\|\phi\|_\rho=\max_{i=1..4}\{\|\phi_i\|_\rho\}$.

In some cases, we will want to disregard the constant part of the Hamiltonians, i.e. we will consider two Hamiltonians H and H' to be equivalent if $H'=H+\text{const}$. On the space of equivalence classes, $\mathcal{A}(\rho)/\sim$, introduce a norm
$$\|H\|^*_\rho=\sum_{\nu\in\mathbb{Z}^2,k\in\mathbb{N},n\in\mathbb{N}} \left|H_{(\nu,k,n)}\right| e^{|\nu\cdot\omega|\rho_1+|\nu\cdot\Omega|\rho_2}\rho_3^k\rho_4^n-\left|H_{((0,0),0,0)}\right|. \tag{3.4}$$

Below, we will not always explicitly distinguish between the space $\mathcal{A}(\rho)$ and its quotient space. Since all Hamiltonian functions in an equivalence class define one and the same flow, we will use the norm on the quotient space mostly for Hamiltonians, while the norm on space $\mathcal{A}(\rho)$ will be often used in estimates of derivatives of the Hamiltonian functions. The norm on the quotient space $\mathcal{A}^4(\rho)/\sim$ will be defined as $\|\cdot\|^*_\rho=\max_{i=1..4}\{\|(\cdot)_i\|^*_\rho\}$.

Open balls (and neighborhoods) in spaces $\mathcal{A}(\rho)$ and $\mathcal{A}'(\rho)$ will be denoted by $\mathcal{B}$ and $\mathcal{B}'$ respectively, their radii, when necessary, will be indicated in parentheses. We will always specify the center of these open balls explicitly.

In what follows we will be using the notion of differentiability of maps on some Banach spaces over the complex numbers. If X and Y are Banach spaces over the complex numbers we denote by $L(X,Y)$ the Banach space of all continuous linear operators from X to Y. Here, for completeness, we state the following

Theorem 3.3. *Let* X *and* Y *be Banach spaces over the complex numbers, and let* F *be a locally bounded map from some open set* $D\subset X$ *to* Y. *Then the following properties are equivalent.*

(a) F *is Gateaux differentiable on* D.
(b) F *is of class* C^∞.

(c) *F is complex analytic on D. That is for every $x \in D$, $h \in X$ and $l \in L(Y,\mathbb{C})$, the function $z \mapsto l(F(x+zh))$ is analytic in a neighborhood of zero.*

Such map F will be referred to as "differentiable" or "analytic".

We also want to mention several straightforward facts about the functional space $\mathcal{A}(\rho)$ and its derivative spaces. The basic estimates from the following Proposition will play a fundamental role in the proofs of practically all results of this work. The proof of this Proposition, however, is elementary and will be omitted here.

Proposition 3.4. *Let ρ and δ be quadruples of real positive numbers and denote $\bar\delta = (\delta_3, \delta_4, \delta_1, \delta_2)$. Consider $f, g \in \mathcal{A}(\rho)$, $(X,Y,U,Z) \in \mathcal{A}^4(\rho)$ and $h \in \mathcal{A}(\rho+\bar\delta)$. Define $\mathcal{U}(x,y,u,z) = (x + X(x,y,u,z), y + Y(x,y,u,z), u + U(x,y,u,z), z + Z(x,y,u,z))$ for all $(x,y,u,z) \in \mathcal{D}(\rho)$. Then*

(a) $|f(x,y,u,z)| \le \|f\|_\rho$ *for all* $(x,y,u,z) \in \mathcal{D}(\rho)$,
(b) $fg \in \mathcal{A}(\rho)$ *and* $\|fg\|_\rho \le \|f\|_\rho \|g\|_\rho$,
(c) $\bar\delta_i \|\partial_i h\|_\rho \le \|h\|_{(\ldots,\rho_i+\bar\delta_i,..)}$, $i = 1,2,3,4$, *where* $(\ldots,\rho_i+\bar\delta_i,..)$ *stands for a quadruple, different from ρ only in the i-th component,*
(d) $\min_{i=1..4}\{\delta_i\}\,\|h\|'_\rho \le \|h\|_{\rho+\bar\delta}$,
(e) $h \circ \mathcal{U} \in \mathcal{A}(\rho)$ *and* $\|h \circ \mathcal{U}\|_\rho \le \|h\|_{\rho+\bar\delta}$, *if* $\|X\|_\rho \le \delta_3$, $\|Y\|_\rho \le \delta_4$, $\|U\|_\rho \le \delta_1$, $\|Z\|_\rho \le \delta_2$.

Finally, we want to specify the notion of irrelevant degrees of freedom in the Fourier–Taylor expansion of the Hamiltonian functions. Let σ, κ_3 and κ_4 be fixed positive real numbers, additional conditions on which will be specified later. These numbers will be called projection parameters and will be used in the following definitions of resonant (slow) and non-resonant (fast) index sets $I^\pm$.

Definition 3.5. *Given real numbers σ, κ_3 and κ_4 define the non-resonant index set*

$$I^- = \left\{(\nu,k,n) \in \mathbb{Z}^2 \times \mathbb{N} \times \mathbb{N} : |\omega\cdot\nu| > \sigma\,|\Omega\cdot\nu|\,, |\omega\cdot\nu| > \kappa_3 k \text{ and } |\omega\cdot\nu| > \kappa_4 n\right\}. \quad (3.5)$$

Define the resonant index set I^+ to be the complement of I^- in $\mathbb{Z}^2 \times \mathbb{N} \times \mathbb{N}$. We will also denote the restriction of the sum in (3.1) to the index sets $I^\pm$ by $\mathbb{I}^\pm H$.

4. Renormalization of Hamiltonians Close to the Fixed Point H^0.

We will start by adopting some of the results of [41] to our setting. In particular, as a first step we would like reproduce the construction of the renormalization operator with a simple fixed point $H^0 = u$. This renormalization group transformation, although not suitable for the studies of the shearless ω-tori, will be used in our proof of the main result; of particular importance will be the existence of the invariant ω-tori for Hamiltonians on the stable manifold at the simple fixed point H^0. We want to emphasize that although the results of this Section are not new, the estimates of individual theorems and propositions are, since our definitions of the relevant functional spaces differ from those in [41]. In the next Definition, we assume for a moment the existence of a canonical transformation $\mathcal{U}_H$ which eliminates the irrelevant degrees of freedom in the Hamiltonians, sufficiently close to H^0; the exact statement about the existence and analyticity property of this transformation is contained in Theorem 4.2. In the next Definition and throughout the rest of the paper we will denote the average of a Hamiltonian in $\mathcal{A}(\rho)$ over the real torus by $\mathbb{E}H$ and its value at a point (u,z) by $\mathbb{E}_{(u,z)}H$.

Definition 4.1. *Define the operator* $\mathcal{R}_\mu$ *on* $\mathcal{A}(\rho)$ *by setting*

$$\mathcal{R}_\mu(H) = \frac{1}{\xi(H)} \left[\tilde{H} - \mathbb{E}_0 \tilde{H} \right], \tag{4.1}$$

where $\tilde{H} = (\vartheta/\mu) H \circ \mathcal{U}_H \circ T_\mu$ *and* $\xi(H) = \mathbb{E}_0 \partial_u \tilde{H}$.

The role of the factor $\xi(H)$ is to normalize the component $(0,0,1,0)$ of the Hamiltonian to one. This normalization does not change the rotation numbers of the orbits of the Hamiltonian and is introduced for convenience.

The renormalization operator (4.1) has a fixed point $H^0(x,y,u,z) = u$. Although the scaling parameter μ is generally H-dependent (this dependence is always nontrivial for critical fixed points), for a renormalization group analysis around the fixed point H^0 almost any choice of μ will do (see [41]). Our immediate objective will be to demonstrate the analyticity and compactness of the renormalization operator $\mathcal{R}_\mu$.

The first step to proving the analyticity of this renormalization operator is showing the existence of a canonical change of coordinates that eliminates the non-resonant components of the Hamiltonians. Define $\hat{H}\phi = \{H, \phi\} = \nabla_{(x,y)} H \cdot \nabla_{(u,z)} \phi - \nabla_{(u,z)} \phi \cdot \nabla_{(x,y)} H$. Given two quadruples ρ and r such that $\rho > r$ componentwise, denote by $\rho - r$ the minimum of $\rho_i - r_i$, $i = 1..4$. As we already mentioned, the next theorems are adaptations of the similar results from [41] and [45]. We will not give their proofs here: an interested reader is referred to [41], [45] and [30].

Theorem 4.2. *Let* ρ *be a quadruple of real positive numbers,* $c > 0$ *and* $\mathcal{H}$ *be some non-empty set of Hamiltonians* $H \in \mathcal{A}(\rho)$, *such that* $\|H\|_\rho < c$. *Also, let* $\mathbb{I}^-$ *be a projection operator on* $\mathcal{A}(\rho')$, *where* ρ' *is a quadruple of real positive numbers, such that* $\rho' < \rho$ *componentwise. Assume that* $\mathbb{I}^-$ *satisfies the following non-resonant conditions.*

There is a choice of $\varrho' \in (\rho', \rho)$ *componentwise, and constants* $a > 0$ *and* e *satisfying* $0 < e < 68[1 + 3\max\{(\rho - \varrho')_3/(\rho - \varrho')_4, (\rho - \varrho')_4/(\rho - \varrho')_3\}]^{-1}$, *such that for all* $r \in [\varrho', \rho - (\rho - \varrho')/4]$ *the following holds:*

(a) *if* $f \in \mathcal{A}(r)$, *then* $\mathbb{I}^- f \in \mathcal{A}(r)$ *and* $\|\mathbb{I}^- f\|_r \le a \|f\|_r$;
(b) $\|\mathbb{I}^- H\|_\rho < e^2 ac/(4(34 + e))$ *for all* $H \in \mathcal{H}$;
(c) *for all* $H \in \mathcal{H}$, *the operator* $\mathbb{I}^- \hat{H}$ *maps* $\mathbb{I}^- \mathcal{A}'(r)$ *onto* $\mathbb{I}^- \mathcal{A}(r)$, *and if* $\phi \in \mathbb{I}^- \mathcal{A}'(r)$ *is non-zero, then* $\|\mathbb{I}^- \hat{H} \phi\|_r > aec/(\rho - \varrho') \|\phi\|'_r$.

Then there exists a map $\mathcal{N}$ *that assigns to each* $H \in \mathcal{H}$ *a canonical transformation* $\mathcal{U}_H$ *from* $\mathcal{D}(\varrho')$ *to* $\mathcal{D}(\varrho)$, *where* $\varrho = \varrho' + (\rho - \varrho')/6$ *componentwise, such that*

$$H \circ \mathcal{U}_H \in \mathcal{A}(\varrho'), \quad \mathbb{I}^- H \circ \mathcal{U}_H = 0. \tag{4.2}$$

The map $\mathcal{N} : H \mapsto H \circ \mathcal{U}_H$ *is analytic from* $\mathcal{H}$ *to* $\mathcal{A}(\varrho')$. *Furthermore, if* $H \in \mathcal{H}$ *satisfies* $\mathbb{I}^- H = 0$, *then* $\mathcal{U}_H = \mathbb{I}$ *and* $D\mathcal{N}(H) = \mathbb{I}^+ - \mathbb{I}^+ \hat{H} (\mathbb{I}^- \hat{H} \mathbb{I}^-)^{-1} \mathbb{I}^-$, *where* $\mathbb{I}^+ = \mathbb{I} - \mathbb{I}^-$.

Define U_ϕ to be the canonical transformation generated by ϕ, i.e. $U_\phi(x,y,u,z) = (x + X, y + Y, u + U, z + Z)$, where

$$(X, Y, U, Z)(x,y,u,z) = (\mathbb{J} \nabla \phi)(x, y, u + U(x,y,u,z), z + Z(x,y,u,z)). \tag{4.3}$$

The next Theorem associates a generating function with the canonical transformation $\mathcal{U}_H$.

Theorem 4.3. *Let* ρ', ρ *and* $\mathcal{H} \subset \mathcal{A}(\rho)$ *be as in Theorem* 4.2, *and let* $\mathbb{I}^-$ *be a projection operator on* $\mathcal{A}(\rho')$, *satisfying conditions of Theorem* 4.2.

Then there exists a quadruple ρ'', $\rho'' < \varrho'$ componentwise, and a differentiable map ψ from $\mathcal{H}$ to $\mathcal{B}'((\rho-\varrho')/16) \in \mathcal{A}'(\rho'')$, such that $U_{\psi(H)} = \mathcal{U}_H$, where $\mathcal{U}_H$ is the canonical transformation, described in Theorem 4.2 *. Furthermore,*

$$\|\psi(H)\|'_{\rho''} \leq \frac{4}{3}\frac{\rho-\varrho'}{aec}\|\mathbb{I}^- H\|_\rho. \tag{4.4}$$

An important ingredient for proving the compactness property of the renormalization operator is compactness of the map $H \mapsto H \circ T_\mu$. The next lemma shows the importance of the elimination of the non-resonant modes: informally, the action of the map T_μ on the resonant subspace of $\mathcal{A}(\rho')$ is not a singular operation, unlike on the whole of $\mathcal{A}(\rho')$.

Lemma 4.4. *Let quadruples ρ' and ρ be given, such that $0 < \rho' < \rho$ componentwise. Assume, that there exist real positive numbers σ, κ_3, κ_4 and a complex number μ, such that*

$$(\vartheta\rho_1 - \rho_1')\sigma < \left(\rho_2' - \frac{\rho_2}{\vartheta}\right), \ \frac{|\mu|}{\vartheta}e^{\kappa_3(\vartheta\rho_1-\rho_1')} < \frac{\rho_3'}{\rho_3}, \ |\mu|\vartheta e^{\kappa_4(\vartheta\rho_1-\rho_1')} < \frac{\rho_4'}{\rho_4}. \tag{4.5}$$

Then, every $H \in \mathbb{I}^+\mathcal{A}(\rho')$ extends analytically to $T_\mu\mathcal{D}(\rho)$, and $H \mapsto H \circ T_\mu$ is a compact linear map from $\mathbb{I}^+\mathcal{A}(\rho')$ to $\mathcal{A}(\rho)$ of norm less than or equal to one.

For any vector $v = (v_1, v_2) \in \mathbb{C}^2$ define a translation of coordinates J_v by setting $J_v(x,y,u,z) = (x+v_1, y+v_2, u, z)$ and a map $\mathcal{J}_v$ on $\mathcal{A}(\rho)$ by $\mathcal{J}_v H = H \circ J_v$. Also, define a scaling transformation $S_\zeta(x,y,u,z) = (x,y,\zeta u, \zeta z)$ and a scaling map $\mathcal{S}_\zeta$ on $\mathcal{A}(\rho)$ by setting $\mathcal{S}_\zeta H = \zeta^{-1} H \circ S_\zeta$.

Lemma 4.5. *Let ρ', ϱ', ϱ and ρ be the quadruples of real positive numbers as in Theorem* 4.2 *and let δ be a pair of positive numbers, such that $\delta_1 < \varrho_1' - \rho_1'$, $\delta_1 < \rho_1 - \varrho_1$ and $\delta_2 < \varrho_2' - \rho_2'$, $\delta_2 < \rho_2 - \varrho_2$. There exists a real $b > 0$, such that for all $v \in \mathcal{D}_1(\delta)$ and all H in an open ball $\mathcal{B}(b)$ around H^0 in $\mathcal{A}(\rho)$ the following identity holds on $\mathcal{D}(\rho_1', \rho_2', \varrho_3', \varrho_4')$:*

$$J_v \circ \mathcal{U}_{\mathcal{J}_v H} = \mathcal{U}_H \circ J_v. \tag{4.6}$$

Theorem 4.6. *Given ρ' and ρ, such that $\rho' < \rho$ componentwise, let real positive numbers σ, κ_3, κ_4 and a complex number μ satisfy conditions* (4.5) *and, additionally, let $\min\{\kappa_3\rho_3, \kappa_4\rho_4\} < \sigma < 1$. Then there exists an open neighborhood $\mathcal{B}_0$ around H^0, such that the operator $\mathcal{R}_\mu$ is well-defined, analytic, and compact, as a map from $\mathcal{B}_0$ to $\mathcal{A}(\rho)$.*

The same holds for the map $\mathcal{R}_\mu \circ \mathcal{J}_v = \mathcal{J}_{T^{-1}v}\mathcal{R}_\mu$, if the imaginary part of $v \in \mathbb{C}^2$ is sufficiently small. Moreover, $\mathcal{R}_\mu$ maps $\mathcal{B}_0 \cap \mathcal{S}_\zeta\mathcal{B}_0$ to $\mathcal{A}(\rho) \cap \mathcal{S}_\zeta\mathcal{A}(\rho)$, for any non-zero complex number ζ, and $\mathcal{S}_\zeta \circ \mathcal{R}_\mu = \mathcal{R}_\mu \circ \mathcal{S}_\zeta$ on $\mathcal{B}_0 \cap \mathcal{S}_\zeta\mathcal{B}_0$.

To better understand the action of the operator $\mathcal{R}_\mu$ near the fixed point H^0, one can look at the derivative of this operator. A relatively straightforward computation gives

$$D\mathcal{R}_\mu(H^0)h = L_\mu h - \mathbb{E}_0 L_\mu h - (\mathbb{E}_0 \partial_u L_\mu h)H^0, \quad L_\mu h = \frac{\vartheta}{\mu}(\mathbb{I}^+ h) \circ T_\mu, \tag{4.7}$$

where h is an element of the tangent space at the point H^0. The corresponding eigenvectors $e_{(k,n)}$ and eigenvalues $\lambda_{(k,n)}$ are given by

$$e_{(k,n)} = u^k z^n, \qquad \lambda_{(k,n)} = (-1)^n \mu^{k+n-1}\vartheta^{n-k+1}, \tag{4.8}$$

for all $k, n \geq 0, (k,n) \neq (0,0), (1,0)$. The eigenvalues corresponding to the constant eigenvector and eigenvector u are equal to zero. Below, we will always assume that

μ satisfies $|\mu| < \vartheta^{-3}$, then all eigenvalues are of modulus less then one except for that corresponding to $(k,n)=(0,1)$. For any $h \in \mathcal{A}(\rho)$ we will define a projection on the subspace spanned by $e_{(0,1)}$ as

$$\mathbb{P}^u h = (\mathbb{E}_0 \partial_z h) e_{(0,1)}, \tag{4.9}$$

and we will also denote $\mathbb{P}^s = \mathbb{I} - \mathbb{P}^u$. This projection operator is used in the next

Definition 4.7. *Define $\mathcal{A}^s(\rho)$ to be the affine space of all functions of the form $H^0 + h$ where $h \in \mathbb{P}^s \mathcal{A}(\rho)$. Define $\mathcal{A}^u(\rho) = \mathbb{P}^u \mathcal{A}(\rho)$.*

Given any two sets $\mathcal{A}, \mathcal{C}$ contained in the domain of the renormalization operator $\mathcal{R}_\mu$, define $\mathcal{W}^0_s(\mathcal{A},\mathcal{C})$ to be the set of all functions $H \in \mathcal{A}$, such that $\mathcal{R}^n(H) \in \mathcal{C}$ for all $n \geq 1$. Also define $\mathcal{W}^0_u(\mathcal{A},\mathcal{C})$ to be the set of all functions $H \in \mathcal{A}$, such that for each $n \geq 1$ there exists a function $H_{-n} \in \mathcal{C}$ satisfying $\mathcal{R}^n_\mu(H_{-n}) = H$.

By the local stable manifold theorem, there exists an open neighborhood $\mathcal{B}$ of H^0 in $\mathcal{A}(\rho)$, such that $\mathcal{W}^0_s(\mathcal{B},\mathcal{B})$ is the graph of an analytic map $W^0_s : \mathcal{B} \cap \mathcal{A}^s(\rho) \mapsto \mathcal{A}^u(\rho)$, whose derivative vanishes at H^0 and is of norm less than one everywhere in its domain.

Similarly, there exists an open neighborhood $\mathcal{A}$ of H^0 in $\mathcal{A}(\rho)$, such that $\mathcal{W}^0_u(\mathcal{A},\mathcal{A})$ is the graph of an analytic map $W^0_u : \mathcal{A} \cap \mathcal{A}^u(\rho) \mapsto \mathcal{A}^s(\rho)$ whose derivative vanishes at H^0 and is of norm less than one everywhere in this domain.

Definition 4.8.

- *Define the local stable manifold of the renormalization operator $\mathcal{R}_\mu$ at the fixed point H^0 to be $\mathcal{W}^0_s = \mathcal{W}^0_s(\mathcal{B},\mathcal{B})$.*
- *Define the local unstable manifold of the renormalization operator $\mathcal{R}_\mu$ at the fixed point H^0 to be $\mathcal{W}^0_u = \mathcal{W}^0_u(\mathcal{A},\mathcal{A})$.*
- *Define the projection Π in the direction of $\mathcal{A}^u(\rho)$ from $\mathcal{B}$ onto $\mathcal{W}^0_s$ as $\Pi(H) = \mathbb{P}^s H + W^0_s(\mathbb{P}^s H)$. We will also use the notation $H^s = \Pi(H)$.*

We want to finish this Section with a result about the existence of invariant ω-tori for Hamiltonians in $\mathcal{W}^0_s$ (see [30] for the proof).

Theorem 4.9. *Let the quadruples ρ', ϱ and ρ be as in Theorem 4.2. Given a real $\chi > 1$ sufficiently close to 1, let r be a pair of real positive numbers, such that $(1-\chi^{-1})\rho_1 > r_1$ and $\rho_2 - \varrho_2 > r_2$. Then, there exists an open neighborhood $\mathcal{B}^\Gamma_0$ of H^0, and for every $H \in \mathcal{B}^\Gamma_0 \cap \mathcal{W}^0_s$ a complex number $\xi'(H)$ and an analytic map Γ_H from $\mathcal{D}_1(r)$ to $\mathcal{D}(\varrho_1, \varrho_2 + r_2, \varrho_3, \varrho_4) \subset \mathcal{D}(\rho)$, such that the following equation holds on $\mathcal{D}_1(r)$:*

$$(\mathbb{J}\nabla H) \circ \Gamma_H = \xi'(H)\ \partial_x \Gamma_H. \tag{4.10}$$

If H depends on (u,z) only, then $\xi'(H) = \mathbb{E}_0 \partial_u H$ and $\Gamma_H(x,y) = (x,y,0,0)$ for all $(x,y) \in \mathcal{D}_1(r)$. Furthermore, the maps $H \mapsto \xi'(H^s)$ and $H \mapsto \Gamma_{H^s} - \Gamma_{H^0}$ are analytic from $\mathcal{B}^\Gamma_0$ to, respectively, $\mathbb{C}$ and some Banach space of analytic maps from $\mathcal{D}_1(r)$ to $\mathcal{D}(\rho)$, 2π-periodic in variables $\omega \cdot (x,y)$ and $\Omega \cdot (x,y)$.

Finally, if γ is any closed contour on the real torus $T^2(\mathbb{R})$, then

$$\oint_\gamma (\omega' \cdot \mathbb{J}\Gamma_H, \Omega' \cdot \mathbb{J}\Gamma_H)(x,y) \cdot d(\omega \cdot \Gamma_H, \Omega \cdot \Gamma_H)(x,y) = 0 \tag{4.11}$$

for any $H \in \mathcal{B}^\Gamma_0 \cap \mathcal{W}^0_s$.

We want to remark that although is is relatively easy to demonstrate that the Hamiltonians on the stable manifold $\mathcal{W}^0_s$ satisfy the equation 4.11, the converse, although probably true (as suggested by Hans Koch in a private communication), is highly non-trivial and difficult to prove. Here we will state it in a form of a conjecture whose proof will be the subject of our future work.

Conjecture 4.10. *The local stable manifold* $\mathcal{W}_s^0$ *consists of all Hamiltonians* $H \in \mathcal{B}_0^\Gamma \subset \mathcal{A}(\rho)$ *which posses a torus map* Γ_H *analytic from* $\mathcal{D}_1(r)$, *where* r *is some pair of real positive numbers, to* $\mathcal{D}(\rho)$ *and satisfying the equations* (4.10) *and* (4.11).

5. **The Modified Renormalization Group Transformation $\mathfrak{R}$.** In this section we shall carry out a construction of a renormalization operator with a simple period two fixed point (H^3_+, H^3_-), where $H^3_\pm(u,z) = u \pm \gamma z^3$ and γ is some real positive number. The rotation vector, $\omega_{H^3_\pm}(z)$, of orbits on invariant tori of the integrable Hamiltonian $H^3_\pm$ is given by $\omega_{H^3_\pm}(z) = \omega \pm 3\gamma z^2 \Omega$, while the rotation number is equal to $(1 \pm 3\gamma z^2 \theta)/(\theta \mp 3\gamma z^2)$ (recall, that the rotation number of an orbit with the rotation vector ω is defined as ω_1/ω_2). One can see right away that the rotation number of the ω-torus of these flows passing through $(u,z) = 0$ is minimal for the plus sign and maximal for the minus sign. In this case the degeneracy is local, which, however, is the generic situation. Globally degenerate dynamical systems are known to be structurally unstable: any globally degenerate Hamiltonian flow becomes locally degenerate under a small perturbation (see e.g. [22]).

The renormalization operator constructed here will be used in one of the steps of the proof of our main result. The fact that iterates of a Hamiltonian in a certain neighborhood of the simple fixed point H^3_+ approach under renormalization the one-parameter families $F^*_\pm = H^3_\pm + \alpha z$ of integrable Hamiltonians, whose tori can be easily described, will be of crucial importance in our construction. However, it can be easily seen that if one fixes the scaling μ in the renormalization operator $\mathcal{R}_\mu$ so that (H^3_+, H^3_-) is the period two fixed point of $\mathcal{R}_\mu$, one necessarily obtains two expanding eigenvectors for the derivative of such renormalization operator at $H^3_\pm$. Hence, for Hamiltonians to approach the abovementioned one-parameter families, it is imperative to reduce the dimension of the unstable manifold of the renormalization operator to one. This will be done by complementing the three ingredients of renormalization described in the Introduction with the fourth one, elimination of the components of Hamiltonians in the second unstable direction, namely, direction spanned by z^2.

To this end, define a map $R_t(x,y,u,z) = (x,y,u,z+t)$ and let $t(H)$ denote the solution of the equation $\left[H \circ R_{t(H)}\right]_{((0,0),0,2)} = 0$, which is the same as requiring that

$$\mathbb{E}_{(0,t(H))} \partial_z^2 H = 0. \tag{5.1}$$

In the rest of this paper we will denote $\tilde{H} = H \circ \mathcal{U}_H$, $\hat{H} = \tilde{H} \circ R_{t(\tilde{H})}$ and $\bar{H} = \hat{H} \circ T_{\vartheta^{-2}}$.

Proposition 5.1. *Let* $\mathcal{U}_H$ *be the canonical transformation from* $\mathcal{D}(\rho')$ *to* $\mathcal{D}(\varrho)$ *described in Theorem* 4.2. *Assume that* γ *is sufficiently small, so that* $H^3_\pm \in \mathcal{H}$, *the set described in Theorem* 4.2. *Then there exist a sufficiently small positive number* b, *and a quadruple* $\tilde{\rho} = (\rho'_1, \rho'_2, \rho'_3, \tilde{\rho}_4)$, $0 < \tilde{\rho}_4 < \rho'_4$, *such that for any* H *in an open ball* $\mathcal{B}_\pm(b) \subset \mathcal{A}(\rho)$, *centered at* $H^3_\pm$, *the equation* $\mathbb{E}_{(0,t(\tilde{H}))} \partial_z^2 \tilde{H} = 0$ *has a unique solution, and the transformation* $\mathcal{V}_H = \mathcal{U}_H \circ R_{t(\tilde{H})}$ *is canonical from* $\mathcal{D}(\tilde{\rho})$ *to* $\mathcal{D}(\varrho)$.

