Rank one perturbations of Jacobi Matrices with Mixed Spectra

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We study spectral properties of rank one perturbations of selfadjoint operators

$$A_\lambda = A + \lambda < \varphi, \cdot > \varphi$$

in relation to their dependence on the real parameter λ. For semi infinite Jacobi matrices criteria is given that guarantees existence of mixed spectral types for sets of positive measure in the coupling constant λ when the unperturbed operator has a potential which vanishes in a finite interval.

1 Introduction

In this paper, spectral properties of rank one perturbations of semi infinite Jacobi Operators are considered. These perturbations depend on a real parameter λ (coupling constant) and our main concern is the behavior of different spectral types as this parameter λ varies. In particular we address the problem of coexistence of distinct kinds of spectra for sets of λ's of positive Lebesgue measure.

It is known that if the support of one type of spectrum has positive measure, then there will be a set of λ's of positive measure too, such that the corresponding perturbed operator A will have this kind of spectrum somewhere in the real numbers $\mathbb{R}$. Thus a support of positive measure gives rise to a set of coupling constants of positive measure. This result is straightforward from the averaging formula (1) below.

Our aim is to find out more about the structure of this set of coupling constants, particularly about its distribution on the real line. For the case of semi infinite Jacobi matrices with potentials vanishing in a finite interval, we were able to give conditions on the size of intervals of coupling constants, which imply existence of a set of positive measure contained in such intervals, so that the operators generated by rank one perturbations corresponding to points in that set, will have mixed singular and a.c. spectra. These conditions will depend on the measure of the support of the absolutely continuous part and on the lenght of the interval where the potential of the unperturbed operator vanishes.

The main tools we use are bounds on averages of spectral measures, which are associated with selfadjoint operators generated by rank one perturbations. Particularly for the bound from below, we need the hypothesis that the potential vanishes in an interval and a result of Chebyshev et al. which follows from the quadrature formula.

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The paper is organized as follows. In Section 2 we prove some preliminary results which give us the bounds we need. It will be important to construct a finite matrix with eigenvalues that do not depend on a real parameter. In Section 3 we obtain conditions which imply mixed spectra. This are related to estimates on the measure of sets where the spectral density is positive. These results are related to the ones in [5] were the Sturm-Liouville case was studied.

2 Preliminary results.

We consider rank one perturbations of self adjoint operators

$$A_\lambda = A + \lambda < \varphi, \cdot > \varphi$$

where φ is a cyclic vector for A.

In particular we are interested in rank one perturbations of Jacobi Matrices. All the Jacobi operators considered in this paper will be assumed to be selfadjoint. We shall denote by ρ_λ the spectral measure for φ and A_λ, so that

$$F_\lambda(z) := < \varphi, (A_\lambda - z)^{-1}\varphi > = \int\limits_{\mathbb{R}} \frac{\mathrm{d}\rho_\lambda(x)}{x - z}$$

we set $F = F_0$.

Define

$$\Lambda_M = \{E / \operatorname{Im} F(E + i0) > M\}$$

where

$$F(E + i0) := \lim_{y \downarrow 0} F(E + iy)$$

which exists and is finite for a.e. E.

For any Borel set M it happens that

$$\int\limits_{\mathbb{R}} \rho_\lambda(M)\mathrm{d}\lambda = |M| \tag{1}$$

(see [9, Theorem 1.8] or [8]), where $|\cdot|$ denotes Lebesgue measure. When the integration takes place over the interval (α, β) then the following result holds (see [9, Theorem 1.12]).

Lemma 2.1 *Let $\alpha < \beta$. Then if B is a Borel set*

$$\int_\alpha^\beta \rho_\theta(B)\mathrm{d}\theta = \frac{1}{\pi} \int\limits_B \arg\left(\frac{1 + \beta F(E + i0)}{1 + \alpha F(E + i0)}\right) \mathrm{d}E$$

We shall need this Lemma in order to prove the following:

Lemma 2.2 *Let $\alpha < \beta$, $\alpha\beta > 0$. Then*

$$\int_\alpha^\beta \rho_\theta\Big(I \cap \Lambda_M\Big)\mathrm{d}\theta \leq \frac{2}{\pi} \arctan\left(\frac{1}{2M}\Big(\frac{1}{\alpha} - \frac{1}{\beta}\Big)\right) \Big|I \cap \Lambda_M\Big|$$