Furthermore, the map $H \mapsto H \circ \mathcal{V}_H$ *is analytic from* $\mathcal{B}_\pm(b)$ *to* $\mathcal{A}(\tilde{\rho})$.

Proof. Simultaneously with proving the claim of the Proposition we will provide a bound on the number $t(\tilde{H})$ in terms of the norm of the difference $\tilde{H} - H^3_\pm$. This bound will be used later for sharper estimates on the size of translation $t(\tilde{H})$, after we establish convergence of families of Hamiltonians to a period two fixed family $\alpha \mapsto (F^*(\alpha)_+, F^*(\alpha)_-)$.

Define $\Phi : \mathcal{B}_\pm(b) \times \mathbb{C} \mapsto \mathbb{C}$ by setting $\Phi(H,t) = \mathbb{E}_{(0,t)} \partial_z^2 \tilde{H}$. The equation $\Phi(H^3_\pm, t) = 0$ has the unique solution $t(H^3_\pm) = 0$. Moreover, $D_2 \Phi(H^3_\pm, t)|_{t=0} = \pm 6\gamma$ and the function Φ satisfies the assumptions of the Implicit Function Theorem. Hence, there exists a constant b, such that the equation $\Phi(H,t) = 0$ has a unique solution for any Hamiltonian H in an open ball $\mathcal{B}_\pm(b) \subset \mathcal{H}$ centered at $H^3_\pm$, and this solution depends analytically on H.

For any $H \in \mathcal{B}_\pm(b)$ write $\tilde{H} = H^3_\pm + (\mathbb{P}^s \tilde{H} - H^3_\pm) + \mathbb{P}^u \tilde{H}$, where $\mathbb{P}^s \tilde{H} = (\mathbb{I} - \mathbb{P}^u)\tilde{H}$ and $\mathbb{P}^u$ is the projection on the unstable subspace spanned by z. Then $\mathbb{E}_{(0,t)} \partial_z^2 \tilde{H} = \pm 6\gamma t + \mathbb{E}_{(0,t)} \partial_z^2 (\mathbb{P}^s \tilde{H} - H^3_\pm)$ for any $t \in \mathbb{C}$. Therefore, if $t(\tilde{H})$ is a solution of the equation (5.1), then $|t(\tilde{H})| \le (1/(6\gamma)) \| \mathbb{E}_{(0,z)} \partial_z^2 (\mathbb{P}^s \tilde{H} - H^3_\pm) \|_r$ for any quadruple r with positive components such that $t(\tilde{H}) \le r_4$. Now, assume that we can choose $r_4 \le \rho'_4/2$, then $|t(\tilde{H})| \le (1/(3\gamma)) \sum_{n \ge 2} n(n-1) 2^{1-n} \left| (\mathbb{P}^s \tilde{H} - H^3_\pm)_{((0,0),0,n)} \right| (\rho'_4)^{n-2}$. It is easily checked that $\max_{n \ge 2} \{ n(n-1) 2^{1-n} \} = 3/2$, and, since by Theorem 4.2 the transformation $\mathcal{U}_H$ is canonical from $\mathcal{D}(\rho')$ to $\mathcal{D}(\varrho)$, one gets the following bound on $t(\tilde{H})$:

$$|t(\tilde{H})| \le \frac{1}{2\gamma(\rho'_4)^2} \| \mathbb{E}(\mathbb{P}^s \tilde{H} - H^3_\pm) \|_{\rho'} \le \frac{1}{2\gamma(\rho'_4)^2} \| \mathbb{P}^s H - H^3_\pm \|_\rho \le \frac{b}{2\gamma(\rho'_4)^2}. \tag{5.2}$$

If b is sufficiently small then our assumption $|t(\tilde{H})| \le \rho'_4/2$ is verified *a posteriori.*

For all $H \in \mathcal{B}_\pm \subset \mathcal{H}$, where the set $\mathcal{H}$ is as in Theorem 4.6, the transformation $\mathcal{V}_H = \mathcal{U}_H \circ R_{t(\tilde{H})}$ is canonical, being a composition of two canonical transformations, from $\mathcal{D}(\tilde{\rho})$ to $\mathcal{D}(\varrho)$, where $\tilde{\rho} = (\rho'_1, \rho'_2, \rho'_3, \tilde{\rho}_4)$ is such that $\tilde{\rho}_4 \le \rho'_4 - b/(2\gamma(\rho'_4)^2)$.

Finally, let $\tilde{\mathcal{B}}_\pm = \{ H \circ \mathcal{U}_H \in \mathcal{A}(\rho') : H \in \mathcal{B}_\pm \subset \mathcal{A}(\rho) \}$. The analyticity of the map $H \mapsto H \circ \mathcal{V}_H$ from $\mathcal{B}_\pm$ to $\mathcal{A}(\tilde{\rho})$ follows from the analyticity of the map $H \mapsto H \circ \mathcal{U}_H$ from $\mathcal{B}_\pm$ to $\tilde{\mathcal{B}}_\pm$ and the analyticity of the map $H \mapsto H \circ R_{t(\tilde{H})}$ from $\tilde{\mathcal{B}}_\pm \subset \mathcal{A}(\rho')$ to $\mathcal{A}(\tilde{\rho})$.

□

We will always assume that $b < \gamma(\rho_4)^3$ which ensures that the open balls $\mathcal{B}_+$ and $\mathcal{B}_-$ from Prop. 5.1 do not intersect. Now, we are ready to define the modified renormalization operator.

Definition 5.2. *Given a quadruple ρ, define for all $H \in \mathcal{B}_\pm(b)$*

$$\mathfrak{R}(H) = \frac{\vartheta^3}{\xi(H)\eta(H)} \left[\hat{H} \circ T_{\vartheta^{-2}} \circ S_{\eta(H)} - \mathbb{E}_0 \left(\hat{H} \circ T_{\vartheta^{-2}} \circ S_{\eta(H)} \right) \right], \tag{5.3}$$

where $\hat{H} = H \circ \mathcal{V}_H$, $\xi(H) = \mathbb{E}_0 \partial_u \hat{H}$, $\eta(H) = \left[(\pm 6\gamma \mathbb{E}_0 \partial_u \hat{H}) / (\mathbb{E}_0 \partial_z^3 \hat{H}) \right]^{1/2}$.

The purpose of the complex parameter $\xi(H)$ is to normalize the component $((0,0), 1, 0)$ of the Hamiltonian $\mathfrak{R}(H)$ to one, while the complex parameter $\eta(H)$ serves the role of normalizing the $((0,0),0,3)$-component of the Hamiltonian $\mathfrak{R}(H)$ to $-\gamma$, if H is in $\mathcal{B}_+$, and to γ, if H is in $\mathcal{B}_-$.

As it can be easily checked, this operator has a fixed point of period two: (H^3_+, H^3_-). For these fixed points $\mathcal{V}_{H^3_\pm} = \mathbb{I}$ and $\eta(H^3_\pm) = \xi(H^3_\pm) = 1$.

The following theorem describes the properties of this operator.

Theorem 5.3. *There exists a sufficiently small $b > 0$, such that the operator $\mathfrak{R}$ is well-defined, analytic and compact from an open ball $\mathcal{B}_\pm(b) \subset \mathcal{A}(\rho)$, centered at $H^3_\pm$, to $\mathcal{A}(\rho)$.*

The same holds for the map $\mathfrak{R}\circ\mathcal{J}_v = \mathcal{J}_{T^{-1}v}\circ\mathfrak{R}$, *if the imaginary part of* v *is sufficiently small.*

Proof. By Prop. 5.1 the map $\mathcal{V} : H \mapsto H\circ\mathcal{V}_H$ is analytic from $\mathcal{B}_\pm(b)$ to $\mathcal{A}(\tilde\rho)$. By Lemma 4.4 the linear operator $L_{\vartheta^{-2}}(H) = \vartheta^3(\mathbb{I}^+H)\circ T_{\vartheta^{-2}}$ is compact from $\mathcal{A}(\rho')$ to $\mathcal{A}(\rho)$ and is certainly compact from $\mathcal{A}(\tilde\rho)$ to $\mathcal{A}(\rho)$.

Let $\hat{\mathcal{B}}_\pm = \{H\circ\mathcal{V}_H \in \mathcal{A}(\tilde\rho) : H \in \mathcal{B}_\pm \subset \mathcal{A}(\rho)\}$. The set $\hat{\mathcal{B}}_\pm$ is clearly open since the map $\mathcal{V}$ is analytic on $\mathcal{B}_\pm$. Define the functionals F_0, F_1 and F_2 on $\hat{\mathcal{B}}_\pm$ by setting $F_0(H) = \mathbb{E}_0 H$, $F_1(H) = \mathbb{E}_0\partial_u H$ and $F_2(H) = \left[(\pm 6\gamma\mathbb{E}_0\partial_u H)/(\mathbb{E}_0\partial_z^3 H)\right]^{1/2}$. Then, for any $H \in \mathcal{B}_\pm(b)$

$$|F_1(H\circ\mathcal{V}_H) - 1 = \left|\mathbb{E}_0\partial_u(H\circ\mathcal{V}_H - H^3_\pm)\right| \le \frac{1}{\tilde\rho_3}\|\mathbb{E}(H - H^3_\pm)\|_\rho \le \frac{b}{\tilde\rho_3}. \tag{5.4}$$

The bound (5.4) implies that the analytic map $H \mapsto \xi(H)$ is bounded on $\mathcal{B}_\pm(b)$, in particular, one obtains for all $H \in \mathcal{B}_\pm(b)$:

$$|\xi(H)| \le 1 + \frac{b}{\tilde\rho_3}. \tag{5.5}$$

Next, for any H in the open ball $\mathcal{B}_\pm(b)$ centered at $H^3_\pm$

$$\left|\mathbb{E}_0\partial_z^3 H\circ\mathcal{V}_H \mp 6\gamma\right| = \left|\mathbb{E}_0\partial_z^3(H\circ\mathcal{V}_H - H^3_\pm)\right| \le \frac{1}{(\tilde\rho_4)^3}\|\mathbb{E}(H - H^3_\pm)\|_\rho \le \frac{b}{(\tilde\rho_4)^3}. \tag{5.6}$$

In its turn, this bound, together with the bound (5.5) and the fact that the map $\mathcal{V} : H \mapsto H\circ\mathcal{V}_H$ is analytic on $\mathcal{B}_\pm(b)$, implies that the map $H \mapsto \eta(H)$ is analytic and bounded on $\mathcal{B}_\pm(b)$, in particular,

$$\left(\frac{1 - \frac{b}{\tilde\rho_3}}{1 + \frac{b}{6\gamma(\tilde\rho_4)^3}}\right)^{1/2} \le |F_2(H\circ\mathcal{V}_H)| = |\eta(H)| \le \left(\frac{1 + \frac{b}{\tilde\rho_3}}{1 - \frac{b}{6\gamma(\tilde\rho_4)^3}}\right)^{1/2}, \tag{5.7}$$

and $|F_2(H\circ\mathcal{V}_H)|$ is bounded away from zero if b is sufficiently small.

The analyticity of the map $H \mapsto F_0(H)$ together with the bounds (5.5) and (5.7) implies that the map $F : H \mapsto (H - F_0(H))/(F_1(H)F_2(H))$ is analytic and bounded from $\hat{\mathcal{B}}_\pm$ to $\mathcal{A}(\rho)$. The scaling map $\mathcal{S} : H \mapsto H\circ S_{\eta(\mathcal{V}^{-1}\circ F^{-1}(H))}$ is also analytic on $\mathcal{B}'_\pm = \{F(H\circ\mathcal{V}_H) \in \mathcal{A}(\tilde\rho) : H \in \mathcal{B}_\pm\}$ since the map $H \mapsto \eta(H)$ is analytic on $\mathcal{B}_\pm$. Finally, the map $\mathfrak{R} = L_{\vartheta^{-2}}\circ\mathcal{S}\circ F\circ\mathcal{V}$ is analytic and compact from $\mathcal{B}_\pm$ to $\mathcal{A}(\rho)$.

The claim about the analyticity of the operator $\mathfrak{R}\circ\mathcal{J}_v$ for a complex v is proved word-by-word as in [41]. ☐

A lengthy but relatively straightforward computation (see [30]) gives the following expression for the derivative of the renormalization operator $\mathfrak{R}$ at the fixed point $H^3_\pm$:

$$D\mathfrak{R}(H^3_\pm)h = (\mathbb{I} - \mathbb{E}_0)L_{\vartheta^{-2}}h - u\mathbb{E}_0\partial_u L_{\vartheta^{-2}}h - \frac{z^2}{2}\mathbb{E}_0\partial_z^2 L_{\vartheta^{-2}}h + \frac{z^3}{6}\mathbb{E}_0\partial_z^3 L_{\vartheta^{-2}}h. \tag{5.8}$$

The eigenvalues of $D\mathfrak{R}(H^3_\pm)$ are the same as those of the linear operator $L_{\vartheta^{-2}}$ with the exception of eigenvectors 1, u, z^2 and z^3, which correspond to the eigenvalue zero. The rest of eigenvectors and eigenvalues are given by

$$e_{(k,n)} = u^k z^n, \qquad \lambda_{(k,n)} = (-1)^n\vartheta^{-3k-n+3}. \tag{5.9}$$

The derivative of the renormalization operator at the point $H^3_\pm$ has one eigenvalue of modulus larger then one, namely $\lambda_{(0,1)} = -\vartheta^2$. At the point $H^3_\pm$ the one-dimensional local unstable manifold of the operator $\mathfrak{R}^2$ coincides with the unstable subspace spanned by z, while the stable manifold is codimension one. The local stable and unstable manifolds of the operator $\mathfrak{R}^2$ at the fixed point H^3_+ will be denoted by $\mathcal{W}^3_s$ and $\mathcal{W}^3_u$ respectively, while the analytic maps, whose graphs these manifolds are, will be denoted by W^3_s and W^3_u. Their precise definitions are analogous to Definition 4.8, which will not be repeated here.

The next lemma is a restatement of a similar result from [41]. Again, the proof of the claim about the contraction rate on a stable manifold of some renormalization operator is independent of the particular choice of the operator and the simple fixed point. The only fact used in the proof presented in [41], specific to this choice, is the largest contracting eigenvalue of the derivative of the renormalization operator at the fixed point of interest. In our case, this is the eigenvalue $\lambda_{(0,4)} = \lambda_{(1,1)}$ whose absolute value is equal to ϑ^{-1}. We want to emphasize that the open ball $\mathcal{B}_\pm$ appearing in the following Lemma is, possibly, smaller than that appearing in Theorem 5.3. Here, however, we chose to keep the same symbol not to make our notation unnecessary cumbersome. In all following sections we will only use the open ball $\mathcal{B}_\pm$ from Lemma 5.4.

Lemma 5.4. *Let $\zeta \in \mathbb{R}$ be given, satisfying $0 < \vartheta^{-1} < \zeta < 1$. Then there exists an open neighborhood $\mathcal{B}_+$ of H^3_+ in $\mathcal{A}(\rho)$, three constants K, $a > 0$ and $c > 1$, and a non-negative integer N, such that*

$$\|\mathfrak{R}^m(H) - H^3_{(-)^m}\|_\rho \le K\zeta^m \|H - H^3_+\|_\rho, \tag{5.10}$$

$$\|(\mathbb{I} - \mathbb{E})\mathfrak{R}^n(H)\| \le e^{-ac^n}\|H - H^3_+\|_\rho + \zeta^{2n}\|H - H^3_+\|^2_\rho, \tag{5.11}$$

whenever $m \ge 0$, $n \ge N$, and $H \in \mathcal{W}^3_s \cap \mathcal{B}_+$.

In the rest of this paper we will consider the number ζ to be fixed together with the the open ball $\mathcal{B}_+$ and its image $\mathcal{B}_-$ under the scaling $I(x,y,u,z) = (x,y,u,-z)$.

6. **Invariant Tori for Hamiltonians Close to the Fixed Family $\mathbf{F}^*_+$.** As it can be easily seen the members of the family F^*_+ undergo a bifurcation of its ω-tori as the family passes through the degenerate Hamiltonian H^3_+. Specifically, let α be the real parameter along the family, then, if $\alpha \ne 0$, each member of F^*_+, $F^*_+(\alpha)$, has precisely two constant torus maps $\Gamma^i_{F^*_+(\alpha)}(x,y) = (x,y,0,t_i(\widetilde{F^*_+(\alpha)}))$, $i = 0,1$, satisfying equation (4.10). Here, $t_i(\widetilde{F^*_+(\alpha)})$ are the two square roots of $-\alpha/3\gamma$. We, notice, however, that for a real α the restriction of the maps $\Gamma^i_{F^*_+(\alpha)}$ to $\mathbb{R}^2$ is real-valued for $\alpha \le 0$ and complex-valued for $\alpha > 0$ (we recall that γ is a real positive number). Thus, the ω-tori of $F^*_+(\alpha)$, viewed as a real parameter family, undergo a bifurcation: for $\alpha < 0$ there are exactly two maps $\Gamma^i_{F^*_+(\alpha)}$ form $T^2(\mathbb{R})$ to $T^2(\mathbb{R}) \times \mathbb{R}^2$, for $\alpha = 0$ there is only one such map, namely $\Gamma_{H^3_+}(x,y) = (x,y,0,0)$, and for $\alpha > 0$ there are none. Furthermore, the rotation number of orbits on torus $\Gamma_{H^3_+}$ is minimal. The same bifurcation happens along the family F^*_- with the parameter regions of existence and non-existence of real tori reversed.

Our goal is to show that a similar result holds for any real-parameter family of Hamiltonians in the vicinity of $F^*_+(\alpha)$. This is a statement about some universal property of transversal families and it is quite natural that we shall use a renormalization group construction to prove this result. Before we start, however, we need to give a rigorous definition of the space of transversal one-parameter families.

Recall, that $\mathcal{A}^u(\rho) = \mathbb{P}^u\mathcal{A}(\rho)$ is the one-dimensional unstable subspace of the renormalization operator $\mathfrak{R}$ at the fixed point H^3_+. Specifically, if $H \in \mathcal{A}(\rho)$ then $\mathbb{P}^u H = \alpha z$, where $\alpha = \mathbb{E}_0 \partial_z H$. The isomorphism $\mathbb{P}^u H \mapsto \alpha$ allows one to identify

$\mathcal{A}^u(\rho)$ with the complex plain $\mathbb{C}$. Below we will abuse notation by writing $\alpha \in \mathcal{A}^u(\rho)$, keeping in mind the mentioned identification.

Definition 6.1. *Given $\delta > 0$, let $\mathcal{D}_\delta = \{\alpha \in \mathcal{A}^u(\rho) : |\alpha| < \delta\}$. Define $\mathcal{F}^\rho_\delta$ to be the Banach space of all analytic one-parameter families $F : \mathcal{D}_\delta \mapsto \mathcal{A}^u(\rho) \oplus \mathcal{A}^s(\rho)$, that extend continuously to the boundary of $\mathcal{D}_\delta$, equipped with the norm*

$$\|F\|_\delta = \sup_{\alpha \in \mathcal{D}_\delta} \|F(\alpha)\|_\rho. \tag{6.1}$$

We will denote an open ball in $\mathcal{F}^\rho_\delta$ centered at F^*_+ by $\mathcal{U}$. Its radius will be indicated in parentheses if necessary. We want to mention that if the radius of $\mathcal{U}$ is sufficiently small, all families in this ball, being analytic and close to the transversal fixed family F^*_+, are automatically transversal to the stable manifold $\mathcal{W}^3_s$.

The following Proposition shows that Hamiltonians, sufficiently close to the fixed family F^*_+ and lying outside of some neighborhood of H^3_+ posses two analytic tori on which the flow is conjugate to the rotation with a rotation vector ω. Such members of the families correspond to the values of parameter α lying outside of some neighborhood of zero. We will denote such annulus by $\mathcal{D}_{\delta,\alpha'}$, i.e. $\mathcal{D}_{\delta,\alpha'} = \{\alpha \in \mathcal{A}^u(\rho) : \alpha' < |\alpha| < \delta\}$.

Proposition 6.2. *Let the quadruples ρ', ϱ and ρ be as in Theorem* 4.2, *and let r be a pair of real positive numbers, such that $r_1 < (1 - \chi^{-1})\rho'_1$ and $r_2 < \rho_2 - \varrho_2$. Then, if $\mathcal{B}_+(b)$, the neighborhood from Lemma* 5.4 *is sufficiently small, there exists a real positive number δ, and for each $0 < \alpha' < \delta$, a real positive $\beta = \beta(\alpha', \delta)$, such that the following holds for every $F \in \mathcal{U}(\beta) \subset \mathcal{F}^\rho_\delta$ and $\alpha \in \mathcal{D}_{\delta,\alpha'}$.*

- (a) *There exist two distinct complex numbers $t_i(\widetilde{F(\alpha)})$, $i = 0, 1$, such that the Hamiltonians $F(\alpha) \circ R_{t_i(\widetilde{F(\alpha)})}$, $i = 0, 1$, are distinct and lie in $\mathcal{B}_+(b) \cap \mathcal{W}^0_s \subset \mathcal{A}(\varrho)$.*
- (b) *There exist two distinct maps $\Gamma^i_{F(\alpha)}$, $i = 0, 1$, analytic from $\mathcal{D}_1(r)$ to $\mathcal{D}(\rho)$ and satisfying*
$$(\mathbb{J}\nabla F(\alpha)) \circ \Gamma^i_{F(\alpha)} = \xi'(F(\alpha) \circ R_{t_i(\widetilde{F(\alpha)})})\partial_x \Gamma^i_{F(\alpha)} \tag{6.2}$$
on $\mathcal{D}_1(r)$, where $\xi'(H)$ is the analytic function on $\mathcal{B}(b)_+$ described in Theorem 4.9.

Proof. Let δ be some real positive number, to be specified later, such that $F^*_+(\alpha) \in \mathcal{B}_+(b)$ for all $\alpha \in \mathcal{D}_\delta$. Consider $F^*_+(\alpha)$ for some non-zero α. Let $t_i(\widetilde{F^*_+(\alpha)})$, $i = 0, 1$, be the two *distinct* square roots of $-\alpha/(3\gamma)$. Then $F^*_+(\alpha) \circ R_{t_i(\widetilde{F^*_+(\alpha)})} \in \mathcal{W}^0_s$. To be more specific,

$$F^*_+(\alpha) \circ R_{t_i(\widetilde{F^*_+(\alpha)})} = u + 3\gamma t_i(\widetilde{F^*_+(\alpha)}) z^2 + \gamma z^3 + \gamma t_i^3(\widetilde{F^*_+(\alpha)}) + \alpha t_i(\widetilde{F^*_+(\alpha)}), \tag{6.3}$$

and a simple computation shows that $\|F^*_+(\alpha) \circ R_{t_i(\widetilde{F^*_+(\alpha)})} - H^3_+\|^*_\rho < b$ if $\delta < b^2/(3\gamma(\rho_4)^4)$, thus $F^*_+(\alpha) \circ R_{t_i(\widetilde{F^*_+(\alpha)})}$ is in $\mathcal{B}_+(b)$ in the norm $\|\cdot\|^*_\rho$.

Fix some $\delta < b^2/(12\gamma(\rho_4)^4)$ (this ensures that $F^*_+(\alpha) \circ R_{t_i(\widetilde{F^*_+(\alpha)})} \in \mathcal{B}_+(b/2)$ in the norm $\|\cdot\|^*_\rho$) and some real positive $\alpha' < \delta$. Equation (6.3) shows that for any $\alpha \in \mathcal{D}_{\delta,\alpha'}$

$$\|F^*_+ \circ R_{t_0(\widetilde{F^*_+(\alpha)})} - F^*_+ \circ R_{t_1(\widetilde{F^*_+(\alpha)})}\|^*_\rho > 2(3\gamma\alpha')^{\frac{1}{2}}\rho_4^2. \tag{6.4}$$

Since $F^*_+ \circ R_{t_0(\widetilde{F^*_+(\alpha)})}$ and $F^*_+ \circ R_{t_1(\widetilde{F^*_+(\alpha)})}$ belong to the stable manifold of the renormalization operator $\mathcal{R}_\mu$ and, for our choice of δ, are contained in $\mathcal{B}_+(b) \subset \mathcal{B}^\Gamma_0$, for each of these two Hamiltonians there exists an analytic torus map, satisfying equation (4.10). Then, given a pair r as in the condition of the Theorem, the maps $\Gamma^i_{F^*_+(\alpha)} = R_{t_i(\widetilde{F^*_+(\alpha)})} \circ \Gamma_{F^*_+(\alpha) \circ R_{t_i(\widetilde{F^*_+(\alpha)}}} = (x, y, 0, t_i(\widetilde{F^*_+(\alpha)})$, $i = 0, 1$, are obviously analytic from $\mathcal{D}(r)$ to $\mathcal{D}(\rho)$ and satisfy

$$(\mathbb{J} \nabla F^*_+(\alpha)) \Gamma^i_{F^*_+(\alpha)} = \xi'(F^*_+(\alpha) \circ R_{t_i(\widetilde{F^*_+(\alpha)})})\, \partial_x \Gamma^i_{F^*_+(\alpha)}, \; i = 0, 1. \tag{6.5}$$

As it can be easily checked, the family $F^*_+(\alpha) \circ R_t$ is transversal to $\mathcal{W}^0_s$ at points $F^*_+(\alpha) \circ R_{t_i(\widetilde{F^*_+(\alpha)})}$. By the stability of transversality, there exists a real positive β, such that all families $F(\alpha) \circ R_t$ with $F \in \mathcal{U}(\beta)$ and $\alpha \in \mathcal{D}_{\delta, \alpha'}$ also intersect $\mathcal{W}^0_s$ only at two points and are transversal to the stable manifold at those points. By the Implicit Function Theorem, for a fixed α, the intersection points are continuous (in fact analytic) functions of the parameter F, i.e. maps $F \mapsto F(\alpha) \circ R_{t_i(\widetilde{F(\alpha)})}$, $i = 0, 1$, are continuous and bounded on $\mathcal{U}(\beta)$, and so are the maps $\tau_i : (F, \alpha) \mapsto t_i(\widetilde{F(\alpha)})$, $i = 0, 1$, on $\mathcal{U} \times \mathcal{D}_{\delta, \alpha'}$. Then, for each ε there is a β, such that $|t_i(\widetilde{F(\alpha)}) - t_i(\widetilde{F^*_+(\alpha)})| < \varepsilon$, whenever $F \in \mathcal{U}(\beta)$. Choose β so that $|t_i(\widetilde{F(\alpha)}) - t_i(\widetilde{F^*_+(\alpha)})| < |\alpha'/12\gamma|^{1/2}$, then $t_0(\widetilde{F(\alpha)}) \neq t_1(\widetilde{F(\alpha)})$ for all $F \in \mathcal{U}(\beta)$ and $\alpha \in \mathcal{D}_{\delta, \alpha'}$. Moreover, for each ε there exists a β, such that $\sup_{F \in \mathcal{U}, \alpha \in \mathcal{D}_{\delta, \alpha'}} ||t_i(\widetilde{F(\alpha)})| - |\delta/\gamma|^{1/2}| < \varepsilon$ for all $F \in \mathcal{U}(\beta)$. Therefore, it is possible to reduce δ and β, so that $\sup_{F \in \mathcal{U}, \alpha \in \mathcal{D}_{\delta, \alpha'}} \{|t_i(\widetilde{F(\alpha)})|\} < \rho_4 - \varrho_4$ for all $\mathcal{U}(\beta) \in \mathcal{F}^\rho_\delta$.

Reduce β, if necessary, so that $\|F^*_+(\alpha) \circ R_{t_i(\widetilde{F^*_+(\alpha)})} - F(\alpha) \circ R_{t_i(\widetilde{F(\alpha)})}\|^*_\rho < (1/2) \min\left\{b, (3\gamma\alpha')^{1/2} \rho_4^2\right\}$. This condition, together with our choice of δ and the condition (6.4), ensures that $F(\alpha) \circ R_{t_i(\widetilde{F(\alpha)})} \in \mathcal{B}_+(b)$ and $\|F \circ R_{t_0(\widetilde{F(\alpha)})} - F \circ R_{t_1(\widetilde{F(\alpha)})}\|^*_\rho > (3\gamma\alpha')^{\frac{1}{2}} \rho_4^2$. Thus, if β is chosen sufficiently small, as described above, for every $F \in \mathcal{U}(\beta)$ and $\alpha \in \mathcal{D}_{\delta, \alpha'}$ the family $F(\alpha) \circ R_t$ intersects the stable manifold $\mathcal{W}^0_s$ transversally at two distinct points, lying within $\mathcal{B}_+(b)$. Therefore, by Theorem 4.9, there exist two maps $\Gamma^i_{F(\alpha)} = R_{t_i(\widetilde{F(\alpha)})} \circ \Gamma_{F(\alpha) \circ R_{t_i(\widetilde{F(\alpha)})}}$, analytic from $\mathcal{D}_1(r)$ to $\mathcal{D}(\rho_1, \rho_2, \varrho_3, \varrho_4 + \epsilon) \subset \mathcal{D}(\rho)$ by our choice of δ and satisfying (6.2). Let $t \mapsto c(t) = (x(t), y(t))$ be any closed contour on $T^2(\mathbb{R})$, such that the winding number of the projection of the curve $t \mapsto \Gamma^i_{F(\alpha) \circ R_{t_i(\widetilde{F(\alpha)})}}(c(t))$ on its y-component is some non-zero integer k. Denote $\mathcal{P}_H = (\omega' \cdot \mathbb{J}\Gamma_H, \Omega' \cdot \mathbb{J}\Gamma_H)$, $\mathcal{Q}_H = (\omega \cdot \Gamma_H, \Omega \cdot \Gamma_H)$ and $t_i = t_i(\widetilde{F(\alpha)})$, then, by Theorem 4.9

$$\begin{aligned} I^i &= \oint_c \mathcal{P}^i_{F(\alpha)}(x, y) \cdot d\mathcal{Q}^i_{F(\alpha)}(x, y) = \oint_c R_{t_i} \circ \mathcal{P}_{F(\alpha) \circ R_{t_i}}(x, y) \cdot d\mathcal{Q}_{F(\alpha) \circ R_{t_i}}(x, y) \\ &= \oint_c (\mathcal{P}_{F(\alpha) \circ R_{t_i}}(x, y) + t_i \Omega) \cdot d\mathcal{Q}_{F(\alpha) \circ R_{t_i}}(x, y) = t_i \frac{\vartheta^2 + 1}{\vartheta^2} 2\pi k. \end{aligned} \tag{6.6}$$

By our choice of β, $t_0(\widetilde{F(\alpha)} \neq t_1(\widetilde{F(\alpha)}$, and, therefore, $I^0 \neq I^1$. Hence, $\Gamma^0_{F(\alpha)} \neq \Gamma^1_{F(\alpha)}$, i.e. the two torus maps are distinct. ☐

The next Corollary shows that there exists a subset of $\mathcal{D}_\delta$ on which the families from the neighborhood $\mathcal{U}(\beta)$ do not posses any real ω-tori. Define for every real and positive $\alpha' < \delta$ the set $\mathcal{D}^+_{\delta, \alpha'} = \{\alpha \in \mathcal{A}^u(\rho) : |\alpha| < \delta, \Re(\alpha) > \alpha'\}$.

Corollary 6.3. *Let $F \in \mathcal{U}(\beta)$, $\alpha \in \mathcal{D}^+_{\delta, \alpha'}$ be as in Prop 6.2, and let r be a pair of real positive numbers, such that $r_1 < (1 - \chi^{-1}) \rho'_1$ and $r_2 < \rho_2 - \varrho_2$. If $\alpha' > \beta/(\rho_4 - \varrho_4)$,* *then any analytic torus map* $\Gamma_{F(\alpha)} : \mathcal{D}_1(r) \mapsto \mathcal{D}(\varrho)$, *satisfying equation* (4.10), *is such that its imaginary part* $\Im(\Gamma_F(\alpha)$ *is non-zero on some non-empty subset of* $T^2(\mathbb{R})$.

Proof. Let $F = F_+^* + f$ be in $\mathcal{U}(\beta)$ and α in $\mathcal{D}^+_{\delta,\alpha'}$, and let $\Gamma_{F(\alpha)}$ be an analytic map from $\mathcal{D}_1(r)$ to $\mathcal{D}(\rho)$, periodic in variables $\omega \cdot (x,y) = q_1$ and $\Omega \cdot (x,y) = q_2$ and satisfying equation (4.10).

Consider a linear flow with the generator ω on $T^2(\mathbb{R})$. The scalar product $s = \omega' \cdot (q - q(0))$ can be chosen as a parameter along an orbit of such flow, starting at the point $q(0)$ (in the following exposition we prefer to use periodic variables (q_1, q_2) which serve the purpose of clarity of our argument better). Since a linear orbit of this flow $\gamma : \mathbb{R} \mapsto T^2(\mathbb{R})$, starting at an arbitrary point $\gamma(0) \in T^2(\mathbb{R})$ and, hence, given by equation $\gamma(s) = (\vartheta s/2 + \gamma_1(0), s/2 + \gamma_2(0)) \mod 2\pi$, has an irrational slope and, thus, covers the real torus $T^2(\mathbb{R})$ densely, for any initial point $\gamma(0)$ and real $\epsilon > 0$ there is a real positive number s_ϵ such that $|\gamma(0) - \gamma(s_\epsilon)| < \epsilon$ (here and in the rest of this Corollary we will use the l_1-norm for vectors in $\mathbb{R}^n$). We want to emphasize that such s_ϵ is independent of the initial point of the orbit $\gamma(0)$.

Furthermore, since the map $\Gamma_{F(\alpha)}$ is analytic and bounded on $\mathcal{D}_1(r)$, it is certainly continuous, in fact uniformly continuous on compact $T^2(\mathbb{R})$. Hence, there is a positive real constant c', such that $|\Gamma_{F(\alpha)}(q) - \Gamma_{F(\alpha)}(q')| < c'|q - q'|$ for any two points q and q' in $T^2(\mathbb{R})$, in particular, $|\Gamma_{F(\alpha)}(\gamma(0)) - \Gamma_{F(\alpha)}(\gamma(s_\epsilon))| < c'\epsilon$. For the sake of brevity in the computations below, we will denote the z-component of the torus map $\Gamma_{F(\alpha)}$ by $\mathcal{Z}_{F(\alpha)}$, and we will also abuse notation by writing ξ' in place of $\xi'(F(\alpha))$.

$$\begin{aligned}
|\xi'|c'\epsilon &> \left| \int_0^{s_\epsilon} \xi'\omega \cdot \nabla_q \Gamma_{F(\alpha)}(\gamma(s)) \; ds \right| \\
&= \left| \int_0^{s_\epsilon} \omega + [\alpha + 3\gamma \mathcal{Z}^2_{F(\alpha)}(\gamma(s))]\Omega + \nabla_p f(\alpha) \circ \Gamma_{F(\alpha)}(\gamma(s)) \; ds \right| \\
&\geq \left| |\gamma(s_\epsilon) - \gamma(0)| - \left| \int_0^{s_\epsilon} [\alpha + 3\gamma \mathcal{Z}^2_{F(\alpha)}(\gamma(s))]\Omega + \nabla_p f(\alpha) \circ \Gamma_{F(\alpha)}(\gamma(s)) ds \right| \right|, \quad (6.7)
\end{aligned}$$

consequently, $\left| \int_0^{s_\epsilon} [\alpha + 3\gamma \mathcal{Z}^2_{F(\alpha)}(\gamma(s))]\Omega + \nabla_p f(\alpha) \circ \Gamma_{F(\alpha)}(\gamma(s)) \; ds \right| < [|\xi'|c' + 1]\epsilon$.

Define $\phi(q) = \Re[\omega' \cdot \nabla_p f \circ \Gamma_{F(\alpha)}(q)]$ and $\psi(q) = \Re[\Omega' \cdot \nabla_p f \circ \Gamma_{F(\alpha)}(q)]$ for all $q \in T^2(\mathbb{R})$. Assume that $\Im \Gamma_{F(\alpha)}(q) = 0$ everywhere on $T^2(\mathbb{R})$, then one gets for every $F \in \mathcal{U}(\beta)$, $\alpha \in \mathcal{D}^+_{\delta,\alpha'}$ and $\gamma(0) \in T^2(\mathbb{R})$:

$$\begin{aligned}
&\left| \int_0^{s_\epsilon} [\alpha + 3\gamma \mathcal{Z}^2_{F(\alpha)}(\gamma(s))]\Omega + \nabla_p f(\alpha) \circ \Gamma_{F(\alpha)}(\gamma(s)) \; ds \right| \\
&\qquad \geq \left| \int_0^{s_\epsilon} \Re[\alpha + 3\gamma \mathcal{Z}^2_{F(\alpha)}(\gamma(s))]\Omega + \phi(\gamma(s))\omega + \psi(\gamma(s))\Omega \; ds \right| \\
&\qquad \geq \left| \int_0^{s_\epsilon} [\alpha' + 3\gamma(\Re \mathcal{Z}_{F(\alpha)}(\gamma(s)))^2 + \psi(\gamma(s))]\Omega + \phi(\gamma(s))\omega \; ds \right| \\
&\qquad \geq \left| \int_0^{s_\epsilon} [\alpha' + \psi(\gamma(s))]\Omega + \phi(\gamma(s))\omega \; ds \right| \\
&\qquad \geq \left| \left| \int_0^{s_\epsilon} [\alpha' + \psi(\gamma(s))]\Omega \; ds \right| - \left| \int_0^{s_\epsilon} \phi(\gamma(s))\omega \; ds \right| \right|. \qquad (6.8)
\end{aligned}$$

In the second inequality we have explicitly used the fact that $\Im \mathcal{Z}_{F(\alpha)}$ is assumed to be zero everywhere on $T^2(\mathbb{R})$. In particular, the inequality (6.8) implies

$$[|\xi'|c'+1]\epsilon + \left| \int_0^{s_\epsilon} \phi(\gamma(s))\omega \ ds \right| > \left| \int_0^{s_\epsilon} \alpha'\Omega + \psi(\gamma(s))\Omega \ ds \right|. \tag{6.9}$$

Since $\sup_{q\in T^2(\mathbb{R})}|\phi(q)| \le \sup_{q\in T^2(\mathbb{R})} |\left(\omega\cdot\nabla_p f\right)\circ\Gamma_{F(\alpha)}(q)| < \|\partial_u f\|_\varrho < \|f\|_\rho/(\rho_3-\varrho_3) < \beta/(\rho_3-\varrho_3)$ and, similarly, $\sup_{q\in T^2(\mathbb{R})}|\psi(q)| < \|\partial_z f\|_\varrho < \beta/(\rho_4-\varrho_4)$, one can weaken the condition (6.9) as

$$\left| \int_0^{s_\epsilon} [\alpha' + \psi(\gamma(s))]\Omega \ ds \right| < |\xi'|c'\epsilon + \epsilon + \frac{\beta\epsilon}{\rho_3-\varrho_3}. \tag{6.10}$$

On the other hand, if β is sufficiently small, in particular, if $\alpha' > \beta/(\rho_4-\varrho_4)$, then

$$\begin{aligned} \left| \int_0^{s_\epsilon} [\alpha' + \psi(\gamma(s))]\Omega \ ds \right| &> \left[\alpha' - \frac{\beta}{\rho_4-\varrho_4}\right] \left| \int_0^{s_\epsilon} \Omega \ ds \right| \\ &= \left[\alpha' - \frac{\beta}{\rho_4-\varrho_4}\right] |\tilde{\gamma}(s_\epsilon) - \tilde{\gamma}(0)|, \end{aligned} \tag{6.11}$$

where we have denoted the linear curve $s \mapsto ((\vartheta^2 s/2), -(s/2\vartheta)) \bmod 2\pi$ by $\tilde{\gamma}$. The difference of curves γ and $\tilde{\gamma}$ is again a linear curve from $\mathbb{R}$ to $T^2(\mathbb{R})$ with an irrational slope. Then, for each sufficiently small ϵ there exists a real positive s_ϵ, such that $|\gamma(s_\epsilon)-\gamma(0)| < \epsilon$, while $|\tilde{\gamma}(s_\epsilon)-\tilde{\gamma}(0)| > \epsilon[(|\xi'|c'+1)(\rho_4-\varrho_4)+\beta]/[\alpha'(\rho_4-\varrho_4)-\beta]$, which contradicts the inequality (6.10).

Thus, if β is sufficiently small, $\Im \mathcal{Z}_{F(\alpha)}$ is non-zero on a subset of the line $\gamma(s)$ of non-zero measure. Since this must be true for any linear orbit $\gamma(s)$ originating at an arbitrary initial point $\gamma(0) \in T^2(\mathbb{R})$, there exists some non-empty set in $T^2(\mathbb{R})$ on which $\Im \Gamma_{F(\alpha)}$ is non-zero.

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7. **One-Parameter Families.**

Denote the parameter value for which a family intersects the stable manifold by α_F, i.e. $F(\alpha_F) \in \mathcal{W}_s^3$. This parameter depends analytically on F as can be seen from the following argument.

Define the map $\Phi(F,\alpha) : \mathcal{U}\times\mathcal{D}_\delta \mapsto \mathcal{A}^u(\rho)$ by setting $\Phi(F,\alpha) = (\Pi - \mathbb{I})F(\alpha)$, where Π is the projection on the stable manifold given by $\Pi H = (W_s^3 + \mathbb{I})\mathbb{P}^s H$. Then, the parameter α_F is a solution of equation $\Phi(F,\alpha)=0$. Since the equation $\Phi(F_+^*,\alpha)=0$ has a unique solution $\alpha_F = 0$ and $D_2\Phi(F_+^*,\alpha)|_{\alpha=0} = 1$, the function $\Phi(F,\alpha)$ satisfies the assumptions of the Implicit Function Theorem, and there exists a sufficiently small neighborhood $\mathcal{U}$, such that the equation $\Phi(F,\alpha)=0$ has a unique solution α_F in $\mathcal{D}_\delta$ for every $F\in\mathcal{U}$, which depends on F analytically.

We will continue with a technical lemma, which will be used in our next result about a contraction mapping on the Hamiltonian families. The proof of this lemma will not be included here: an interested reader is referred to [41] for details.

Lemma 7.1. *The eigenvectors $e_{k,n}$ of the operator $L_{\vartheta^{-2}}$, defined by $L_{\vartheta^{-2}}h = \vartheta^3(\mathbb{I}^+h)\circ T_{\vartheta^{-2}}$, span a dense subspace of $\mathbb{E}\mathcal{A}(\rho)$. The spectral projection corresponding to the eigenvalue zero of $L_{\vartheta^{-2}}$ is $\mathbb{I}-\mathbb{E}$. In particular, there are real positive constants a and $s>1$, such that for every positive integer m,*

$$\|L_{\vartheta^{-2}}^{m+1}\left(\mathbb{I}-\mathbb{E}\right)\| \le |\vartheta^3| e^{-as^m} \|L_{\vartheta^{-2}}^{m}\left(\mathbb{I}-\mathbb{E}\right)\|. \tag{7.1}$$

In the next Proposition we introduce a contraction mapping on this space of families. We will denote the image of a family F in $\mathcal{U}(\beta)$ under reflection in the last variable $F(\alpha) \mapsto F(\alpha) \circ I$, where $I(x, y, u, z) = (x, y, u, -z)$, by $\mathfrak{I}(F)$, i.e. $\mathfrak{I}(F)(\alpha) = F(\alpha) \circ I$.

Proposition 7.2. *Let a real positive number δ be sufficiently small, and let a real positive ζ be such that $\vartheta^{-1} < \zeta < 1$. Define the maps M on $\mathcal{F}_\delta^\rho$ and Y_F on $\mathcal{A}^u(\rho)$ by setting*

$$M(F) = \mathfrak{R} \circ F \circ Y_F, \tag{7.2}$$

$$Y_F = (\mathbb{P}^u \circ \mathfrak{R} \circ F)^{-1}. \tag{7.3}$$

Then there exists a real positive number β, such that $\mathfrak{J} \circ M$ is a contraction mapping on an open neighborhood $\mathcal{U}(\beta)$, centered at F_+^*, with a fixed point F_+^* and a contraction rate $c/\vartheta < \zeta$ for some fixed positive real $1 < c < \zeta/\vartheta$.

In particular, there exist real positive constants c_1 and $d < \vartheta^2$, such that

$$\|M^n(F) - F^*_{(-)^n}\|_\delta \leq c_1 \left(\frac{c}{\vartheta}\right)^n \|F - F_+^*\|_{\left(\frac{d}{\vartheta^2}\right)^n \delta}, \tag{7.4}$$

$$\|M^n(F)(\alpha) - F^*_{(-)^n}(\alpha)\|_\rho \leq c_1 \ \beta \left(\frac{c}{\vartheta}\right)^n \tag{7.5}$$

for all $n \geq 0$ and all $F \in \mathcal{U}(\beta)$. Furthermore, there exist a positive integer N and real positive constants c_2, c_3, a and $s > 1$, such that for all $F \in \mathcal{U}(\beta)$ and $n \geq N$

$$\|(\mathbb{I} - \mathbb{E}) M^{2n}(F)\|_\delta \leq \left(c_2 e^{-a s^{2n}} + c_3 \zeta^{4n}\right) \beta \tag{7.6}$$

where the averaging projection $\mathbb{E}$ is defined by $(\mathbb{E}F)(\alpha) = \mathbb{E}F(\alpha)$.

Proof. The map $\mathbb{P}^u \circ \mathfrak{R} \circ F_+^*$ agrees with the restriction of $D\mathfrak{R}$ to the unstable subspace, thus the operator norm of its derivative is larger than one, and certainly bounded away from zero. The map $Y_{F_+^*}$ is linear and contracting: $Y_{F_+^*}(\alpha) = -\vartheta^{-2}\alpha$. By the inverse function theorem $F \mapsto Y_F$ is well-defined and analytic near F_+^*. In particular, there exists a neighborhood $\mathcal{U}'$ of F_+^* and a real positive $d < \vartheta^2$, such that, for all $F \in \mathcal{U}'$, the map Y_F is a contraction on $\mathcal{D}_\delta$ of rate d/ϑ^2. Denote the map $F \mapsto Y_F$ by Y. Then, as the calculation below shows, the operator norm of $DM(F_+^*)$ is less than one.

$$\begin{aligned}
(DM(F_+^*)f)(\alpha) &= D\mathfrak{R}(F_+^*(Y_{F_+^*}(\alpha))) \left[f(Y_{F_+^*}(\alpha)) + DF_+^*(Y_{F_+^*}(\alpha))(DY_{F_+^*}f)(\alpha)\right] \\
&= D\mathfrak{R}(F_+^*(Y_{F_+^*}(\alpha))) \left[f(Y_{F_+^*}(\alpha)) + (DY(F_+^*)f)(\alpha) z\right] \\
&= D\mathfrak{R}(H_+^3) \left[f(Y_{F_+^*}(\alpha)) - Y_{F_+^*}(\mathbb{P}^u D\mathfrak{R}(H_+^3) f(Y_{F_+^*}(\alpha))) z\right] \\
&= D\mathfrak{R}(H_+^3) \left[f(Y_{F_+^*}(\alpha)) - \mathbb{P}^u f(Y_{F_+^*}(\alpha))\right] \\
&= \mathbb{P}^s D\mathfrak{R}(H^3) f\left(-\frac{\alpha}{\vartheta^2}\right),
\end{aligned} \tag{7.7}$$

and, again, there exist a real positive $c < \vartheta$ and a neighborhood $\mathcal{U}''$ of F_+^*, on which $\mathfrak{J} \circ M$ is a contraction of rate c/ϑ. Let $\mathcal{U}(\beta) \subset \mathcal{U}' \cap \mathcal{U}''$. The claim of the Proposition and the bound (7.4) follow immediately from the above argument, while the bound (7.5) from the following estimate:

$$\|M^n(F)(\alpha) - F^*_{(-)^n}(\alpha)\|_\rho \leq \|M^n(F) - F^*_{(-)^n}\|_\delta \leq c_1 \left(\frac{c}{\vartheta}\right)^n \beta. \tag{7.8}$$

To obtain the bound (7.6), choose a real positive $\vartheta^{-1} < \zeta < 1$ and fix β, so that $c/\vartheta < \zeta$. Consider $F' = F_+^* + f' \in \mathcal{U}(\beta)$ and write $F'' = M^2(F') = F_+^* + f'' =$

$F^*_+ + h'' + g''$, where

$$h'' = DM^2(F^*_+)f', \quad g'' = M^2(F^*_+ + f') - DM^2(F^*_+)f'. \tag{7.9}$$

As we have shown above $\|h''\|_\delta \le \left(\frac{c}{\vartheta}\right)^2 \|f'\|_\delta$. Also, since M^2 is a differentiable map, being a composition of such, there exists a real positive constant b, such that $\|M^2(F^*_+ + f) - DM^2(F^*_+)f\|_\delta \le b\|f\|^2_\delta$ for all $F = F^*_+ + f$ in $\mathcal{U}(\beta)$, then $\|g''\|_\delta \le b\|f'\|^2_\delta$.

At the next step, define $F''' = M^2(F'') = F^*_+ + f''' = F^*_+ + h''' + g'''$, where $h''' = DM^2(F^*_+)h'' = [DM^2(F^*_+)]^2 f'$ and $g''' = M^2(F^*_+ + f'') - DM^2(F^*_+)f'' + DM^2(F^*_+)g''$, then

$$\|h'''\|_\delta \le \left(\frac{c}{\vartheta}\right)^2 \|h''\|_\delta, \quad \|g'''\|_\delta \le b\|h'' + g''\|^2 + \left(\frac{c}{\vartheta}\right)^2 \|g''\|_\delta. \tag{7.10}$$

Now, since $DM^2(F^*_\pm)f(\alpha) = [D\mathfrak{R}(H^3_\pm)]^2 \mathbb{P}^s f(\vartheta^{-4}\alpha)$, and $(\mathbb{I} - \mathbb{E})L_{\vartheta^{-2}} = (\mathbb{I} - \mathbb{E})D\mathfrak{R}(H^3_\pm)$, by Lemma 7.1, there exist constants $a > 0$ and $s > 1$, a sufficiently large integer N, and a sequence of integers $k_n < \epsilon n$ (a real positive number ϵ will be specified later), such that $n - k_n - 1 > 0$ and

$$\|(\mathbb{I} - \mathbb{E})[DM^2(F^*_\pm)]^{k_n}\| \le e^{-a s^{2n}}, \tag{7.11}$$

whenever $n \ge N$.

Next, given such n and an arbitrary $F \in \mathcal{U}$, consider $F' = M^{2m}(F)$, where $m = n - k_n - 1$. As we have already shown, there exists a positive real constant c_1, such that $\|f'\|_\delta \le c_1 (c/\vartheta)^{2m} \beta = \varepsilon$. Then, the pair (h'', g'') given by the equations (7.9) satisfies $\|h''\|_\delta \le \varepsilon$ and $\|g''\|_\delta \le b\varepsilon^2$. Now, iterate k_n times applying the inequalities (7.10). At each step, if $\|g''\|_\delta \le b\varepsilon^2$, then $\|g'''\|_\delta \le b\varepsilon^2 + (c/\vartheta)^2 b\varepsilon^2 \le (1 + (c/\vartheta)^2) b\varepsilon^2$, and we obtain after k_n iterations

$$h^{k_n} = [DM^2(F^*_+)]^{k_n} h'', \quad \|g^{k_n}\| \le (1 + (c/\vartheta)^2)^{k_n} b\varepsilon^2. \tag{7.12}$$

Choose ϵ, so that $\zeta^4 \ge (1+(c/\vartheta)^2)^\epsilon (c/\vartheta)^{4(1-\epsilon)}$, then $(1+(c/\vartheta)^2)^{k_n}\varepsilon^2 = (c_1\beta)^2(1+(c/\vartheta)^2)^{k_n}(c/\vartheta)^{4(n-k_n)} \le \zeta^{4n}$. Finally, notice, that $M^{2n} = F^*_+ + h^{k_n} + g^{k_n}$, then one can find real positive constants c_2 and c_3 such that for all $n > N$

$$\|(\mathbb{I} - \mathbb{E})M^{2n}(F)\|_\delta \le \|(\mathbb{I} - \mathbb{E})h^{k_n}\|_\delta + \|(\mathbb{I} - \mathbb{E})g^{k_n}\|_\delta \le \left(c_2 e^{-a s^{2n}} + c_3 \zeta^{4n}\right)\beta. \tag{7.13}$$

□

Corollary 7.3. *Let ρ' and ρ be as in Theorem* 4.3, *and let c_1, c, d and $\mathcal{U}(\beta) \subset \mathcal{F}^\rho_\delta$ be as in Prop* 7.2. *For every $n \ge 0$, denote $\delta_n = \delta\left(d/\vartheta^2\right)^n$. Let $t(H)$ be the solution of the equation* (5.1), *then for every $F \in \mathcal{U}$ and every $\alpha \in \mathcal{D}_{\delta_n}$*

$$|t(\mathfrak{R}^n(F(\alpha)) \circ \mathcal{U}_{\mathfrak{R}^n(F(\alpha))})| \le \frac{1}{2\gamma} \frac{c_1}{(\rho'_4)^2} \beta \left(\frac{c}{\vartheta}\right)^n. \tag{7.14}$$

Proof. Let, as before, $\tilde{H} = H \circ \mathcal{U}_H$. Then, by Prop. 5.1, we must have $|t(\tilde{H})| \le (1/(2\gamma(\rho'_4)^2))\|\mathbb{E}(\mathbb{P}^s H - H^3_\pm)\|_\rho$ for all $H \in \mathcal{H}$. For any $n \ge 0$, $F \in \mathcal{U}$ and $\alpha \in \mathcal{D}_{\delta_n}$, denote $F_n = M^n(F)$ and $(\alpha)_n = (Y_{F_0} \circ Y_{F_1} \circ \ldots \circ Y_{F_{(n-1)}})^{-1}\alpha$. By Prop. 7.2, $(\alpha)_n \in \mathcal{D}_\delta$. Furthermore,

$$\begin{aligned} |t(\widetilde{\mathfrak{R}^n(F(\alpha))}| &\le \frac{1}{2\gamma(\rho'_4)^2} \|\mathbb{E}(\mathbb{P}^s \mathfrak{R}^n(F(\alpha)) - H^3_{(-)^n}\|_\rho \\ &\le \frac{1}{2\gamma(\rho'_4)^2} \|\mathbb{P}^s[M^n(F)((\alpha)_n) - F^*_{(-)^n}((\alpha)_n)]\|_\rho \end{aligned}$$

$$\le \frac{c_1\beta}{2\gamma(\rho_4')^2}\left(\frac{c}{\vartheta}\right)^n, \tag{7.15}$$

where we have again used Prop. 7.2. ☐

8. **Invariant Tori for Hamiltonians on the Stable Manifold at the Fixed Point $\mathbf{H^3}_+$.** This Section presents a construction of the invariant tori for the isoenergetically degenerate Hamiltonians on the stable manifold $\mathcal{W}_s^3$ at the fixed point H_+^3.