Proof. Let

$$w = Tz = \frac{\beta z + 1}{\alpha z + 1}$$

For each $M > 0$, T maps the half plane $\operatorname{Im} z > M$ onto the disk

$$\left(\frac{\beta}{\alpha} - x\right)^2 + \left(y - \frac{\beta - \alpha}{2\alpha^2 M}\right)^2 < \left(\frac{\beta - \alpha}{2\alpha^2 M}\right)^2$$

From here it follows that if $\operatorname{Im} z > M$ then $\arg w \leq 2 \arctan\left(\frac{1}{2M}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\right)$

Using Lemma 2.1 we obtain

$$\begin{aligned}
\int_{\alpha}^{\beta} \rho_\theta\Big(I \cap \Lambda_M\Big) \mathrm{d}\theta &= \frac{1}{\pi} \int_{I \cap \Lambda_M} \arg T\Big((F(E + i0)\Big) \mathrm{d}E \\
&\leq \frac{2}{\pi} \int_{I \cap \Lambda_M} \arctan\left(\frac{1}{2M}\Big(\frac{1}{\alpha} - \frac{1}{\beta}\Big)\right) \mathrm{d}E \\
&= \frac{2}{\pi} \arctan\left(\frac{1}{2M}\Big(\frac{1}{\alpha} - \frac{1}{\beta}\Big)\right) \Big|I \cap \Lambda_M\Big|
\end{aligned}$$

□

The estimate given in Lemma 2.2, holds for general selfadjoint operators generated by rank one perturbations. To get the lower bound we need (Lemma 2.4), we shall have to consider particular Jacobi operators. First we will find a finite tridiagonal matrix which has eigenvalues that do not depend on the first entry.

Let us consider the linear transformation $J_N^q : \mathbb{R}^N \longrightarrow \mathbb{R}^N, q \in \mathbb{R}$ given by

$$J_N^q := \begin{pmatrix} q & 1 & & & & \\ 1 & 0 & 1 & & & \bigcirc \\ & 1 & 0 & 1 & & \\ & & & \ddots & & \\ \bigcirc & & & 1 & 0 & 1 \\ & & & & 1 & \frac{1}{q} \end{pmatrix} \tag{2}$$

The eigenvectors and the corresponding eigenvalues are

$$\begin{pmatrix} 1 \\ q^{-1} \\ q^{-2} \\ \vdots \\ q^{-(N-1)} \end{pmatrix} \longleftrightarrow q + \frac{1}{q}\ , \quad P_j := \begin{pmatrix} P_j(0) \\ P_j(1) \\ \vdots \\ P_j(N-1) \end{pmatrix} \longleftrightarrow 2\cos\frac{\pi j^*}{N}$$

where

$$P_j(n) := \frac{\sin\frac{\pi j^*}{N}(n+1) - q \sin\frac{\pi}{N} j^* n}{\sin\frac{\pi}{N} j^*}$$

$$j = 1, 2, \ldots, N-1 \quad n = 0, 1, 2, \ldots, N-1 \text{ and } j^* = N - j$$

We have normalized so that $P_j(0) = 1$. One way of finding the eigenvectors above, is to look for solutions of the form

$$P(n) = Az^n + Bz^{-n}$$

for complex z where the eigenvalues t will be related to z by $t = \Big(z + \frac{1}{z}\Big)$.

Notice that all eigenvalues of J_N^q with the exception of one, do not depend on q. This will be relevant for what follows.

We shall need to calculate the norm of the vector P_j. Using the identity $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, after some elementary calculations we obtain

$$\Big\|P_j\Big\|^2 = \sum_{n=0}^{N-1} P_j^2(n) = \frac{N}{2\Big(\sin \frac{\pi}{N} j^*\Big)^2} \Big| e^{i\frac{\pi}{N}j^*} - q \Big|^2 \tag{3}$$

$$j = 1, \ldots, N-1 \quad , j^* = N - j$$

We shall state now an old and interesting result.

Let J_N be the finite Jacobi Matrix

$$J_N := \begin{pmatrix} a_0 & b_0 & & & & \bigcirc \\ b_0 & a_1 & b_1 & & & \\ & b_1 & a_2 & b_2 & & \\ & & & \ddots & & \\ \bigcirc & & & & a_{N-2} & b_{N-2} \\ & & & & b_{N-2} & a_{N-1} \end{pmatrix}$$

Let us denote the eigenvectors by $P_j = \big(P_j(0), P_j(1), \ldots, P_j(N-1)\big)^T$, and the corresponding eigenvalues by $\lambda_j, j = 0, 1, \ldots, N-1$. We shall assume the P_j normalized so that $P_j(0) = 1$ for all j and that $\lambda_0 < \lambda_1 < \ldots < \lambda_{N-1}$.

As a consequence of the quadrature formula the following result of Chebyshev, Markov, Stieltjes holds, see [1, Theorem 2.5.4], [6].

Theorem 2.3 *Let σ be the spectral funcion of any semi infinite Jacobi Matrix with fixed given entries $a_o, \ldots, a_{N-2}, b_0, \ldots, b_{N-2}$. Let $\|P_\ell\| = \Big(\sum_{n=0}^{N-1} P_\ell^2(n)\Big)^{1/2}$ where P_ℓ are the eigenfunctions of J_N as defined above with corresponding eigenvalues λ_ℓ. Then the following inequalities hold:*

$$\sigma\Big(\Big(\lambda_j, \lambda_{i+1}\Big)\Big) \geq \sum_{\ell=j+1}^{i} \Big\|P_\ell\Big\|^{-2} \geq \sigma\left(\Big[\lambda_{j+1}, \lambda_i\Big]\right)$$

$j < i, \quad i, j \in \{0, 1, \ldots, N-2\}$.

We remark that it is not necessary for the semi infinite matrix to have the same entry a_{N-1} as J_N. In [1] this is stated as an hypotheses of the theorem but it is not used in the proof.

We are ready now to prove the bound from below:

Lemma 2.4 *Let J be a semi infinite Jacobi Matrix*

$$J = \begin{pmatrix} 0 & 1 & & & & & & & \\ 1 & 0 & 1 & & & & & & \\ & \cdots & & & & & & & \bigcirc \\ & & \cdots & & & & & & \\ & & & \cdots & & & & & \\ & & & 1 & 0 & 1 & & & \\ & & & & 1 & a_{N-1} & b_{N-1} & & \\ & & & & & b_{N-1} & a_N & b_N & \\ & & & & & & \cdots & & \\ \bigcirc & & & & & & & \cdots & \\ & & & & & & & & \cdots \end{pmatrix}$$

$$J : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$$

such that

$$a_i = 0 \text{ and } b_i = 1 \text{ for } i = 0, 1, 2, \dots, N-2$$

Let $I = \left(2\cos\frac{\pi(j^*+1)}{N}, 2\cos\frac{\pi(j^*-1)}{N}\right)$ *where* $j^* \in \{2, 3, \dots, N-2\}$ *is fixed.*
Then

$$\int_\alpha^\beta \rho_q(I)\mathrm{d}q \geq \frac{2}{N}\sin\frac{\pi j^*}{N} \int_{\frac{\alpha - \cos\frac{\pi}{N}j^*}{\sin\frac{\pi}{N}j^*}}^{\frac{\beta - \cos\frac{\pi}{N}j^*}{\sin\frac{\pi}{N}j^*}} \frac{\mathrm{d}y}{y^2+1}$$

where $\rho_q(I)$ is the spectral measure of the matrix

$$J_q = J + q < \delta_0, \cdot > \delta_0$$

applied to the interval I.

Proof. Consider the perturbed operator J_q and apply Theorem 2.3. Then we know that

$$\rho_q\Big(\Big(\lambda_{j-1}, \lambda_{j+1}\Big)\Big) \geq \Big\|P_j\Big\|^2$$

In our case, since $a_0, \dots, a_{N-2}$ and $b_0, \dots, b_{N-2}$ coincide with the entries of J_N^q see (2), we can take λ_j and P_j to be the eigenvalues and the corresponding eigenvectors of J_N^q. If $j^* = N - j$ then $I = (\lambda_{j-1}, \lambda_{j+1})$.
According to (3) we have

$$\int_\alpha^\beta \rho_q\Big(\Big(\lambda_{j-1}, \lambda_{j+1}\Big)\Big)\mathrm{d}q \geq \frac{2\Big(\sin\frac{\pi}{N}j^*\Big)^2}{N} \int_\alpha^\beta \frac{\mathrm{d}q}{\Big|e^{i\frac{\pi}{N}j^*} - q\Big|^2} = \frac{2}{N}\sin\frac{\pi j^*}{N} \int_{\frac{\alpha - \cos\frac{\pi}{N}j^*}{\sin\frac{\pi}{N}j^*}}^{\frac{\beta - \cos\frac{\pi}{N}j^*}{\sin\frac{\pi}{N}j^*}} \frac{\mathrm{d}y}{y^2+1}$$

□

Notice that the operator J_q is the same as J, just with a change in the first entry.
Now we can prove next theorem.