Definition 8.1. *Consider a Hamiltonian $H \in \mathcal{B}_+$. Let $H_k = \mathfrak{R}^n(H)$ and $\tilde{H}_k = H_k \circ U_{\psi(H_k)}$. For every $k \ge 0$ and $n \ge 1$ define*

$$W_k(H) = T_{\vartheta^{-2}}^{k} \circ U_{\psi(H_k)} \circ R_{t(\tilde{H}_k)} \circ T_{\vartheta^{-2}}^{-k} \circ S_{\eta(H_k)}, \tag{8.1}$$

$$W_n'(H) = W_0(H) \circ W_1(H) \circ \ldots \circ W_{n-1}(H). \tag{8.2}$$

We will continue with a series of results about the properties of the transformation $T_{\vartheta^{-2}}^{k} \circ U_{\psi(H_k)} \circ T_{\vartheta^{-2}}^{-k}$. First, however, we will mention the following fundamental result concerning canonical transformations in $\mathcal{A}^4(\rho)$.

Proposition 8.2. *Let r and δ be quadruples of real positive numbers, ϵ — a real number, such that $0 < \epsilon < 2/3\,[1+\max\{\delta_1/\delta_2, \delta_2/\delta_1\}]^{-1}$. Denote, as before, $\bar{\delta} = (\delta_3, \delta_4, \delta_1, \delta_2)$. Also, denote by $\mathcal{B}'$ the set of all functions $\phi \in \mathcal{A}'(r)$, such that $\|\partial_i(\phi)\|_{r+\bar{\delta}} < \epsilon\delta_i$, $1 \le i \le 4$. Then for $\forall \phi \in \mathcal{B}'$ the equation*

$$(X, Y, U, Z)\,(x, y, u, z) = \mathbb{J}\nabla\phi(x, y, u + U(x, y, u, z), z + Z(x, y, u, z)) \tag{8.3}$$

has a unique solution $(X, Y, U, Z)\,(x, y, u, z)$, satisfying $\|X\|_r \le \epsilon\delta_3$, $\|Y\|_r \le \epsilon\delta_4$, $\|U\|_r \le \epsilon\delta_1$, $\|Z\|_r \le \epsilon\delta_2$. The corresponding canonical transformation U_ϕ is analytic from $\mathcal{D}(r)$ to $\mathcal{D}(r+\bar{\delta})$.

Proof. See [30]. ☐

We will use the notation $T_\mu(r) = (\vartheta r_1, \vartheta^{-1} r_2, |\mu|\vartheta^{-1} r_3, |\mu|\vartheta r_4')$ and $\hat{T}_\mu(r) = (r_1, \vartheta^{-1} r_2, |\mu|\vartheta^{-1} r_3, |\mu|\vartheta r_4)$ in the next Lemma.

Lemma 8.3.

(a) *The linear operator $\phi \mapsto \phi \circ T_\mu^{-1}$ is continuous from $\mathcal{A}(r)$ to $\mathcal{A}(r')$ if $r' \le T_\mu(r)$ componentwise. It is also continuous from $\mathcal{A}'(r)$ to $\mathcal{A}'(r')$ of norm $\vartheta/|\mu|$. Furthermore, if $\phi \in \mathcal{A}'(r)$ is such that $\|\partial_i\phi\|_r < \delta_i$, then $\|\partial_i\phi'\|_{r'} < \delta_i'$, where $\delta' = (|\mu|\delta_1/\vartheta, |\mu|\vartheta\delta_2, \vartheta\delta_3, \delta_4/\vartheta)$ and $\phi' = \mu\phi \circ T_\mu^{-1}$, and one has the following identity on $\mathcal{D}(r' - 3\bar{\delta}')$:*

$$T_\mu \circ U_\phi \circ T_\mu^{-1} = U_{\phi'}. \tag{8.4}$$

(b) *If $r'' \le \hat{T}_\mu(r)$ componentwise, then the linear operator $\phi \mapsto \phi \circ T_\mu^{-1}$ is continuous from $\mathbb{I}^-\mathcal{A}'(r)$ to $\mathcal{A}'(r'')$ of norm $(\vartheta/|\mu|)e^{c_0(\vartheta^{-1}-1)r_1}$, where $c_0 = \sigma(\theta+\theta^{-1})/(1+\sigma\theta)$. Furthermore, if $\phi \in \mathbb{I}^-\mathcal{A}'(r)$ is such that $\|\partial_i\phi\|_r < \delta_i$ then $\|\partial_i\phi'\|_{r''} < e^{c_0(\vartheta^{-1}-1)r_1}\delta_i'$, and one has the following identity on $\mathcal{D}(r'' - 3e^{c_0(\vartheta^{-1}-1)r_1}\bar{\delta}')$:*

$$T_\mu \circ U_\phi \circ T_\mu^{-1} = U_{\phi'}. \tag{8.5}$$

Proof.
(a) Continuity follows from the estimate:

$$\|\phi \circ T_\mu^{-1}\|_{r'} = \sum_{(\nu,k,n)} |\phi_{(\nu,k,n)} e^{|\omega\cdot\nu|\vartheta^{-1} r'_1 + |\Omega\cdot\nu|\vartheta r'_2} \left(\frac{\vartheta r'_3}{|\mu|}\right)^k \left(\frac{r'_4}{\vartheta|\mu|}\right)^n \le \|\phi\|_r. \quad (8.6)$$

To prove the second part of the claim, we obtain the following bounds on derivatives:

$$\|\partial_x(\phi \circ T_\mu^{-1})\|_{r'} = \|\vartheta^{-1}\partial_x\phi \circ T_\mu^{-1}\|_{r'} \le \vartheta^{-1}\|\partial_x\phi\|_r, \quad (8.7)$$

$$\|\partial_y(\phi \circ T_\mu^{-1})\|_{r'} = \|\vartheta\partial_y\phi \circ T_\mu^{-1}\|_{r'} \le \vartheta\|\partial_y\phi\|_r, \quad (8.8)$$

$$\|\partial_u(\phi \circ T_\mu^{-1})\|_{r'} = \|\frac{\vartheta}{\mu}\partial_u\phi \circ T_\mu^{-1}\|_{r'} \le \frac{\vartheta}{|\mu|}\|\partial_u\phi\|_r, \quad (8.9)$$

$$\|\partial_z(\phi \circ T_\mu^{-1})\|_{r'} = \|\frac{1}{\vartheta\mu}\partial_z\phi \circ T_\mu^{-1}\|_{r'} \le \frac{1}{\vartheta|\mu|}\|\partial_z\phi\|_r, \quad (8.10)$$

and $\|\partial_i\phi'\|_{r'} \le \delta'_i$ as claimed. By Prop. 8.2 the generating functions ϕ and ϕ' correspond to the canonical transformations $U_\phi : \mathcal{D}(r - 3\bar{\delta}) \mapsto \mathcal{D}(r)$ and $U_{\phi'} : \mathcal{D}(r' - 3\bar{\delta}) \mapsto \mathcal{D}(r')$. T_μ^{-1} maps the domain of U_ϕ into the domain of $U_{\phi'}$.
(b) If $(\nu, k, n) \in I^-$, then $|\omega \cdot \nu| > \sigma|\Omega \cdot \nu|$, $|\omega \cdot \nu| > \kappa_3 k$ and $|\omega \cdot \nu| > \kappa_4 n$. A simple computation using the first of these conditions shows that $|\omega \cdot \nu|$ is bounded away from zero: $|\omega\cdot\nu| > \sigma(\theta + \theta^{-1})/(1 + \sigma\theta)$ for any $(\nu, k, n) \in I^-$. Now, the claim follows from the argument of part (a) and the following estimate for any function $f \in \mathbb{I}^-\mathcal{A}(r)$:

$$\begin{aligned}\|f \circ T_\mu^{-1}\|_{r''} &= \sum_{(\nu,k,n)\in I^-} |f_{(\nu,k,n)}| e^{|\omega\cdot\nu|\vartheta^{-1} r''_1 + |\nu\cdot\Omega|\vartheta r''_2} \left(\frac{\vartheta r''_3}{|\mu|}\right)^k \left(\frac{r''_4}{\vartheta|\mu|}\right)^n \\ &\le \sum_{(\nu,k,n)\in I^-} |f_{(\nu,k,n)}| e^{|\omega\cdot\nu|\vartheta^{-1} r_1 + |\nu\cdot\Omega| r_2} r_3^k r_4^n \\ &\le \sum_{(\nu,k,n)\in I^-} |f_{(\nu,k,n)}| e^{|\omega\cdot\nu|(\vartheta^{-1}-1) r_1} e^{|\omega\cdot\nu| r_1 + |\nu\cdot\Omega| r_2} r_3^k r_4^n \\ &\le e^{c_0(\vartheta^{-1}-1) r_1}\|f\|_r. \end{aligned} \quad (8.11)$$

□

Remark 8.4. The bounds on the generating function ϕ will play an important role in our construction of analytic torus maps for the Hamiltonians on the stable manifolds at the fixed point H^3_+. Although transformation (8.4) is naturally defined on $\mathcal{D}(T_\mu(r))$, in what follows below we shall restrict it to $\mathcal{D}(\hat{T}_\mu(r))$ for the purpose of obtaining estimates on domains which do not expand.

In the next Proposition, and throughout the rest of this Section, the following symbols will be used for the sake of brevity of our exposition:

$$r_{(n)} = (\vartheta^n r_1, \vartheta^{-n} r_2, \vartheta^{-3n} r_3, \vartheta^{-n} r_4), \quad (8.12)$$

$$r_{\{n\}} = (r_1, \vartheta^{-n} r_2, \vartheta^{-3n} r_3, \vartheta^{-n} r_4), \quad (8.13)$$

$$r_{\{n,\tilde{\eta}\}} = (r_1, \vartheta^{-n} r_2, \vartheta^{-3n}\tilde{\eta}^{-1} r_3, \vartheta^{-n}\tilde{\eta}^{-1} r_4), \quad (8.14)$$

and we will also denote $e^{c_0(\vartheta^{-1}-1)\rho''_1}$ by ϖ.

Proposition 8.5. *Let $\tilde{\rho}$, ρ', ρ'' and ρ be as in Theorem 4.3 and Prop. 5.1. We will say that the canonical transformation $U_\psi \circ R_t$ is associated with the pair (ψ, t).*

Then, for every $H \in \mathcal{B}_+$, $T^k_{\vartheta^{-2}} U_{\psi(H)} \circ R_{t(\tilde{H})} \circ T^{-k}_{\vartheta^{-2}}$ *is a canonical transformation from* $\mathcal{D}(\tilde{\rho}_{\{k\}})$ *to* $\mathcal{D}(\varrho_{(k)})$ *associated with a pair* $(\psi_k(H), t'(H))$, *where* $\psi_k(H) = \vartheta^{-2k}\psi(H) \circ T^{-k}_{\vartheta^{-2}}$ *and* $t'(\tilde{H}) = -\vartheta^{-k} t(H)$.

Moreover, if $\psi(H) \in \mathcal{A}'(\rho'')$ *is such that* $\|\psi(H)\|'_{\rho''} < \delta$, *then, if* δ *is sufficiently small,* $\psi_k(H) \in \mathcal{A}'(\rho''_{\{k\}})$ *and is such that* $\|\partial_i \psi_k(H)\|'_{\rho''_{\{k\}}} < \varpi^k (\delta_k)_i$, *where* $\delta_k = (\vartheta^{-3k}\delta, \vartheta^{-k}\delta, \vartheta^k\delta, \vartheta^{-k}\delta)$.

Proof. Observe, that if $t(\tilde{H})$ is a solution of the equation $\mathbb{E}_{(0,t(\tilde{H}))}\partial_z^2 \tilde{H} = 0$, then $t'(\tilde{H}) = -\vartheta^{-k} t(\tilde{H})$ is a solution of $\mathbb{E}_{(0,t'(\tilde{H}))}\partial_z^2 \tilde{H} \circ T^{-k}_{\vartheta^{-2}} = 0$. Since by Prop. 5.1, $U_{\psi(H)} \circ R_{t(\tilde{H})}$ is a canonical transformation from $\mathcal{D}(\tilde{\rho})$ to $\mathcal{D}(\varrho)$, the following two identities hold on $\mathcal{D}(\tilde{\rho}_{(k)})$, and on $\mathcal{D}(\tilde{\rho}_{\{k\}})$ which is a subset of $\mathcal{D}(\tilde{\rho}_{(k)})$:

$$H \circ U_{\psi(H)} \circ R_{t(\tilde{H})} \circ T^{-k}_{\vartheta^{-2}} = H \circ U_{\psi(H)} \circ T^{-k}_{\vartheta^{-2}} \circ R_{-\vartheta^{-k} t(\tilde{H})}, \tag{8.15}$$

$$T^k_{\vartheta^{-2}} U_{\psi(H)} \circ R_{t(\tilde{H})} \circ T^{-k}_{\vartheta^{-2}} = T^k_{\vartheta^{-2}} U_{\psi(H)} \circ T^{-k}_{\vartheta^{-2}} \circ R_{-\vartheta^{-k} t(\tilde{H})}. \tag{8.16}$$

By Lemma 8.3, $T^k_{\vartheta^{-2}} U_{\psi(H)} \circ T^{-k}_{\vartheta^{-2}}$ is a canonical transformation on $\mathcal{D}(\rho''_{\{k\}} - 3\varpi^k \bar{\delta}_k)$ and is generated by $\psi_k(H) = \vartheta^{-2k}\psi(H) \circ T^{-k}_{\vartheta^{-2}}$. If δ is sufficiently small and ρ''_1 is sufficiently large, then $\mathcal{D}(\rho'_{\{k\}}) \subset \mathcal{D}(\rho''_{\{k\}} - 3\varpi^k \bar{\delta}_k)$. Then for every $H \in \mathcal{B}_+$, $T^k_{\vartheta^{-2}} U_{\psi(H)} \circ T^{-k}_{\vartheta^{-2}}$ is a well-defined canonical transformation from $\mathcal{D}(\rho'_{\{k\}})$ to $\mathcal{D}(\varrho_{(k)})$. By Prop. 5.1, for every $H \in \mathcal{B}_+$ the translation $R_{-\vartheta^{-k} t(\tilde{H})}$ is canonical from $\mathcal{D}(\tilde{\rho}_{\{k\}})$ to $\mathcal{D}(\rho'_{\{k\}})$ which implies that $U_{\psi_k(H)} \circ R_{-\vartheta^{-k} t(\tilde{H})}$ is canonical from $\mathcal{D}(\tilde{\rho}_{\{k\}})$ to $\mathcal{D}(\varrho_{(k)})$.

The claim of the Proposition about the bounds on the derivative of the generating function is also a direct consequence of Lemma 8.3. □

The "natural" domain of the transformation $W_k(H)$ is $S^{-1}_{\eta(H_k)}\mathcal{D}(\tilde{\rho}_{\{k\}})$. However, to make these domains H_k-independent we shall reduce them to $S^{-1}_{\tilde{\eta}}\mathcal{D}(\tilde{\rho}_{\{k\}})$, where $\tilde{\eta} = \sup_{H \in \mathcal{B}_+} |\eta(H)|$. By Theorem 5.3 $\tilde{\eta} \le (1 + (b/\tilde{\rho}_3))^{1/2}/(1 - (b/6\gamma(\tilde{\rho}_4)^3))^{1/2}$.

The next two Propositions prepare the proof of the existence of the torus maps for Hamiltonians $F(\alpha_F)$ by providing the bounds on the transformation $W_k(F(\alpha_F))$ for F in the neighborhood $\mathcal{U}(\beta)$ of the fixed family F^*_+.

Proposition 8.6. *Let* $\tilde{\rho}$, ρ' *and* ρ'' *be as in Theorem* 4.3 *and Prop.* 5.1, *the real positive number* $\zeta < 1$ *be as in Lemma* 5.4, *and let an open ball* $\mathcal{U}(\beta)$ *be as in Prop.* 7.2. *Then, for every* $F \in \mathcal{U}(\beta)$ *the maps* $F \mapsto W_k(F(\alpha_F)) - \mathbb{I}$ *and* $F \mapsto \xi(\mathfrak{R}^k(F(\alpha_F)))$ *are analytic from* $\mathcal{U}(\beta)$ *to* $\mathcal{A}^4(\tilde{\rho}_{\{n,\tilde{\eta}\}})$ *and* $\mathbb{C}$ *respectively. Furthermore, there are real positive constants* c_4, c_5 *and* c_6 *such that the following bounds hold:*

$$\| [W_k(F(\alpha_F))]_x - \mathbb{I}\|_{\tilde{\rho}_{\{k,\tilde{\eta}\}}} \le c_4 (\zeta\vartheta\varpi)^k \|F(\alpha_F) - H^3_+\|_\rho, \tag{8.17}$$

$$\| [W_k(F(\alpha_F))]_y - \mathbb{I}\|_{\tilde{\rho}_{\{k,\tilde{\eta}\}}} \le c_4 \left(\frac{\zeta}{\vartheta}\varpi\right)^k \|F(\alpha_F) - H^3_+\|_\rho, \tag{8.18}$$

$$\| [W_k(F(\alpha_F))]_u - \mathbb{I}\|_{\tilde{\rho}_{\{k,\tilde{\eta}\}}} \le \left[c_4\varpi^k + c_5\tilde{\rho}_3\right] \left(\frac{\zeta}{\vartheta^3}\right)^k \|F(\alpha_F) - H^3_+\|_\rho, \tag{8.19}$$

$$\| [W_k(F(\alpha_F))]_z - \mathbb{I}\|_{\tilde{\rho}_{\{k,\tilde{\eta}\}}} \le \left[c_4\varpi^k + c_5\tilde{\rho}_4\right] \left(\frac{\zeta}{\vartheta}\right)^k \|F(\alpha_F) - H^3_+\|_\rho, \tag{8.20}$$

$$|\xi(\mathfrak{R}^k(F(\alpha_F))) - 1| \le c_6 \zeta^k \|F(\alpha_F) - H^3_+\|_\rho. \tag{8.21}$$

Proof. Denote, for brevity, $H_k = \mathfrak{R}^k(F(\alpha_F))$ and $\psi_k(H_k) = \vartheta^{-2k}\psi(H_k) \circ T^{-k}_{\vartheta-2}$. One can use Prop. 8.5 to get

$$\begin{aligned}
\| \left[W_k(H_0)\right]_x - \mathbb{I}\|_{\tilde{\rho}_{\{k,\tilde{\eta}\}}} &\leq \left\| \left[T^k_{\vartheta^{-2}} U_{\psi(H_k)} \circ R_{t(\tilde{H}_k)} \circ T^{-k}_{\vartheta^{-2}}\right]_x - \mathbb{I} \right\|_{\tilde{\rho}_{\{k\}}} \\
&\leq \left\| \left[U_{\psi_k(H_k)} \circ R_{-\vartheta^{-k} t(\tilde{H}_k)}\right]_x - \mathbb{I} \right\|_{\tilde{\rho}_{\{k\}}} \\
&\leq \left\| \left[U_{\psi_k(H_k)}\right]_x - \mathbb{I} \right\|_{\rho'_{\{k\}}} \\
&\leq \|\partial_u \psi_k(H_k)\|_{\rho''_{\{k\}}} \leq \varpi^k \vartheta^k \|\psi(H_k)\|'_{\rho''}. \qquad (8.22)
\end{aligned}$$

Moreover, by Theorem 4.3, there is a positive real constant c', such that

$$\begin{aligned}
\|\psi(H_k)\|'_{\rho''} &\leq c' \|\mathbb{I}^- H_k\|_\rho \\
&= c' \left\| \mathbb{I}^- \left[\mathfrak{R}^k(F(\alpha_F)) - H^3_{(-)^k}\right] \right\|_\rho \\
&\leq c' K \zeta^k \|F(\alpha_F) - H^3_+\|_\rho, \qquad (8.23)
\end{aligned}$$

where we have used Lemma 5.4. Combining estimates (8.22) and (8.23), we obtain the bound (8.17). The bound (8.18) follows by a similar argument (see [30] for details).

Next, we proceed to obtain the bound (8.21). Notice, that by Theorem 5.3

$$|\xi(H_k) - 1| \leq \frac{1}{\tilde{\rho}_3} \|\mathbb{E}(H_k - H^3_{(-)^k})\|_\rho \leq \frac{bK}{\tilde{\rho}_3} \zeta^k \|F(\alpha_F) - H^3_+\|_\rho, \qquad (8.24)$$

which, in particular, implies the bound (8.21). The analyticity of the function ξ on $\mathcal{B}_+$ is a simple consequence of its definition. Combined with the analyticity of the map $F \mapsto \alpha_F$ on $\mathcal{U}$, the analyticity of the renormalization operator $\mathfrak{R}$ on $\mathcal{B}_+$ and the fact that $\mathfrak{R}^k(F(\alpha_F)) \in \mathcal{B}_+$, it gives the analyticity of the map $F \mapsto \xi(\mathfrak{R}^k(F(\alpha_F)))$ on $\mathcal{U}$.

Since $\eta(H_k) = \left(\gamma \xi(H_k)/[\hat{H}_k]_{((0,0),0,3)}\right)^{1/2}$, the estimates (8.24) and (5.6) imply that there exists a real positive constant c_5, such that

$$|1 - |\eta(H_k)|| \leq \frac{\left| |\gamma \xi(H_K)|^{\frac{1}{2}} - \left|[\hat{H}_k]_{((0,0),0,3)}\right|^{\frac{1}{2}} \right|}{\left|[\hat{H}_k]_{((0,0),0,3)}\right|^{\frac{1}{2}}} \leq c_5 \tilde{\eta} \zeta^k \|F(\alpha_F) - H^3_+\|_\rho. \qquad (8.25)$$

This estimate on $\eta(H_k)$ combined with a calculation similar to that leading to (8.17) can be used to obtain the bound (8.19):

$$\begin{aligned}
\| \left[W_k(H_0)\right]_u - \mathbb{I}\|_{\tilde{\rho}_{\{k,\tilde{\eta}\}}} &\leq \| \left[W_k(H_0)\right]_u - (\mathcal{S}_{\eta(H_k)})_u \|_{\tilde{\rho}_{\{k,\tilde{\eta}\}}} + \frac{\tilde{\rho}_3}{\vartheta^{3k}} c_5 \zeta^k \|F(\alpha_F) - H^3_+\|_\rho \\
&\leq \frac{\varpi^k}{\vartheta^{3k}} c' \|\mathbb{I}^- H_k\|_\rho + c_5 \tilde{\rho}_3 \left(\frac{\zeta}{\vartheta^3}\right)^k \|F(\alpha_F) - H^3_+\|_\rho \\
&\leq \frac{\varpi^k}{\vartheta^{3k}} c' \left\| \mathbb{I}^- \left[H_k - H^3_{(-)^k}\right] \right\|_\rho + c_5 \tilde{\rho}_3 \left(\frac{\zeta}{\vartheta^3}\right)^k \|F(\alpha_F) - H^3_+\|_\rho \\
&\leq \left[c_4 \varpi^k + c_5 \tilde{\rho}_3\right] \left(\frac{\zeta}{\vartheta^3}\right)^k \|F(\alpha_F) - H^3_+\|_\rho. \qquad (8.26)
\end{aligned}$$

The bound (8.20) follows by a similar computation (see [30] for details). □

As we already pointed out, the size of the domains $S_{\tilde{\eta}}^{-1}\mathcal{D}(\tilde{\rho}_{\{k\}})$, which we used above, is constant in the variable x. The fact that these domains were not expanding in the first variable was essential in our argument: it allowed us to introduce an additional factor, of an absolute value less than one, into the bounds, namely, the factor $\varpi = e^{c_0(\vartheta^{-1}-1)\rho_1''}$. Although not important for the convergence of the estimates (8.18)–(8.20) as k increases, where this convergence is guaranteed because the factors ζ/ϑ^3 and ζ/ϑ are less than one, the factor ϖ is necessary to overcome the exponential growth of $(\zeta\vartheta)^k$ in estimate (8.17) . In the next result we will consider domains $\mathcal{D}(r_{\{n,\tilde{\eta},\chi\}})$, where

$$r_{\{n,\tilde{\eta},\chi\}} = \left(\frac{r_1}{\sum_{i=0}^{n-1}\chi^{-i}}, \vartheta^{-n}r_2, \vartheta^{-3n}\tilde{\eta}^{-1}r_3, \vartheta^{-n}\tilde{\eta}^{-1}r_4\right). \tag{8.27}$$

These domains shrink in the variable x to a strip $|\Im(x)| < (1-\chi^{-1})\tilde{\rho}_1$ of some non-zero width.

Proposition 8.7. *Let $\tilde{\rho}$, ρ' and ρ'' be as in Theorem* 4.3 *and Prop.* 5.1*, the real positive number $\zeta < 1$ be as in Lemma* 5.4*, and let the open ball $\mathcal{U}(\beta)$ be as in Prop.* 7.2*. If β is sufficiently small and ρ_1'' is sufficiently large, then for every $F \in \mathcal{U}(\beta)$ and an arbitrary fixed $\chi > 1$ the maps $F \mapsto W_n'(F(\alpha_F)) - \mathbb{I}$ and $F \mapsto \xi_n'(F(\alpha_F)) = \prod_{i=0}^{n-1}\xi\left(\mathfrak{R}^i(F(\alpha_F))\right)$ are analytic from $\mathcal{U}(\beta)$ to $\mathcal{A}^4(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$ and $\mathbb{C}$ respectively, while the transformation $W_n'(F(\alpha_F))$ is analytic from $\mathcal{D}(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$ to $\mathcal{D}(\varrho)$. Furthermore, there are real positive constants c_7, c_8, c_9 and c_{10} such that the following bounds hold for all $F \in \mathcal{U}(\beta)$:*

$$\|W_{n+1}'(F(\alpha_F)) - W_n'(F(\alpha_F))\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} \le \left[c_7 + \frac{c_8}{\chi^n}\right](\zeta\vartheta\chi\varpi)^n\|F(\alpha_F) - H_+^3\|_\rho, \tag{8.28}$$

$$\|W_{n+1}'(F(\alpha_F)) - \mathbb{I}\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} \le c_9\|F(\alpha_F) - H_+^3\|_\rho, \tag{8.29}$$

$$|\xi_n'(F(\alpha_F)) - 1| \le c_{10}\|F(\alpha_F) - H_+^3\|_\rho. \tag{8.30}$$

Proof. Define

$$G_n(z,F) = z\left[W_n(F(\alpha_F)) - \mathbb{I}\right], \tag{8.31}$$

$$G_k(z,F) = W_k(F(\alpha_F))\circ\ldots\circ W_{n-1}(F(\alpha_F))\circ(\mathbb{I} + z\left[W_n(F(\alpha_F)) - \mathbb{I}\right]) - \mathbb{I}. \tag{8.32}$$

As it was shown in the previous Proposition, the map $F \mapsto W_n(F(\alpha_F)) - \mathbb{I}$ is analytic from $\mathcal{U}(\beta)$ to $\mathcal{A}^4(\tilde{\rho}_{\{n,\tilde{\eta}\}})$. Notice, that $\mathcal{D}(\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}) \subset \mathcal{D}(\tilde{\rho}_{\{n,\tilde{\eta}\}})$. Then, the map G_k is analytic from $\mathcal{D}_{\epsilon_n} \times \mathcal{U}(\beta)$ to $\mathcal{A}^4(\tilde{\rho}_{\{n,\tilde{\eta},\chi\}})$ for some real positive ϵ_n, if the range of $\mathbb{I} + z[W_n(F(\alpha_F)) - \mathbb{I}]$ is contained in the domain of $W_{n-1}(F(\alpha_F))$ for all $z \in \mathcal{D}_{\epsilon_n}$, in particular, if the following conditions hold:

$$\frac{\tilde{\rho}_1}{\sum_{i=0}^{n-2}\chi^{-i}} \ge \frac{\tilde{\rho}_1}{\sum_{i=0}^{n-1}\chi^{-i}} + |z|\,\|(W_n(F(\alpha_F)))_x - \mathbb{I}\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} \tag{8.33}$$

$$\frac{\tilde{\rho}_2}{\vartheta^{n-1}} \ge \frac{\tilde{\rho}_2}{\vartheta^n} + |z|\,\|(W_n(F(\alpha_F)))_y - \mathbb{I}\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}}, \tag{8.34}$$

$$\frac{\tilde{\rho}_3}{\tilde{\eta}\vartheta^{3(n-1)}} \ge \frac{\tilde{\rho}_3}{\tilde{\eta}\vartheta^{3n}} + |z|\,\|(W_n(F(\alpha_F)))_u - \mathbb{I}\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}}, \tag{8.35}$$

$$\frac{\tilde{\rho}_4}{\tilde{\eta}\vartheta^{n-1}} \ge \frac{\tilde{\rho}_4}{\tilde{\eta}\vartheta^n} + |z|\,\|(W_n(F(\alpha_F)))_z - \mathbb{I}\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}}. \tag{8.36}$$

Since $\mathcal{D}(\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}) \subset \mathcal{D}(\tilde{\rho}_{\{n,\tilde{\eta}\}})$, by Prop. 8.6, it is enough to ensure that

$$\frac{\tilde{\rho}_1}{\sum_{i=0}^{n-2}\chi^{-i}} \ge \frac{\tilde{\rho}_1}{\sum_{i=0}^{n-1}\chi^{-i}} + |z|c_4\beta(\zeta\vartheta\varpi)^n, \tag{8.37}$$

$$\frac{\tilde{\rho}_2}{\vartheta^{n-1}} \ge \frac{\tilde{\rho}_2}{\vartheta^n} + |z| c_4 \beta \left(\varpi \frac{\zeta}{\vartheta}\right)^n, \tag{8.38}$$

$$\frac{\tilde{\rho}_3}{\tilde{\eta}\vartheta^{3(n-1)}} \ge \frac{\tilde{\rho}_3}{\tilde{\eta}\vartheta^{3n}} + |z|\beta \left[c_4 \varpi^n + c_5 \tilde{\rho}_3\right] \left(\frac{\zeta}{\vartheta^3}\right)^n, \tag{8.39}$$

$$\frac{\tilde{\rho}_4}{\tilde{\eta}\vartheta^{n-1}} \ge \frac{\tilde{\rho}_4}{\tilde{\eta}\vartheta^n} + |z|\beta \left[c_4 \varpi^n + c_5 \tilde{\rho}_4\right] \left(\frac{\zeta}{\vartheta}\right)^n, \tag{8.40}$$

or require an even stronger set of conditions on $|z|$:

$$|z| \le \frac{\chi(1-\chi^{-1})^2 \tilde{\rho}_1}{c_4 \beta} \left(\zeta \vartheta \chi \varpi\right)^{-n}, \quad |z| \le \frac{(\vartheta - 1)\tilde{\rho}_2}{c_4 \beta} \left(\zeta \varpi\right)^{-n}, \tag{8.41}$$

$$|z| \le \frac{(\vartheta^3 - 1)\tilde{\rho}_3}{\beta \tilde{\eta}(c_4 \varpi + c_5 \tilde{\rho}_3)} \zeta^{-n}, \quad |z| \le \frac{(\vartheta - 1)\tilde{\rho}_4}{\beta \tilde{\eta}(c_4 \varpi + c_5 \tilde{\rho}_4)} \zeta^{-n}. \tag{8.42}$$

Let $c' = \beta^{-1} \min\{\chi(1-\chi^{-1})^2 \tilde{\rho}_1 c_4^{-1}, (\vartheta-1)\tilde{\rho}_2 c_4^{-1}, (\vartheta^3-1)\tilde{\rho}_3 \tilde{\eta}^{-1} \left[c_4 \varpi + c_5 \tilde{\rho}_3\right]^{-1}, (\vartheta-1)\tilde{\rho}_4 \tilde{\eta}^{-1} [c_4 \varpi + c_5 \tilde{\rho}_4]^{-1}\}$, and let $\epsilon_n = c' \left(\zeta \vartheta \chi \varpi\right)^{-n}$. Then, the map G_k is analytic from $\mathcal{D}_{\epsilon_n} \times \mathcal{U}(\beta)$ to $\mathcal{A}^4(\tilde{\rho}_{\{n,\tilde{\eta},\chi\}})$ for all $k \ge 0$. If β is sufficiently small, and ρ_1'' is sufficiently large, so that $\{z = 1\} \in \mathcal{D}_{\epsilon_n}$ for all $n \ge 1$, then the analyticity of the map $F \mapsto W_n'(F(\alpha_F)) - \mathbb{I}$ from $\mathcal{U}(\beta)$ to $\mathcal{A}^4(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$ is an immediate consequence of the identity $W_{n+1}'(F(\alpha_F)) - \mathbb{I} = G_0(1, F)$.

To prove the bounds (8.28)–(8.29) we use the identity

$$G_k(z, F) = \left[W_k(F(\alpha_F)) - \mathbb{I}\right] \circ \left[\mathbb{I} + G_{k+1}(z, F)\right] + G_{k+1}(z, F), \tag{8.43}$$

and the following inductive estimate on $G_k(z, F)$ for all $z \in \mathcal{D}_{\epsilon_n}$:

$$\begin{aligned}
\|G_k(z,F)\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} &\le \|W_k(F(\alpha_F)) - \mathbb{I}\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} + \|G_{k+1}(z,F)\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} \\
&\le \sum_{i=k}^{n-1} \|W_i(F(\alpha_F)) - \mathbb{I}\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} + |z| \|W_n(F(\alpha_F)) - \mathbb{I}\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} \\
&\le c_4 \left[\sum_{i=k}^{n-1} (\zeta \vartheta \varpi)^i + c' \chi^{-n}\right] \|F(\alpha_F) - H_+^3\|_\rho \\
&< c_4 \left[\frac{(\zeta\vartheta\varpi)^k - (\zeta\vartheta\varpi)^n}{1 - \zeta\vartheta\varpi} + \frac{c'}{\chi^n}\right] \|F(\alpha_F) - H_+^3\|_\rho, \qquad (8.44)
\end{aligned}$$

which implies (8.29). The bound (8.28) can be obtained from the following Cauchy estimate:

$$\begin{aligned}
\|W_{n+1}'(F(\alpha_F)) - W_n'(H)\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} &= \|G_0(1,F) - G_0(0,F)\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} \\
&\le \left\| \frac{1}{2\pi i} \oint_{w=\epsilon_n} \frac{G_0(w,F)dw}{(w-1)\zeta} \right\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} \\
&\le c_4 \left(\frac{1}{1 - \zeta\vartheta\varpi} + \frac{c'}{\chi^n}\right) \frac{\|F(\alpha_F) - H_+^3\|_\rho}{\epsilon_n - 1} \\
&\le c_7 \left(1 + \frac{c_8}{\chi^n}\right) (\zeta \vartheta \chi \varpi)^n \|F(\alpha_F) - H_+^3\|_\rho. \qquad (8.45)
\end{aligned}$$

Finally, the analyticity of the map $F \mapsto \xi_n'(F(\alpha_F))$ on $\mathcal{U}(\beta)$ follows from the analyticity of the maps $F \mapsto \xi(\mathfrak{R}^k(F(\alpha_F)))$ on $\mathcal{U}(\beta)$ (by Prop. 8.6) and the definition of $\xi_n'(F(\alpha_F))$. We notice that $\prod_{k=0}^{n-1} \left[1 - c_6 \zeta^k \|F(\alpha_F) - H_+^3\|_\rho\right] \le |\xi_n'(H_0)| \le \prod_{k=0}^{n-1} \left[1 + c_6 \zeta^k \|f(\alpha_F) - H_+^3\|_\rho\right]$ by Prop. 8.6, which immediately gives the bound (8.30). ☐

Our next goal is to show that the transformation $W'_n(F(\alpha_F))$ is, in a sense, an approximation of the desired torus map, specifically, we want to show that it satisfies the equation (4.10) "approximately". Observe, that from the definition of the operator $\mathfrak{R}$

$$H_0 \circ W'_n(H) = \frac{\prod_{i=0}^{n-1} \xi(H_i)\eta(H_i)}{\vartheta^{3n}} H_n \circ T^{-n}_{\vartheta^{-2}} + \mathbb{E}_0 H_0 \circ W'_n(H). \tag{8.46}$$

This representation will be used in the next

Proposition 8.8. *Let $\tilde{\rho}$ be as in Theorem 4.6, and let a real $\chi > 1$. Define $\varepsilon_n = (\chi^{-n}\epsilon_1, \vartheta^{-n}\epsilon_2, \tilde{\eta}^{-1}\vartheta^{-3n}\epsilon_3, \tilde{\eta}^{-1}\vartheta^{-n}\epsilon_4)$, where the real positive ϵ_i are such that $(1-\chi^{-1})\tilde{\rho}_1 \ge \epsilon_1$ and $\rho_i > \epsilon_i$, $i = 2..4$. Then, given a real $\zeta > 0$ as in Lemma 5.4, there exist real positive numbers c_{11} and c_{12}, such that the following holds for all $F \in \mathcal{U}(\beta)$ and $n \ge 1$:*

$$\begin{aligned} &\|\{\mathbb{I}, F(\alpha_F)\} \circ W'_n(F(\alpha_F)) - \xi'_n(F(\alpha_F))\{W'_n(F(\alpha_F)), H^3_+\}\|_{\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}}-\varepsilon_{n-1}} \\ &\qquad\qquad \le c_{11}\left(\chi^n + c_{12}\right)\zeta^n \|F(\alpha_F) - H^3_+\|_\rho. \end{aligned} \tag{8.47}$$

Proof. Let $H_n = \mathfrak{R}^n_\eta(F(\alpha_F))$. Notice, that the number ϵ_1 is chosen so that $\tilde{\rho}_1 / \sum_{i=0}^{n-1} \chi^{-i} - \epsilon_1/\chi^n > (1-\chi^{-1})\tilde{\rho}_1$ for any $\epsilon_1 \le (1-\chi^{-1})\tilde{\rho}_1$. Also, notice, that by Prop. 8.7, $|\xi'_n(H_0)| \le 1 + c_{10}\beta$. Using this together with the identity $(1/\vartheta^3)H^3_+ \circ T^{-1}_{\vartheta^{-2}} = H^3_-$, the bound (8.29) and Lemma 5.4, one gets the following estimate:

$$\begin{aligned} &\|\{\mathbb{I}, H_0\} \circ W'_n(H_0) - \xi'_n(H_0)\{W'_n(H_0), H^3_+\}\|_{\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}}-\varepsilon_{n-1}} \\ &\quad \le \frac{\xi'_n(H_0)}{\vartheta^{3n}} \left\| \{W'_n(H_0), \left[H_n - H^3_{(-)^n}\right] \circ T^{-n}_{\vartheta^{-2}}\} \right\|_{\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}}-\varepsilon_{n-1}} \\ &\quad \le \frac{\xi'_n(H_0)}{\vartheta^{3n}} \left[\left\| \{W'_n(H_0) - \mathbb{I}, \left[H_n - H^3_{(-)^n}\right] \circ T^{-n}_{\vartheta^{-2}}\} \right\|_{\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}}-\varepsilon_{n-1}} \right. \\ &\quad \left. + \left\| \nabla \left[H_n - H^3_{(-)^n}\right] \circ T^{-n}_{\vartheta^{-2}} \right\|_{\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}}-\varepsilon_{n-1}} \right] \\ &\quad \le \frac{2(1+c_{10}\beta)\tilde{\eta}}{\min\{\epsilon_1\epsilon_3, \epsilon_2\epsilon_4\}} \left(\chi^n + \frac{1}{\vartheta^n}\right) \|W'_n(H_0) - \mathbb{I}\|_{\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}}} \|H_n - H^3_{(-)^n}\|_{\tilde{\rho}} \\ &\quad + \frac{1+c_{10}\beta}{\vartheta^{3n}\min_{i=1..4}\epsilon_i} \left(\chi^n + \vartheta^n + \vartheta^{3n}\tilde{\eta} + \vartheta^n\tilde{\eta}\right) \zeta^n \|H_0 - H^3_+\|_\rho \\ &\quad \le \frac{2(1+c_{10}\beta)\tilde{\eta}}{\min\{\epsilon_1\epsilon_3, \epsilon_2\epsilon_4\}} \left(\chi^n + \frac{1}{\vartheta^n}\right) c_9 \zeta^n \|H_0 - H^3_+\|^2_\rho \\ &\quad + \frac{1+c_{10}\beta}{\vartheta^{3n}\min_{i=1..4}\epsilon_i} \left(\chi^n + \vartheta^n + \vartheta^{3n}\tilde{\eta} + \vartheta^n\tilde{\eta}\right) \zeta^n \|H_0 - H^3_+\|_\rho \\ &\quad \le c_{11}(\chi^n + c_{12})\zeta^n \|H_0 - H^3_+\|_\rho, \end{aligned} \tag{8.48}$$

where $1/\vartheta < \zeta < 1$ is as in Lemma 5.4. ☐

Now we are ready to give an explicit construction of invariant ω-tori for members of the Hamiltonian families in $\mathcal{U}(\beta)$. First, we stop to define the functional space in which the expression $\{\mathbb{I}, F(\alpha_F)\} \circ W'_n(F(\alpha_F)) - \xi'_n(F(\alpha_F))\left\{W'_n(F(\alpha_F)), H^3_+\right\}$ converges.

Definition 8.9. *Denote $\mathcal{D}(r) = \{z \in \mathbb{C} : |\Im(z)| < r\}$. For any triple $l = (l_2, l_3, l_4)$ of non-negative integers denote $\mathcal{C}^l_r(T^2)$ the vector space of all functions from $\mathcal{D}(r) \times \mathbb{R}^3$ to $\mathbb{C}^4$ that are a) analytic on $\mathcal{D}(r)$ in the first variable; b) l_2, l_3 and l_4-times continuously differentiable in variables y, u and z respectively; c) 2π-periodic in* *variables* $\omega \cdot (x,y)$ *and* $\Omega \cdot (x,y)$ *and constant in variables* u *and* z. *If* $g \in \mathcal{C}^l_r(T^2)$, *define its norm by*

$$\|g\|_{(l,r)} = \max_{k_i \le l_i} \left\{ \sup_{x \in \mathcal{D}(r), y \in \mathbb{R}} \left| (\partial_y^{k_2} \partial_u^{k_3} \partial_z^{k_4} g)(x,y,0,0,) \right| \right\}. \tag{8.49}$$

If G is some function in $\mathcal{A}(r)$, then the restriction of $G \circ S_0$ to $\mathbb{C} \times \mathbb{R}^3$, where S_0 is defined by $S_0(x,y,u,z) = (x,y,0,0)$, is in $\mathcal{C}^l_{r_1}(T^2)$ for any l. Moreover,

$$\partial_y^{k_2} \partial_u^{k_3} \partial_z^{k_4} G \circ S_0 = \sum_{\nu \in \mathbb{Z}^2} G_{(\nu, k_3, n_4)} \partial_y^{k_2} e^{i(\nu \cdot \omega x + \nu \cdot \Omega y)} k_3! k_4!, \tag{8.50}$$

and for any $0 < \epsilon < r_2$ the restriction of $G \circ S_0$ to $\mathbb{C} \times \mathbb{R}^3$ satisfies the following bound:

$$\|G \circ S_0\|_{(l,r_1)} \le \frac{l_3!\ l_4!}{(r_3)^{l_3} (r_4)^{l_4}} \|\partial^{\ l_2}{}_y G\|_{(r_1,\epsilon,r_3,r_4)} \le \frac{(l_2)^{l_2}\ l_3!\ l_4!}{(r_2-\epsilon)^{l_2} (r_3)^{l_3} (r_4)^{l_4}} \|G\|_r. \tag{8.51}$$

Since this inequality holds for any real positive ϵ, we must have

$$\|G \circ S_0\|_{(l,r_1)} \le \frac{(l_2)^{l_2}\ l_3!\ l_4!}{(r_2)^{l_2} (r_3)^{l_3} (r_4)^{l_4}} \|G\|_r. \tag{8.52}$$

In particular, since the transformation $W'_n(H) \in \mathcal{A}^4(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$, the restriction of $W'_n(H) \circ S_0$ to $\mathbb{C} \times \mathbb{R}^3$ is in $\mathcal{C}^0_{(1-\chi^{-1})\rho'_1} \times \mathcal{C}^0_{(1-\chi^{-1})\rho'_1} \times \mathcal{C}^0_{(1-\chi^{-1})\rho'_1} \times \mathcal{C}^0_{(1-\chi^{-1})\rho'_1}$ for all $n \ge 0$, and $\|W'_n(H) \circ S_0\|_{(0,(1-\chi^{-1})\rho'_1)} \le \|W'_n(H)\|_{\rho'_{\{n-1,\tilde{\eta},\chi\}}}$.

In most of the following results we will not need to consider norms with l different from zero, however some of the results can and will be proved for spaces $\mathcal{C}^l_r(T^2)$ with a non-zero triple l.

Proposition 8.10. *Let a real positive* ρ''_1, *sufficiently large, and a real* $\chi > 1$, *sufficiently close to one, be given. Then for every* $F \in \mathcal{U}(\beta)$ *the limits*

$$W'(F(\alpha_F)) = S_0 + \lim_{n\to\infty} [W'_n(F(\alpha_F)) - \mathbb{I}] \circ S_0 \tag{8.53}$$

$$\xi'(F(\alpha_F)) = \lim_{n\to\infty} \xi'_n(F(\alpha_F)) \tag{8.54}$$

converge uniformly in $\mathcal{C}^0_{(1-\chi^{-1})\tilde{\rho}_1}(T^2)$ *and* $\mathbb{C}$, *respectively. The maps* $F \mapsto \xi'(F(\alpha_F))$ *and* $F \mapsto W'(F(\alpha_F)) - S_0$ *are analytic on* $\mathcal{U}(\beta)$. *Furthermore, there exist real positive numbers* c_{13} *and* c_{14}, *such that*

$$\|W'(F(\alpha_F)) - S_0\|_{(0,(1-\chi^{-1})\tilde{\rho}_1)} \le c_{13} \|F(\alpha_F) - H^3_+\|_\rho, \tag{8.55}$$

$$|\xi'(F(\alpha_F)) - 1| \le c_{14} \|F(\alpha_F) - H^3_+\|_\rho, \tag{8.56}$$

and, moreover,

$$\mathbb{J} \nabla H \circ W'(F(\alpha_F)) = \xi'(F(\alpha_F)) \partial_x W'(F(\alpha_F)). \tag{8.57}$$

Proof. The uniform convergence and the analyticity follow from Prop. 8.8 and inequality (8.52), and so do the bounds (8.55) and (8.56).

Define, $r_n = ((1-\chi^{-1})\tilde{\rho}_1, \vartheta^{-n}\tilde{\rho}_2, \tilde{\eta}^{-1}\vartheta^{-3n}\tilde{\rho}_3, \tilde{\eta}^{-1}\vartheta^{-n}\tilde{\rho}_4)$. Then, one can use the bound (8.28) and Prop. 3.4 to obtain the following estimate:

$$\begin{aligned} \|\partial_x [W'_{n+1}(F(\alpha_F)) - W'_n(F(\alpha_F))]\|_{r_n} &\le \frac{\chi^{n+1}}{(\chi-1)\tilde{\rho}_1} \|W'_{n+1}(F(\alpha_F)) - W'_n(F(\alpha_F))\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} \\ &\le c_7 \frac{\chi + \frac{c_8}{\chi^{n-1}}}{(\chi-1)\rho'_1} \left(\chi^2 \zeta \vartheta \varpi\right)^n \|F(\alpha_F) - H^3_+\|_\rho, \end{aligned} \tag{8.58}$$

and, therefore, if $\chi > 1$ is chosen sufficiently close to one and ρ_1'' is sufficiently large, the sequence $\{L_n\} = \left\{\xi_n'(F(\alpha_F))\ \left(\partial_x W_n'(F(\alpha_F)) + 3\gamma z^2 \partial_y W_n'(F(\alpha_F))\right)\right\}$, converges in $\mathcal{C}^0_{(1-\chi^{-1})\tilde{\rho}_1}(T^2)$ to $\xi'(F(\alpha_F))\ \partial_x W'(F(\alpha_F))$, while, by Prop. 8.7, $\{R_n\} = \{\mathbb{J}\nabla F(\alpha_F) \circ W_n'(F(\alpha_F))\}$ converges in $\mathcal{C}^0_{(1-\chi^{-1})\tilde{\rho}_1}(T^2)$ to $\mathbb{J}\nabla F(\alpha_F) \circ W'(F(\alpha_F))$. The bounds (8.47) and (8.51) show that $L_n - R_n \to 0$ in $\mathcal{C}^0_{(1-\chi^{-1})\tilde{\rho}_1}(T^2)$ and, hence, (8.57) holds. □

For any quadruple r with positive components define the evaluation functional $\mathcal{E}_{(x,y,u,z)}$ on $\mathcal{A}^4(r)$ by setting $\mathcal{E}_{(x,y,u,z)} G = G(x,y,u,z)$ for any $G \in \mathcal{A}^4(r)$.

Lemma 8.11. *Let $\tilde{\rho}$, ϱ and ρ be as in Theorem 4.2 and Prop. 5.1, and let r be a pair of real positive numbers, such that $(1-\chi^{-1})\tilde{\rho}_1 > r_1$ and $\rho_2 - \varrho_2 > r_2$. Then, there exists an open neighborhood $\mathcal{U}'$ of F_+^* in $\mathcal{F}_\delta^\rho$, such that for every $F \in \mathcal{U}'$ and any $(x,y) \in \mathcal{D}_1(r)$, and $(u,z) \in \mathbb{C}^2$ the equation*

$$\mathcal{W}_F(x,y,u,z) = \mathcal{E}_{(x,0,0,0)} J_{(0,y)} \circ W'(\mathcal{J}_{(0,y)} F(\alpha_F)) \tag{8.59}$$

defines an analytic map $\mathcal{W}_F$ on $\mathcal{D}_1(r) \times \mathbb{C}^2$ whose restriction to $\mathbb{R}^4$ coincides with $W'(F(\alpha_F))$. Furthermore, the map $F \mapsto \mathcal{W}_F - S_0$ is analytic from $\mathcal{U}'$ to a space of analytic functions from $\mathcal{D}_1(r) \times \mathbb{C}^2$ to $\mathcal{D}(\varrho_1, \varrho_2 + r_2, \varrho_3, \varrho_4) \subset \mathcal{D}(\rho)$.

Proof. Choose a number r_2', so that $r_2 < r_2' < \rho_2 - \varrho_2$. The map $\mathcal{J}_{(0,y)} : \mathcal{A}(\varrho_1, \varrho_2 + r_2', \varrho_3, \varrho_4) \mapsto \mathcal{A}(\varrho)$ is bounded, of norm less or equal to one, for all $y \in \mathcal{D}(r_2') = \{y \in \mathbb{C} : |\Im y| < r_2'\}$. Furthermore, for any function $G \in \mathcal{A}(\varrho_1, \varrho_2 + r_2', \varrho_3, \varrho_4)$ we have $\|\partial_i G\|_{\varrho_1, \varrho_2 + r_2, \varrho_3, \varrho_4} \le \|G\|_{\varrho_1, \varrho_2 + r_2', \varrho_3, \varrho_4} / (r_2' - r_2)$, and differentiation is bounded from $\mathcal{A}(\varrho_1, \varrho_2 + r_2', \varrho_3, \varrho_4)$ to $\mathcal{A}(\varrho_1, \varrho_2 + r_2, \varrho_3, \varrho_4)$. Then, for any $H \in \mathcal{A}(\rho)$, in particular for $F(\alpha_F)$ with $F \in \mathcal{F}_\delta^\rho$, the map $(y, H) \mapsto \mathcal{J}_{(0,y)} H$ is differentiable from $\mathcal{D}(r_2') \times \mathcal{A}(\rho)$ to $\mathcal{A}(\varrho)$.

Define $\mathcal{W} : \mathcal{F}_\delta^\rho \mapsto \mathcal{C}^0_{(1-\chi^{-1})\tilde{\rho}_1}(T^2)$ by setting $\mathcal{W}(F) = W'(F(\alpha_F)) - S_0$ for all $F \in \mathcal{U}(\beta) \subset \mathcal{F}_\delta^\rho$. By Prop. 8.10 $\mathcal{W}$ is analytic from $\mathcal{U}(\beta) \subset \mathcal{F}_\delta^\rho$ to $\mathcal{C}^0_{(1-\chi^{-1})\tilde{\rho}_1}(T^2)$. Since $\mathcal{J}_{(0,y)}$ is bounded from $\mathcal{A}(\varrho_1, \varrho_2 + r_2', \varrho_3, \varrho_4)$ to $\mathcal{A}(\varrho)$ for all $y \in \mathcal{D}(r_2')$, there is an open neighborhood $\mathcal{U}' \subset \mathcal{F}_\delta^\rho$ of F_+^* such that for all $F \in \mathcal{U}'$, both F and $\mathcal{J}_{(0,y)} F$ belong to $\mathcal{U}(\beta) \subset \mathcal{F}_\delta^\rho$. Here, $\mathcal{J}_v F$ is defined by $(\mathcal{J}_v F)(\alpha) = \mathcal{J}_v F(\alpha)$ for any $v \in \mathbb{C}^2$. Now, since the map $\mathcal{W}$ is analytic on $\mathcal{U}(\beta) \subset \mathcal{F}_\delta^\rho$, and since the evaluation functional $\mathcal{E}_{(x,0,0,0)}$ is bounded on $\mathcal{C}^0_{(1-\chi^{-1})\tilde{\rho}_1}(T^2)$, the map $((x,y), F) \mapsto \mathcal{E}_{(x,0,0,0)} J_{(0,y)} \circ \mathcal{W}(\mathcal{J}_{(0,y)} F)$ is analytic from $\mathcal{D}_1(r_1, r_2') \times \mathcal{U}'$, to $\mathcal{C}^0_{(1-\chi^{-1})\tilde{\rho}_1}(T^2)$. The analyticity of the transformation $\mathcal{W}_F$ on $\mathcal{D}_1(r) \times \mathbb{C}^2$ follows from the fact that the identity $[\mathcal{W}_F - S_0](x,y,u,z) = \mathcal{E}_{(x,0,0,0)} J_{(0,y)} \circ \mathcal{W}(\mathcal{J}_{(0,y)} F)$ holds for every $(x,y,u,z) \in \mathcal{D}_1(r) \times \mathbb{C}^2$.