3 Conditions which imply mixed spectra and examples.

With the tools developed in last section it is not hard to prove the following result.

Theorem 3.1 *Let J_q be a semi infinite Jacobi Matrix as in the lemma above and let*

$$I = \left(2\cos\pi\frac{(j^*+1)}{N}, 2\cos\pi\frac{(j^*-1)}{N}\right)$$

$$j^* \in \{2, 3, \ldots, N-2\}$$

If

$$\frac{2}{N}\sin\frac{\pi j^*}{N}\int\limits_{\frac{\alpha-\cos\frac{\pi}{N}j^*}{\sin\frac{\pi}{N}j^*}}^{\frac{\beta-\cos\frac{\pi}{N}j^*}{\sin\frac{\pi}{N}j^*}}\frac{\mathrm{d}y}{y^2+1} > \frac{2}{\pi}|I\cap\Lambda_M|\int\limits_{0}^{\frac{1}{2M}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)}\frac{\mathrm{d}y}{y^2+1}$$

then

$$\int_\alpha^\beta \rho_q(I\cap\Lambda_M^C)\mathrm{d}q > 0$$

Proof. The upper bound given by Lemma 2.2 together with the lower bound of Lemma 2.4 and the hypotheses of the theorem give us

$$\int_\alpha^\beta \rho_q(I)\mathrm{d}q > \int_\alpha^\beta \rho_q(I\cap\Lambda_M)\mathrm{d}q$$

and since

$$\int\limits_\alpha^\beta \rho_q(I)\mathrm{d}q = \int_\alpha^\beta \rho_q(I\cap\Lambda_M^C)\mathrm{d}q + \int_\alpha^\beta \rho_q(I\cap\Lambda_M)\mathrm{d}q$$

we obtain

$$\int_\alpha^\beta \rho_q(I\cap\Lambda_M^C)\mathrm{d}q > 0$$

and the theorem is proved.

□

Examples

a) Take $j^* = 2, N = 4, M = 0$. In this case the condition

$$\int_\alpha^\beta \frac{\mathrm{d}y}{y^2+1} > 2|I\cap\Lambda_0|$$

where $I = \left(-\sqrt{2},\sqrt{2}\right)$, implies singular spectrum in I for a set of coupling constants of positive measure which lies in the interval (α, β).

b) Let us take a set $B \subset [-2, 2] = I$ with $|B| = 1$ such that B and B^C are essentially dense in I, that is, for every subinterval $J \subset I$ we have $|B \cap J| > 0$ and the same for B^C. Now consider the measure $\mathrm{d}\rho(x) = \chi_B(x)\mathrm{d}x$ where χ denotes the characteristic function of B, ($\chi_B(x) = 1$ if $x \in B$ and zero otherwise).

   Once we have the measure ρ, we can construct a Jacobi Matrix J_ρ such that ρ is the spectral function of J_ρ. In fact if we orthonormalize the monomials $1, \lambda, \lambda^2, \lambda^3, \ldots$ with respect to the scalar product

$$\langle f, g\rangle_\rho = \int\limits_{\mathbb{R}} f(x)\overline{g(x)}\mathrm{d}\rho(x)$$

   to obtain the orthonormal family of polynomials $1 \equiv P_0(\lambda), P_1(\lambda), \ldots, P_n(\lambda), \ldots$ where P_n is of degree n, then the coefficients a_k and b_k of J_ρ can be written as

$$\begin{array}{rcl} a_k & = & \langle xP_k(x), P_k(x)\rangle_\rho \\ b_k & = & \langle xP_k(x), P_{k+1}(x)\rangle_\rho \end{array}$$

   (see [2]).

From [4, Example 1] we know that $J_q := \;J_\rho + q < \delta_0, \cdot >$ has mixed spectrum for every $q \neq 0$ in every subinterval of I.

If we add a finite rank perturbation R such that the Jacobi Matrix

$$J_R = J_\rho + R$$

satisfies $a_i = 0$ and $b_i = 1$ for $i = 1, 2, \ldots, N-2$, then the absolutely continuous part of J will be unitarily equivalent to the absolutely continuous part of J_ρ by Kato Rosenblum's theorem see [7, Chapter 10, Theorem 4.3].