Since $\mathcal{J}_{(0,y)} F(\alpha_F)$ is periodic in the variables $\omega \cdot (0,y)$ and $\Omega \cdot (0,y)$, $\mathcal{E}_{(x,0,0,0)} J_{(0,y)} \circ \mathcal{W}(\mathcal{J}_{(0,y)}(F(\alpha_F))$ is periodic in $q_1 = \omega \cdot (x,y)$ and $q_2 = \Omega \cdot (x,y)$, and for all $F \in \mathcal{U}'$ the transformation $\mathcal{W}_F - S_0$ belongs to a space of analytic functions from $\mathcal{D}_1(r) \times \mathbb{C}^2$ to $\mathcal{D}(\varrho_1, \varrho_2 + r_2, \varrho_3, \varrho_4)$ with the mentioned periodicity. All derivatives with respect to the second argument of the map $((x,y), F) \mapsto \mathcal{E}_{(x,0,0,0)} J_{(0,y)} \circ \mathcal{W}(\mathcal{J}_{(0,y)} F(\alpha_F))$, at any $F \in \mathcal{U}'$, are bounded uniformly in (x,y) in the sup-norm on the space of analytic functions from $\mathcal{D}_1(r) \times \mathbb{C}^2$ to $\mathcal{D}(\varrho_1, \varrho_2 + r_2, \varrho_3, \varrho_4)$, hence the map $F \mapsto \mathcal{W}_F - S_0$ is analytic on $\mathcal{U}'$.

Lastly, we prove the claim about the restriction of the map $\mathcal{W}_F$. Consider an arbitrary but fixed real x and y. Since $\mathcal{J}_{(0,y)}$ is an isometry on $\mathcal{A}(\rho)$, given any open neighborhood $\mathcal{U}'' \in \mathcal{F}_\delta^\rho$ around F_+^*, the Hamiltonians $F'(\alpha_F) = \mathfrak{R}^k(\mathcal{J}_{(0,y)} F(\alpha_F))$ and $F_k(\alpha_F) = \mathfrak{R}^k(F(\alpha_F))$ belong to $\mathcal{U}''$ for all $k \ge 0$, if $\|F - F_+^*\|_\delta$ is sufficiently small. By Theorem 5.3, $F'_k(\alpha_F) = \mathcal{J}_{T^{-k}(0,y)} F_k(\alpha_F)$. By the definition of the space $\mathcal{F}^{\rho}_{\delta}$, the Hamiltonians $F(\alpha_F)$, for all $F \in \mathcal{U}$, are contained in the neighborhood of Lemma 4.5, where $J_{(0,y)} \circ \mathcal{U}_{\mathcal{J}_{(0,y)} F(\alpha_F)} = \mathcal{U}_{F(\alpha_F)} \circ J_{(0,y)}$. Combining this with the identity $T^k_{\vartheta^{-2}} \circ J_{T^{-k}(0,y)} = J_{(0,y)} \circ T^k_{\vartheta^{-2}}$ we obtain $W_k(F'(\alpha_F)) = J^{-1}_{(0,y)} \circ W_k(F(\alpha_F)) \circ J_{(0,y)}$ for all $k \geq 0$. Repeating the argument of Prop. 8.10 we see that the limit

$$\begin{aligned} W'(F'(\alpha_F)) &= S_0 + \lim_{n \to \infty} [W'_n(F'(\alpha_F)) - \mathbb{I}] \circ S_0 \\ &= S_0 + \lim_{n \to \infty} [W_0(F'(\alpha_F)) \circ W_1(F'(\alpha_F)) \circ \ldots \circ W_{n-1}(F'(\alpha_F)) - \mathbb{I}] \circ S_0 \\ &= S_0 + \lim_{n \to \infty} \left[J^{-1}_{(0,y)} \circ W_0(F(\alpha_F)) \circ \ldots \circ W_{n-1}(F(\alpha_F)) \circ J_{(0,y)} - \mathbb{I} \right] \circ S_0 \\ &= S_0 + J^{-1}_{(0,y)} \circ \lim_{n \to \infty} [W_n(F(\alpha_F)) - \mathbb{I}] \circ S_0 \circ J_{(0,y)} \\ &= J^{-1}_{(0,y)} \circ W'(F(\alpha_F)) \circ J_{(0,y)} \end{aligned} \tag{8.60}$$

converges in $\mathcal{C}^0_{(1-\chi^{-1})\tilde{\rho}_1}(T^2)$, which means that for all real x,y,u and z

$$\begin{aligned} W'(F(\alpha_F))(x,y,u,z) &= \mathcal{E}_{(x,y,0,0)} W'(F(\alpha_F)) \\ &= \mathcal{E}_{(x,0,0,0)} \left(W'(F(\alpha_F)) \circ J_{(0,y)} \right) \\ &= \mathcal{E}_{(x,0,0,0)} J_{(x,y)} W'(\mathcal{J}_{(0,y)} F(\alpha_F)), \end{aligned} \tag{8.61}$$

and the last claim follows. □

Theorem 8.12. *Let the quadruples $\tilde{\rho}$, ϱ and ρ be as in Theorem 4.2 and Prop. 5.1, and let r be a pair of real positive numbers, such that $(1-\chi^{-1})\tilde{\rho}_1 > r_1$ and $\rho_2 - \varrho_2 > r_2$. Then, there exist an open neighborhood $\mathcal{U}$ of F^*_+ in $\mathcal{F}^{\rho}_{\delta}$, and for every $F \in \mathcal{U}$ a complex number $\xi'(F(\alpha_F))$ and an analytic map $\Gamma_{F(\alpha_F)}$ from $\mathcal{D}_1(r)$ to $\mathcal{D}(\varrho_1, \varrho_2 + r_2, \varrho_3, \varrho_4) \subset \mathcal{D}(\rho)$, such that the following equation holds on $\mathcal{D}_1(r)$:*

$$(\mathbb{J} \nabla F(\alpha_F)) \circ \Gamma_{F(\alpha_F)} = \xi'(F(\alpha_F)) \; \partial_x \Gamma_{F(\alpha_F)}. \tag{8.62}$$

The maps $F \mapsto \xi'(F(\alpha_F))$ and $F \mapsto \Gamma_{F(\alpha_F)} - \Gamma_{H^3_+}$ are analytic from $\mathcal{U}$ to, respectively, $\mathbb{C}$ and some Banach space of analytic functions from $\mathcal{D}_1(r)$ to $\mathcal{D}(\rho)$, 2π-periodic in variables $\omega \cdot (x,y)$ and $\Omega \cdot (x,y)$. Finally, if $F(\alpha_F)$ depends on the variables (u,z) only, then $\xi'(F(\alpha_F)) = (\partial_u \mathbb{E} H)(0)$ and $\Gamma_{F(\alpha_F)}(x,y) = (x,y,0,0)$ for all $(x,y) \in \mathcal{D}_1(r)$.

Proof. Define $\Gamma_{F(\alpha_F)}(x,y) = \mathcal{W}_F(x,y,0,0)$ for all $(x,y) \in \mathcal{D}_1(r)$ and for all $F \in \mathcal{U}' \in \mathcal{F}^{\rho}_{\delta}$, described in Lemma 8.11. First, assume that $F(\alpha_F)$ depends on (u,z) only, which means that $\mathcal{J}_{(0,y)} F(\alpha_F) = F(\alpha_F)$ for all x and y. Then, $\mathcal{U}_{\mathfrak{R}^k(F(\alpha_F))} = \mathbb{I}$ for all $k \geq 0$ and, therefore, by the definition of $\xi'(F(\alpha_F)) = \xi(F(\alpha_F)) = (\partial_u \mathbb{E} F(\alpha_F))(0,0)$, while $W'(F(\alpha_F)) = S_0$. Thus, by Lemma 8.11, for all $(x,y) \in \mathcal{D}_1(r)$

$$\begin{aligned} \Gamma_{F(\alpha_F)}(x,y) &= \mathcal{E}_{(x,0,0,0)} J_{(0,y)} \circ W'(\mathcal{J}_{(0,y)} F(\alpha_F)) \\ &= \mathcal{E}_{(x,0,0,0)} J_{(0,y)} \circ S_0 = (x,y,0,0). \end{aligned} \tag{8.63}$$

The claims about the analyticity and domains of $\Gamma_{F(\alpha_F)}$ and the map $F \mapsto \Gamma_{F(\alpha_F)} - \Gamma_{F^*_+(0)}$ follow from Lemma 8.11.

By Lemma 8.11 the restriction of the transformation $\mathcal{W}_F$ to $\mathbb{R}^4$ equals $W'(F(\alpha_F))$ and thus, by Prop. 8.10, satisfies equation (8.57), which means that we have on $\mathbb{R}^2$

$$\mathbb{J} F(\alpha_F) \circ \Gamma_{F(\alpha_F)} = \xi'(F(\alpha_F)) \; \partial_x \Gamma_{F(\alpha_F)}. \tag{8.64}$$

Finally, since $\Gamma_{F(\alpha_F)}$ is an analytic function on $\mathcal{D}_1(r)$, so are both sides of (8.64) and this equation holds on all of $\mathcal{D}_1(r)$.

The claim about the analyticity of $\xi'(F(\alpha_F))$ is just a restatement of the same claim from Prop. 8.10, possibly on a smaller neighborhood $\mathcal{U}$.

$\square$

We have proved constructively that the Hamiltonian functions on the stable manifold of the renormalization operator at the fixed point H^3_+ do posses analytic invariant ω-tori. One more claim, advertised in the beginning of this section, remains to be verified, specifically, the fact that the invariant ω-torus is shearless. This claim will be only demonstrated for Hamiltonians on the stable manifold, independent of the variable u. We will describe the difficulties of extending the proof to a generic Hamiltonian at the end of this Section. Although, here we will include only the proof of a partial result, we will nevertheless state the following

Conjecture 8.13. *Let ρ and $\tilde{\rho}$ be as in Theorem* 8.12. *Then there exists a neighborhood $\mathcal{U}$ of F^*_+ in $\mathcal{F}^{\rho}_{\delta}$, a quadruple r of real positive numbers, and a transformation $G_{F(\alpha_F)}$, analytic from $\mathcal{D}(r)$ to $\mathcal{D}(\rho)$ and satisfying $G_{F(\alpha_F)}(x,y,0,0) = \Gamma_{F(\alpha)}(x,y)$ for all $(x,y) \in \mathcal{D}_1(r_1,r_2)$, such that the following equation holds for all $(x,y,u,z) \in \mathcal{D}(r)$ and $F \in \mathcal{U}$:*

$$\begin{aligned}\mathbb{J}\nabla F(\alpha_F) \circ G_{F(\alpha_F)}(x,y,u,z) &= \xi'(F(\alpha_F)\ (\partial_x + 3\gamma z^2 \partial_y) G_{F(\alpha_F)}(x,y,u,z) \\ &\quad + O(|z|^3, |u|^3, |u|^2|z|, |u||z|^2). \qquad (8.65)\end{aligned}$$

We will say that the Hamiltonian $F(\alpha_F)$ for which such analytic transformation exists has a shearless ω-torus $\Gamma_{F(\alpha_F)}$. Although, this definition of "shearlessness" is different from its conventional definition appearing in the KAM theory (see, for example [33]), it intuitively makes sense: the equation (8.65) means that close to the ω-torus the flow of the Hamiltonian $F(\alpha_F)$ is conjugate to a rotation with a rotation number $\rho_{F(\alpha_F)} = (1+3\gamma z^2 \vartheta)/(\vartheta - 3\gamma z^2)$ up to terms of the third order of magnitude. In this sense, the rotation number on the torus $\Gamma_{F(\alpha_F)}$ is minimal. A similar definition of a shearless torus (circle) has been used, for example, in [25] to prove the result about two-parameter families of non-twist maps, described in the Introduction. As we have already pointed out, we will prove this conjecture only for a subclass of Hamiltonians on the stable manifold.

Proposition 8.14. *Conjecture* 8.13 *holds for all $F \in \mathcal{U}$, such that $F(\alpha) - H^*_+$ is constant in the variable u for all $\alpha \in \mathcal{D}_\delta$.*

Proof. First, we will show that the limits of some derivatives of $W'_n\ (F(\alpha_F)) - \mathbb{I}$ are well-defined and analytic on $\mathcal{D}(\tilde{\rho}_{\{n,\tilde{\eta},\chi\}} - \tilde{\varepsilon}_n)$, where $\tilde{\varepsilon} = (0,0,0,\epsilon_4 \tilde{\eta}^{-1} \vartheta^{-n})$ and $\epsilon_4 < \tilde{\rho}_4$.

$$\begin{aligned}\|\partial_z [W'_{n+1}(F(\alpha_F)) - W'_n(F(\alpha_F))]\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}} - \tilde{\varepsilon}_n} &\le \frac{\tilde{\eta}\vartheta^n}{\epsilon_4} \|W'_{n+1}(F(\alpha_F)) - W'_n(F(\alpha_F))\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}}} \\ &\le c_{15} (\zeta \vartheta^2 \chi \varpi)^n \left\| F(\alpha_F) - H^3_+ \right\|_{\rho}. \qquad (8.66)\end{aligned}$$

where we have used the bound (8.28). Similarly,

$$\|\partial^2_z [W'_{n+1}(F(\alpha_F)) - W'_n(F(\alpha_F))]\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}} - \tilde{\varepsilon}_n} \le c_{16} (\zeta \vartheta^3 \chi \varpi)^n \|F(\alpha_F) - H^3_+\|_{\rho}. \qquad (8.67)$$

Therefore, there exist real positive constants c_{17} and c_{18}, such that

$$\begin{aligned}\|\partial_z [W'_{n+1}(F(\alpha_F)) - \mathbb{I}]\|_{\tilde{\rho}_{\{n,\tilde{\eta},\chi\}} - \tilde{\varepsilon}_n} &< \sum_{k=0}^{n+1} \|\partial_z [W'_k(F(\alpha_F)) - W'_{k-1}(F(\alpha_F))]\|_{\tilde{\rho}_{\{k,\tilde{\eta},\chi\}} - \tilde{\varepsilon}_k} \\ &\le c_{17} \|F(\alpha_F) - H^3_+\|_{\rho}, \qquad (8.68)\end{aligned}$$

$$\begin{aligned}\|\partial_z^2[W'_{n+1}(F(\alpha_F))-\mathbb{I}]\|_{\tilde\rho_{\{n,\tilde\eta,\chi\}}-\tilde\varepsilon_n} &< \sum_{k=0}^{n+1}\|\partial_z^2[W'_k(F(\alpha_F))-W'_{k-1}(F(\alpha_F))]\|_{\tilde\rho_{\{n,\tilde\eta,\chi\}}-\tilde\varepsilon_n}\\ &\le c_{18}\|F(\alpha_F)-H^3_+\|_\rho. \qquad (8.69)\end{aligned}$$

Similarly, if one defines $\breve\varepsilon=(0,0,\epsilon_3\tilde\eta^{-1}\vartheta^{-3n},0)$ and $\epsilon_3<\tilde\rho_3$, one can also obtain an estimate on the y-derivative of $W'_n(F(\alpha_F))-\mathbb{I}$ which will be used in a later computation:

$$\|\partial_y[W'_{n+1}(F(\alpha_F))-\mathbb{I}]\|_{\rho_{\{n,\tilde\eta,\chi\}}-\breve\varepsilon_n}\le c_{19}\|F(\alpha_F)-H^3_+\|_\rho. \qquad (8.70)$$

The bounds (8.66)-(8.67) show that if ρ''_1 is large enough, then, for a given family $F\in\mathcal{U}$, the following limits converge in $\mathcal{C}^{(1,0,0)}_{(1-\chi^{-1})\tilde\rho_1}(T^2)$:

$$W'_z(F(\alpha_F))=\lim_{n\to\infty}[\partial_z(W'_n(F(\alpha_F))-\mathbb{I})]\circ S_0, \qquad (8.71)$$

$$W'_{zz}(F(\alpha_F))=\lim_{n\to\infty}\left[\partial_z^2(W'_n(F(\alpha_F))-\mathbb{I})\right]\circ S_0. \qquad (8.72)$$

One can analytically continue these maps to a complex neighborhood of the real torus, as in Lemma 8.11. More specifically, there exists a neighborhood $\mathcal{U}''$ of F^*_+ in $\mathcal{F}^\rho_\delta$, such that for every $F\in\mathcal{U}''$ and any pair of real positive numbers $r'_1<(1-\chi^{-1})\tilde\rho_1$ and $r'_2<\rho_2-\varrho_2$ the equations

$$\mathcal{W}^z_F(x,y,u,z)=\mathcal{E}_{(x,0,0,0)}J_{(0,y)}\circ W'_z(\mathcal{J}_{(0,y)}F(\alpha_F)), \qquad (8.73)$$

$$\mathcal{W}^{zz}_F(x,y,u,z)=\mathcal{E}_{(x,0,0,0)}J_{(0,y)}\circ W'_{zz}(\mathcal{J}_{(0,y)}F(\alpha_F)) \qquad (8.74)$$

define analytic bounded maps $\mathcal{W}^z_F$ and $\mathcal{W}^{zz}_F$ on $\mathcal{D}_1(r')\times\mathbb{C}^2$ whose restriction to $\mathbb{R}^4$ coincides with $W'_z(F(\alpha_F))$ and $W'_{zz}(F(\alpha_F))$. Define $\Gamma^z_F(\alpha_F)(x,y)=\mathcal{W}^z_F(x,y,0,0)$ and $\Gamma^{zz}_F(\alpha_F)(x,y)=\mathcal{W}^{zz}_F(x,y,0,0)$ for all $(x,y)\in\mathcal{D}_1(r')$.

As we have shown in Theorem 8.12, the flow of the Hamiltonian $F(\alpha_F)$ on the invariant ω- torus is a rotation with a rotation vector ω, as expressed by the equation (8.62). To show that the rotation number (if it exists) of any nearby orbit is larger than ϑ^{-1}, we want to extend the equation (8.62) to a small neighborhood of $\Gamma_{F(\alpha)}\mathcal{D}_1(r')$. Specifically, we want to show that there exist a quadruple r of real positive numbers and an analytic transformation $G_{F(\alpha_F)}$ from $\mathcal{D}(r)$ to $\mathcal{D}(\rho)$, which satisfies the equation (8.65) and coincides with $\Gamma_{F(\alpha)}$ whenever $(u,z)=(0,0)$.

We will show that this can be done by choosing $G_{F(\alpha_F)}(x,y,u,z)=\Gamma_{F(\alpha_F)}(x,y)+z\Gamma^z_{F(\alpha_F)}(x,y)+(z^2/2)\Gamma^{zz}_{F(\alpha_F)}(x,y)$. Recall, that by Theorem 8.12 there exists an open neighborhood $\mathcal{U}'$ of F^*_+ in $\mathcal{F}^\rho_\delta$, such that for any $F\in\mathcal{U}'$ the torus map $\Gamma_{F(\alpha)}$ is analytic from $\mathcal{D}_1(r')$ to $\mathcal{D}(\varrho_1,\varrho_2+r'_2,\varrho_3,\varrho_4)$ for any pair r' of real positive numbers $r'_1<(1-\chi^{-1})\tilde\rho_1$ and $r'_2<\rho_2-\varrho_2$. Since for any $F\in\mathcal{U}''$ the maps $\Gamma^z_{F(\alpha_F)}$ and $\Gamma^{zz}_{F(\alpha_F)}$ are analytic and bounded on $\mathcal{D}_1(r')$, there exists a quadruple r of sufficiently small positive real numbers and a neighborhood $\mathcal{U}\subset\mathcal{U}'\cap\mathcal{U}''$, such that for all $F\in\mathcal{U}$ the transformation $G_{F(\alpha_F)}$ is analytic from $\mathcal{D}(r)$ to $\mathcal{D}(\rho)$. Next,

$$\begin{aligned}\mathbb{J}\nabla F(\alpha_F)\circ&\left(\Gamma_{F(\alpha_F)}(x,y)+z\Gamma^z_{F(\alpha_F)}(x,y)+\frac{z^2}{2}\Gamma^{zz}_{F(\alpha_F)}(x,y)\right)=\\ &=\mathbb{J}\nabla F(\alpha_F)\circ\Gamma_{F(\alpha_F)}(x,y)+z(\Gamma^z_{F(\alpha_F)}(x,y)\cdot\nabla\mathbb{J}\nabla F(\alpha_F))\circ\Gamma_{F(\alpha_F)}(x,y)\\ &+\frac{z^2}{2}((\Gamma^z_{F(\alpha_F)}(x,y)\cdot\nabla)^2\mathbb{J}\nabla F(\alpha_F))\circ\Gamma_{F(\alpha_F)}(x,y)\\ &+\frac{z^2}{2}(\Gamma^{zz}_{F(\alpha_F)}(x,y)\cdot\nabla\mathbb{J}\nabla F(\alpha_F))\circ\Gamma_{F(\alpha_F)}(x,y)+O(|z|^3). \qquad (8.75)\end{aligned}$$

On the other hand

$$\begin{aligned}
&[W'_z(F(\alpha_F))\cdot\nabla\mathbb{J}\nabla F(\alpha_F)]\circ W'(F(\alpha_F)) = \\
&= \lim_{n\to\infty}[\partial_z(W'_n(F(\alpha_F))-\mathbb{I})]\circ S_0\nabla\{\mathbb{I},F(\alpha_F)\}\circ\Big(S_0+\lim_{n\to\infty}[W'_n(F(\alpha_F))-\mathbb{I}]\circ S_0\Big)\\
&= \lim_{n\to\infty}\partial_z[\{\mathbb{I},F(\alpha_F)\}\circ W'_n(F(\alpha_F))]\circ S_0, \qquad (8.76)\\
&[W'_{zz}(F(\alpha_F))\cdot\nabla\mathbb{J}\nabla F(\alpha_F)+(W'_z(F(\alpha_F))\cdot\nabla)^2\mathbb{J}\nabla F(\alpha_F)]\circ W'(F(\alpha_F)) = \\
&= \lim_{n\to\infty}[\partial_z^2(W'_n(F(\alpha_F))-\mathbb{I})]\circ S_0\cdot\nabla\{\mathbb{I},F(\alpha_F)\}\circ\Big(S_0+\lim_{n\to\infty}[W'_n(F(\alpha_F))-\mathbb{I}]\circ S_0\Big)+\\
&+\Big(\lim_{n\to\infty}[\partial_z(W'_n(F(\alpha_F))-\mathbb{I})]\circ S_0\cdot\nabla\Big)^2\{\mathbb{I},F(\alpha_F)\}\circ\Big(S_0+\lim_{n\to\infty}[W'_n(F(\alpha_F))-\mathbb{I}]\circ S_0\Big)\\
&= \lim_{n\to\infty}\partial_z^2[\{\mathbb{I},F(\alpha_F)\}\circ W'_n(F(\alpha_F)]\circ S_0. \qquad (8.77)
\end{aligned}$$

In the string of the equalities above we were able to interchange limits and differentiation because the bounds (8.28), (8.66) and (8.67) are uniform on $\mathcal{U}$ and so is convergence of the corresponding transformations in $\mathcal{C}^{(1,0,0)}_{(1-\chi^{-1})\tilde\rho_1}(T^2)$. Fix real positive numbers ϵ_1, ϵ_2 and ϵ_4, such that $\epsilon_1<\tilde\rho_1$, $\epsilon_2<\tilde\rho_2$ and $3\epsilon_4<\tilde\rho_4$, and define $\bar\varepsilon_n=(\epsilon_1\chi^{-n},\epsilon_2\vartheta^{-n},0,2\epsilon_4\tilde\eta^{-1}\vartheta^{-n})$ and $\hat\varepsilon_n=(\epsilon_1\chi^{-n},\epsilon_2\vartheta^{-n},0,3\epsilon_4\tilde\eta^{-1}\vartheta^{-n})$. Then, using the shorthanded notation $H_n=\mathfrak{R}(F(\alpha_F))$ as before, we get

$$\begin{aligned}
&\Big\|\partial_z\big[\{\mathbb{I},H_0\}\circ W'_n(H_0)-\xi'_n(H_0)\{W'_n(H_0),H^3_+\}\big]\Big\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}} = \\
&= \Big\|\partial_z\big[\{W'_n(H_0)-\mathbb{I},H_0\circ W'_n(H_0)-\xi'_n(H)H^3_+\}\big]+\{\mathbb{I},\partial_z(H\circ W'_n(H))\}\Big\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}}\\
&= \frac{\xi'_n(H)}{\vartheta^{3n}}\Big\|\partial_z\Big[\{W'_n(H_0)-\mathbb{I},\Big[H_n-H^3_{(-)^n}\Big]\circ T^{-n}_{\vartheta^{-2}}\}\Big]\Big\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}}\\
&+\frac{\xi'_n(H)}{\vartheta^3 n}\Big|\nabla\partial_z\Big[\Big[H_n-H^3_{(-)^n}\Big]\circ T^{-n}_{\vartheta^{-2}}\Big]\Big|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}} = I^1_n+I^2_n. \qquad (8.78)
\end{aligned}$$

We will estimate the two norms in the above expression separately:

$$\begin{aligned}
I^1_n &= \frac{\xi'_n(H_0)}{\vartheta^{3n}}\Big\|\partial_z\Big[\{W'_n(H_0)-\mathbb{I},\Big[H_n-H^3_{(-)^n}\Big]\circ T^{-n}_{\vartheta^{-2}}\}\Big]\Big\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}}\\
&\le \frac{(1+c_{10}\beta)\tilde\eta}{\epsilon_4\vartheta^{2n}}\Big\|\{W'_n(H_0)-\mathbb{I},\Big[H_n-H^3_{(-)^n}\Big]\circ T^{-n}_{\vartheta^{-2}}\}\Big\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\bar\varepsilon_{n-1}}\\
&\le \frac{(1+c_{10}\beta)\tilde\eta}{\epsilon_4\vartheta^{2n}}\Big\|\partial_y\Big[W'_n(H_0)-\mathbb{I}]\partial_z\Big[[H_n-H^3_{(-)^n}\Big]\circ T^{-n}_{\vartheta^{-2}}\Big]\Big\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\bar\varepsilon_{n-1}}\\
&+\frac{(1+c_{10}\beta)\tilde\eta}{\epsilon_4\vartheta^{2n}}\Big\|\partial_z\,[W'_n(H_0)-\mathbb{I}]\,\partial_y\Big[\Big[H_n-H^3_{(-)^n}\Big]\circ T^{-n}_{\vartheta^{-2}}\Big]\Big\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\bar\varepsilon_{n-1}}\\
&\le \frac{2(1+c_{10}\beta)\tilde\eta^2}{\epsilon_2\epsilon_4^2}\|W'_n(H_0)-\mathbb{I}\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}}\|H_n-H^3_{(-)^n}\|_\rho\\
&< \frac{2c_9(1+c_{10}\beta)\tilde\eta^2}{\epsilon_2\epsilon_4^2}\zeta^n\left\|F(\alpha_F)-H^3_+\right\|^2_\rho, \qquad (8.79)
\end{aligned}$$

for some real positive ζ, such that $1/\vartheta<\zeta<1$.

$$\begin{aligned}
I^2_n &= \frac{\xi'_n(H)}{\vartheta^3 n}\Big|\nabla\partial_z\Big[\Big[H_n-H^3_{(-)^n}\Big]\circ T^{-n}_{\vartheta^{-2}}\Big]\Big|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}}\\
&= \frac{\xi'_n(H)}{\vartheta^3 n}\Big[\Big\|\partial_z^2\Big[\Big[H_n-H^3_{(-)^n}\Big]\circ T^{-n}_{\vartheta^{-2}}\Big]\Big\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}}\\
&+\Big\|\partial_x\partial_z\Big[\Big[H_n-H^3_{(-)^n}\Big]\circ T^{-n}_{\vartheta^{-2}}\Big]\Big\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}}
\end{aligned}$$

$$+\left\|\partial_y\partial_z\left[\left[H_n-H^3_{(-)^n}\right]\circ T^{-n}_{\vartheta^{-2}}\right]\right\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}}\Bigg]$$
$$\leq\frac{\xi'_n(H)}{\epsilon_4\vartheta^n}\left[\frac{\tilde\eta^2}{\epsilon_4}+\frac{\tilde\eta\chi^n}{\epsilon_1\vartheta^n}+\frac{\tilde\eta}{\epsilon_2}\right]\left\|H_n-H^3_{(-)^n}\right\|_\rho$$
$$<\frac{c_{20}}{\vartheta^n}\left[\frac{\chi^n}{\vartheta^n}+c_{21}\right]\zeta^n\|F(\alpha_F)-H^3_+\|_\rho. \qquad (8.80)$$