Therefore a support for the a.c. spectrum of J_R^q is B, for every $q \in \mathbb{R}$, and a support for the singular spectrum of J_R^q is B^C for a.e. $q \in \mathbb{R}$, where

$$J_R^q = J_R + q < \delta_0, \cdot > \delta_0.$$

Since $|B^C| > 0$, it follows from (1) the existence of a set C of positive Lebesgue measure such that if $q \in C$, then J_N^q has singular spectrum somewhere in $\mathbb{R}$ (see [4, Cororally 2.7] for example).

Using Theorem 3.1 above, much more can be said about the set C and the location of singular spectrum. If an interval I of the type considered in the theorem is fixed, then we obtain an estimate on the length of (α, β) which implies $|(\alpha, \beta) \cap C| > 0$. This estimate depends on the size of B. Moreover for $q \in (\alpha, \beta) \cap C$ the operator J_R^q will have singular spectrum in I.

Remarks

1) The theorem allow us to conclude existence of singular spectrum in some intervals I. However it does not tell us that this kind of spectra is everywhere in I. To give conditions which guarantee this is an open question as far as we know.

2) The examples given above use only a particular case of Theorem 3.1, namely when $M = 0$.

Now we turn to the more general case of an abstract selfadjoint operator A and its family of rank one perturbations

$$A_\lambda = A + \lambda < \varphi, \cdot > \varphi \ , \quad \lambda \in \mathbb{R} \quad \|\varphi\| < \infty$$

where φ is a cyclic vector for A.

Using just Lemma 2.1 it is possible to give a simple criterion to have singular spectrum somewhere in $\mathbb{R}$.

From Lemma 2.1 we have, if $0 < \alpha < \beta$

$$\begin{aligned} \int_\alpha^\beta \rho_\theta(\mathbb{R}) \mathrm{d}\theta &= \frac{1}{\pi} \int\limits_{\mathbb{R}} \arg\Big(1 + \beta F(E + i0)\Big) - \arg\Big(1 + \alpha F(E + i0)\Big) \mathrm{d}E \\ &= |A_{\alpha\beta}| + \frac{1}{\pi} \int\limits_{\Lambda} f_{\alpha\beta}(E) \mathrm{d}E \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_{\alpha\beta} &:= \Big\{ E / -\frac{1}{\alpha} < F(E + i0) < -\frac{1}{\beta} \Big\} \\ \Lambda &:= \Big\{ E / \operatorname{Im} F(E + i0) > 0 \Big\} \\ f_{\alpha\beta}(E) &= \arg\Big(1 + \beta F(E + i0)\Big) - \arg\Big(1 + \alpha F(E + i0)\Big) \end{aligned}$$

and as before

$$F(z) = < \varphi, (A - z)^{-1} \varphi >$$

Observe that Λ supports the a.c. part of A_λ for every λ and $A_{\alpha\beta}$ supports the singular part of A_λ for a.e. $\lambda \in (\alpha, \beta)$. See [9] or [3] for example.

Since $\rho_\theta(\Delta) = < E_\theta(\Delta)\varphi, \varphi >$ we know that $\rho_\theta(\mathbb{R}) = \|\varphi\|^2$ for all $\theta \in \mathbb{R}$ and from (4) we obtain

$$\begin{aligned} |A_{\alpha\beta}| &= (\beta - \alpha)\|\varphi\|^2 - \frac{1}{\pi} \int\limits_{\Lambda} f_{\alpha\beta}(E) \mathrm{d}E \\ &\geq (\beta - \alpha)\|\varphi\|^2 - |\Lambda| \end{aligned} \tag{5}$$

The next result then follows

Theorem 3.2 *If*

$$\beta - \alpha > \frac{|\Lambda|}{\|\varphi\|^2}$$

then the family of rank one perturbations

$$A_\lambda = A + \lambda < \varphi, \cdot > \varphi$$

has some singular spectrum for $\lambda \in B$, *where* $|B| > 0$ *and* $B \subset (\alpha, \beta)$

Proof. The condition $\beta - \alpha > \frac{|\Lambda|}{\|\varphi\|}^2$ implies $|A_{\alpha\beta}| > 0$ by (5). Since $\int_\alpha^\beta \rho_\theta(A_{\alpha\beta})\mathrm{d}\theta = |A_{\alpha\beta}|$ and $A_{\alpha\beta}$ supports the singular part, the results follows.

□

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