Similarly,

$$\left\|\,\partial_z^2\left[\{\mathbb{I},H_0\}\circ W'_n(H_0)-\xi'_n(H_0)\{W'_n(H_0),H^3_+\}\right]\right\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}}$$
$$=\frac{\xi'_n(H)}{\vartheta^{3n}}\left\|\partial_z^2\left[\{W'_n(H_0)-\mathbb{I},\left[H_n-H^3_{(-)^n}\right]\circ T^{-n}_{\vartheta^{-2}}\}\right]\right\|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}}$$
$$+\frac{\xi'_n(H)}{\vartheta^{3}n}\left|\nabla\partial_z^2\left[\left[H_n-H^3_{(-)^n}\right]\circ T^{-n}_{\vartheta^{-2}}\right]\right|_{\tilde\rho_{\{n-1,\tilde\eta,\chi\}}-\hat\varepsilon_{n-1}}=J^1_n+J^2_n. \qquad (8.81)$$

Again, consider the two norms separately. Then a computation similar to those for I^1_n and I^2_n gives the following bounds:

$$J^1_n\leq\left(\frac{2c_{17}(1+c_{10}\beta)\tilde\eta^2}{\epsilon_2\epsilon_4^2}+\frac{2c_{19}(1+c_{10}\beta)\tilde\eta^3}{\epsilon_4^3}\right)\zeta^n\|F(\alpha_F)-H^3_+\|^2_\rho, \qquad (8.82)$$
$$J^2_n<c_{22}\left(\frac{\chi^n}{\vartheta^n}+c_{23}\right)\zeta^n\|F(\alpha_F)-H^3_+\|_\rho. \qquad (8.83)$$

The bounds above show that the expression $\partial_z[\{\mathbb{I},F(\alpha_F)\}\circ W'_n(F(\alpha_F))]\circ S_0$ converges to $[W'_z(F(\alpha_F))\cdot\nabla\mathbb{J}\nabla F(\alpha_F)]\circ W'(F(\alpha_F))$ in $\mathcal{C}^0_{(1-\chi^{-1})\tilde\rho_1}(T^2)$ on one hand, and to $\xi'(F(\alpha_F))\partial_x W'_z(F(\alpha_F))$ on the other, also in $\mathcal{C}^0_{(1-\chi^{-1})\tilde\rho_1}(T^2)$. At the same time, the expression $\partial_z^2[\{\mathbb{I},F(\alpha_F)\}\circ W'_n(F(\alpha_F))]\circ S_0$ converges in $\mathcal{C}^0_{(1-\chi^{-1})\tilde\rho_1}(T^2)$ to $\left[W'_{zz}(F(\alpha_F))\cdot\nabla\mathbb{J}\nabla F(\alpha_F)+(W'_z(F(\alpha_F))\cdot\nabla)^2\,\mathbb{J}\nabla F(\alpha_F)\right]\circ W'(F(\alpha_F))$ and to $\xi'(F(\alpha_F))\partial_x W'_{zz}(F(\alpha_F))+\xi'(F(\alpha_F))6\gamma\partial_y W'(F(\alpha_F))$. Therefore, the equation (8.75) can be rewritten in $\mathbb{R}^4$ as

$$\mathbb{J}\nabla F(\alpha_F)\circ\left(W'(F(\alpha_F))+zW'_z(F(\alpha_F))+\frac{z^2}{2}W'_{zz}(F(\alpha_F))\right)(x,y)=$$
$$=\xi'(F(\alpha_F))\left[(\partial_x+3\gamma z^2\partial_y)W'(F(\alpha_F))+z\partial_x W'_z(F(\alpha_F))+\frac{z^2}{2}\partial_x W'_{zz}(F(\alpha_F))\right](x,y)+O(|z|^3)$$
$$=\xi'(F(\alpha_F))(\partial_x+3\gamma\partial_y)\left[W'(F(\alpha_F))+zW'_z(F(\alpha_F))+\frac{z^2}{2}W'_{zz}(F(\alpha_F))\right](x,y)+O(|z|^3), \qquad (8.84)$$

and the same equation should hold for the analytic continuations of the maps $W'(F(\alpha_F))$, $W'_z(F(\alpha_F))$ and $W'_{zz}(F(\alpha_F))$:

$$\mathbb{J}\nabla F(\alpha_F)\circ\left(\Gamma_{F(\alpha_F)}+z\Gamma^z_{F(\alpha_F)}+\frac{z^2}{2}\Gamma^{zz}_{F(\alpha_F)}\right)(x,y)=$$
$$=\xi'(F(\alpha_F))(\partial_x+3\gamma\partial_y)\left(\Gamma_{F(\alpha_F)}+z\Gamma^z_{F(\alpha_F)}+\frac{z^2}{2}\Gamma^{zz}_{F(\alpha_F)}\right)+O(|z|^3). \qquad (8.85)$$

☐

Remark 8.15. The difficulty of extending this proof to the general case of $F(\alpha_F)=F(\alpha_F)(x,y,u,z)$ is proving the bounds (8.79) -(8.83). In particular, one has to estimate the u-derivatives of the transformation $W'_n(F(\alpha_F))-\mathbb{I}$ and the Hamiltonian

$H_n - H^3_{(-)^n}$. In doing so, one acquires additional powers of ϑ^n which can not be suppressed by the rate of convergence of either $\mathfrak{R}^n(F(\alpha_F))$ to $H^3_{(-)^n}$ or $\partial_u \mathfrak{R}^n(F(\alpha_F))$ to one.

9. **Bifurcations and Shearless Invariant Tori.** In this Section we will proof our main bifurcation result. We will start by showing that the transformation $W'_n(F(\alpha))$ for every family $F \in \mathcal{U}$ and an arbitrary $\alpha \in \mathcal{D}_\delta$, is analytic from $\mathcal{D}(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$ to $\mathcal{D}(\rho)$ as long as all iterates $\mathfrak{R}^k(F(\alpha))$, $0 \le k \le n$, are in the domain of the analyticity of the renormalization operator $\mathfrak{R}$, and not only in the case of $\alpha = \alpha_F$ which has been already considered. Recall the definitions of the contraction mappings Y_F and M on $\mathcal{D}_\delta$ and $\mathcal{U}$, respectively, from Prop. 7.2: $Y_F = (\mathbb{P}^u \circ \mathfrak{R} \circ F)^{-1}$ and $M(F) = \mathfrak{R} \circ F \circ Y_F$. Given a positive integer n, define, as in Prop. 7.2, $F_n = M^n(F)$ and $(\alpha)_n = \left(Y_{F_0} \circ Y_{F_1} \circ \ldots \circ Y_{F_{n-1}}\right)^{-1}(\alpha)$.

Proposition 9.1. *Let $\tilde{\rho}$, ρ' and ρ'' be as in Theorem* 4.3 *and Prop.* 5.1, *and let an open ball $\mathcal{U}(\beta)$, an open disk $\mathcal{D}_\delta$ and a real positive number c be as in Prop.* 7.2. *Then, if β is sufficiently small and ρ''_1 is sufficiently large, the following holds for every $F \in \mathcal{U}(\beta)$ and $\alpha \in \mathcal{D}_\delta$, such that $\alpha_k \in \mathcal{D}_\delta$ for all $1 \le k \le n$.*

(a) *The map $(\alpha, F) \mapsto W_k(F(\alpha)) - \mathbb{I}$ is analytic from $\mathcal{D}_\delta \times \mathcal{U}(\beta)$ to $\mathcal{A}^4(\tilde{\rho}_{\{k,\tilde{\eta}\}})$, and there exists a real positive constant c_{24}, such that*

$$\|W_k(F(\alpha)) - \mathbb{I}\|_{\tilde{\rho}_{\{k,\tilde{\eta}\}}} \le c_{24} \beta c^k \varpi^k. \tag{9.1}$$

(b) *The map $(\alpha, F) \mapsto W'_n(F(\alpha)) - \mathbb{I}$ is analytic from $\mathcal{D}_\delta \times \mathcal{U}(\beta)$ to $\mathcal{A}^4(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$, while the transformation $W'_n(F(\alpha))$ is analytic from $\mathcal{D}(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$ to $\mathcal{D}(\varrho)$.*

Proof. The proof follows those of Prop. 8.6 and Prop. 8.7 and will be omitted here. An interested reader is referred to [30] for details. ☐

The above result will be used to prove that a Hamiltonian $F(\alpha)$, where F and α satisfy the assumptions of Prop. 9.1, has an invariant ω-torus if and only if its n-th image under the renormalization operator does.

Proposition 9.2. *Let the quadruples $\tilde{\rho}$, ρ, ϱ and a pair r be as in Theorem* 8.12, *and let the neighborhood $\mathcal{U}(\beta)$ and the disk $\mathcal{D}_\delta$ be as in Prop.* 7.2. *Then, the following holds for all $F \in \mathcal{U}(\beta)$ and all $\alpha \in \mathcal{D}_\delta$, such that $(\alpha)_k \in \mathcal{D}_\delta$ for all $1 \le k \le n$.*

(a) *Let $\Gamma_{\mathfrak{R}^n(F(\alpha))}$ be a torus map, analytic from $\mathcal{D}_1(r)$ to $\mathcal{D}((\vartheta^n \sum_{i=0}^{n-2} \chi^{-1})^{-1} \rho'_1, \rho_2, \rho_3, \rho_4)$ and satisfying the equation*

$$\mathbb{J} \nabla \mathfrak{R}^n(F(\alpha)) \circ \Gamma_{\mathfrak{R}^n(F(\alpha))} = c \; \partial_x \Gamma_{\mathfrak{R}^n(F(\alpha))} \tag{9.2}$$

on $\mathcal{D}_1(r)$, where c is some complex number. Then there exist a torus map $\Gamma_{F(\alpha)} = W'_n(F(\alpha)) \circ T^n_{\vartheta^{-2}} \Gamma_{\mathfrak{R}^n(F(\alpha))} \circ T^{-n}_{\vartheta^{-2}}$ analytic from $T^n_{\vartheta^{-2}} \mathcal{D}_1(r)$ to $\mathcal{D}(\rho)$ and satisfying

$$\mathbb{J} \nabla F(\alpha) \circ \Gamma_{F(\alpha)} = c \; \xi'_n(F(\alpha)) \; \partial_x \Gamma_{F(\alpha)}, \tag{9.3}$$

where $W'_n(F(\alpha))$ is the transformation described in Prop. 9.1 *and $\xi'_n(H)$ is the function, defined in Prop.* 8.7.

(b) *Let $\Gamma_{F(\alpha)}$ be a torus map, analytic from $\mathcal{D}_1(r)$ to the image of the domain $\mathcal{D}((\sum_{i=0}^{n-2} \chi^{-i})^{-1} \rho'_1, \rho_2 \vartheta^{-n}, \rho_3 \vartheta^{-3n}, \rho_4 \vartheta^{-n})$ under $W'_n(F(\alpha))$ and satisfying the equation*

$$\mathbb{J} \nabla F(\alpha) \circ \Gamma_{F(\alpha)} = c \; \partial_x \Gamma_{F(\alpha)} \tag{9.4}$$

on $\mathcal{D}_1(r)$, where c is some complex number. Then, there also exists a torus map $\Gamma_{\mathfrak{R}^n(F(\alpha))} = T_{\vartheta^{-2}}^{-n} W_n'^{-1}(F(\alpha)) \circ \Gamma_{F(\alpha)} \circ T_{\vartheta^{-2}}^{n}$, analytic from $T_{\vartheta^{-2}}^{-n} \mathcal{D}_1(r)$ to $\mathcal{D}((\vartheta^n \sum_{i=0}^{n-2} \chi^{-i})^{-1} \rho_1', \rho_2, \rho_3, \rho_4)$ and satisfying

$$\mathbb{J} \nabla \mathfrak{R}^n(F(\alpha)) \circ \Gamma_{\mathfrak{R}^n(F(\alpha))} = \frac{c}{\xi_n'(F(\alpha))} \, \partial_x \Gamma_{\mathfrak{R}^n(F(\alpha))}. \tag{9.5}$$

Proof.
(a) Denote $H_n = \mathfrak{R}_\eta^n(F(\alpha))$. Let Γ_{H_n} be a torus map, analytic from $\mathcal{D}_1(r)$ to $\mathcal{D}(\rho)$ and satisfying

$$\mathbb{J} \nabla H_n \circ \Gamma_{H_n} = c \, \partial_x \Gamma_{H_n}, \tag{9.6}$$

and let $W_n'(H)$ be the transformation described in Prop. 9.1, analytic from the domain $\mathcal{D}(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$ to $\mathcal{D}(\varrho)$. Notice that $\mathbb{J} T_{\vartheta^{-2}}^{-n} = \vartheta^{2n} T_{\vartheta^{-2}}^{n} \mathbb{J}$ and that $W_k(H) \circ S^{-1}_{\eta(\bar{H}_k)} = T_{\vartheta^{-2}}^{k} U_{\psi(H_k)} \circ R_{t(\tilde{H}_k)} \circ T_{\vartheta^{-2}}^{-k}$, is a canonical transformation for all $1 \le k \le n$. We use this together with the identity (8.46) and the fact, that $\{f, g\} \circ u = \{f \circ u, g \circ u\}$ for any two analytic maps f and g and a canonical transformation u, to obtain the following equation:

$$\begin{aligned}
\mathbb{J} \nabla H_0 \circ W_n'(H_0) &= \{\mathbb{I}, H_0\} \circ W_n'(H_0) \\
&= \{\mathbb{I}, H_0\} \circ U_{\psi(H_0)} \circ R_{t(\tilde{H}_0)} \circ S_{\eta(H_0)} \circ W_1(H_0) \circ \ldots \circ W_{n-1}(H_0) \\
&= \frac{1}{\eta(H_0)} \{W_0(H_0), H_0 \circ W_0(H_0)\} \circ W_1(H_0) \circ \ldots \circ W_{n-1}(H_0) \\
&= \prod_{i=0}^{n-1} \frac{1}{\eta(H_i)} \{W_n'(H_0), H_0 \circ W_n'(H_0)\} \\
&= \frac{\xi_n'(H_0)}{\vartheta^{3n}} \left\{W_n'(H_0), H_n \circ T_{\vartheta^{-2}}^{-n}\right\} \\
&= \frac{\xi_n'(H_0)}{\vartheta^{3n}} \mathbb{J} T_{\vartheta^{-2}}^{-n} \nabla H_n \circ T_{\vartheta^{-2}}^{-n} \cdot \nabla W_n'(H_0) \\
&= \frac{\xi_n'(H_0)}{\vartheta^{n}} T_{\vartheta^{-2}}^{n} \mathbb{J} \nabla H_n \circ T_{\vartheta^{-2}}^{-n} \cdot \nabla W_n'(H_0).
\end{aligned} \tag{9.7}$$

Thus,

$$\begin{aligned}
&\mathbb{J} \nabla H_0 \circ W_n'(H_0) \circ T_{\vartheta^{-2}}^{n} \Gamma_{H_n} \circ T_{\vartheta^{-2}}^{-n} = \\
&\quad = \xi_n'(H_0) \vartheta^{-n} T_{\vartheta^{-2}}^{n} \mathbb{J} \nabla H_n \circ \Gamma_{H_n} \circ T_{\vartheta^{-2}}^{-n} \cdot (\nabla W_n'(H_0)) \circ T_{\vartheta^{-2}}^{n} \Gamma_{H_n} \circ T_{\vartheta^{-2}}^{-n} \\
&\quad = c \vartheta^{-n} \xi_n'(H_0) T_{\vartheta^{-2}}^{n} \partial_x \Gamma_{H_n} \circ T_{\vartheta^{-2}}^{-n} \cdot (\nabla W_n'(H_0)) \circ T_{\vartheta^{-2}}^{n} \Gamma_{H_n} \circ T_{\vartheta^{-2}}^{-n} \\
&\quad = c \xi_n'(H_0) \partial_x \left(T_{\vartheta^{-2}}^{n} \Gamma_{H_n} \circ T_{\vartheta^{-2}}^{-n}\right) \cdot (\nabla W_n'(H_0)) \circ T_{\vartheta^{-2}}^{n} \Gamma_{H_n} \circ T_{\vartheta^{-2}}^{-n} \\
&\quad = c \, \xi_n'(H_0) \, \partial_x (W_n'(H_0) \circ T_{\vartheta^{-2}}^{n} \Gamma_{H_n} \circ T_{\vartheta^{-2}}^{-n}).
\end{aligned} \tag{9.8}$$

Notice that, if Γ_{H_n} is as in the condition of the Proposition, then $T_{\vartheta^{-2}}^{n} \Gamma_{H_n} \circ T_{\vartheta^{-2}}^{-n}$ maps $T_{\vartheta^{-2}}^{n} \mathcal{D}_1(r)$ to $\mathcal{D}((\sum_{i=0}^{n-2} \chi^{-1})^{-1} \rho_1', \rho_2 \vartheta^{-n}, \rho_3 \vartheta^{-3n}, \rho_4 \vartheta^{-n})$. Also, by Prop. 8.7 the transformation $W_n'(F(\alpha)) - \mathbb{I}$ is well-defined and analytic on $\mathcal{D}(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$. Assume that the neighborhood $\mathcal{B}_+(b)$ is sufficiently small (and so is $|1 - \tilde{\eta}|$), so that $\vartheta \tilde{\eta} > 1$, then $\mathcal{D}((\sum_{i=0}^{n-2} \chi^{-1})^{-1} \rho_1', \rho_2 \vartheta^{-n}, \rho_3 \vartheta^{-3n}, \rho_4 \vartheta^{-n}) \subset \mathcal{D}(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$, and $W_n'(F(\alpha)) \circ T_{\vartheta^{-2}}^{n} \Gamma_{H_n} \circ T_{\vartheta^{-2}}^{-n}$ is analytic from $T_{\vartheta^{-2}}^{n} \mathcal{D}_1(r)$ to $\mathcal{D}(\rho)$.
(b) First, notice that $U_{\psi(H^3)} = \mathbb{I}$ and, by Theorem 4.2, $D\mathcal{N}(H_+^3) = \mathbb{I}^+$, where $\mathcal{N} : H \mapsto H \circ \mathcal{U}_H$. Then, there exists an open neighborhood $\mathcal{B}'$ of H_+^3 such that for every $H \in \mathcal{B}'$ the inverse of the transformation $U_{\psi(H)}$ exists and is analytic and bounded on the image of $\mathcal{D}(\varrho')$ under $U_{\psi(H)}$. Assume that b is small enough, so

that $\mathcal{B}_+(b) \in \mathcal{B}'$. Then, the inverse of the transformation $W'_n(F(\alpha))$ exists and is analytic on the image of $\mathcal{D}\left(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}}\right)$ under $W'_n(F(\alpha))$. Therefore,

$$\begin{aligned}
(\mathbb{J}\nabla H_n) \circ T^{-n}_{\vartheta^{-2}} W'^{-1}_n(H_0) &= \{\mathbb{I}, H_n\} \circ T^{-n}_{\vartheta^{-2}} W'^{-1}_n(H_0) \\
&= \prod_{i=0}^{n-1} \eta(H_i)\vartheta^{-2n} \left\{T^{-n}_{\vartheta^{-2}} W'^{-1}_n(H_0), H_n \circ T^{-n}_{\vartheta^{-2}} W'^{-1}_n(H_0)\right\} \\
&= \xi'_n(H_0)^{-1}\vartheta^n \left\{T^{-n}_{\vartheta^{-2}} W'^{-1}_n(H_0), H_0\right\} \\
&= \xi'_n(H_0)^{-1}\vartheta^n \mathbb{J}\nabla H_0 \cdot \nabla(T^{-n}_{\vartheta^{-2}} W'^{-1}_n(H_0)), \qquad (9.9)
\end{aligned}$$

and, finally, we obtain

$$\begin{aligned}
&(\mathbb{J}\nabla H_n) \circ T^{-n}_{\vartheta^{-2}} W'^{-1}_n(H_0) \circ \Gamma_{H_0} \circ T^{n}_{\vartheta^{-2}} = \\
&= \frac{\vartheta^n}{\xi'_n(H_0)} (\mathbb{J}\nabla H_0) \circ \Gamma_{H_0} \circ T^{n}_{\vartheta^{-2}} \cdot \nabla(T^{-n}_{\vartheta^{-2}} W'^{-1}_n(H_0)) \circ \Gamma_{H_0} \circ T^{n}_{\vartheta^{-2}} \\
&= \frac{c\vartheta^n}{\xi'_n(H_0)} \partial_x \Gamma_{H_0} \circ T^{n}_{\vartheta^{-2}} \cdot \nabla(T^{-n}_{\vartheta^{-2}} W'^{-1}_n(H_0)) \circ \Gamma_{H_0} \circ T^{n}_{\vartheta^{-2}} \\
&= \frac{c}{\xi'_n(H_0)} \partial_x (T^{-n}_{\vartheta^{-2}} W'^{-1}_n(H_0) \circ \Gamma_{H_0} \circ T^{n}_{\vartheta^{-2}}). \qquad (9.10)
\end{aligned}$$

By assumption, the torus map Γ_{H_0} is analytic from $\mathcal{D}_1(r)$ to the image of the domain $\mathcal{D}((\sum_{i=0}^{n-2} \chi^{-i})^{-1}\rho'_1, \rho_2\vartheta^{-n}, \rho_3\vartheta^{-3n}, \rho_4\vartheta^{-n}) \subset \mathcal{D}(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$ under $W'_n(F(\alpha))$. The inverse transformation $W'^{-1}_n(F(\alpha))$ is analytic from this image back to the domain $\mathcal{D}((\sum_{i=0}^{n-2} \chi^{-i})^{-1}\rho'_1, \rho_2\vartheta^{-n}, \rho_3\vartheta^{-3n}, \rho_4\vartheta^{-n})$, whose image under $T^{-n}_{\vartheta^{-2}}$ is equal to $\mathcal{D}((\vartheta^n \sum_{i=0}^{n-2} \chi^{-i})^{-1}\rho'_1, \rho_2, \rho_3, \rho_4)$.

$\square$

Our main result will be stated for one-(real-)parameter families of Hamiltonians, which are real-valued, when evaluated at real arguments. Such families will be called real. We will also call a torus map Γ_H real, if it is real-valued on $T^2(\mathbb{R})$. Obviously, if H is a member of a real family, then $H_{(\nu,k,n)} = H^*_{(-\nu,k,n)}$. We will denote the subset of such Hamiltonians in $\mathcal{B}_+$ by $\mathscr{B}_+$. We want to emphasize several simple consequences of this property of the Hamiltonians in $\mathscr{B}_+$. First, the generating function $\phi(H_n)$, introduced in Theorem 3.3 is real-valued when restricted to the real arguments, if $H_n \in \mathscr{B}_+$. A consequence of this real property of the generating functions $\phi(H_n)$ is the fact that the canonical transformation $\mathcal{U}_H$ is also real-valued when restricted to real arguments. Second, the image H' of a Hamiltonian H from $\mathscr{B}_+$ under both the renormalization operator $\mathcal{R}_\mu$, defined in (4.1), and the operator $\mathfrak{R}$, introduced in (5.3), is also a Hamiltonian with $H'_{(\nu,k,n)} = H'^*_{(-\nu,k,n)}$. This follows from the abovementioned real property of $\mathcal{U}_H$, the fact that $T_{\vartheta^{-2}}$ maps $\mathbb{R}^4$ to itself and the fact that the normalizing functions $\xi(H)$ and $\eta(H)$, given by certain combinations of (real) coefficients of the Hamiltonian H, are also real-valued. Third, the transformations $W_n(H)$, $W'_n(H)$ and $W'(H)$ are similarly real-valued when restricted to real arguments, if $H \in \mathscr{B}_+$. The function $\xi'(H)$, introduced in (8.53)–(8.54), is also real. Lastly, for every $H \in \mathscr{B}_+$ the torus map, described in Theorem 8.12, coincides with the transformation $W'(H) \circ S_0$ when restricted to real arguments, and, hence, is real, and so is the torus map from Theorem 4.9.

Theorem 9.3. *There exists an open disk $\mathcal{D}_\delta$ around zero in $\mathbb{C}$, a neighborhood $\mathcal{U}$ of the family F^*_+ in $\mathcal{F}^\rho_\delta$, and an analytic function $F \mapsto \alpha_F$ from $\mathcal{U}$ to $\mathcal{D}_\delta$, such that the following holds for every $F \in \mathcal{U}$ and $\alpha \in \mathcal{D}_\delta$.*

(a) *There exists a pair r of real positive numbers, and two torus maps $\Gamma^0_{F(\alpha)}$ and $\Gamma^1_{F(\alpha)}$, analytic from $\mathcal{D}_1(r)$ to $\mathcal{D}(\rho)$ and satisfying*

$$(\mathbb{J}\nabla F(\alpha)) \circ \Gamma^i_{F(\alpha)} = c\ \partial_x \Gamma^i_{F(\alpha)} \tag{9.11}$$

for some complex number c.

The torus maps are distinct if $\alpha \neq \alpha_F$ and coincide if $\alpha = \alpha_F$. The maps $(\alpha, F) \mapsto \Gamma^i_{F(\alpha)}$ are analytic on $\mathcal{D}_\delta \times \mathcal{U} \setminus \{(\alpha_F, F) : F \in \mathcal{U}\}$. Moreover, given a family $F \in \mathcal{U}$ the map $\alpha \mapsto \Gamma^i_{F(\alpha)}$ is continuous at α_F, and there exist real positive constants C and d, such that

$$\sup_{(x,y)\in\mathcal{D}(r)} \left| \Gamma^i_{F(\alpha)}(x,y) - \Gamma_{F(\alpha_F)}(x,y) \right| < C\, |\alpha - \alpha_F|^d . \tag{9.12}$$

(b) *If Conjecture* 8.13 *is assumed to hold for every $F \in \mathcal{U}$, or the assumption on $F(\alpha_F)$ of Prop.* 8.14 *is satisfied, then the torus $\Gamma^0_{F(\alpha_F)}$ is shearless.*

(c) *If F is real and α belongs to the interval $(-\delta, \alpha_F]$ the torus maps $\Gamma^i_{F(\alpha)}$ are real.*

(d) *If F is real and α belongs to the interval (α_F, δ) the torus maps $\Gamma^i_{F(\alpha)}$ are not real. Furthermore, for every such F and α there exists a set $\mathfrak{D} = \mathfrak{D}(\alpha, F) \subset \mathcal{D}(\rho)$, such that the range of both $\Gamma^0_{F(\alpha)}$ and $\Gamma^1_{F(\alpha)}$ is contained in $\mathfrak{D}$ and there exist no real torus map satisfying the equation* (9.11) *with the range in $\mathfrak{D}$.*

Proof.
(a) Let δ be the minimum of the δ's appearing in Prop. 7.2 and Prop. 6.2. Then, the map $Y_F^{-1} = \mathbb{P}^u \circ \mathfrak{R} \circ F$ is well-defined and analytic on $\mathcal{D}_\delta$ for every $F \in \mathcal{U}(\beta)$. Given a positive integer n, define $F_n = M^n(F)$ and $(\alpha)_n = \left(Y_{F_0} \circ Y_{F_1} \circ ... \circ Y_{F_{n-1}}\right)^{-1}(\alpha)$ as in Prop. 7.2. The linear map $Y^{-1}_{F^*_+}$ is an expansion of rate ϑ^2 on $\mathcal{D}_\delta \setminus \{0\}$ and $(\alpha)_n(\alpha, F^*_+) = (-\vartheta^2)^n \alpha$. By the argument in the beginning of Section 7, for each $F \in \mathcal{U}(\beta)$ there exists a unique point $\alpha = \alpha_F \in \mathcal{D}_\delta$, such that the sequence $\{(\alpha)_n\}$ converges in $\mathcal{D}_\delta$ (to the point $\alpha = 0$), and the map $F \mapsto \alpha_F$ is analytic from $\mathcal{U}(\beta)$ to $\mathcal{D}_\delta$. The map $Y^{-1} : F \mapsto Y_F^{-1}$ is clearly analytic and bounded on $\mathcal{U}(\beta)$. Therefore, there exists a real positive $b < \vartheta^2$, such that $|(\alpha)_n - (\alpha_F)_n| < (\vartheta^2/b)^n |\alpha - \alpha_F|$ for all $\alpha \in \mathcal{D}_\delta \setminus \{\alpha_F\}$ and every $F \in \mathcal{U}(\beta)$.

Now, fix a family $F \in \mathcal{U}(\beta)$ and let $\alpha \in \mathcal{D}_\delta \setminus \{\alpha_F\}$. By the definition of Y_F^{-1}, for every such α and every real positive α', there exists a positive integer $N' = N'(\alpha, \alpha', F)$, such that $|(\alpha)_{2N'}| > \alpha'$. Choose $\alpha' < (b/\vartheta^2)^2 \delta$, then for every $F \in \mathcal{U}(\beta)$ and every $\alpha \in \mathcal{D}_\delta \setminus \{\alpha = \alpha_F\}$ there exists an integer $N \geq N'$, such that $(\alpha)_{2N} \in \mathcal{D}_{\delta,\alpha'} = \{\alpha \in \mathcal{A}^u(\rho) : \alpha' < |\alpha| < \delta\}$.

Choose a $\beta \leq \beta(\delta, \alpha')$ as in Prop. 6.2. If $N = 0$, then the claim of part (a) follows immediately from Prop. 6.2. Assume, $N \geq 1$, and consider the two distinct torus maps $\Gamma^i_{\mathfrak{R}^{2N}(F(\alpha))} = R_{t_i(\widetilde{\mathfrak{R}^{2N}(F(\alpha))})} \circ \Gamma_{\mathfrak{R}^{2N}(F(\alpha)) \circ R_{t_i(\widetilde{\mathfrak{R}^{2N}(F(\alpha))})}}$, $i = 0, 1$, whose existence is again guaranteed by Prop. 6.2. Let $\mathcal{Q}_H$ signify the projection of the torus map Γ_H on the coordinates (x, y), let $\chi > 1$ be as in Theorem 4.9, and let $|\cdot|$ be the max-norm for vectors in $\mathbb{C}^d$ or $\mathbb{R}^d$. Then, given a pair r of real positive numbers $r_1 < (1 - \chi)\rho_1$ and $r_2 < \rho_2 - \varrho_2$, by Theorem 4.9 there exists a constant c', such that

$$\sup_{(x,y)\in\mathcal{D}_1(r')} \left| \Im \mathcal{Q}^i_{\mathfrak{R}^{2N}(F(\alpha))}(x,y) \right| = \sup_{(x,y)\in\mathcal{D}_1(r')} \left| \Im \mathcal{Q}^i_{\mathfrak{R}^{2N}(F(\alpha)) \circ R_{t_i(\widetilde{\mathfrak{R}^{2N}(F(\alpha))})}}(x,y) \right|$$
$$< |r'| + c' \left\| (\mathbb{I} - \mathbb{E}) \mathfrak{R}^{2N}(F(\alpha)) \circ R_{t_i(\widetilde{\mathfrak{R}^{2N}(F(\alpha))})} \right\|_{(\rho_1, \rho_2, \rho_3, \varrho_4)}$$

$$< |r'| + c' \left\| (\mathbb{I} - \mathbb{E}) \mathfrak{R}^{2N}(F(\alpha)) \right\|_\rho , \tag{9.13}$$

where we have used that by Prop. 6.2 the map $H \mapsto H \circ R_{t(\tilde{H})}$ is bounded of norm less or equal to one from $\mathcal{A}(\rho)$ to $\mathcal{A}(\rho_1, \rho_2, \rho_3, \varrho_4)$ for our choice of δ. By Prop. 7.2, for any ζ, such that $\vartheta^{-1} < \zeta < 1$, there exists a $\beta' = \beta'(\zeta)$, a positive integer K and real positive constants c_2, c_3, a and $s > 1$ such that

$$\left\| (\mathbb{I} - \mathbb{E}) M^{2n}(F)((\alpha)_{2n}) \right\|_\rho < \left(c_2 e^{-a s^{2n}} + c_3 \zeta^{4n} \right) \beta', \tag{9.14}$$

for all $n \geq K$. Choose ζ, so that $\vartheta \zeta^2 < 1$, and increase c_2 and c_3, so that the inequality (9.14) holds for all $n \geq 1$. Reduce β, so that it is smaller than the minimum of the numbers $\beta(\delta, \alpha')$ and $\beta'(\zeta)$ described above. Then,

$$\sup_{(x,y) \in \mathcal{D}_1(r')} \left| \Im \mathcal{Q}^i_{\mathfrak{R}^{2N}(F(\alpha))}(x, y) \right| < |r'| + c' \left(c_2 e^{-a s^{2N}} + c_3 \zeta^{4N} \right) \beta < \frac{\rho'_1}{\vartheta^{2N} \sum_{i=0}^{2N-2} \chi^{-1}} \tag{9.15}$$

for an arbitrary fixed $N \geq 1$, if the numbers r'_1 and r'_2 and β are chosen sufficiently small. Therefore, $\Gamma^i_{\mathfrak{R}^{2N}(F(\alpha))}$ is analytic from $\mathcal{D}_1(r')$ to $\mathcal{D}((\vartheta^n \sum_{i=0}^{n-2} \chi^{-i})^{-1} \rho'_1, \rho_2, \rho_3, \rho_4)$. Then, by part a) of Prop. 9.2, there exist two distinct torus maps $\Gamma^i_{F(\alpha)}$, $i = 0, 1$, analytic from $\mathcal{D}_1(r)$ to $\mathcal{D}(\rho)$ for any $r \leq (r'_1 \vartheta^{2N}, r'_2 \vartheta^{-2N})$ and satisfying the equation (9.11).

Next, we proceed to demonstrate that under the conditions of the Theorem the torus maps constructed above are independent of N; i.e. we want to show that if $\tilde{N}$ is another integer (without loss of generality, $\widetilde{N} > N$), such that $(\alpha)_{2\tilde{N}} \in \mathcal{D}_{\delta,\alpha'}$ then the torus maps $\widetilde{\Gamma}^i_{F(\alpha)} = W'_{2\tilde{N}}(F(\alpha)) \circ T^{2\widetilde{N}}_{\vartheta^{-2}} \Gamma^i_{\mathfrak{R}^{2\widetilde{N}}(F(\alpha))} \circ T^{-2\widetilde{N}}_{\vartheta^{-2}}$ coincide (up to a possible reparametrization of the periodic coordinates q and a reordering of indices i) with the maps $\Gamma^i_{F(\alpha)} = W'_{2N}(F(\alpha)) \circ T^{2N}_{\vartheta^{-2}} \Gamma^i_{\mathfrak{R}^{2N}(F(\alpha))} \circ T^{-2N}_{\vartheta^{-2}}$ on the intersection of their analyticity domains. As it was the case with the proof of Conjecture 8.13, we will show this independence only for the Hamiltonians in $\mathcal{A}(\rho)$ linear in the variable u. We notice that if an integer $\widetilde{N}$ described above exists, then the condition (9.15) with N substituted by $\widetilde{N}$ is satisfied automatically and the existence of the torus maps $\widetilde{\Gamma}^i_{F(\alpha)}$, analytic from $\mathcal{D}_1(r'')$ to $\mathcal{D}(\rho)$ for any $r'' \leq (r'_1 \vartheta^{2\tilde{N}}, r'_2 \vartheta^{-2\tilde{N}})$ and satisfying the equation (9.11), is guaranteed. To demonstrate the claim, it is enough to show that $\Gamma^i_{\mathfrak{R}^{2N}(F(\alpha))} = W'_{2(\tilde{N}-N)}(\mathfrak{R}^{2N}(F(\alpha))) \circ T^{2(\tilde{N}-N)}_{\vartheta^{-2}} \Gamma^i_{\mathfrak{R}^{2\widetilde{N}}(F(\alpha))} \circ T^{-2(\tilde{N}-N)}_{\vartheta^{-2}}$ (up to a possible reordering and a reparametrization) on $\mathcal{D}_1(r)$. Assume the contrary, then by Prop. 9.2 $\mathfrak{R}^{2N}(F(\alpha))$ posses four torus maps: the two distinct maps $\Gamma^i_{\mathfrak{R}^{2N} F(\alpha)}$, analytic from $\mathcal{D}_1(r')$ to $\mathcal{D}((\vartheta^n \sum_{i=0}^{n-2} \chi^{-1})^{-1} \rho'_1, \rho_2, \rho_3, \rho_4)$, and the two distinct maps $\tilde{\Gamma}^i_{\mathfrak{R}^{2N}(F(\alpha))} = W'_{2(\tilde{N}-N)}(\mathfrak{R}^{2N}(F(\alpha))) \circ T^{2(\tilde{N}-N)}_{\vartheta^{-2}} \Gamma^i_{\mathfrak{R}^{2\widetilde{N}}(F(\alpha))} \circ T^{-2(\tilde{N}-N)}_{\vartheta^{-2}}$, analytic from $\mathcal{D}_1(r'_1 \vartheta^{2(\tilde{N}-N)}, r'_2 \vartheta^{-2(\tilde{N}-N)})$ to $\mathcal{D}(\rho)$. By Prop. 6.2 there exist two distinct numbers $t_i(\widetilde{\mathfrak{R}^{2N}(F}(\alpha))$ such that the maps $R_{-t_i(\widetilde{\mathfrak{R}^{2N}(F}(\alpha))} \circ \Gamma^i_{\mathfrak{R}^{2N} F(\alpha)}$ are centered at zero, while $\mathfrak{R}^{2N}(F(\alpha)) \circ R_{t_i(\widetilde{\mathfrak{R}^{2N}(F}(\alpha))} \in \mathcal{W}^0_s$. Obviously, there also exist two translations $\tilde{t}_i(\widetilde{\mathfrak{R}^{2N}(F}(\alpha))$ such that the maps $R_{-\tilde{t}_i(\widetilde{\mathfrak{R}^{2N}(F}(\alpha))} \circ \widetilde{\Gamma}^i_{\mathfrak{R}^{2N} F(\alpha)}$ are centered at zero and, hence, if Conjecture 4.10 is assumed to be true, $\mathfrak{R}^{2N}(F(\alpha)) \circ R_{\tilde{t}_i(\widetilde{\mathfrak{R}^{2N}(F}(\alpha))} \in \mathcal{W}^0_s$. The last two translations $\tilde{t}_i(\widetilde{\mathfrak{R}^{2N}(F}(\alpha))$ must be distinct and different from $t_i(\widetilde{\mathfrak{R}^{2N}(F}(\alpha))$: for if any of the two coincide, for example,

$\tilde{t}_0(\widetilde{\mathfrak{R}^{2N}(F(\alpha)}) = t_1(\widetilde{\mathfrak{R}^{2N}(F(\alpha)})$, then the images of the real torus under the translates $R_{-\tilde{t}_0(\widetilde{\mathfrak{R}^{2N}(F(\alpha)}))} \circ \widetilde{\Gamma}^0_{\mathfrak{R}^{2N}F(\alpha)}$ and $R_{-t_1(\widetilde{\mathfrak{R}^{2N}(F(\alpha)}))} \circ \Gamma^1_{\mathfrak{R}^{2N}F(\alpha)}$, viewed as two-dimensional smooth manifolds embedded in the cylinder $T^2(\mathbb{R}) \times \{0\} \times \mathbb{R}$ (recall that we consider Hamiltonians linear in u), are both centered at zero and, therefore, have to intersect. Hence, these maps, as well as, $\widetilde{\Gamma}^0_{\mathfrak{R}^{2N}F(\alpha)}$ and $\Gamma^1_{\mathfrak{R}^{2N}F(\alpha)}$ coincide (up to a reparametrization) on the intersection of their analyticity domains, which contradicts our assumption. Therefore, there exist four distinct translations — $R_{t_i(\widetilde{\mathfrak{R}^{2N}(F(\alpha)}))}$ and $R_{\tilde{t}_i(\widetilde{\mathfrak{R}^{2N}(F(\alpha)}))}$ — putting the Hamiltonian $\mathfrak{R}^{2N}(F(\alpha))$ on the stable manifold $\mathcal{W}^0_s$. However, this is impossible, since by the Prop. 6.2 the translation families $F(\alpha) \circ R_t$ intersect the stable manifold $\mathcal{W}^0_s$ at no more than two points, if F is in a sufficiently small neighborhood $\mathcal{U}$ of F^*_+, and if α is sufficiently bounded away from zero (which our choice of β and α' above is assumed to guarantee).

The existence and analyticity from $\mathcal{D}_1(r)$ to $\mathcal{D}(\rho)$ of one torus map $\Gamma^0_{F(\alpha)}$ for $\alpha = \alpha_F$ follows immediately from Theorem 8.12.

The maps $(\alpha, F) \mapsto \Gamma^i_{F(\alpha)}$, $i = 0, 4$, are not analytic on the whole of $\mathcal{D}_\delta \times \mathcal{U}$, in particular, it is not analytic at the point $(0, F^*_+)$ as a straightforward application of the definition of the analyticity from Theorem 3.3 shows (with the linear functional chosen as the evaluation functional). However, the analyticity on $\mathcal{D}_\delta \times \mathcal{U} \setminus \{(\alpha_F, F) : F \in \mathcal{U}\}$ is easy to see: it is a consequence of the analyticity of the map $(\alpha, F) \mapsto W'_n(F(\alpha)) - \mathbb{I}$ on $\mathcal{D}_\delta \times \mathcal{U}$ for all n, such that $(\alpha)_k \in \mathcal{D}_\delta$ for all $1 \le k \le n$ (Prop. 9.1), the analyticity of the map $H \mapsto \Gamma_{H^s} - \Gamma_{H^0}$ on $\mathcal{B}^\Gamma_0$ (Theorem 4.3), the analyticity of the map $\alpha \mapsto F(\alpha)$ on $\mathcal{D}_\delta$ for all $F \in \mathcal{U}$ (Definition 6.1) and the analyticity of the map $(\alpha, F) \mapsto R_{t_i(\widetilde{F(\alpha)})}$ on $\mathcal{D}_\delta \times \mathcal{U} \setminus \{(\alpha_F, F) : F \in \mathcal{U}\}$.

To show the continuity of the map $\alpha \mapsto \Gamma^i_{F(\alpha_F)}$ at α_F and the bound (9.12) we proceed as follows:

$$
\begin{aligned}
\sup_{(x,y)\in\mathcal{D}(r)} & \left|\Gamma^i_{F(\alpha)}(x,y) - \Gamma_{F(\alpha_F)}(x,y)\right| = \\
&= \sup_{(x,y)\in\mathcal{D}(r)} \Big| W'_{2N}(F(\alpha)) \circ T^{2N}_{\vartheta^{-2}} \Gamma^i_{\mathfrak{R}^{2n}(F(\alpha))} \circ T^{-2N}_{\vartheta^{-2}}(x,y) \\
&- W'_{2N}(F(\alpha_F)) \circ T^{2N}_{\vartheta^{-2}} \Gamma_{\mathfrak{R}^{2n}(F(\alpha_F))} \circ T^{-2N}_{\vartheta^{-2}}(x,y) \Big| \\
&\le \sup_{(x,y)\in\mathcal{D}(r)} \left| [W'_{2N}(F(\alpha_F)) - W'_{2N}(F(\alpha))] \circ T^{2N}_{\vartheta^{-2}} \Gamma_{\mathfrak{R}^{2N}(F(\alpha_F))} \circ T^{-2N}_{\vartheta^{-2}}(x,y) \right| \\
&+ \sup_{(x,y)\in\mathcal{D}(r)} \Big| W'_{2N}(F(\alpha)) \circ T^{2N}_{\vartheta^{-2}} \Gamma^i_{\mathfrak{R}^{2N}(F(\alpha))} \circ T^{-2N}_{\vartheta^{-2}}(x,y) \\
&- W'_{2N}(F(\alpha)) \circ T^{2N}_{\vartheta^{-2}} \Gamma_{\mathfrak{R}^{2N}(F(\alpha_F))} \circ T^{-2N}_{\vartheta^{-2}}(x,y) \Big| = I_1 + I_2.
\end{aligned} \tag{9.16}
$$

We will estimate the expressions I_1 and I_2 separately.

$$
\begin{aligned}
I_1 &\le \sup_{(x,y)\in\mathcal{D}(r')} \left| [W'_{2N}(F(\alpha_F)) - W'_{2N}(F(\alpha))] \circ T^{2N}_{\vartheta^{-2}} \Gamma_{\mathfrak{R}^{2N}(F(\alpha_F))}(x,y) \right| \\
&\le \left\| [W'_{2N}(F(\alpha_F)) - W'_{2N}(F(\alpha))] \circ T^{2N}_{\vartheta^{-2}} \right\|_{\left(\left(\vartheta^{2N} \sum_{i=0}^{2N-2} \chi^{-i}\right)^{-1} \rho'_1, \rho_2, \rho_3, \rho_4\right)} \\
&\le \| W'_{2N}(F(\alpha_F)) - W'_{2N}(F(\alpha)) \|_{\tilde{\rho}_{\{2N-1, \tilde{\eta}, \chi\}}} .
\end{aligned}
$$

Since by Prop. 9.1 the map $(\alpha, F) \mapsto W'_n(F(\alpha)) - \mathbb{I}$ is analytic (and bounded) from $\mathcal{D}_\delta \times \mathcal{U}(\beta)$ to $\mathcal{A}^4(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}})$, there exists a real positive constant c_{25}, such that $I_1 < c_{25} \, |\alpha_F - \alpha|$. Next, by the same Proposition, the transformation $W'_n(F(\alpha))$ is analytic from $\mathcal{D}\left(\tilde{\rho}_{\{n-1,\tilde{\eta},\chi\}}\right)$ to $\mathcal{D}(\varrho)$, therefore, there exists a real positive constant

c_{26}, such that

$$\begin{aligned}
I_2 &\leq \sup_{(x,y)\in\mathcal{D}(r')} \left| [W'_{2N}(F(\alpha)) \circ T^{2N}_{\vartheta^{-2}} \Gamma^i_{\mathfrak{R}^{2N}(F(\alpha))} - W'_{2N}(F(\alpha)) \circ T^{2N}_{\vartheta^{-2}} \Gamma_{\mathfrak{R}^{2N}(F(\alpha_F))}](x,y) \right| \\
&\leq c_{26} \sup_{(x,y)\in\mathcal{D}(r')} \left| \, T^{2N}_{\vartheta^{-2}} \Gamma^i_{\mathfrak{R}^{2N}(F(\alpha))}(x,y) - T^{2N}_{\vartheta^{-2}} \Gamma_{\mathfrak{R}^{2N}(F(\alpha_F))}(x,y) \right| \\
&\leq c_{26} \sup_{(x,y)\in\mathcal{D}(r')} \left\{ \left| T^{2N}_{\vartheta^{-2}} \left[\Gamma^i_{\mathfrak{R}^{2N}(F(\alpha))} - \Gamma_{H^3_+} \right](x,y) \right| + \left| T^{2N}_{\vartheta^{-2}} \left[\Gamma_{\mathfrak{R}^{2N}(F(\alpha_F))} - \Gamma_{H^3_+} \right](x,y) \right| \right\} \\
&\leq c_{26} \sup_{(x,y)\in\mathcal{D}(r')} \left\{ \vartheta^{2N} \left| \left[\mathcal{Q}^i_{\mathfrak{R}^{2N}(F(\alpha))} - \mathbb{I} \right](x,y) \right| + \vartheta^{-2N} \left| \mathcal{P}^i_{\mathfrak{R}^{2N}(F(\alpha))}(x,y) - (0,0) \right| \right\} \\
&+ c_{26} \sup_{(x,y)\in\mathcal{D}(r')} \left| T^{2N}_{\vartheta^{-2}} \left[\Gamma_{\mathfrak{R}^{2N}(F(\alpha_F))} - \Gamma_{H^3_+} \right](x,y) \right| \\
&\leq 2c_{26}\vartheta^{2N} c' \left(c_2 e^{-as^{2N}} + c_3 \zeta^{4N} \right) \beta' + c_{26} \frac{\max\{\rho_3, \rho_4\}}{\vartheta^{2N}}. \qquad (9.17)
\end{aligned}$$

As it was mentioned in the beginning of the proof, there exists a positive real constant $b < \vartheta^2$, such that $|(\alpha)_{2N} - (\alpha_F)_{2N}| < \left(\vartheta^2/b\right)^{2N} |\alpha - \alpha_F|$, which implies

$$N > \frac{\log\left(\alpha' - |(\alpha_F)_{2N}|\right) - \log|\alpha - \alpha_F|}{2\log\left(\frac{\vartheta^2}{b}\right)}, \qquad (9.18)$$

and, therefore, there exists a $c_{27} > 0$, such that $I_2 \leq c_{27} |\alpha - \alpha_F|^d$, where $d = \min\left\{\log\vartheta, \log\left(\vartheta\zeta^2\right)^{-1}\right\} / \log\left(\vartheta^2/b\right)$ is positive if ζ is chosen so that $\vartheta\zeta^2 < 1$. Combining this with the estimate for I_1, one obtains the bound (9.12).

(b) Let F be in $\mathcal{U}(\beta)$. If Conjecture 8.13 is assumed to hold, then the torus map $\Gamma_{F(\alpha_F)}$ is shearless in the sense of Section 8.

Alternatively, if $F(\alpha_F)$ is linear in u, then by Prop. 8.14 the map $\Gamma_{F(\alpha_F)}$ is shearless.

(c) Let F be real and $\alpha \in (-\delta, \alpha_F)$. To show, that the analytic torus maps $\Gamma^i_{F(\alpha)}$, $i = 0, 1$, of part (a) are real, define a function $\Phi : \mathcal{U} \times \mathcal{D}_\delta \times \mathbb{R} \mapsto \mathcal{A}^u(\rho)$ by setting $\Phi(F, \alpha, t) = (\Pi - \mathbb{I})F(\alpha) \circ R_t$, where $\Pi = (\mathbb{I} + W^0_s) \circ \mathbb{P}^s$ is the projection on the stable manifold, the graph of the map W^0_s. Recall that $t_i(\widetilde{F(\alpha)})$ are the two distinct (generally) complex non-zero solutions of the equation $\Phi(F, \alpha, t) = 0$ and, in particular, $\Phi(F^*_+, \alpha, t_i(\widetilde{F^*_+(\alpha)})) = 0$, where $t_i(\widetilde{F^*_+(\alpha)} = (-1)^i \sqrt{-\alpha/(3\gamma)} \in \mathbb{R}$, $i = 0, 1$. Notice, that $D_3\Phi(F^*_+, \alpha, t)|_{t = t_i(\widetilde{F^*_+(\alpha)})} = 6\gamma t_i(\widetilde{F^*_+(\alpha)}) = (-1)^i 2\sqrt{-3\alpha\gamma}$ and that $\Phi(F, \alpha, t) \in \mathbb{R}$ for every real F, α and t. Then, if β is sufficiently small, by the Implicit Function Theorem for every real $F \in \mathcal{U}(\beta)$ and $\alpha \in (-\delta, -\alpha')$ the equation $\Phi(F, \alpha, t) = 0$ has two distinct *real* non-zero solution (in fact, in the first order, the two real roots are $t_i(\widetilde{F(\alpha)}) = t_i(\widetilde{F^*_+(\alpha)}) + (-1)^i/(2|\alpha|)\mathbb{P}^u(F^*_+(\alpha) - F(\alpha)))$. Therefore, by Prop. 6.2, $F(\alpha) \circ R_{t_i(\widetilde{F(\alpha)})} \in \mathcal{W}^0_s$ is real for all real $F \in \mathcal{U}$ and $\alpha \in (-\delta, -\alpha')$. Finally, if N is the positive integer described in part (a), the maps $\Gamma^i_{\mathfrak{R}^{2N}(F(\alpha))} = R_{t_i(\widetilde{\mathfrak{R}^{2N}(F(\alpha))})} \circ \Gamma_{\mathfrak{R}^{2N}(F(\alpha)) \circ R_{t_i(\widetilde{\mathfrak{R}^{2N}(F(\alpha))})}}$, $i = 0, 1$, are real for all real $F \in \mathcal{U}$ and $\alpha \in (-\delta, \alpha_F)$, and so are the two distinct torus maps $\Gamma^i_{F(\alpha)}$, constructed in part (a).

The reality of the map $\Gamma_{F(\alpha_F)}$ for a real $F \in \mathcal{U}$ is straightforward.

(d) Let F be real and $\alpha \in (\alpha_F, \delta)$. Then, if $N \geq 0$ is the positive integer described in part (a), $(\alpha)_{2N} \in \mathcal{D}^+_{\delta,\alpha'} = \{\alpha \in \mathcal{A}^u(\rho) : |\alpha| < \delta, \Re(\alpha) > \alpha'\}$, and, by Cor. 6.3, the Hamiltonian $F(\alpha)$ has no real ω-torus maps. In particular, the torus maps

$\Gamma^i_{\mathfrak{R}^{2N}(F(\alpha))} = R_{t_i(\mathfrak{R}^{2N}\widetilde{(F(\alpha)))}} \circ \Gamma_{\mathfrak{R}^{2N}(F(\alpha)) \circ R_{t_i(\mathfrak{R}^{2N}\widetilde{(F(\alpha)))}}}$, $i = 0, 1$, are not real. Now, the claim follows by the contrapositive of part (b) of Prop. 9.2: there is no real torus map for the flow of the Hamiltonian $F(\alpha)$, analytic from a complex neighborhood of the real torus to the image of $\mathcal{D}((\sum_{i=0}^{2N-2} \chi^{-i})^{-1}\rho'_1, \rho_2\vartheta^{-2N}, \rho_3\vartheta^{-6N}, \rho_4\vartheta^{-2N})$ under $W'_{2N}(F(\alpha))$ and satisfying the equation (9.11), since there is no real ω-torus map for the flow of $\mathfrak{R}^{2N}(F(\alpha))$ analytic from some complex neighborhood of the real torus to $\mathcal{D}(\rho)$. ☐

This proves our main result.

As a concluding remark, we want to mention that all claims of the the Main Theorem hold in the vicinity of the fixed family F^*_- if one interchanges the parameter intervals of existence and non-existence of the real ω-tori.

10. **Conclusion.**

As it has been demonstrated in earlier works ([41], [45]), and as we have shown in the present paper, the renormalization group transformation can be successfully used to prove KAM-like results for Hamiltonian flows. A salient feature of the renormalization proofs is that they do not rely on various versions of the twist (non-degeneracy) conditions appearing in standard KAM-proofs, usually used to show the existence of a unique solution for a system of algebraic equations for the Fourier coefficients of a conjugating transformation. In our approach, such existence results depend on the possibility of translating a Hamiltonian onto the stable manifold of the renormalization: this manifold plays an outstanding role in the renormalization proofs.

However, it should be noted that the characterization of the stable manifold is far from trivial. It is our belief that its possible description is given by Conjecture 4.10 which will be the subject of our future work, in particular, the proof of this Conjecture would justify the uniqueness of our construction of the invariant ω-tori.

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